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Singularly perturbed partly dissipative reaction–diffusion systems in case of exchange of stabilities

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1. Introduction

We consider initial–boundary value problems for singularly perturbed partly dissipative reaction–diffusion systems of the type

$$\varepsilon^2\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right)=g(u,v,x,t,\varepsilon),$$
$$\frac{\partial v}{\partial t}=f(u,v,x,t,\varepsilon), \tag{1.1}$$

where $u, v, x \in R$, ε is a small positive parameter. Partly dissipative systems can be used to model reaction–diffusion processes in different fields (chemical kinetics, biology, astrophysics) when the effect of diffusion of one of the species is negligible (see, e.g., [4–7,10–12]).

If we assume that the so-called *degenerate equation* to (1.1)

$$g(u,v,x,t,0)=0 \tag{1.2}$$

has an isolated simple root with respect to u, then, according to the standard theory of singularly perturbed systems (see, e.g., [13,14]), this root essentially

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determines the behavior of the u-solution component (fast component) of the initial–boundary value problem under consideration provided some additional conditions are satisfied.

In this paper we assume that the degenerate equation has two roots with respect to u which intersect in some smooth surface. Such situation is quite natural in applications, especially when we look for a positive solution under the assumptions that $u \equiv 0$ is a trivial solution (see [2,9]).

As a motivating example we consider the following initial–boundary value problem:

$$
\begin{aligned}
&\varepsilon^2\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right)=g(u,v,x,t,\varepsilon)\equiv -u(u-v+x+t+2)+\varepsilon I(x,t),\\
&\frac{\partial v}{\partial t}=f(u,v,x,t,\varepsilon)\equiv u+2,\\
&(x,t)\in Q:=\left\{(x,t)\in R^2\colon\ 0<x<1,\ 0<t\leqslant T\right\},\quad T>2,\\
&\frac{\partial u}{\partial x}(0,t,\varepsilon)=\frac{\partial u}{\partial x}(1,t,\varepsilon)=0\quad \text{for } 0<t\leqslant T,\\
&u(x,0,\varepsilon)=u^0(x)>0,\quad v(x,0,\varepsilon)=v^0(x)\equiv 1\quad \text{for } 0\leqslant x\leqslant 1,
\end{aligned}
\tag{1.3}
$$

where $I:\overline{Q}\to R$ is smooth and positive, u^0 is a smooth function on $0\leqslant x\leqslant 1$. Here, u can be considered as the concentration of some reacting species ($u\geqslant 0$), while v is some auxiliary variable (sometimes the difference of two species) which can be positive and negative, the term $u(u-v+x+t+2)/\varepsilon^2$ characterizes the reaction rate (very fast reaction), $I(x,t)/\varepsilon$ represents the input rate of the species u.

The degenerate equation to (1.3)

$$-u(u-v+x+t+2)=0$$

has two roots

$$u=\varphi_1(v,x,t)\equiv 0\quad \text{and}\quad u=\varphi_2(v,x,t)\equiv v-x-t-2 \tag{1.4}$$

intersecting in the smooth surface

$$v=s(x,t)\equiv x+t+2. \tag{1.5}$$

Thus, the standard theory of singularly perturbed systems cannot be applied near this surface.

Any root $u=\varphi(v,x,t)$ of the degenerate equation (1.2) represents a family of equilibria of the so-called *associated equation* to (1.1)

$$\frac{du}{d\tau}=g(u,v,x,t,0),$$

where v,x,t have to be considered as parameters. Hence, the assumption of the existence of two intersecting roots of the degenerate equation implies an exchange of stabilities for the corresponding families of equilibria of the associated equation.

This paper is concerned with the existence and asymptotic behavior in ε of the solution of some initial–boundary value problem to system (1.1) in case of exchange of stabilities. The proof of our results is based on the method of asymptotic lower and upper solutions. To construct these solutions we exploit the structure of the solution set of the degenerate equation and their stability properties as equilibria of the associated equation.

The goal of this paper is to derive conditions which imply the phenomenon of immediate exchange of stabilities, that is, the behavior of the fast solution component (u-component) is determined at any time by the stable root of the degenerate equation (1.2). This excludes the occurrence of interior layers (spikes) as well as a delayed exchange of stabilities where the u-component follows for some $O(1)$-time interval the unstable root of the degenerate equation. The results of this paper are extensions of corresponding results in [2,3,8,9] for ordinary and parabolic differential equations.

The paper is organized as follows: In Section 2 we formulate our assumptions and construct the so-called *composed stable solution* which plays a crucial role for the formulation as well as for the proof of our main result. At the same time we consider a simple motivating example where all assumptions can be checked analytically and where the composed stable solution can be constructed explicitly. The definition of ordered lower and upper solutions will also be given in Section 2. Section 3 contains the detailed proof of our result.

2. Formulation of the problem. Assumptions

We study the singularly perturbed nonlinear initial–boundary value problem

$$
\begin{aligned}
&\varepsilon^2\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right)=g(u,v,x,t,\varepsilon),\\
&\frac{\partial v}{\partial t}=f(u,v,x,t,\varepsilon),\\
&(x,t)\in Q:=\left\{(x,t)\in R^2\colon\ 0<x<1,\ 0<t\leqslant T\right\},\\
&\varepsilon\in I_{\varepsilon_0}:=\{\varepsilon\in R\colon\ 0<\varepsilon\leqslant\varepsilon_0\ll 1\},\qquad (2.1)\\
&\frac{\partial u}{\partial x}(0,t,\varepsilon)=\frac{\partial u}{\partial x}(1,t,\varepsilon)=0\quad\text{for } 0<t\leqslant T,\\
&u(x,0,\varepsilon)=u^0(x),\quad v(x,0,\varepsilon)=v^0(x)\quad\text{for } 0\leqslant x\leqslant 1\qquad (2.2)
\end{aligned}
$$

under the following assumptions:

(A_0) $f,g\in C^2(\overline{D},R)$, where $D:=R\times R\times Q\times I_{\varepsilon_0}$, $u^0,v^0\in C^2([0,1],R)$.

If we set $\varepsilon = 0$ in (2.1), then we get the *degenerate system*

$$\begin{aligned} &0 = g(u, v, x, t, 0), \\ &\frac{dv}{dt} = f(u, v, x, t, 0). \end{aligned} \tag{2.3}$$

Concerning the solution set of the *degenerate equation*

$$g(u, v, x, t, 0) = 0 \tag{2.4}$$

we assume

(A_1) Equation (2.4) has exactly two roots $u = \varphi_1(v, x, t)$ and $u = \varphi_2(v, x, t)$ defined for $(v, x, t) \in \bar{I}_v \times \overline{Q}$, where I_v is some open bounded interval, φ_1 and φ_2 are twice continuously differentiable.

From assumption (A_1) we get that the relations

$$\begin{aligned} &g\big(\varphi_i(v, x, t), v, x, t, 0\big) \equiv 0, \\ &g_u\big(\varphi_i(v, x, t), v, x, t, 0\big)\frac{\partial \varphi_i}{\partial v}(v, x, t) + g_v\big(\varphi_i(v, x, t), v, x, t, 0\big) \equiv 0 \end{aligned} \tag{2.5}$$

hold for $(v, x, t) \in \bar{I}_v \times \overline{Q}$, and for $i = 1, 2$.

The following assumption expresses the property that the surfaces $u = \varphi_1(v, x, t)$ and $u = \varphi_2(v, x, t)$ intersect in a smooth surface whose projection into the (v, x, t)-space can be described by $v = s(x, t)$:

(A_2) There exists a smooth function $s : \overline{Q} \to I_v$ such that

$$\begin{aligned} &\varphi_1(v, x, t) = \varphi_2(v, x, t) \quad \text{for } v = s(x, t), \\ &\varphi_1(v, x, t) > \varphi_2(v, x, t) \quad \text{for } v < s(x, t), \\ &\varphi_1(v, x, t) < \varphi_2(v, x, t) \quad \text{for } v > s(x, t). \end{aligned}$$

We note that the case of intersecting roots of the degenerate equation does not fit into the standard theory of singularly perturbed systems (see, e.g., [13,14]).

The differential equation

$$\frac{du}{d\tau} = g(u, v, x, t, 0), \tag{2.6}$$

where v, x, t are considered as parameters, is said to be the associated equation to (2.1). It follows from hypothesis (A_1) that $u = \varphi_i(v, x, t)$, $i = 1, 2$, are families of equilibria of (2.6). The families φ_i are stable (unstable) if $g_u(\varphi_i, v, x, t, 0)$ is negative (positive). For definiteness we assume the following stability behavior:

(A_3) For $(x, t) \in \overline{Q}$ it holds

$$g_u\big(\varphi_1(v,x,t),v,x,t,0\big)<0, \qquad g_u\big(\varphi_2(v,x,t),v,x,t,0\big)>0$$
$$\text{for } v<s(x,t),$$
$$g_u\big(\varphi_1(v,x,t),v,x,t,0\big)>0, \qquad g_u\big(\varphi_2(v,x,t),v,x,t,0\big)<0$$
$$\text{for } v>s(x,t).$$

From assumption (A_3) we get that $g_u(u,v,x,t,0)$ changes its sign when the point (v,x,t) crosses the surface $v=s(x,t)$ where $u=\varphi_1(v,x,t)$ and $u=\varphi_2(v,x,t)$ intersect. This sign change of g_u implies an exchange of stabilities of the families of equilibria of the associated Eq. (2.6). Moreover, we have for $(x,t)\in\overline{Q}$

$$g_u\big(\varphi_1\big(s(x,t),x,t\big),s(x,t),x,t,0\big)$$
$$\equiv g_u\big(\varphi_2\big(s(x,t),x,t\big),s(x,t),x,t,0\big)\equiv 0.$$

Assumptions (A_1)–(A_3) express our key hypothesis: the roots of the degenerate equation (2.4) intersect transversally which implies an exchange of stabilities of the families of equilibria of the associated equation (2.6).

Now we consider our example (1.3) and verify the hypotheses (A_0)–(A_3). It is obvious that the assumptions (A_0) and (A_1) are fulfilled. From (1.4) and (1.5) it follows that the inequalities $\varphi_1(v,x,t)>\varphi_2(v,x,t)$ and $\varphi_1(v,x,t)<\varphi_2(v,x,t)$ hold for $v<s(x,t)$ and $v>s(x,t)$, respectively, that is, assumption (A_2) is valid. From (1.3) and (1.4) we get

$$g_u\big(\varphi_1(v,x,t),x,t,0\big)\equiv v-x-t-2\equiv -g_u\big(\varphi_2(v,x,t),x,t,0\big).$$

Obviously, we have for $(x,t)\in\overline{Q}$

$$g_u\big(\varphi_1(v,x,t),x,t,0\big)<0, \qquad g_u\big(\varphi_2(v,x,t),x,t,0\big)>0$$
$$\text{for } v<s(x,t),$$
$$g_u\big(\varphi_1(v,x,t),x,t,0\big)>0, \qquad g_u\big(\varphi_2(v,x,t),x,t,0\big)<0$$
$$\text{for } v>s(x,t),$$

i.e., assumption (A_3) holds.

In the sequel we construct the so-called *composed stable solution* to the degenerate system (2.3) which will be used to construct lower and upper solutions to the initial–boundary value problem (2.1)–(2.2).

The function $\varphi(v,x,t)$ defined by means of the stable roots $\varphi_1(v,x,t)$ and $\varphi_2(v,x,t)$,

$$\varphi(v,x,t)=\begin{cases}\varphi_1(v,x,t) & \text{for } v\leqslant s(x,t),\\ \varphi_2(v,x,t) & \text{for } v\geqslant s(x,t),\end{cases} \tag{2.7}$$

is called the *stable root of Eq.* (2.4) *in* $\bar{I}_v\times\overline{Q}$.

If we replace u in the second equation of the degenerate system (2.3) by $\varphi(v,x,t)$ we get the *reduced equation* to system (2.1)

$$\frac{\partial v}{\partial t} = f\big(\varphi(v,x,t), v, x, t, 0\big), \tag{2.8}$$

where x has to be considered as a parameter.

In what follows we consider for Eq. (2.8) the initial value problem

$$v(x,0) = v^0(x), \tag{2.9}$$

where we assume $v^0(x) \neq s(x,0)$ for $0 \leqslant x \leqslant 1$. First we consider the case

$$v^0(x) < s(x,0) \quad \text{for } 0 \leqslant x \leqslant 1. \tag{2.10}$$

Then, according to (2.7), the reduced initial value problem (2.8), (2.9) reads

$$\frac{\partial v}{\partial t} = f\big(\varphi_1(v,x,t), v, x, t, 0\big), \qquad v(x,0) = v^0(x). \tag{2.11}$$

Concerning this initial value problem we suppose

(A_4) There exists a function $t_c \in C^2([0,1],(0,T))$ such that for $x \in [0,1]$ the initial value problem (2.11), where $v^0(x)$ satisfies (2.10), has a unique solution $v = v_1(x,t)$ defined on $0 \leqslant t \leqslant t_c(x)$ with values in I_ν and satisfying

$$\begin{aligned} &v_1(x,t) < s(x,t) \quad \text{for } 0 \leqslant t < t_c(x), \\ &v_1(x,t) = s(x,t) \quad \text{for } t = t_c(x). \end{aligned} \tag{2.12}$$

Assumption (A_4) says that the surfaces $v = v_1(x,t)$ and $v = s(x,t)$ intersect in a curve whose projection into $\overline{Q}$ can be described by $t = t_c(x)$. We denote this curve by $\mathcal{C}$ which decomposes $\overline{Q}$ into the subsets Q_1 and Q_2 where Q_1 consists of all points $(x,t) \in \overline{Q}$ satisfying $t < t_c(x)$, $Q_2 = \overline{Q} \backslash \overline{Q}_1$ (see Fig. 1).

Next, for $0 \leqslant x \leqslant 1$, we consider the initial value problem

$$\begin{aligned} &\frac{\partial v}{\partial t} = f\big(\varphi_2(v,x,t), v, x, t, 0\big) \quad \text{for } t_c(x) < t \leqslant T, \\ &v\big(x, t_c(x)\big) = s\big(x, t_c(x)\big). \end{aligned} \tag{2.13}$$

Concerning (2.13) we assume

(A_5) For $x \in [0,1]$, the initial value problem (2.13) has a unique solution $v = v_2(x,t)$ defined on $t_c(x) \leqslant t \leqslant T$ with values in I_ν such that

$$v_2(x,t) > s(x,t) \quad \text{for } (x,t) \in Q_2. \tag{2.14}$$

Now we define the function $\hat{v}(x,t)$ by

$$\hat{v}(x,t) = \begin{cases} v_1(x,t) & \text{for } (x,t) \in \overline{Q}_1, \\ v_2(x,t) & \text{for } (x,t) \in \overline{Q}_2. \end{cases} \tag{2.15}$$

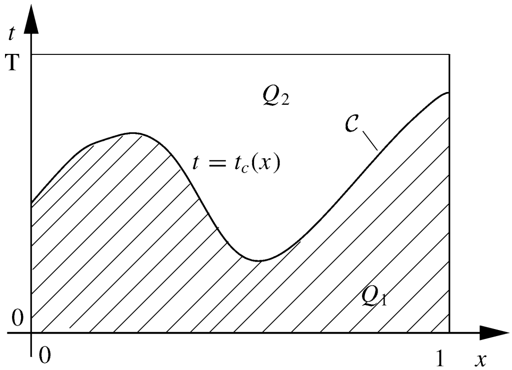

Fig. 1. Decomposition of $\overline{Q}$ into Q_1 and Q_2 by the curve $\mathcal{C}$.

Remark 2.1. The case $v^0(x) > s(x,0)$ can be treated analogously. In that case we have to use the function $\varphi_2(v,x,t)$ to construct $v_1(x,t)$ and the function $\varphi_1(v,x,t)$ to construct $v_2(x,t)$. The case when $v^0(x) = s(x,0)$ for some x requires a special treatment.

Furthermore, we introduce the function $\hat{u}(x,t)$ by

$$\hat{u}(x,t) = \varphi\big(\hat{v}(x,t),x,t\big) = \begin{cases} \varphi_1(\hat{v}_1(x,t),x,t) \equiv \psi_1(x,t) & \text{for } (x,t) \in \overline{Q}_1, \\ \varphi_2(\hat{v}_2(x,t),x,t) \equiv \psi_2(x,t) & \text{for } (x,t) \in \overline{Q}_2. \end{cases} \tag{2.16}$$

The pair of functions $(\hat{u}(x,t), \hat{v}(x,t))$ defined by (2.16), (2.15) is referred to as the *composed stable solution* of the degenerate system (2.3).

From assumption (A_2) and from the identities

$$v_1\big(x,t_c(x)\big) \equiv s\big(x,t_c(x)\big) \equiv v_2\big(x,t_c(x)\big) \quad \text{for } 0 \leqslant x \leqslant 1$$

we obtain

$$\psi_1(x,t) \equiv \psi_2(x,t) \quad \text{on } \mathcal{C}. \tag{2.17}$$

Let us illustrate the composed stable solution by means of example (1.3). Note that $1 \equiv v^0(x) < s(x,0) = x+2$ for $x \in [0,1]$ and $f(\varphi_1(v,x,t),v,x,t,0) \equiv 2$. Therefore, the initial value problem for $v_1(x,t)$ reads

$$\frac{dv_1}{dt} = 2, \quad 0 < t \leqslant T, \qquad v_1(x,0) = 1.$$

It has the solution

$$v_1(x,t) = 2t+1.$$

The equation

$$v_1(x,t) = s(x,t), \quad \text{i.e.,} \quad 2t+1 = x+t+2$$

defines the curve $\mathcal{C}$:

$$t = t_c(x) := x + 1.$$

It is obvious that

$$v_1(x,t) < s(x,t) \quad \text{for } 0 \leqslant t < t_c(x),$$

i.e., assumption (A_4) is fulfilled.

From $f(\varphi_2(v,x,t), v, x, t, 0) \equiv v - x - t$ and $v_1(x, t_c(x)) = 2x + 3$ it follows that the initial value problem for $v_2(x,t)$ reads

$$\frac{dv_2}{dt} = v_2 - x - t, \qquad v_2\big(x, t_c(x)\big) = 2x + 3.$$

Its solution is

$$v_2(x,t) = \exp(t - x - 1) + x + t + 1.$$

It is easy to check that

$$v_2(x,t) > s(x,t) \quad \text{for } t_c(x) < t \leqslant T \ \text{(i.e., in } Q_2).$$

Therefore, assumption (A_5) holds and the composed stable solution has the form

$$\hat{u}(x,t) = \begin{cases} \psi_1(x,t) \equiv 0 & \text{in } \overline{Q}_1, \\ \psi_2(x,t) \equiv \exp(t - x - 1) - 1 & \text{in } \overline{Q}_2, \end{cases} \tag{2.18}$$

$$\hat{v}(x,t) = \begin{cases} v_1(x,t) \equiv 2t + 1 & \text{in } \overline{Q}_1, \\ v_2(x,t) \equiv \exp(t - x - 1) + x + t + 1 & \text{in } \overline{Q}_2. \end{cases} \tag{2.19}$$

Let us return to the composed stable solution defined in (2.15), (2.16). The function $\hat{v}(x,t)$ is obviously continuously differentiable with respect to t. But $\hat{u}(x,t)$ is in general not smooth on the curve $\mathcal{C}$, since we get from (2.12), (2.14) and (2.15)

$$\frac{\partial \psi_1}{\partial t} \leqslant \frac{\partial \psi_2}{\partial t} \quad \text{on } \mathcal{C}.$$

For the sequel it is convenient to introduce the following notation: the symbol ˆ over g and f or some derivative of g and f denotes that we have to consider the arguments (u, v, ε) at $(\hat{u}(x,t), \hat{v}(x,t), 0)$.

It follows from assumption (A_1) that

$$\hat{g}(x,t) := g\big(\hat{u}(x,t), \hat{v}(x,t), x, t, 0\big) \equiv 0 \quad \text{in } \overline{Q}, \tag{2.20}$$

by assumption (A_3) we have

$$\hat{g}_u(x,t) < 0 \quad \text{in } \overline{Q} \backslash \mathcal{C}, \tag{2.21}$$

$$\hat{g}_u(x,t) \equiv 0 \quad \text{on } \mathcal{C}. \tag{2.22}$$

In what follows we prove that under the hypotheses (A_0)–(A_5) and under some additional assumptions (see (A_6)–(A_8) below) problem (2.1), (2.2) has a unique solution $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ satisfying

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon) = \hat{u}(x,t) \quad \text{in } \overline{Q}\backslash\{t=0,\ 0\leqslant x\leqslant 1\},$$
$$\lim_{\varepsilon\to 0} v(x,t,\varepsilon) = \hat{v}(x,t) \quad \text{in } \overline{Q}. \tag{2.23}$$

Concerning the initial condition $u^0(x)$ for $u(x,t,\varepsilon)$ we assume as in the standard theory:

(A_6) For $x\in[0,1]$, $u^0(x)$ lies in the basin of attraction of the equilibrium point $\varphi_1(v^0(x),x,0)$ of the associated equation (2.6) for $v=v^0(x)$, $t=0$.

Assumption (A_6) implies that for $0\leqslant x\leqslant 1$ the initial value problem

$$\frac{du}{d\tau} = g\big(u, v^0(x), 0, 0\big), \qquad u(x,0) = u^0(x)$$

has a unique solution $u=\bar{u}(x,\tau)$ defined for $\tau\geqslant 0$, and such that

$$\lim_{\tau\to\infty} \bar{u}(x,\tau) = \varphi_1\big(v^0(x),x,0\big).$$

Finally, we assume

(A_7) $\hat{g}_{uu}(x,t) := g_{uu}(\hat{u}(x,t),\hat{v}(x,t),x,t,0) < 0$ on $\mathcal{C}$.
(A_8) $\hat{g}_\varepsilon(x,t) > 0$ on $\mathcal{C}$.

Concerning assumption (A_8) we would like to mention that the sign of $\hat{g}_\varepsilon(x,t)$ on $\mathcal{C}$ plays an important role (see [1–3]).

Let us return to example (1.3) and verify the hypotheses (A_6)–(A_8). The associated equation (2.6) to (1.3) reads in case $v=v^0(x)\equiv 1$, $t=0$

$$\frac{du}{d\tau} = -u(u+x+1), \quad \tau>0.$$

It is easy to see that for $0\leqslant x\leqslant 1$ the solution $\bar{u}(x,\tau)$ of this equation with the initial condition

$$\bar{u}(x,0) = u^0(x) > 0$$

exists for $\tau>0$ and tends to $\varphi_1(v^0(x),x,0)=0$ as $\tau\to\infty$. Hence, assumption (A_6) is fulfilled.

Assumptions (A_7) and (A_8) are obviously satisfied since we have for $(x,t)\in\overline{Q}$

$$g_{uu}\equiv -2<0, \qquad g_\varepsilon \equiv I(x,t) > 0.$$

Our approach to prove the asymptotic behavior of the solution of problem (2.1) is based on the concept of ordered lower and upper solutions. Before we recall its definition (see, e.g., [10]), we introduce the following notation. Let the operators L_v and M_u be defined by

$$(L_v w)(x,t,\varepsilon) := \varepsilon^2\left(\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2}\right) - g(w,v,x,t,\varepsilon), \tag{2.24}$$

$$(M_u w)(x,t,\varepsilon) := \frac{\partial w}{\partial t} - f(u,w,x,t,\varepsilon). \tag{2.25}$$

Definition 2.1. Let the vector-functions $\alpha(x,t,\varepsilon) := (\alpha^u(x,t,\varepsilon), \alpha^v(x,t,\varepsilon))$ and $\beta(x,t,\varepsilon) := (\beta^u(x,t,\varepsilon), \beta^v(x,t,\varepsilon))$ be defined for $(x,t,\varepsilon) \in \overline{Q} \times \overline{I}_{\varepsilon_1}, \varepsilon_1 \leqslant \varepsilon_0$ and satisfy the smoothness conditions $\alpha^u, \beta^u \in C^{2,1,0}_{x,t,\varepsilon}(Q \times I_{\varepsilon_1}) \cap C^{1,0,0}_{x,t,\varepsilon}(\overline{Q} \times \overline{I}_{\varepsilon_1})$, $\alpha^v, \beta^v \in C^{0,1,0}_{x,t,\varepsilon}(Q \times I_{\varepsilon_1}) \cap C^{0,0,0}_{x,t,\varepsilon}(\overline{Q} \times \overline{I}_{\varepsilon_1})$. Then $\alpha(x,t,\varepsilon))$ and $\beta(x,t,\varepsilon))$ are called ordered lower and upper solutions to the initial–boundary value problem (2.1), (2.2) in $\overline{Q}$ for $\varepsilon \in I_{\varepsilon_1}$, respectively, if they satisfy for $\varepsilon \in I_{\varepsilon_1}$ the conditions

$$\alpha^u(x,t,\varepsilon) \leqslant \beta^u(x,t,\varepsilon), \qquad \alpha^v(x,t,\varepsilon) \leqslant \beta^v(x,t,\varepsilon)$$
$$\text{for } (x,t) \in \overline{Q}, \tag{2.26}$$

$$(L_v\alpha^u)(x,t,\varepsilon) \leqslant 0 \leqslant (L_v\beta^u)(x,t,\varepsilon)$$
$$\text{for } (x,t) \in Q,\ \alpha^v \leqslant v \leqslant \beta^v, \tag{2.27}$$

$$(M_u\alpha^v)(x,t,\varepsilon) \leqslant 0 \leqslant (M_u\beta^v)(x,t,\varepsilon)$$
$$\text{for } (x,t) \in Q,\ \alpha^u \leqslant u \leqslant \beta^u, \tag{2.28}$$

$$\frac{\partial\alpha^u}{\partial x}(0,t,\varepsilon) \geqslant 0 \geqslant \frac{\partial\beta^u}{\partial x}(0,t,\varepsilon), \qquad \frac{\partial\alpha^u}{\partial x}(1,t,\varepsilon) \leqslant 0 \leqslant \frac{\partial\beta^u}{\partial x}(1,t,\varepsilon)$$
$$\text{for } 0 \leqslant t \leqslant T, \tag{2.29}$$

$$\alpha^u(x,0,\varepsilon) \leqslant u^0(x) \leqslant \beta^u(x,0,\varepsilon), \qquad \alpha^v(x,0,\varepsilon) \leqslant v^0(x) \leqslant \beta^v(x,0,\varepsilon)$$
$$\text{for } 0 \leqslant x \leqslant 1. \tag{2.30}$$

This definition can be obviously adapted to any subdomain of $\overline{Q}$. It is known (see, e.g., [10]) that the existence of ordered lower and upper solutions to (2.1), (2.2) implies the existence of a unique solution $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ of (2.1), (2.2) satisfying for $(x,t,\varepsilon) \in \overline{Q} \times I_{\varepsilon_1}$

$$\alpha^u(x,t,\varepsilon) \leqslant u(x,t,\varepsilon) \leqslant \beta^u(x,t,\varepsilon),$$
$$\alpha^v(x,t,\varepsilon) \leqslant v(x,t,\varepsilon) \leqslant \beta^v(x,t,\varepsilon).$$

The goal of the following investigations is to characterize the asymptotic behavior of the solution of (2.1), (2.2), in particular, we prove the limit behavior (2.23) by constructing lower and upper solutions to the initial–boundary value problem (2.1), (2.2).

3. Existence and asymptotic behavior of the solution

In this section we will prove that the initial–boundary value problem (2.1), (2.2) has a unique solution. Taking into account an initial layer correction we can

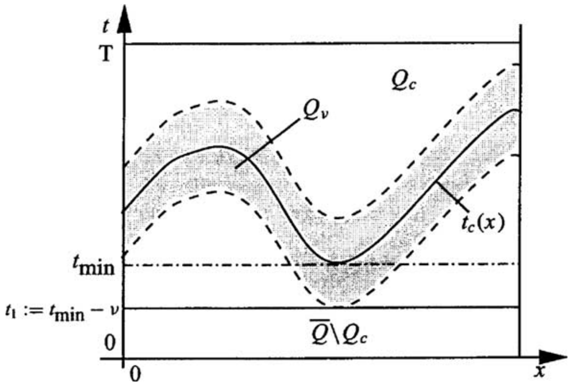

Fig. 2. Decomposition of $\overline{Q}$.

show that for small ε the solution of (2.1), (2.2) is close to the composed stable solution $(\hat{u}(x,t), \hat{v}(x,t))$.

In order to be able to formulate our main result we decompose the domain $\overline{Q}$ and introduce a function which represents an approximation of the initial layer correction.

First we decompose $\overline{Q}$. Let $t_{\min}$ be the minimum of the function $t_c(x)$ in [0,1], let ν be any small positive number such that $t_1 := t_{\min} - \nu$ is positive. Let Q_c be the domain defined by $Q_c := \{(x,t) \in R^2\colon\ 0 < x < 1,\ t_1 < t \leqslant T\}$ (see Fig. 2).

Next we introduce an initial layer correction. According to [14] we define the zeroth-order initial layer function $\Pi_0(x,\tau)$ $(\tau = t/\varepsilon^2)$ as the solution of the initial value problem where $x \in [0,1]$ has to be considered as a parameter

$$\begin{aligned} &\frac{d\Pi_0}{d\tau} = g\bigl(\psi_1(x,0) + \Pi_0, v^0(x), x, 0, 0\bigr), \quad \tau > 0,\\ &\Pi_0(x,0) = u^0(x) - \psi_1(x,0). \end{aligned} \tag{3.1}$$

By (2.16) we have $\psi_1(x,0) = \varphi_1(v^0(x), x, 0)$. Thus, from assumption (A_6) and from (2.21) it follows that the initial value problem (3.1) has a solution which satisfies the estimate $|\Pi_0(x,\tau)| < c\exp(-\kappa\tau)$, $\tau \geqslant 0$, for some positive constants c and κ.

Concerning our example (1.3) the initial value problem (3.1) reads

$$\begin{aligned} &\frac{d\Pi_0}{d\tau} = -\Pi_0(\Pi_0 + x + 1), \quad \tau > 0,\\ &\Pi_0(x,0) = u^0(x). \end{aligned}$$

Its solution can be found in the explicit form

$$\begin{aligned} \Pi_0(x,\tau) = {} & u^0(x)(x+1)\bigl[u^0(x)\bigl(1 - \exp\bigl(-(x+1)\bigr)\tau\bigr) + x + 1\bigr]^{-1}\\ & \times \exp\bigl(-(x+1)\tau\bigr). \end{aligned}$$

Now we formulate our main result.

Theorem 3.1. *Assume hypotheses* (A_0)–(A_8) *to be valid. Then, for sufficiently small* ε*, the initial-boundary value problem* (2.1), (2.2) *has a unique solution* $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ *satisfying*

$$u(x,t,\varepsilon) = \begin{cases} \hat{u}(x,t) + \Pi_0(x,\tau) + O(\varepsilon) & \text{for } (x,t) \in \overline{Q \setminus Q_c}, \\ \hat{u}(x,t) + O(\sqrt{\varepsilon}) & \text{for } (x,t) \in \overline{Q}_c, \end{cases} \tag{3.2}$$

$$v(x,t,\varepsilon) = \begin{cases} \hat{v}(x,t) + O(\varepsilon) & \text{for } (x,t) \in \overline{Q \setminus Q_c}, \\ \hat{v}(x,t) + O(\sqrt{\varepsilon}) & \text{for } (x,t) \in \overline{Q}_c. \end{cases} \tag{3.3}$$

Corollary 3.1. *From* (3.2), (3.3) *it is obvious that the relations* (2.23) *hold.*

Proof of Theorem 3.1. The proof consists of two steps. In the first step we consider the initial–boundary value problem (2.1), (2.2) in the subdomain $\overline{Q \setminus Q_c}$. From our assumptions it follows that the exchange of stabilities takes place in Q_c. Therefore, we can apply the standard theory [14] to solve the initial–boundary value problem in $\overline{Q \setminus Q_c}$. We get the following result.

Lemma 3.1. *Assume hypotheses* (A_0)–(A_6) *to be valid. Then, for sufficiently small* ε ($\varepsilon \in I_{\varepsilon_1} \subset I_{\varepsilon_0}$)*, the initial boundary value problem* (2.1), (2.2) *has a unique solution* $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ *in* $\overline{Q \setminus Q_c}$ *satisfying*

$$u(x,t,\varepsilon) = \hat{u}(x,t) + \Pi_0(x,\tau) + O(\varepsilon),$$
$$v(x,t,\varepsilon) = \hat{v}(x,t) + O(\varepsilon). \tag{3.4}$$

Let $u^1(x,\varepsilon) := u(x,t_1,\varepsilon)$, $v^1(x,\varepsilon) := v(x,t_1,\varepsilon)$. Now we consider the initial–boundary value problem for (2.1) in Q_c with the initial conditions

$$u(x,t_1,\varepsilon) = u^1(x,\varepsilon), \quad v(x,t_1,\varepsilon) = v^1(x,\varepsilon) \quad \text{for } 0 \leqslant x \leqslant 1 \tag{3.5}$$

and the boundary conditions

$$\frac{\partial u}{\partial x}(0,t,\varepsilon) = \frac{\partial u}{\partial x}(1,t_1,\varepsilon) = 0 \quad \text{for } t_1 < t \leqslant T \tag{3.6}$$

for sufficiently small ε. Our approach to study this problem is based on the method of ordered lower and upper solutions. We construct these solutions for (2.1), (3.5), (3.6) by means of the composed stable solution $(\hat{u}(x,t), \hat{v}(x,t))$ defined in (2.15), (2.16).

As we noticed above, in general $\hat{u}(x,t)$ is not smooth on the curve $\mathcal{C}$. In order to be able to use $\hat{u}(x,t)$ for the construction of lower and upper solutions we have to smooth $\hat{u}(x,t)$ near the curve $\mathcal{C}$. To this end we extend smoothly the functions $\psi_1(t,x)$ and $\psi_2(t,x)$ into the regions $\overline{Q}_2$ and $\overline{Q}_1$, respectively. Using the function

$$\omega(\xi) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\xi} \exp(-s^2)\, ds,$$

where

$$\xi := \big(t - t_c(x)\big)/\varepsilon^a, \quad a \in (1/2, 1),$$

we introduce the function $\tilde{u}$ by

$$\tilde{u}(x,t,\varepsilon) := \psi_1(x,t)\omega(-\xi) + \psi_2(x,t)\omega(\xi). \tag{3.7}$$

Let Q_ν be defined by $Q_\nu := \{(x,t) \in \overline{Q}\colon\ |t - t_c(x)| < \nu,\ 0 \leqslant x \leqslant 1\}$, where ν is any sufficiently small positive number such that Q_ν has no common point with $t = T$ (see Fig. 2).

It is easy to show that $\tilde{u}$ is smooth in $\overline{Q}_c$ and satisfies

$$\tilde{u}(x,t,\varepsilon) = \hat{u}(x,t) + \eta(x,t,\varepsilon), \tag{3.8}$$

where

$$\eta(x,t,\varepsilon) = \begin{cases} O(\varepsilon^a) & \text{for } (x,t) \in Q_\nu, \\ O(\exp -(\nu/\varepsilon)) & \text{for } (x,t) \in \overline{Q} \setminus Q_\nu \end{cases} \tag{3.9}$$

(see [1]).

Now we construct lower and upper solutions for the initial–boundary value problem (2.1), (3.5), (3.6) in $\overline{Q}_c$ by using the smooth function $\tilde{u}$ as follows:

$$\begin{aligned}
\beta^u(x,t,\varepsilon) &:= \tilde{u}(x,t,\varepsilon) + \sqrt{\varepsilon}\gamma h(x,t) + \varepsilon^a z(x,\varepsilon),\\
\alpha^u(x,t,\varepsilon) &:= \tilde{u}(x,t,\varepsilon) - \sqrt{\varepsilon}\sigma h(x,t) - \varepsilon^a z(x,\varepsilon),\\
\beta^v(x,t,\varepsilon) &:= \hat{v}(x,t) + \sqrt{\varepsilon}\sigma^2 h(x,t),\\
\alpha^v(x,t,\varepsilon) &:= \hat{v}(x,t) - \sqrt{\varepsilon}\sigma^2 h(x,t),
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
h(x,t) &:= \exp\big(\lambda\big(t - t_c(x)\big)\big),\\
z(x,\varepsilon) &:= \exp(-kx/\varepsilon^a) + \exp\big(-k(1-x)/\varepsilon^a\big)
\end{aligned} \tag{3.11}$$

are positive functions in $\overline{Q}_c \times I_{\varepsilon_1}$, $\gamma, \sigma, \lambda, k$ are positive numbers. We will determine these numbers in such a way that α and β will be ordered lower and upper solutions, i.e., they will satisfy all conditions of Definition 2.1 in $\overline{Q}_c$.

It is obvious that for any choice of γ, σ, λ and k we have

$$\alpha^u(x,t,\varepsilon) \leqslant \beta^u(x,t,\varepsilon), \qquad \alpha^v(x,t,\varepsilon) \leqslant \beta^v(x,t,\varepsilon)$$
$$\text{for } (x,t,\varepsilon) \in \overline{Q}_c \times I_{\varepsilon_1};$$

hence, the relations (2.26) are fulfilled.

Taking into account the exponential decay of $\Pi_0(x,\tau)$ we get from (3.10), (3.4) for sufficiently small ε

$$\begin{aligned}
\alpha^u(x,t_1,\varepsilon) &\leqslant u(x,t_1,\varepsilon) = u^1(x,\varepsilon) \leqslant \beta^u(x,t_1,\varepsilon),\\
\alpha^v(x,t_1,\varepsilon) &\leqslant v(x,t_1,\varepsilon) = v^1(x,\varepsilon) \leqslant \beta^v(x,t_1,\varepsilon).
\end{aligned}$$

Consequently, the inequalities (2.30) for the initial data hold.

Now we check that $\alpha^u(x,t,\varepsilon)$ and $\beta^u(x,t,\varepsilon)$ satisfy the inequalities (2.27) in Q_ν for sufficiently small ε.

From (2.17) we obtain

$$\psi_2(x,t) - \psi_1(x,t) = O\big(\big|t - t_c(x)\big|\big).$$

Using this relation it can be shown (see [1,3]) that

$$\varepsilon^2\left(\frac{\partial \tilde u}{\partial t} - \frac{\partial^2 \tilde u}{\partial x^2}\right) = \begin{cases} O(\varepsilon^{2-a}) & \text{for } (x,t) \in Q_\nu, \\ O(\varepsilon^2) & \text{for } (x,t) \in Q_c \backslash Q_\nu. \end{cases} \tag{3.12}$$

From (3.11) we get

$$\varepsilon^2\varepsilon^{1/2}\left(\frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2}\right) = O(\varepsilon^{5/2}) \quad \text{for } (x,t) \in Q_c,$$

$$\varepsilon^2\varepsilon^{a}\left(\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2}\right) = O(\varepsilon^{2-a}) \quad \text{for } (x,t) \in Q_c. \tag{3.13}$$

Thus, because of $1/2 < a < 1$, we obtain from (3.10)–(3.13)

$$\varepsilon^2\left(\frac{\partial \beta^u}{\partial t} - \frac{\partial^2 \beta^u}{\partial x^2}\right) = O(\varepsilon^{2-a}) = o(\varepsilon) \quad \text{for } (x,t) \in Q_c, \tag{3.14}$$

$$\varepsilon^2\left(\frac{\partial \alpha^u}{\partial t} - \frac{\partial^2 \alpha^u}{\partial x^2}\right) = O(\varepsilon^{2-a}) = o(\varepsilon) \quad \text{for } (x,t) \in Q_c. \tag{3.15}$$

To treat the expression $g(\beta^u(x,t,\varepsilon), v, x, t, \varepsilon)$ in $L_v\beta^u$ we use the relations

$$\tilde u(x,t,\varepsilon) = \hat u(x,t) + O(\varepsilon^a) \quad \text{for } (x,t) \in Q_\nu$$

which follows from (3.8) and (3.9), and

$$\varepsilon^a z(x,\varepsilon) = O(\varepsilon^a) \quad \text{for } (x,t) \in Q_\nu$$

due to (3.11). Moreover, we note that the set of all v satisfying $\alpha^v(x,t,\varepsilon) \leqslant v \leqslant \beta^v(x,t,\varepsilon)$ can be represented in the form

$$v = \hat v(x,t) + \sqrt{\varepsilon}\sigma^2 h(x,t)\theta, \quad |\theta| \leqslant 1.$$

Thus, we have

$$\begin{aligned} &g\big(\beta^u(x,t,\varepsilon), v, x, t, \varepsilon\big) \\ &\quad = g\big(\hat u(x,t) + \sqrt{\varepsilon}\gamma h(x,t) + O(\varepsilon^a), \hat v + \sqrt{\varepsilon}\sigma^2 h(x,t)\theta, x, t, \varepsilon\big) \\ &\quad = \hat g(x,t) + \sqrt{\varepsilon}\big[\hat g_u(x,t)\big(\gamma + O(\varepsilon^{a-1/2})\big) + \hat g_v(x,t)\sigma^2\theta\big]h(x,t) \\ &\qquad + \frac{1}{2}\varepsilon\big[\hat g_{uu}(x,t)\gamma^2 + 2\hat g_{uv}(x,t)\gamma\sigma^2\theta + \hat g_{vv}(x,t)\sigma^4\theta^2\big]h^2(x,t) \\ &\qquad + \varepsilon\hat g_\varepsilon(x,t) + o(\varepsilon). \end{aligned} \tag{3.16}$$

Our goal is to prove $g(\beta^u(x,t,\varepsilon), v, x, t, \varepsilon) = -c\varepsilon + o(\varepsilon)$ for $(x,t) \in Q_\nu$ and some positive constant c.

From (2.5) we get

$$\hat{g}_v(x,t) = -\hat{g}_u(x,t)\hat{\varphi}_v(x,t), \tag{3.17}$$

where

$$\hat{\varphi}_v(x,t) = \begin{cases} \varphi_{1v}(v_1(x,t), x, t) & \text{for } (x,t) \in Q_1, \\ \varphi_{2v}(v_2(x,t), x, t) & \text{for } (x,t) \in Q_2. \end{cases}$$

Since $\hat{\varphi}_v(x,t)$ is uniformly bounded in $\overline{Q}$, $|\theta| \leqslant 1$, we have by (3.17) and (2.21), (2.22) for any fixed σ and for sufficiently large γ

$$\begin{aligned} &\hat{g}_u(x,t)\big(\gamma + O(\varepsilon^{a-1/2})\big) + \hat{g}_v(x,t)\sigma^2\theta \\ &\quad = \hat{g}_u(x,t)\big[\gamma + O(\varepsilon^{a-1/2}) - \hat{\varphi}_v(x,t)\sigma^2\theta\big] \leqslant 0. \end{aligned} \tag{3.18}$$

According to assumption (A_7) there is a positive constant c_ν such that for sufficiently small ν

$$\hat{g}_{uu}(x,t) \leqslant -c_\nu < 0 \quad \text{in } Q_\nu. \tag{3.19}$$

Hence, for sufficiently large γ, we have for $(x,t) \in Q_\nu$

$$\gamma\big[\hat{g}_{uu}(x,t)\gamma + 2\hat{g}_{uv}(x,t)\sigma^2\theta + \gamma^{-1}\hat{g}_{vv}(x,t)\sigma^4\theta^2\big] < -2\gamma\bar{c}, \tag{3.20}$$

where $\bar{c}$ is some positive constant.

Now we set $\lambda = 1/\nu$. Then, by (3.11), it holds

$$e^{-1} \leqslant h(x,t) \leqslant e \quad \text{for } (x,t) \in \overline{Q}_\nu. \tag{3.21}$$

Under our smoothness assumption there is a positive constant c_g such that

$$\big|\hat{g}_\varepsilon(x,t)\big| \leqslant c_g \quad \text{for } (x,t) \in \overline{Q}_\nu. \tag{3.22}$$

By (2.20), (3.17)–(3.22) we get from (3.16)

$$g\big(\beta^u(x,t,\varepsilon), v, x, t, \varepsilon\big) < -(\gamma\bar{c}e^{-2} - c_g)\varepsilon + o(\varepsilon). \tag{3.23}$$

Taking into account (3.14) and (3.23) we have for sufficiently small ν and ε and for sufficiently large γ

$$\begin{aligned} (L_v\beta^u)(x,t,\varepsilon) &\equiv \varepsilon^2\left(\frac{\partial\beta^u}{\partial t} - \frac{\partial^2\beta^u}{\partial x^2}\right) - g\big(\beta^u(x,t,\varepsilon), v, x, t, \varepsilon\big) \\ &> (\gamma\bar{c}e^{-2} - c_g)\varepsilon + o(\varepsilon) \geqslant 0 \\ &\text{for } (x,t) \in Q_\nu,\ \alpha^v(x,t,\varepsilon) \leqslant v \leqslant \beta^v(x,t,\varepsilon), \end{aligned}$$

i.e., the inequality (2.27) holds for β^u in Q_ν.

Now we verify the inequality (2.27) for α^u in Q_ν. Using (3.10), (3.15), and a representation for $g(\alpha^u(x,t,\varepsilon), v, x, t, \varepsilon)$ similar to (3.16) we get

$$\begin{aligned}L_v\alpha^u(x,t,\varepsilon) &\equiv \varepsilon^2\left(\frac{\partial\alpha^u}{\partial t}-\frac{\partial^2\alpha^u}{\partial x^2}\right)-g\big(\alpha^u(x,t,\varepsilon),v,x,t,\varepsilon\big)\\ &= \sqrt{\varepsilon}\,\hat g_u(x,t)\big[\sigma+O(\varepsilon^{a-1/2})+\hat\varphi_v(x,t)\sigma^2\theta\big]h(x,t)\\ &\quad -\frac{1}{2}\sigma^2\varepsilon\big[\hat g_{uu}(x,t)-2\hat g_{uv}(x,t)\sigma\theta\\ &\qquad +\hat g_{vv}(x,t)\sigma^2\theta^2\big]h^2(x,t)\\ &\quad -\varepsilon\hat g_\varepsilon(x,t)+o(\varepsilon).\end{aligned}\tag{3.24}$$

There is a sufficiently small σ_0 such that for $0<\sigma\leqslant\sigma_0$

$$1+\sigma\hat\varphi_v(x,t)\theta\geqslant 1/2\quad \text{for } (x,t)\in Q_\nu,\ |\theta|\leqslant 1.$$

Thus, because of $a-1/2>0$ and taking into account (2.21), (2.22) and (3.11), we have for sufficiently small ε

$$\hat g_u(x,t)\big[\sigma+O(\varepsilon^{a-1/2})+\hat\varphi_v(x,t)\sigma^2\theta\big]h(x,t)\leqslant 0.\tag{3.25}$$

By assumption (A_8) there is a positive constant k_g such that for sufficiently small ν

$$-\hat g_\varepsilon(x,t)\leqslant -k_g<0\quad \text{for } (x,t)\in Q_\nu.$$

Now we choose σ_0 so small that for $0<\sigma\leqslant\sigma_0$

$$\frac{1}{2}\sigma^2\big|\hat g_{uu}(x,t)-2\hat g_{uv}(x,t)\sigma\theta+\hat g_{vv}(x,t)\sigma^2\theta^2\big|h^2(x,t)\leqslant k_g/2 \quad \text{for } (x,t)\in Q_\nu.\tag{3.26}$$

Therefore, for $0<\sigma\leqslant\sigma_0$, and for sufficiently small ε we get from (3.24), (3.25), and (3.26)

$$\begin{aligned}&(L_v\alpha^u)(x,t,\varepsilon)\leqslant 0\quad \text{for } (x,t)\in Q_\nu,\\ &\alpha^v(x,t,\varepsilon)\leqslant v\leqslant \beta^v(x,t,\varepsilon),\end{aligned}$$

i.e., inequality (2.27) is satisfied for α^u in Q_ν.

Now we will prove that α^u and β^u satisfy the inequalities (2.27), (2.28) in $Q_c\backslash Q_\nu$. From (3.16) we get

$$\begin{aligned}&g\big(\beta^u(x,t,\varepsilon),v,x,t,\varepsilon\big)\\ &\quad=\sqrt{\varepsilon}\,\big[\hat g_u(x,t)\gamma+\hat g_v(x,t)\sigma^2\theta\big]h(x,t)+o\big(\sqrt{\varepsilon}\big).\end{aligned}\tag{3.27}$$

It follows from (2.21) that there is a positive constant c_1 such that for sufficiently large γ

$$\hat g_u(x,t)\gamma+\hat g_v(x,t)\sigma^2\theta\leqslant -c_1\quad \text{for } (x,t)\in Q_c\backslash Q_\nu.\tag{3.28}$$

Therefore, by (2.24), (3.14), (3.27), and (3.28) we have for γ sufficiently large and ε sufficiently small

$$(L_v\beta^u)(x,t,\varepsilon) \geqslant 0 \quad \text{for } (x,t) \in Q_c \backslash Q_\nu,\ \alpha^v(x,t,\varepsilon) \leqslant v \leqslant \beta^v(x,t,\varepsilon).$$

Analogously, we get from (3.24) for σ and ε sufficiently small

$$(L_v\alpha^u)(x,t,\varepsilon) = \sqrt{\varepsilon}\hat{g}_u(x,t)\big(\sigma + \hat{\varphi}_v(x,t)\sigma^2\theta\big)h(x,t) + o\big(\sqrt{\varepsilon}\big) \leqslant 0$$
$$\text{for } (x,t) \in Q_c \backslash Q_\nu,\ \alpha^v(x,t,\varepsilon) \leqslant v \leqslant \beta^v(x,t,\varepsilon).$$

Thus, the inequalities (2.27) for α^u, β^u hold in $Q_c \backslash Q_\nu$.

Now we verify the inequality (2.28) in Q_c. For u we use the representation

$$u = \hat{u}(x,t) + \sqrt{\varepsilon}\kappa h(x,t) + O(\varepsilon^a), \quad -\sigma \leqslant \kappa \leqslant \gamma.$$

By (2.25) and (3.10) we have

$$\begin{aligned}(M_u\beta^v)(x,t,\varepsilon) &\equiv \frac{\partial\beta^v}{\partial t} - f\big(u, \beta^v(x,t,\varepsilon), x, t, \varepsilon\big)\\ &= \frac{\partial\hat{v}}{\partial t} + \sqrt{\varepsilon}\frac{\sigma^2}{\nu}h(x,t)\\ &\quad - f\big(\hat{u}(x,t) + \sqrt{\varepsilon}\kappa h(x,t) + O(\varepsilon^a),\\ &\qquad \hat{v} + \sqrt{\varepsilon}\sigma^2 h(x,t), x, t, \varepsilon\big).\end{aligned} \tag{3.29}$$

Using the representation

$$\begin{aligned}&f\big(\hat{u}(x,t) + \sqrt{\varepsilon}\kappa h(x,t) + O(\varepsilon^a), \hat{v} + \sqrt{\varepsilon}\sigma^2 h(x,t), x, t, \varepsilon\big)\\ &\quad = f(\hat{u}, \hat{v}, x, t, 0)\sqrt{\varepsilon}\big[\hat{f}_u(x,t)\kappa + \hat{f}_v(x,t)\sigma^2\big]h(x,t) + o\big(\sqrt{\varepsilon}\big)\end{aligned}$$

and taking into account

$$\frac{\partial\hat{v}}{\partial t} - f(\hat{u}, \hat{v}, x, t, 0) \equiv 0$$

we get from (3.29)

$$(M_u\beta^v)(x,t,\varepsilon) = \sqrt{\varepsilon}\left[\frac{\sigma^2}{\nu} - \hat{f}_u(x,t)\kappa - \hat{f}_v(x,t)\sigma^2\right]h(x,t) + o\big(\sqrt{\varepsilon}\big). \tag{3.30}$$

To given $\sigma > 0$ we choose ν so small such that

$$\left[\frac{\sigma^2}{\nu} - \hat{f}_u(x,t)\kappa - \hat{f}_v(x,t)\sigma^2\right]h(x,t) \geqslant c_2 \quad \text{for } (x,t) \in Q_c,$$

where c_2 is some positive number. Thus, for sufficiently small ε, we have

$$(M_u\beta^v)(x,t,\varepsilon) \geqslant 0$$
$$\text{for } (x,t) \in Q_c,\ \alpha^v(x,t,\varepsilon) \leqslant u \leqslant \beta^u(x,t,\varepsilon).$$

Similarly we can verify the inequality (2.28) for α^v.

Finally, we verify the inequalities (2.29). If we differentiate β^u with respect to x at $x = 0$ and $x = 1$, we get from (3.10)

$$\frac{\partial\beta^u}{\partial x}(0,t,\varepsilon) = \frac{\partial\tilde{u}}{\partial x}(0,t,\varepsilon) - k + O\big(\sqrt{\varepsilon}\big),$$
$$\frac{\partial\beta^u}{\partial x}(1,t,\varepsilon) = \frac{\partial\tilde{u}}{\partial x}(1,t,\varepsilon) + k + O\big(\sqrt{\varepsilon}\big).$$

Using (3.7) it can be shown that there exists a positive constant c_3 such that

$$\left|\frac{\partial\tilde{u}}{\partial x}(x,t,\varepsilon)\right| \leqslant c_3 \quad \text{for } (x,t) \in \overline{Q}.$$

Consequently, the inequalities (2.29) for β^u in Definition 2.1 are satisfied if we choose k sufficiently large. The inequalities (2.29) for α^u can be verified in a similar way.

From our considerations above it follows that the functions $\alpha(x,t,\varepsilon)$, $\beta(x,t,\varepsilon)$ fulfill all conditions in Definition 2.1, and we can conclude that for sufficiently small ε there exists a unique solution $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ of problem (2.1), (2.2) satisfying for $(x,t) \in \overline{Q_c}$

$$\alpha^u(x,t,\varepsilon) \leqslant u(x,t,\varepsilon) \leqslant \beta^u(x,t,\varepsilon),$$
$$\alpha^v(x,t,\varepsilon) \leqslant v(x,t,\varepsilon) \leqslant \beta^v(x,t,\varepsilon).$$

From these inequalities and from (3.10) it follows that the representations (3.2) and (3.3) for $u(x,t,\varepsilon)$ and $v(x,t,\varepsilon)$ in $\overline{Q_c}$ are valid. This completes the proof of Theorem 3.1. □

Remark 3.1. We have considered (2.1), (2.2) in the case when u and v are scalars. Our approach can obviously be extended to the case that u is a scalar and v is a vector.

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