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A de Montessus de Ballore Theorem for Best Rational Approximation over the Whole Plane

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We prove a de Montessus de Ballore type theorem for rational functions R_{nq} of type (n, q) formed by best approximation over the whole plane to functions $f(z)$ meromorphic in the plane with exactly q poles. This resolves a question raised by Lubinsky and Shisha (*J. Approx. Theory* **36** (1982), 277–293).

1. INTRODUCTION

In [4], Lubinsky and Shisha considered best approximations of complex functions $f(z)$ by rational functions formed by minimizing a metric which involves values of $f(z)$ throughout the plane. They proved existence, non-uniqueness, and that sequences of best approximations converge in planar Lebesgue measure under general conditions. They also raised the question as to whether there is an analog for these approximations of the classical de Montessus de Ballore theorem for Padé approximations (see Baker [1], Wallin [5]). It is the purpose of this paper to answer their question in the affirmative.

I shall now state a special case of our main result, for meromorphic functions of finite order. Let $\mathscr{R}_{nq}$ denote the class of rational functions with numerator and denominator of degrees at most n and q, respectively. Let $D(z, u)$ denote the chordal metric on the Riemann sphere, that is,

$$D(z, u) = \frac{|z-u|}{\{(1+|z|^2)(1+|u|^2)\}^{1/2}}, \qquad z, u \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \tag{1.1}$$

and let

$$\rho_D(f, g) = \iint_{\mathbb{C}} D(f(z), g(z)) \exp(-\exp(|z|))\, dx\, dy, \tag{1.2}$$

where $z = x + iy$ for measurable functions $f, g: \mathbb{C} \to \bar{\mathbb{C}}$.

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THEOREM 1.1. *Let f be a meromorphic function of finite order in $\mathbb{C}$, with exactly q poles (counting multiplicity). For each positive integer n, let R^*_{nq} be a rational satisfying*

$$\rho_D(f, R^*_{nq}) = \min\{\rho_D(f, R) : R \in \mathscr{R}_{nq}\}. \tag{1.3}$$

Then

$$\lim_{n \to \infty} R^*_{nq}(z) = f(z)$$

uniformly in compact subsets of $\mathbb{C}$ not containing the poles of f. Further,

$$\limsup_{n \to \infty} \rho_D(f, R^*_{nq})^{1/n \log n} < 1, \tag{1.4}$$

and uniformly in compact subsets of $\mathbb{C}$ not containing the poles, we have

$$\limsup_{n \to \infty} |f(z) - R^*_{nq}(z)|^{1/n \log n} < 1. \tag{1.5}$$

We remark that $\exp(-\exp(|z|))$ in (1.2) can be replaced by $\exp(-Q(|z|))$, where $Q(r)$ is any positive, continuous function defined on $[0, \infty)$ such that

$$\lim_{r \to \infty} Q(r)\, r^{-\alpha} = \infty \qquad \text{for all} \quad \alpha > 0.$$

Regarding the proofs, use is made of a lemma by Goncar [2] establishing uniform convergence of a sequence of analytic functions given that they converge in one dimensional Hausdorff measure.

In Section 3, we prove a lemma which relates the uniform norm of a polynomial in a disc to the size of the set on which the absolute value of the polynomial is bounded by 1. Use is also made of Cartan's lemma on small values of polynomials, of standard measure-theoretic techniques, and of the elements of complex analysis.

The paper is divided as follows: Section 2 clarifies the notation used in this paper, and lists the main results. Section 3 proves lemmas needed for the major results of this paper. Section 4 proves the theorem on uniform convergence, while Section 5 proves a theorem relating to the rate of convergence.

2. NOTATION AND STATEMENT OF RESULTS

(1) Throughout, meas will denote planar Lebesgue measure, and μ will denote a measure on the finite complex plane $\mathbb{C}$, normalized so that

$$\iint d\mu = \iint_{\mathbb{C}} d\mu(z) = 1. \tag{2.1}$$

Further, we shall assume that we are given a function $Q(r)$, defined in $[0, \infty)$, such that

$Q(r)$ is positive in $[0, \infty)$ and $Q'(r)/r$ is non-decreasing in $[0, \infty)$. (2.2)

The measure μ will be assumed to have density function e^{-Q}. More precisely, we assume that for every measurable set E in the plane

$$\mu(E) = \iint_E \exp(-Q(|z|))\, dx\, dy, \tag{2.3}$$

where $z = x + iy$. Note that from (2.2) and (2.3)

$$\mu(E) \leqslant \text{meas}(E) \tag{2.4}$$

for any measurable set E. Further if $E \subseteq \{z : |z| \leqslant r\}$,

$$\mu(E) \geqslant e^{-Q(r)} \qquad \text{meas}(E). \tag{2.5}$$

(2) Throughout, as in Lubinsky and Shisha [4], $D(z, u)$ will denote a fixed function, defined and continuous on $\mathbb{C} \times \mathbb{C}$, satisfying

$$\begin{aligned} &D(z, u) \in [0, 1] \\ &D(z, u) = D(u, z) \\ &D(z, u) = 0 \Leftrightarrow z = u. \end{aligned} \tag{2.6}$$

We also assume that for each $z \in \mathbb{C}$,

$$D(z, \infty) = \lim_{\substack{|u| \to \infty \\ z_0 \to z}} D(z_0, u) \qquad \text{exists and is positive,} \tag{2.7}$$

and we set

$$D(\infty, \infty) = 0.$$

Finally, we shall need three more restrictions on D: Given a compact set $K \subseteq \mathbb{C}$, there exist positive constants C_1 and α such that

$$C_1 |z - w|^\alpha \leqslant D(z, w), \qquad z, w \in K. \tag{2.8A}$$

Further, we shall assume that there exist positive constants C_2 and β such that

$$D(z, w) \leqslant C_2 |z - w|^\beta, \qquad z, w \in \mathbb{C}. \tag{2.8B}$$

Given a bounded set $K \subseteq \mathbb{C}$,

$$\liminf_{|w| \to \infty} \min\{D(z, w): z \in K\} > 0. \tag{2.9}$$

We note that (2.6), (2.7), (2.8A), (2.8B), and (2.9) are satisfied by the chordal metric in (1.1).

(3) Corresponding to D, we define a distance between (Borel) measurable functions $f, g: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ by

$$\rho_D(f, g) = \iint D(f, g)\, d\mu. \tag{2.10}$$

(4) Throughout $C, C_1, C_2, C_3, \ldots$ denote positive constants independent of n and z. The same symbol C may denote different constants from line to line. $\{\Gamma_n\}_{n=1}^{\infty}$ will denote a sequence of positive numbers such that Γ_n/n increases as n increases, and

$$\lim_{n \to \infty} \Gamma_n/n = \infty. \tag{2.11}$$

The most important example is

$$\Gamma_n = n \log n, \qquad n = 2, 3, 4, \ldots,$$

which arises in considering meromorphic functions of finite order.

(5) Given non-negative integers n and q, the class of rational functions with complex coefficients and numerator and denominator degrees at most n and q, respectively, will be denoted by $\mathcal{R}_{nq}$.

We can now state our main results. The following is an analog of the de Montessus de Ballore theorem for Padé approximants (Baker [1, p. 139]). The distinguishing feature of such a result is that without any a priori assumptions about the poles of the rational functions we obtain uniform convergence of these functions.

THEOREM 2.1. *Let q be a non-negative integer. Let f be meromorphic in $\mathbb{C}$ with exactly q poles, counting multiplicity. Let ρ_D be a distance function satisfying* (2.1)–(2.10), *and let $\{\Gamma_n\}$ satisfy* (2.11). *Let there be given rational functions $R_{nq} \in \mathcal{R}_{nq}$, $n = 1, 2, 3, \ldots$, such that*

$$\limsup_{n \to \infty} \rho_D(f, R_{nq})^{1/\Gamma_n} < 1. \tag{2.12}$$

Then the only limit points of the poles of $\{R_{nq}\}_{n=1}^{\infty}$ *are the poles of* f *and*

$$\limsup_{n\to\infty} |f(z)-R_{nq}(z)|^{1/\Gamma_n}<1, \tag{2.13}$$

uniformly in compact subsets of $\mathbb{C}$ *not containing the poles of* f.

The next result shows that, under certain additional assumptions, one can find a sequence of rational functions satisfying (2.12).

THEOREM 2.2. *Let* $f(z)=g(z)/P(z)$, *where* $P(z)$ *is a monic polynomial of degree* q, *and* $g(z)=\sum_{j=0}^{\infty} a_j z^j$ *is entire, with*

$$\limsup_{j\to\infty} |a_j|^{1/\Gamma_j}<1. \tag{2.14}$$

Assume further that

$$\liminf_{n\to\infty} Q(\rho^{-\Gamma_n/n})/\Gamma_n>0, \tag{2.15}$$

for all $0<\rho<1$. *Let*

$$R_{nq}(z)=\left(\sum_{j=0}^{n} a_j z^j\right)\Big/P(z), \qquad n=1,2,3,..., \tag{2.16}$$

then (2.12) *holds.*

Remarks. (i) The condition (2.15) can be substantially weakened, but we omit the more cumbersome formulation.

(ii) Theorem 1.1 is an easy consequence of Theorem 2.1 and 2.2—see Section 5.

(iii) It is easy to see that for a given function $f(z)$ that is meromorphic in $\mathbb{C}$ with exactly q poles, one can always represent f in the form $f=g/P$, as in Theorem 2.2, and one can find $\{\Gamma_n\}$ and Q satisfying (2.2), (2.11), (2.14), and (2.15). Hence, Theorems 2.1 and 2.2 may be applied in a wide variety of situations.

(6) In some proofs, we shall use the concept of one dimensional Hausdorff content. Given $E\subseteq\mathbb{C}$, we set

$$m_1(E)=\inf\left\{\sum_{i=1}^{\infty} d(B_i)\colon E\subseteq\bigcup_{i=1}^{\infty} B_i\right\}.$$

Here the inf is over all countable sequences of balls $\{B_i\}$ with diameters $\{d(B_i)\}$ whose union covers E. Given functions $f, f_1, f_2, ...,$ and a bounded measurable set K, we say that f_n converges to f in m_1-measure in K if for each $\varepsilon > 0$

$$m_1\{z \in K: |f_n(z) - f(z)| > \varepsilon\} \to 0 \qquad \text{as} \quad n \to \infty.$$

Further we say that f_n converges to f almost everywhere with respect to m_1 in K if

$$m_1\{z \in K: \limsup_{n \to \infty} |f_n(z) - f(z)| > 0\} = 0.$$

3. Preliminary Lemmas

Lemma 3.1 (Cartan's lemma for planar Lebesgue measure). *Let $P(z)$ be a monic polynomial of degree $n \geqslant 1$. Let $H > 0$. Then the inequality*

$$|P(z)| > H^n$$

holds outside at most n balls, whose union has planar measure at most $4\pi e H^2$. Further, the sum of diameters of these balls is at most $4\pi e H$.

Proof. See Baker [1, p. 194]. ∎

Lemma 3.2 (Goncar). *Suppose that the sequence $\{f_n\}$ converges to the function f in m_1 measure inside the domain Ω. Assume also that each of the functions f_n $(n = 1, 2, ...)$ is meromorphic in Ω and has at most q $(< \infty)$ poles in Ω. Then*

(a) *f is meromorphic and has at most q poles in Ω.*

(b) *If f has exactly q poles in Ω, then for n large enough, we have that f_n also has exactly q poles in Ω, and the poles of the functions f_n tend to the poles $z_1, z_2, ..., z_q$ (taking account of multiplicity) of f, and the sequence $\{f_n\}$ converges uniformly on the compact subsets of $\Omega' = \Omega / \{z_i\}_{i=1}^{q}$.*

Proof. See Goncar [2, p. 507]. ∎

Lemma 3.3. *Let $0 < \varepsilon < 1$ and $r > 0$. Assume $P(z)$ is a polynomial of degree at most n, satisfying*

$$\operatorname{meas}\{z: |z| \leqslant r \quad and \quad |P(z)| \leqslant 1\} \geqslant \varepsilon. \tag{3.1}$$

Then for $|z| \leqslant r$,

$$|P(z)| \leqslant (c \max\{1, r\}/\varepsilon)^n, \tag{3.2}$$

where c is independent of n, P, ε, and r.

Proof. We note that the result is trivial if $P \equiv 0$. So assume $P \not\equiv 0$. Then we may write

$$|P(z)| = \delta \prod_{i=1}^{m} |z - z_i| \prod_{i=m+1}^{n'} |1 - z/z_i|, \tag{3.3}$$

where $\delta > 0$, $0 \leqslant m \leqslant n' \leqslant n$, $z_1, z_2, \ldots, z_m$ lie in $\{z: |z| \leqslant 2r\}$ and $z_{m+1}, \ldots, z_{n'}$ lie in $\{z: |z| > 2r\}$. Suppose first $m = 0$. Then

$$\begin{aligned} |P(z)| &= \delta \prod_{i=1}^{n'} |1 - z/z_i| \\ &\geqslant \delta/2^{n'} \qquad \text{for} \quad |z| \leqslant r. \end{aligned} \tag{3.4}$$

Since the set in (3.1) is non-empty, we deduce

$$\delta/2^{n'} \leqslant 1$$

so that, by (3.4), for $|z| \leqslant r$,

$$\begin{aligned} |P(z)| &\leqslant 2^{n'} \prod_{i=1}^{n} |1 - z/z_i| \\ &\leqslant 2^{n'}(3/2)^{n'} \\ &\leqslant 3^n. \end{aligned}$$

This establishes (3.2) in the case $m = 0$. Suppose next $m > 0$. By (3.3), for $|z| \leqslant r$,

$$|P(z)| \geqslant \delta \prod_{i=1}^{m} |z - z_i| \left(\tfrac{1}{2}\right)^{n' - m},$$

so that, by (3.1),

$$\begin{aligned} \varepsilon &\leqslant \text{meas}\left\{z: |z| \leqslant r \quad \text{and} \quad \delta \prod_{i=1}^{m} |z - z_i| \left(\tfrac{1}{2}\right)^{n'-m} \leqslant 1\right\} \\ &\leqslant \text{meas}\left\{z: \prod_{i=1}^{m} |z - z_i| \leqslant \delta^{-1} 2^{n'-m}\right\} \\ &\leqslant 4e\pi(\delta^{-1} 2^{n'-m})^{2/m} \qquad \text{by Lemma 3.1.} \end{aligned}$$

Therefore,

$$\delta \leqslant (4e\pi/\varepsilon)^{m/2} 2^{n'-m} \leqslant (c/\varepsilon)^n,$$

where c is independent of n, P, ε, and r. Then, by (3.3), for $|z| \leqslant r$,

$$|P(z)| \leqslant (c/\varepsilon)^n (3r)^m (3/2)^{n'-m}$$
$$\leqslant (c \max\{1, r\}/\varepsilon)^n. \quad \blacksquare$$

4. Proof of Theorem 2.1

We divide the proof of Theorem 2.1 into a series of lemmas. Throughout this section we assume that f and $\{R_{nq}\}$ are as in Theorem 2.1, and that, in particular, $\{R_{nq}\}$ satisfies (2.12).

LEMMA 4.1. *Let* $r \geqslant 1$, $\{z_i\}_{i=1}^q$ *be the poles of* f. *Let* $0 < \eta < \min\{1, \pi r^2/(2q)\}$. *Then there exists* $0 < \delta < 1 < S$ *such that for all* n *large enough,*

$$\operatorname{meas}\{z\colon |z| \leqslant r \text{ and } |z - z_i| \geqslant \eta, \quad i = 1, 2, ..., q \text{ and } |f - R_{nq}|(z) \geqslant \delta^{\Gamma_n}\} \leqslant S^{-\Gamma_n}. \tag{4.1}$$

Proof. By (2.10) and (2.12), there exists $0 < \theta < 1$ such that for large enough n,

$$\iint D(f, R_{nq})\, du < \theta^{\Gamma_n}. \tag{4.2}$$

Let $K > 1$ be such that $K\theta < 1$, and let

$$\mathscr{G}_n = \{z\colon |z| \leqslant r \text{ and } |z - z_i| \geqslant \eta,\ i = 1, 2, ..., q \text{ and } D(f(z), R_{nq}(z)) \geqslant (K\theta)^{\Gamma_n}\}.$$

Now, by (4.2),

$$\mu(\mathscr{G}_n)(K\theta)^{\Gamma_n} \leqslant \theta^{\Gamma_n}.$$

Therefore,

$$\mu(\mathscr{G}_n) \leqslant K^{-\Gamma_n}. \tag{4.3}$$

Hence, by (2.5) and (4.3),

$$\operatorname{meas}(\mathscr{G}_n) \leqslant e^{Q(r)} \mu(\mathscr{G}_n) \leqslant (K')^{-\Gamma_n} \tag{4.4}$$

for n large enough, where $K' > 1$. Now let

$$F = \{z\colon |z| \leqslant r \text{ and } |z - z_i| \geqslant \eta,\ i = 1, 2, ..., q\}.$$

Then $f(F) = \{f(z)\colon z \in F\}$ is bounded.

Further, if $z \in F \backslash \mathscr{G}_n$, then for any $\varepsilon > 0$, we have

$$D(f(z), R_{nq}(z)) < (K\theta)^{\Gamma_n} < \varepsilon \tag{4.5}$$

for n large enough. It follows from (2.9) that there exists a positive constant C and a positive integer n_0 such that

$$|R_{nq}(z)| \leqslant C, \qquad z \in F \backslash \mathscr{G}_n, \ n \geqslant n_0.$$

Then by (2.8A), there exists C_1 and $\alpha > 0$ independent of n such that

$$C_1 |f(z) - R_{nq}(z)|^{\alpha} \leqslant D(f(z), R_{nq}(z)), \qquad z \in F \backslash \mathscr{G}_n, \ n \geqslant n_0.$$

Hence, by (4.5), for $n \geqslant n_0$ and $z \in F \backslash \mathscr{G}_n$,

$$\begin{aligned} |f(z) - R_{nq}(z)| &< C_2 (K\theta)^{\Gamma_n \cdot \alpha} \\ &\leqslant \delta^{\Gamma_n}, \end{aligned} \tag{4.6}$$

where $\delta \in (0, 1)$. Finally, if

$$z \in \mathbb{H}_n = \{z : |z| \leqslant r \text{ and } |z - z_i| \geqslant \eta, \ i = 1, 2, ..., q, \\ \text{and } |f - R_{nq}|(z) \geqslant \delta^n\},$$

then $z \in F$. But $z \notin F \backslash \mathscr{G}_n$ by (4.6). Hence,

$$\mathbb{H}_n \subseteq F \cap \mathscr{G}_n.$$

In particular, it follows from (4.4) that

$$\text{meas}(\mathbb{H}_n) \leqslant \text{meas}(\mathscr{G}_n) \leqslant (K')^{-\Gamma_n}.$$

Hence the result. ∎

LEMMA 4.2. *Under the conditions of Lemma* 4.1 *there exists* $0 < \delta' < 1 < S'$ *such that, for all n large enough,*

$$\text{meas}\{z : |z| \leqslant r \text{ and } |z - z_i| \geqslant \eta, \ i = 1, 2, ..., q, \\ \text{and } |R_{n+1,q}(z) - R_{n,q}(z)| \geqslant (\delta')^{\Gamma_n}\} \leqslant (S')^{-\Gamma_n}. \tag{4.7}$$

Proof. If

$$|R_{n+1,q}(z) - R_{n,q}(z)| \geqslant 2\delta^{\Gamma_n}$$

then either

$$|f(z) - R_{nq}(z)| \geqslant \delta^{\Gamma_n}$$

or

$$|f(z)-R_{n+1,q}(z)|\geqslant \delta^{\Gamma_{n+1}}.$$

Hence, by Lemma 4.1

$$\text{meas}\{z: |z|\leqslant r \text{ and } |z-z_i|\geqslant \eta,\ i=1,2,\ldots,q, \\ \text{and } |R_{n+1,q}(z)-R_{nq}(z)|\geqslant 2\delta^{\Gamma_n}\}\leqslant 2S^{-\Gamma_n}.$$

The lemma follows by choosing suitable δ' and S'. ∎

LEMMA 4.3. *Assume the conditions of Lemmas* 4.1 *and* 4.2. *Further, write*

$$R_{nq}(z)=P_n(z)/Q_n(z), \tag{4.8}$$

where P_n and Q_n are polynomials of degree at most n and q, respectively, and Q_n is normalized so that

$$Q_n(z)=\prod_{|z_{ni}|\leqslant 2r}(z-z_{ni})\prod_{|z_{ni}|>2r}(1-z/z_{ni}), \tag{4.9}$$

and $z_{n1}, z_{n2},\ldots, z_{nq}$ are the zeros of Q_n. Let $\Delta>1$ and

$$E_n=\left\{z: \prod_{|z_{ni}|\leqslant 2r}|z-z_{ni}|\leqslant n^{-\Delta q}\right\}, \qquad n=1,2,3,\ldots. \tag{4.10}$$

Then there exists $0<\delta_1<1$ such that for n large enough,

$$|R_{n+1,q}(z)-R_{nq}(z)|\leqslant \delta_1^{\Gamma_n} \qquad \textit{for} \quad |z|\leqslant r, z\notin E_n\cup E_{n+1}. \tag{4.11}$$

Proof. For n large enough, let

$$H_n(z)=P_{n+1}(z)\,Q_n(z)-P_n(z)\,Q_{n+1}(z) \tag{4.12}$$

$$=(R_{n+1,q}(z)-R_{nq}(z))\,Q_n(z)\,Q_{n+1}(z). \tag{4.13}$$

From (4.9), and as $r>1$,

$$|Q_n(z)|\leqslant (3r)^q, \qquad \text{for} \quad |z|\leqslant r.$$

Therefore for $|z|\leqslant r$,

$$|H_n(z)|\leqslant (3r)^{2q}\,|R_{n+1,q}(z)-R_{nq}(z)|.$$

From Lemma 4.2,

$$\text{meas}\{z: |z|\leqslant r \text{ and } |z-z_i|\geqslant \eta,\ (i=1,2,\ldots,q) \\ \text{and } |H_n(z)|\geqslant (3r)^{2q}\,(\delta')^{\Gamma_n}\}\leqslant (S')^{-\Gamma_n}.$$

Let $\delta' < \delta'' < 1$. Then for n large enough,

$$\begin{aligned}\operatorname{meas}\{z: |z| \leqslant r \text{ and } |H_n(z)| \leqslant (\delta'')^{\Gamma_n}\} &\geqslant \pi r^2 - q\eta - (S')^{-\Gamma_n} \\ &\geqslant \pi r^2/2 - (S')^{-\Gamma_n} \\ &\geqslant \tfrac{1}{2},\end{aligned}$$

as $q\eta \leqslant \pi r^2/2$ and if n is large enough. Applying Lemma 3.3 to the polynomial

$$P(z) = H_n(z)(\delta'')^{-\Gamma_n},$$

which is of degree at most $n+q+1$, we obtain

$$\max\{|H_n(z)|: |z| \leqslant r\} \leqslant (\delta'')^{\Gamma_n} (c2r)^{n+1+q}, \tag{4.14}$$

where c is independent of n. Next, if $|z_{ni}| \geqslant 2r$ and $|z| \leqslant r$,

$$|1 - z/z_{ni}| \geqslant \tfrac{1}{2},$$

so from (4.9)

$$|Q_n(z)| \geqslant 2^{-q} \mathrm{n}^{-\Delta q}, \qquad |z| \leqslant r, z \notin E_n. \tag{4.15}$$

Then (4.13), (4.14), and (4.15) show that for $|z| \leqslant r$ and $z \notin E_n \cup E_{n+1}$.

$$|R_{n+1,q}(z) - R_{nq}(z)| \leqslant (\delta'')^{\Gamma_n} (c2r)^{n+1+q}\, 2^{2q} n^{\Delta q}(n+1)^{\Delta q}.$$

Clearly (4.11) follows with a suitable choice of δ_1. ∎

LEMMA 4.4. *Assume the conditions of Lemmas* 4.1, 4.2, *and* 4.3. *Then there exists a function g, defined everywhere except possibly on a set F of m_1-measure zero, such that R_{nq} converges to g in m_1-measure in $|z| \leqslant r$ as $n \to \infty$, and*

$$\lim_{n \to \infty} R_{nq}(z) = g(z), \qquad z \notin F.$$

Proof. For n large enough, let

$$F_n = \bigcup_{k=n}^{\infty} E_k. \tag{4.16}$$

Now by Cartan's lemma (Lemma 3.1), and by (4.10),

$$m_1(E_n) \leqslant 4e\pi n^{-\Delta}.$$

Hence,

$$m_1(F_n) \leqslant 4e\pi \sum_{k=n}^{\infty} k^{-\Delta}$$

$$\to 0 \qquad \text{as } n \to \infty, \text{ since } \Delta > 1. \tag{4.17}$$

Now if $|z| \leqslant r$, $z \notin F_n$, then by (4.11),

$$\sum_{k=n}^{\infty} |R_{k+1,q}(z) - R_{k,q}(z)| \leqslant \sum_{k=n}^{\infty} \delta_1^{\Gamma_k}.$$

Hence, for $|z| \leqslant r$, $z \notin F = \bigcap_{k=1}^{\infty} F_k$,

$$g(z) = \lim_{k \to \infty} R_{kq}(z)$$

exists. Note that for all $k = 1, 2, 3, \ldots$

$$m_1(F) \leqslant m_1(F_k) \to 0 \qquad \text{as} \quad k \to \infty$$

by (4.17). Hence $m_1(F) = 0$. Further, uniformly for $|z| \leqslant r$ and $z \notin F_n$,

$$|g(z) - R_{nq}(z)| = \left| \sum_{k=n}^{\infty} (R_{k+1,q}(z) - R_{k,q}(z)) \right|$$

$$\leqslant \sum_{k=n}^{\infty} \delta_1^{\Gamma_k}$$

$$\to 0 \qquad \text{as} \quad n \to \infty. \tag{4.18}$$

Hence, $R_{nq} \to g$ in m_1-measure in $|z| \leqslant r$, as $n \to \infty$. ∎

Proof of Theorem 2.1. We first show that the poles of R_{nq} tend to the poles of f. Let $r > 1$, and g be as in Lemma 4.4. By Lemmas 4.4 and 3.2(a), $g(z)$ must be meromorphic in $|z| < r$, with at most q poles there. Now (4.1) shows that R_{nq} converges in planar Lebesgue measure to f in compact subsets of $\{z: |z| < r\}$ not containing the poles of f. Further, every subsequence of a sequence which converges in planar Lebesgue measure contains a subsequence converging almost everywhere with respect to planar Lebesgue measure.

From Lemma 4.4 we deduce that $f = g$ in $|z| \leqslant r$, except possibly on a set of planar Lebesgue measure zero. As f and g are meromorphic in $|z| < r$, we deduce that $f(z) = g(z)$ in $|z| \leqslant r$. Since $r > 1$ is arbitrary and both f and R_{nq} have at most q poles in $\mathbb{C}$, Lemma 3.2(b) yields the result.

We next establish (2.13). Let $r \geqslant 1$ and let K be a compact subset of $\{z: |z| \leqslant r\}$ not containing the poles of f. Now the poles of R_{nq} tend to the poles of f as $n \to \infty$. Further, the set E_n, defined by (4.10), is contained in at most q circles of radius $n^{-\Delta}$ centered on the poles of R_{nq}. It follows that for n large enough, E_n does not intersect K, and hence F_n, defined by (4.16), does not intersect K. Then (4.18) shows that for $z \in K$ and for n large enough,

$$|f(z) - R_{nq}(z)| \leqslant \sum_{k=n}^{\infty} \delta_1^{\Gamma_k}$$
$$\leqslant 2\delta_1^{\Gamma_n}$$

for n large enough. This proves (2.13). ∎

5. Proof of Theorem 2.2

LEMMA 5.1. *Let $f(z) = g(z)/P(z)$, where $P(z)$ is a monic polynomial of degree q and $g(z) = \sum_{j=0}^{\infty} a_j z^j$ is entire, with*

$$\limsup_{j \to \infty} |a_j|^{1/\Gamma_j} < \theta < 1. \tag{5.1}$$

Let $0 < \varepsilon < 1 < r$ and

$$\eta_n = \theta^{\Gamma_n/n}, \quad n = 1, 2, \ldots. \tag{5.2}$$

Let β be as in (2.8B) and R_{nq} as in (2.16). Then there exists C and n_0 independent of ε and r such that

$$\rho_D(f, R_{nq}) \leqslant C(\varepsilon^{-1}\theta^{\Gamma_n} r^n)^{\beta} (1 - \eta_n r)^{\beta} + C\varepsilon^{1/q} + \mu\{z : |z| \geqslant r\} \tag{5.3}$$

for all $n \geqslant n_0$ for which

$$\eta_n r < 1. \tag{5.4}$$

Proof. Let

$$E_1 = \{z : |z| \leqslant r \text{ and } |P(z)| \geqslant \varepsilon\} \tag{5.5}$$

$$E_2 = \{z : |z| \leqslant r \text{ and } |P(z)| < \varepsilon\}. \tag{5.6}$$

Then as $D(z, u) \leqslant 1$ (by (2.6)),

$$\rho_D(f, R_{nq}) = \iint D(f, R_{nq})\, d\mu$$

$$\leqslant \iint_{E_1} D(f, R_{nq})\, d\mu + \mu(E_2) + \mu\{z : |z| \geqslant r\}. \tag{5.7}$$

Next, by (2.16) and (5.5), for $z \in E_1$,

$$|f(z) - R_{nq}(z)| \leqslant \left(\sum_{j=n+1}^{\infty} |a_j| \, r^j \right) \Big/ \varepsilon$$
$$\leqslant \left(\sum_{j=n+1}^{\infty} \theta^{\Gamma_j} r^j \right) \Big/ \varepsilon \tag{5.8}$$

(by (5.1) for n large enough, say for $n \geqslant n_1$). Now, by (2.11),

$$\Gamma_j/j \geqslant \Gamma_n/n \qquad \text{for} \quad j \geqslant n,$$
$$\Rightarrow \Gamma_j - \Gamma_n \geqslant \Gamma_n(j/n - 1) = (\Gamma_n/n)(j - n).$$

Hence, by (5.8), for $z \in E_1$,

$$|f(z) - R_{nq}(z)| \leqslant \theta^{\Gamma_n} r^n \varepsilon^{-1} \sum_{j=n}^{\infty} \theta^{\Gamma_j - \Gamma_n} r^{j-n}$$
$$\leqslant \theta^{\Gamma_n} r^n \varepsilon^{-1} \sum_{j=n}^{\infty} \theta^{(\Gamma_n/n)(j-n)} r^{j-n}$$
$$= \theta^{\Gamma_n} r^n \varepsilon^{-1} (1 - \eta_n r)^{-1} \tag{5.9}$$

by (5.2), for n satisfying (5.4). By (2.8B), for $z \in E_1$,

$$D(f(z), R_{nq}(z)) \leqslant C_2 |f(z) - R_{nq}(z)|^{\beta}$$
$$\leqslant C_2 (\varepsilon^{-1} \theta^{\Gamma_n} r^n)^{\beta} (1 - \eta_n r)^{-\beta}$$

for $n \geqslant n_0$ and $\eta_n r < 1$, by (5.9). Hence, by (2.1),

$$\iint_{E_1} D(f, R_{nq}) \, d\mu \leqslant C_2 (\varepsilon^{-1} \theta^{\Gamma_n} r^n)^{\beta} (1 - \eta_n r)^{-\beta} \tag{5.10}$$

for n large enough. Next, by Cartan's lemma (Lemma 3.1)

$$\text{meas}(E_2) \leqslant (4\pi e) \, \varepsilon^{1/q}$$

and then by (2.4),

$$\mu(E_2) \leqslant (4\pi e) \, \varepsilon^{1/q}. \tag{5.11}$$

Finally (5.7), (5.10), and (5.11) yield the result. ∎

LEMMA 5.2. *Let $r > 0$. Then there exists C independent of r such that*

$$\mu\{z : |z| \geqslant r\} \leqslant C \exp(-Q(r)). \tag{5.12}$$

Proof. By (2.3) and converting to polar coordinates, we obtain

$$\begin{aligned}\mu\{z : |z| \geqslant r\} &= 2\pi \int_r^\infty s \exp(-Q(s))\, ds \\ &= 2\pi \int_r^\infty (s/Q'(s)) \exp(-Q(s))\, Q'(s)\, ds \\ &\leqslant (2\pi r/Q'(r)) \int_r^\infty \exp(-Q(s))\, Q'(s)\, ds \qquad \text{(by (2.2))} \\ &\leqslant (2\pi/Q'(1)) \exp(-Q(r)). \quad \blacksquare\end{aligned}$$

Proof of Theorem 2.2. With the notation of Lemma 5.1, let

$$r_n = \eta_n^{-1/2}, \qquad n = 1, 2, \ldots, \tag{5.13}$$

so that

$$\eta_n r_n = \eta_n^{1/2} \to 0 \qquad \text{as} \quad n \to \infty$$

by (5.2), (2.11), and as $0 < \theta < 1$. Then (5.4) holds for $r = r_n$ and n large enough. Further, by (5.2) and (5.13),

$$r_n^n = \theta^{-\Gamma_n/2}. \tag{5.14}$$

Then by (5.3), (5.12), and (5.14) with $r = r_n$

$$\begin{aligned}\rho_D(f, R_{nq}) \leqslant{} & C_1(\varepsilon^{-1}\theta^{\Gamma_n/2})^\beta (1 - \eta_n^{1/2})^\beta \\ & + C_1 \varepsilon^{1/q} + C \exp(-Q(\eta_n^{-1/2})).\end{aligned} \tag{5.15}$$

We note that the choice of $r = r_n$ is possible as the constants in (5.3) are independent of r and ε. Now let

$$\varepsilon = \varepsilon_n = \theta^{\Gamma_n/4}. \tag{5.16}$$

By (5.2) and (2.15) with $\rho = \theta^{-1/2}$, there exists $C_2 > 0$ such that

$$Q(\eta_n^{-1/2}) \geqslant C_2 \Gamma_n, \qquad n \text{ large enough.} \tag{5.17}$$

Then by (5.15), (5.16), and (5.17) there exists C_3 such that

$$\rho_D(f, R_{nq}) \leqslant C_3(\theta^{\Gamma_n/4})^\beta + C_3 \theta^{\Gamma_n/(4q)} + C_3 \exp(-C_2 \Gamma_n).$$

Then (2.12) follows. $\blacksquare$

Proof of Theorem 1.1. Let f be meromorphic in $\mathbb{C}$, of finite order, with poles of total multiplicity q. Let $P(z)$ be the monic polynomial of degree q

whose roots are the poles of f, taking account of multiplicity. Then $g(z)=f(z)\,P(z)$ is entire, and by the elementary theory of meromorphic functions (Hayman [3]), g has the same order as f. Further, we can write

$$g(z)=\sum_{j=0}^{\infty} a_j z^j,$$

where, as g has finite order,

$$\limsup_{j\to\infty} |a_j|^{1/(j\log j)}<1. \tag{5.18}$$

If we choose

$$\Gamma_j=j\log j, \qquad j=2, 3,...$$

then (2.14) follows fro (5.18). We can apply Theorem 2.2 provided we can verify (2.15). In our case, $Q(z)=\exp|z|$, so (2.15) is equivalent to

$$\liminf_{n\to\infty} \exp(\rho^{-\log n})/(n\log n)>0$$

for all $0<\rho<1$. This is easy to verify. Hence R_{nq} given by (2.16) satisfies (2.12). Consequently if R^*_{nq} is a best approximation (as in (1.3)) (2.12) implies (1.4). Finally (1.5) is implied by (2.13). ∎

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References

1. G. A. Baker, Jr, "Essentials of Padé Approximants," Academic Press, New York, 1975.
2. A. A. Goncar, On the convergence of generalized Padé approximants of meromorphic functions, *Math. USSR-Sb.* **27** (1975), 503–514.
3. W. K. Hayman, "Meromorphic Functions," Oxford Univ. Press, London/New York, 1964.
4. D. S. Lubinsky and O. Shisha, Best approximation over the whole complex plane, *J. Approx. Theory* **36** (1982), 277–293.
5. H. Wallin, Potential theory and approximation of analytic functions by rational interpolation, *in* "Proceedings, Colloq. on Complex Analysis at Joensuu," pp. 434–450, Lecture Notes in Math. Vol. 747, Springer-Verlag, Berlin/New York, 1979.