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Invariant tori for commuting Hamiltonian PDEs

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ABSTRACT

We generalize to some PDEs a theorem by Eliasson and Nekhoroshev on the persistence of invariant tori in Hamiltonian systems with r integrals of motion and n degrees of freedom, $r \leqslant n$. The result we get ensures the persistence of an r-parameter family of r-dimensional invariant tori. The parameters belong to a Cantor-like set. The proof is based on the Lyapunov–Schmidt decomposition and on the standard implicit function theorem. Some of the persistent tori are resonant. We also give an application to the nonlinear wave equation with periodic boundary conditions on a segment and to a system of coupled beam equations. In the first case we construct 2-dimensional tori, while in the second case we construct 3-dimensional tori.

1. Introduction

In the nineties Eliasson [13] and independently Nekhoroshev [18] proved a theorem on persistence of invariant tori in Hamiltonian systems with symmetry (see also [3]). Such a theorem interpolates between Poincaré's theorem on persistence of periodic orbits and Arnold–Liouville theorem on the structure of the phase space of integrable systems.

More precisely Eliasson–Nekhoroshev's theorem pertains a system of r commuting Hamiltonians $H^{(1)}, \dots, H^{(r)}$ in a $2n$-dimensional phase space, $n \geqslant r$; assume (i) that there exists one r-dimensional torus $\mathbb{T}_0^r$ invariant under the flow of each of the Hamiltonians $H^{(l)}$ and (ii) that a suitable non-resonance condition is fulfilled. Then Eliasson–Nekhoroshev's theorem ensures that there exists a $2r$-dimensional symplectic manifold N on which each system is integrable; in particular N is foliated in r-dimensional tori which are invariant under the flow of each system. An application to some infinite-dimensional systems was given in [8] where the existence of quasiperiodic breathers in some chains of particles with symmetry was proved.

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In the present paper we extend Eliasson–Nekhoroshev's theorem to a suitable class of Hamiltonian PDEs and give applications to a nonlinear Klein–Gordon equation with periodic boundary conditions on a segment and to a system of coupled beam equations. In the first case we construct 2-dimensional tori, while in the second case we construct 3-dimensional tori.

The extension is far from trivial since the case of PDEs involves small denominators. To overcome such a problem we have to impose (i) a quantitative nonresonance condition that generalizes the assumption of Eliasson–Nekhoroshev's theorem and (ii) a further nondegeneracy condition similar to that used in KAM theory. Moreover, the proof we give here is completely different from the one of [3] which was based on the use of generalized Poincaré sections. On the contrary the present proof is based on the Lyapunov–Schmidt decomposition and on the use of a strong nonresonance condition (see [2]) which allows to avoid KAM-type techniques. Indeed we only use the standard implicit function theorem.

Related results were obtained in the papers [1,7,19] in which the authors exploited translation invariance in order to construct some quasiperiodic solutions of the nonlinear wave equation. While the theory of [1,7,19] is based on the structure of the wave equation, the theory we develop here applies to the general case of commuting Hamiltonian systems, including the case where all the Hamiltonians are nonlinear. Our theory however does not apply directly to the wave equation treated in [1,7,19] since in that case our nonresonance assumption is violated.

We also recall that persistence of invariant tori in systems with symmetries is often studied by passing to the reduced system and then using continuation techniques for periodic orbits of such a reduced system (see e.g. [9,17]). Such techniques do not seem to be applicable to the case of PDEs where often the orbits of the symmetry group are continuous but not differentiable (they are orbits of a partial differential equation).

Persistence of invariant tori in Hamiltonian PDEs is usually obtained through KAM techniques (see e.g. [4,5,10–12,14]). The main difference with the previous papers is that we deal here with systems with symmetry. This allows to obtain persistence of invariant tori using just the standard implicit function theorem based on the contraction mapping principle. Moreover the flow on the tori whose existence is proved here is not necessarily ergodic. Actually, in the case of the wave equation we prove the existence of some tori on which the flow is rational.

The paper is organized as follows. In Section 2 we state the main theorem of the paper, namely the generalization to some PDEs of Eliasson–Nekhoroshev's theorem. In Section 3 we give the proof of the main theorem. Such a section is split into a few subsections. Finally in Section 4 we give the application to the nonlinear wave and the nonlinear beam equation.

2. Statement of the main result

Fix $r \geqslant 1$; consider the spaces $\ell^2_{s,\sigma}$ of the sequences $p = \{p_j\}_{j \geqslant r+1}$ s.t.

$$\|p\|^2_{s,\sigma} := \sum_j [j]^{2s} e^{2\sigma j} |\hat{p}_j|^2 < \infty, \quad [j] := \max\{1, j\}, \tag{2.1}$$

and define the phase spaces as

$$\mathcal{P}_{s,\sigma} := \mathcal{U} \times \mathbb{T}^r \times \ell^2_{s,\sigma} \times \ell^2_{s,\sigma} \ni (I, \phi, p, q) \equiv z \tag{2.2}$$

where $\mathcal{U} \subset \mathbb{R}^r$ is open, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We endow $\mathcal{P}_{s,\sigma}$ with the norm (on the tangent space of $\mathcal{P}_{s,\sigma}$)

$$\|z\|^2_{s,\sigma} = \|I\|^2 + \|\phi\|^2 + \|p\|^2_{s,\sigma} + \|q\|^2_{s,\sigma}. \tag{2.3}$$

We will also use the weak scalar product

$$\big\langle (I, \phi, p, q); (I', \phi', p', q') \big\rangle := \sum_{j=1}^{r} \big(I_j I'_j + \phi_j \phi'_j\big) + \sum_{j \geqslant r+1} \big(p_j p'_j + q_j q'_j\big). \tag{2.4}$$

The open ball of radius R and center 0 in $\mathbb{R}^r$, $\mathcal{P}_{s,\sigma}$ or $\ell^2_{s,\sigma}$ will be denoted by B_R. Define the Poisson tensor $\mathcal{J}$ by

$$\mathcal{J}(I,\phi,p,q)=(-\phi,I,-q,p). \tag{2.5}$$

Then, given a Hamiltonian function H on $\mathcal{P}_{s,\sigma}$, we will denote by ∇H the gradient of H with respect to the scalar product (2.4), namely

$$\nabla H:=\left(\frac{\partial H}{\partial I},\frac{\partial H}{\partial \phi},\frac{\partial H}{\partial p},\frac{\partial H}{\partial q}\right), \tag{2.6}$$

and by $X_H:=\mathcal{J}\nabla H$ the Hamiltonian vector field.

In $\mathcal{P}_{s,\sigma}$ consider r Hamiltonian functions $H^{(l)}_\mu$, $l=1,\dots,r$, of the form

$$H^{(l)}_\mu=I_l+\sum_{j\geqslant r+1}\Omega^{(l)}_j\frac{p_j^2+q_j^2}{2}+\mu F^{(l)}_\mu(z). \tag{2.7}$$

When $\mu=0$ the submanifold

$$N:=\mathcal{U}\times\mathbb{T}^r\times\{0\}\times\{0\} \tag{2.8}$$

is foliated in tori which are invariant under the flow of each of the Hamiltonians. We are interested in the persistence of such tori as invariant tori of each of the Hamiltonians $H^{(l)}_\mu$, when μ is different from zero.

Remark 2.1. Consider r commuting Hamiltonians $K^{(1)},\dots,K^{(r)}$, and assume that there exists a torus which is invariant under the flow of each of the fields $X_{K^{(l)}}$. Assume also that the fields $X_{K^{(l)}}$ are linearly independent on such a torus. Then, working as in the proof of Arnold–Liouville theorem, one can prove that, on the torus, the flow of each of the systems is quasiperiodic. *Moreover, the torus is invariant under the Hamiltonian flow of any Hamiltonian which is a linear combination of the $K^{(l)}$'s.*

Remark 2.2. In the above situation, assume also that the invariant torus is elliptic for each flow. If the phase space is finite-dimensional, then it can be shown [15] that there always exist coordinates in which the Hamiltonians have the form

$$K^{(l)}=\sum_{j=1}^{r}\omega^{(l)}_j I_j+\sum_{j\geqslant r+1}\omega^{(l)}_j\frac{p_j^2+q_j^2}{2}+\text{h.o.t.}, \tag{2.9}$$

where h.o.t. are terms which are of higher order in I, p, q and depend on ϕ also. Then, one can construct linear combination of the $K^{(l)}$'s having the form (2.7). So, in this case, it is equivalent to consider the Hamiltonians (2.9) or the Hamiltonians (2.7). In [16] Kuksin studied in detail the case of PDEs, showing that coordinates in which (2.9) holds exist also in quite general PDE cases. For this reason here we will directly assume the form (2.7).

In what follows we will often omit the index μ of the Hamiltonians and of the functions $F^{(l)}_\mu$ provided this does not create ambiguities.

We assume that:

(A.1) For any μ small enough one has

$$\left\{H^{(l_1)}_\mu;H^{(l_2)}_\mu\right\}=0,\quad \forall l_1,l_2=1,\dots,r. \tag{2.10}$$

(A.2) There exist $d \geqslant 0$ and $s_* \in \mathbb{R}$ such that $\nabla F_\mu^{(l)} \in C^\infty(\mathcal{V}_{s,\sigma}, \mathcal{P}_{s+d,\sigma})$ for all $s > s_*$ and for some fixed $\sigma \geqslant 0$; here $\mathcal{V}_{s,\sigma} \subset \mathcal{P}_{s,\sigma}$ is an open neighborhood of N.

Remark 2.3. In (A.2) we are assuming that the nonlinear part of the vector field is smoothing. Precisely we assume that the nonlinearity allows to gain d derivatives. This is typical when the system comes from a second order in time equation.

We come now to the nonresonance assumption that generalizes to the case of PDEs Eliasson–Nekhoroshev's nonresonance assumption.

For $\gamma > 0$, $\tau \in \mathbb{R}$, and $\mathbf{n} := (n_1, \dots, n_r) \in \mathbb{Z}^r$ consider the set $\mathcal{N}(\gamma, \tau, \mathbf{n})$ of the $(\epsilon_1, \dots, \epsilon_r) \in \mathbb{R}^r$ such that

$$\left| k - \sum_{l=1}^{r} (n_l + \epsilon_l) \Omega_j^{(l)} \right| \geqslant \frac{\gamma}{j^\tau}, \quad \forall k \in \mathbb{Z},\ j \geqslant r+1. \tag{2.11}$$

(A.3) There exist $\mathbf{n} \in \mathbb{Z}^r$, $\tau \leqslant d$ and $\gamma > 0$ such that, for any open set $\mathcal{O} \subset \mathbb{R}^r$, whose closure contains zero, one has that $\mathcal{N}(\gamma, \tau, \mathbf{n}) \cap \mathcal{O}$ has zero as an accumulation point.

Remark 2.4. Assumption (A.3) means that, for some $\mathbf{n}$ the set of the frequencies which are sufficiently nonresonant accumulates at the origin from any direction. Moreover we need $\tau \leqslant d$, with d the amount of smoothing of the nonlinear part of the vector field.

Finally we need a nondegeneracy condition. In order to state it we consider the average of the nonlinearity under a suitable periodic flow. Let $\mathbf{n} \in \mathbb{Z}^r$ be an integer vector. Define

$$F_{\mathbf{n}} := \sum_{l=1}^{r} n_l F_\mu^{(l)} \Big|_{\mu=0}, \tag{2.12}$$

and

$$\langle F_{\mathbf{n}} \rangle (a, \psi) := \frac{1}{2\pi} \int_0^{2\pi} F_{\mathbf{n}}(a, \mathbf{n}t + \psi, 0, 0)\, dt. \tag{2.13}$$

First of all we have the following lemma which will play an important role in the proof of our main result.

Lemma 2.1. *For any $\mathbf{n} \in \mathbb{Z}^r$ the function $\langle F_{\mathbf{n}} \rangle (a, \psi)$ is independent of ψ.*

Proof. The proof is based on a representation formula for the functions $F_0^{(l)}$ that we are now going to derive. First remark that assumption (A.1), written explicitly, takes the form

$$0 = \mu \left[\frac{\partial F_\mu^{(l_1)}}{\partial \phi_{l_2}} - \frac{\partial F_\mu^{(l_2)}}{\partial \phi_{l_1}} \right] + \mu \left\{ H_T^{(l_1)}; F_\mu^{(l_2)} \right\} + \mu \left\{ F_\mu^{(l_1)}; H_T^{(l_2)} \right\} + \mu^2 \left\{ F_\mu^{(l_1)}; F_\mu^{(l_2)} \right\} \tag{2.14}$$

where we denoted

$$H_T^{(l)} := \sum_{j \geqslant r+1} \Omega_j^{(l)} \frac{p_j^2 + q_j^2}{2}.$$

Restrict Eq. (2.14) to the manifold N. The terms containing $H_T^{(l)}$ vanish since they are at least linear in p and q. Thus on this submanifold Eq. (2.14) takes the form

$$\frac{\partial F_\mu^{(l_1)}}{\partial \phi_{l_2}} - \frac{\partial F_\mu^{(l_2)}}{\partial \phi_{l_1}} = -\mu\left\{F_\mu^{(l_1)}; F_\mu^{(l_2)}\right\}. \tag{2.15}$$

Evaluating at $\mu = 0$ one gets

$$\frac{\partial F_0^{(l_1)}}{\partial \phi_{l_2}} - \frac{\partial F_0^{(l_2)}}{\partial \phi_{l_1}} = 0. \tag{2.16}$$

Using the fact that the generators of the first cohomological class of the torus are $d\phi_l$, $l = 1, \dots, r$, one has that there exists a function V on $\mathbb{T}^r$ parametrically depending on I, and a set of functions $c_l(I)$, independent of ϕ, such that

$$F_0^{(l)}(I, \phi) = \frac{\partial V}{\partial \phi_l} + c_l(I). \tag{2.17}$$

Inserting such a representation formula in the definition of $\langle F_{\mathbf{n}}\rangle$ one gets that such a function is the sum of two terms, one of which is independent of ϕ. The other term contains V and is proportional to

$$\sum_l \int_0^{2\pi} n_l \frac{\partial V}{\partial \phi_l}(\mathbf{n}t + \psi, I)\, dt = \int_0^{2\pi} \frac{d}{dt} V(\mathbf{n}t + \psi, I)\, dt = V(\mathbf{n}2\pi + \psi) - V(\psi) = 0 \tag{2.18}$$

which implies that $\langle F_{\mathbf{n}}\rangle(I, \psi) = \sum_l n_l c_l(I)$. □

For this reason we will simply write $\langle F_{\mathbf{n}}\rangle(a, \psi) \equiv \langle F_{\mathbf{n}}\rangle(a)$.

(A.4) There exists $\mathbf{n} \in \mathbb{Z}^r$ such that (A.3) is fulfilled and

$$\det\left(\frac{\partial^2 \langle F_{\mathbf{n}}\rangle}{\partial a^2}(a)\right) \neq 0, \quad \forall a \in \mathcal{U}, \tag{2.19}$$

and $\frac{\partial \langle F_{\mathbf{n}}\rangle}{\partial a} : \mathcal{U} \to \mathbb{R}^r$ is a 1–1 map.

Remark 2.5. In applications the frequencies often depend on an external parameter m. Maybe assumption (A.4) can be substituted by a weak nondegeneracy assumption for such a dependence (see e.g. [20]). We did not succeed in completing the proof with such a weaker assumption. This is due to the fact that the solution of Eq. (3.23) below is usually not a Lipschitz function of $\epsilon \in \mathcal{N}$.

Fix a set $\mathcal{U}'$ whose closure is contained in $\mathcal{U}$. Define

$$\mathcal{A}(\mathbf{n}) := \bigcup_{|\mu|<1} \bigcup_{a \in \mathcal{U}'} \left\{-\mu \frac{\partial \langle F_{\mathbf{n}}\rangle}{\partial a}(a)\right\} \subset \mathbb{R}^r \tag{2.20}$$

and assume that this is an open set. Since it has zero as an accumulation point, it has a nonempty intersection with $\mathcal{N} := \mathcal{N}(\gamma, \tau, \mathbf{n})$. For any μ consider the set $\mathcal{A}(\mathbf{n}) \cap \mathcal{N} \cap B_\mu$. Moreover, for any $\epsilon \in \mathcal{A}(\mathbf{n}) \cap \mathcal{N} \cap B_\mu$ define $a_*(\epsilon/\mu)$ as the solution of

$$\frac{\epsilon}{\mu} = -\frac{\partial \langle F_{\mathbf{n}}\rangle}{\partial a}(a_*). \tag{2.21}$$

We use it to define a reference torus

$$\mathbb{T}_{\epsilon,0} := \bigcup_{\psi \in \mathbb{T}^r} \{(a_*, \psi, 0, 0)\}. \tag{2.22}$$

Then we have the following theorem.

Theorem 2.1. *Fix σ, then for any $s > s_*$ there exist two constants μ_* and C such that the following holds. For any $|\mu| < \mu_*$, any $\epsilon \in \mathcal{A} \cap \mathcal{N} \cap B_\mu$, there exists a smooth torus $\mathbb{T}_{\epsilon,\mu} \subset \mathcal{P}_{s,\sigma}$ which is invariant under the flow of the Hamiltonian vector fields of each of the Hamiltonians $H_\mu^{(l)}$. Moreover one has*

$$d(\mathbb{T}_{\epsilon,\mu}, \mathbb{T}_{\epsilon,0}) < C\mu \tag{2.23}$$

where $d(.,.)$ denotes the Hausdorff distance in $\mathcal{P}_{s,\sigma}$.

Remark 2.6. As in the first step of the proof of Arnold–Liouville's theorem one has that the flow of each of the Hamiltonians on the torus $\mathbb{T}_{\epsilon,\mu}$ is the Kronecker linear flow.

Remark 2.7. From the proof it will also turn out that the flow of

$$\tilde{H}_\mu := \sum_l (n_l + \epsilon_l) H_\mu^{(l)} \tag{2.24}$$

on $\mathbb{T}_{\epsilon,\mu}$ is periodic of period 2π and has trajectories homotopic to the curve $\phi(t) = \mathbf{n}t$.

In all the applications we know one is interested in the dynamics of one system while the other Hamiltonians are just the generators of the linear symmetries of the system. In such a case we also compute the frequencies of the dynamics on the torus.

Precisely assume that there exist $r-1$ independent linear combinations $K^{(2)}, \dots, K^{(r)}$ of the functions $H_\mu^{(l)}$ which are of the form

$$K^{(l)} = \sum_{j=1}^{r} \omega_j^{(l)} I_j + \sum_{j \geqslant r+1} \omega_j^{(l)} \frac{p_j^2 + q_j^2}{2} \tag{2.25}$$

(no nonlinear part) and have the property that the flow of each of these Hamiltonians is periodic with period 2π. Since these functions have no nonlinear part, the validity of assumption (A.4) implies that $\sum_l n_l H_\mu^{(l)}$ and the functions $K^{(l)}$ are independent. It follows that a system of r independent Hamiltonians in involution is given by $\tilde{H}_\mu, K^{(2)}, \dots, K^{(l)}$. It is thus immediate to obtain the following corollary.

Corollary 2.2. *Given an arbitrary vector $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$ consider the Hamiltonian $H_\mu^\alpha := \alpha_1 \tilde{H}_\mu + \sum_{l=2}^r \alpha_l K^{(l)}$. The flow of H_μ^α on $\mathbb{T}_{\epsilon,\mu}$ is a Kronecker flow with frequency vector α.*

Remark 2.8. In application this property will allow to give a simple proof of the fact that tori corresponding to different values of ϵ are actually different.

3. Proofs

Remark 3.1. Fix an integer vector $\mathbf{n}$ such that assumptions (A.3), (A.4) hold. Make the change of variables

$$\hat{I}_1 := \sum_l n_l I_l, \qquad \hat{I}_j = I_j, \quad j = 2, \ldots, r,$$

and complete it to a canonical transformation. Define

$$\hat{H}^{(1)} := \sum_{l=1}^{r} n_l H^{(l)} \tag{3.1}$$

$$= \sum_{l=1}^{r} n_l I_l + \sum_{j \geqslant r+1} \sum_{l=1}^{r} n_l \Omega_j^{(l)} \frac{p_j^2 + q_j^2}{2} + \mu F_n \tag{3.2}$$

$$= \hat{I}_1 + \sum_{j \geqslant r+1} \hat{\Omega}_j^{(1)} \frac{p_j^2 + q_j^2}{2} + \mu \hat{F}_1. \tag{3.3}$$

Thus the system of Hamiltonians $\hat{H}^{(1)}, H^{(2)}, \ldots, H^{(r)}$ fulfills the assumptions (A.3) and (A.4) with $\mathbf{n} = \mathbf{e}_1$.

For this reason the proof will be carried out in the case where (A.3) and (A.4) are fulfilled with $\mathbf{n} = \mathbf{e}_1$. *Moreover, from now on we fix the values of γ and τ, and we will simply write $\mathcal{N}$ instead of $\mathcal{N}(\gamma, \tau, \mathbf{e}_1)$.*

In the following we will denote by $\Phi_{\mu,l}^{t_l}$ the time t_l flow of the Hamiltonian vector field $X_{H_\mu^{(l)}}$. Moreover, following the standard notation we will use the notation

$$\Phi_\mu^{(t_1,\ldots,t_r)} := \Phi_{\mu,1}^{t_1} \circ \cdots \circ \Phi_{\mu,r}^{t_r}. \tag{3.4}$$

Φ^t is an action of $\mathbb{R}^n$ on $\mathcal{P}_{s,\sigma}$.

The first (and main) step of the proof consists in looking for r-tuples $(\epsilon_1, \ldots, \epsilon_r)$ and for initial data z_0 such that the function

$$z_\mu(t) := \Phi_\mu^{(1+\epsilon_1)t, \epsilon_2 t, \ldots, \epsilon_r t}(z_0) \tag{3.5}$$

is periodic of period 2π. Moreover, we will assume that (3.5) is contained in a neighborhood of order μ of a suitable reference periodic function.

A function of the form (3.5) is a solution of the Hamiltonian system

$$\tilde{H} := (1+\epsilon_1) I_1 + \epsilon_2 I_2 + \cdots + \epsilon_r I_r + \sum_{j \geqslant r+1} \tilde{\Omega}_j(\epsilon) \frac{p_j^2 + q_j^2}{2} + \mu \tilde{F}_\mu(z) \tag{3.6}$$

where

$$\tilde{\Omega}_j(\epsilon) := (1+\epsilon_1) \Omega_j^{(1)} + \sum_{l=2}^{r} \epsilon_l \Omega_j^{(l)}, \qquad \tilde{F}_\mu := (1+\epsilon_1) F_\mu^{(1)} + \sum_{l=2}^{r} \epsilon_l F_\mu^{(l)};$$

so *we will actually look for a periodic solution of such a Hamiltonian system.*

We will show that such periodic solutions form a torus, and subsequently we will show that such a torus is actually invariant under the flow of each one of the Hamiltonian systems $H_{\mu}^{(l)}$.

To come to a precise statement consider again $a_*(\epsilon/\mu)$, namely the solution of (2.21) with $\mathbf{n} = \mathbf{e}_1$. Correspondingly we define the reference periodic function

$$z_{0,\psi}(t) = (a_*, \mathbf{e}_1 t + \psi, 0, 0) \subset N. \tag{3.7}$$

Then we need to measure the distance between periodic functions. Thus consider the space $\mathcal{H}_{s,\sigma} := H^1(\mathbb{T}; \mathbb{R}^r \times \mathbb{R}^r \times \ell^2_{s,\sigma} \times \ell^2_{s,\sigma})$ of the H^1 periodic functions of period 2π taking values in the covering space of the phase space. In such a space we will use the norm

$$\|\zeta\|^2_{T,s,\sigma} := \int_{-\pi}^{\pi} \|\zeta(t)\|^2_{s,\sigma}\, dt + \int_{-\pi}^{\pi} \|\dot{\zeta}(t)\|^2_{s,\sigma}\, dt. \tag{3.8}$$

Then we consider the space of the functions of the form

$$z(t) = \big(I(t), \mathbf{e}_1 t + \phi(t), p(t), q(t)\big) = (0, \mathbf{e}_1 t, 0, 0) + \zeta(t), \tag{3.9}$$

with $\zeta(t) = (I(t), \phi(t), p(t), q(t)) \in \mathcal{H}_{s,\sigma}$. The space of the functions of the form (3.9) will be symbolically denoted by $\mathbf{e}_1 t + \mathcal{H}_{s,\sigma}$. Moreover in $\mathbf{e}_1 t + \mathcal{H}_{s,\sigma}$ we will use the topology induced by the distance introduced by the norm of $\mathcal{H}_{s,\sigma}$.

Lemma 3.1. *Fix σ, then, for any $s > s_*$ there exist two constants μ_* and C such that the following holds. For any $|\mu| < \mu_*$, any $\epsilon \in \mathcal{A}(\mathbf{e}_1) \cap \mathcal{N} \cap B_{\mu}$, and any $\psi \in \mathbb{T}^r$, there exists a* unique *2π periodic function $z^{\epsilon}_{\mu,\psi}(t) \in \mathbf{e}_1 t + \mathcal{H}_{s,\sigma}$ of the form* (3.5) *fulfilling*

$$\big\|z_{0,\psi} - z^{\epsilon}_{\mu,\psi}\big\|_{T,s,\sigma} \leqslant C\mu \tag{3.10}$$

and

$$z^{\epsilon}_{\mu,\psi}(0) = \big(I(0), \psi, p(0), q(0)\big),$$

i.e. the initial datum for the component on $\mathbb{T}^r$ is exactly ψ. Moreover the map $(\mu, \psi) \mapsto z^{\epsilon}_{\mu,\psi} \in \mathbf{e}_1 t + \mathcal{H}_{s,\sigma}$ is C^{∞}.

The proof of this lemma consists of several steps and is contained in the next subsection.

3.1. Proof of Lemma 3.1

As usual when dealing with the construction of periodic solutions in systems having some integrals of motion the situation is quite delicate. Thus, instead of looking for periodic solutions of $\tilde{H}$, cf. (3.6), we look for constants β_l and for periodic solutions of the system

$$\dot{z} = X_{\tilde{H}}(z) + \sum_{l=1}^{r} \beta_l \nabla \mathcal{I}^{(l)}(z), \tag{3.11}$$

where $\mathcal{I}^{(l)}(z) := I_l$. This is possible in view of the following lemma.

Lemma 3.2. *Consider a function $z \in \mathbf{e}_1 t + \mathcal{H}_{s,\sigma}$, $s > s_*$. Then there exists a $\mu_* > 0$ such that, if $z(t) \in \mathcal{V}_{s,\sigma}$ (cf. (A.2)), $\forall t$, and if $|\mu| < \mu_*$, then $(z(t), \beta)$ is a solution of the system* (3.11) *if and only if $\beta = 0$ and $z(t)$ is a periodic solution of the system*

$$\dot{z} = X_{\tilde{H}}(z).$$

Proof. Assume that $z(t)$ is a periodic solution of (3.11), and apply $dH^{(l_1)}(z(t))$ to (3.11). One has

$$\begin{aligned} dH^{(l_1)}\big(z(t)\big)\dot{z} &= dH^{(l_1)} X_{\tilde{H}} + \sum_{l=1}^{r} \beta_l dH^{(l_1)} \nabla \mathcal{I}^{(l)}(z) \\ &= \big\{H^{(l_1)}; \tilde{H}\big\} + \sum_{l=1}^{r} \beta_l \big\langle \nabla H^{(l_1)}; \nabla \mathcal{I}^{(l)} \big\rangle \\ &= \sum_{l=1}^{r} \beta_l \big\langle \nabla H^{(l_1)}; \nabla \mathcal{I}^{(l)} \big\rangle = \sum_{l=1}^{r} \beta_l \frac{\partial H^{(l_1)}}{\partial I_l} \\ &= \sum_{l=1}^{r} \beta_l \left(\delta_{l,l_1} + \mu \frac{\partial F^{(l_1)}}{\partial I_l} \right). \end{aligned}$$

Take the integral over 2π of such an equation, and notice that the left-hand side is the time derivative of $H^{(l_1)}(z(t))$, which by assumption is a periodic function, thus we get

$$0 = \sum_{l=1}^{r} \beta_l \left[\int_0^{2\pi} \left(\delta_{l,l_1} + \mu \frac{\partial F^{(l_1)}}{\partial I_l} \big(z(t)\big) \right) dt \right]. \tag{3.12}$$

Then, for μ small enough the square bracket is an invertible matrix, and therefore (3.12) implies $\beta = 0$. □

We look now for β, ϵ and a periodic solution $z(t) \in \mathbf{e}_1 t + \mathcal{H}_{s,\sigma}$ of (3.11). Write $z(t) = (0, \mathbf{e}_1 t, 0, 0) + \zeta(t)$ with $\zeta \equiv (I(t), \phi(t), p(t), q(t)) \in \mathcal{H}_{s,\sigma}$; substituting in (3.11) one gets the system:

$$L_\epsilon \zeta = \Theta(\zeta, \epsilon, \beta, \mu), \tag{3.13}$$

where the operator L_ϵ is defined by

$$L_\epsilon \begin{pmatrix} I_j \\ \phi_j \\ p_j \\ q_j \end{pmatrix} = \begin{bmatrix} \frac{d}{dt} & 0 & 0 & 0 \\ 0 & \frac{d}{dt} & 0 & 0 \\ 0 & 0 & \frac{d}{dt} & \tilde{\Omega}_j(\epsilon) \\ 0 & 0 & -\tilde{\Omega}_j(\epsilon) & \frac{d}{dt} \end{bmatrix} \begin{pmatrix} I_j \\ \phi_j \\ p_j \\ q_j \end{pmatrix} \tag{3.14}$$

and Θ is a nonlinear operator

$$\Theta : \mathcal{H}_{s,\sigma} \times \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R} \to \mathcal{H}_{s+d,\sigma} \tag{3.15}$$

$$(\zeta, \epsilon, \beta, \mu) \mapsto \zeta' \equiv \Theta(\zeta, \epsilon, \beta, \mu) \tag{3.16}$$

with $\zeta'(t) = (I'(t), \phi'(t), p'(t), q'(t))$ defined by

$$I'_j(t) = -\mu \frac{\partial \tilde{F}}{\partial \phi_j}\big(\zeta(t)\big) + \beta_j, \qquad \phi'_j(t) = \epsilon_j + \mu \frac{\partial \tilde{F}}{\partial I_j}\big(\zeta(t)\big), \tag{3.17}$$

$$p'_j(t) = -\mu \frac{\partial \tilde{F}}{\partial q_j}\big(\zeta(t)\big), \qquad q'_j = \mu \frac{\partial \tilde{F}}{\partial p_j}\big(\zeta(t)\big). \tag{3.18}$$

Remark 3.2. Since $\mathcal{H}_{s,\sigma}$ is an algebra, by assumption (A.2) the operator Θ is a smooth map from $\mathcal{H}_{s,\sigma} \to \mathcal{H}_{s+d,\sigma}$ for all $s > s_*$.

We solve the system (3.13) by using Lyapunov–Schmidt decomposition. It is easy to see that the kernel of $L_0 := L_\epsilon|_{\epsilon=0}$ is given by the space of the constant functions of the form $(a, \psi, 0, 0)$, while its range is the space $R_{s,\sigma}$ of the functions $w(t) \equiv (J(t), \theta(t), p(t), q(t)) \in \mathcal{H}_{s,\sigma}$ with $J(t), \theta(t)$ having zero average.

We project our system of equations on the two subspaces $R_{s,\sigma}$ and K. Correspondingly we also decompose the unknown as $\zeta(t) = (a, \psi, 0, 0) + w(t)$, $w \in R_{s,\sigma}$.

Denote by P the projector on $R_{s,\sigma}$, and by Q the projector on K. Explicitly Q acts by putting equal to zero the p and q components of ζ and by taking the average of the first two components. Thus the system (3.13) decomposes into the range equation

$$L_\epsilon w = P\Theta\big(w + (a, \psi, 0, 0), \epsilon, \beta, \mu\big), \tag{3.19}$$

and the kernel system

$$0 = -\mu\left\langle \frac{\partial \tilde{F}}{\partial \phi_j}\big(w(t) + (a, \mathbf{e}_1 t + \psi, 0, 0)\big)\right\rangle + \beta_j, \tag{3.20}$$

$$0 = \epsilon_j + \mu\left\langle \frac{\partial \tilde{F}}{\partial I_j}\big(w(t) + (a, \mathbf{e}_1 t + \psi, 0, 0)\big)\right\rangle, \tag{3.21}$$

where we denoted by $\langle . \rangle$ the operation of taking the average, for example

$$\langle J_j \rangle = \frac{1}{2\pi} \int_0^{2\pi} J_j(t)\, dt. \tag{3.22}$$

First fix $\epsilon \in \mathcal{N}$ and an open set $\mathcal{U}'' \supset \mathcal{U}'$, with closure contained in $\mathcal{U}$. We fix $a \in \mathcal{U}''$, β in a neighborhood of zero and $\psi \in \mathbb{T}^r$ and we solve the system (3.19) getting a solution $w_\epsilon(\mu, a, \psi)$ which turns out to be independent of β. Then we substitute in the system (3.20)–(3.21) and we solve it.

Consider (3.19); we have that under the assumption (A.3) the operator L_ϵ admits an inverse which is bounded as an operator from $R_{s+\tau,\sigma}$ to $R_{s,\sigma}$. Precisely the following lemma holds.

Lemma 3.3. *Let $w \in R_{s+\tau,\sigma}$, fix $\epsilon \in \mathcal{N}$ then there exists a unique $w' \in R_{s,\sigma}$ such that*

$$L_\epsilon w' = w, \tag{3.23}$$

moreover one has

$$\|w'\|_{T,s,\sigma} \leqslant \frac{1}{\gamma} \|w\|_{T,s+\tau,\sigma}. \tag{3.24}$$

Proof. Notice that the operator L_ϵ is block diagonal. Denote $w = (J_j, \theta_j, p_j, q_j)$ and similarly for w', expand each of the components of w in time Fourier series, i.e. write

$$J_j(t) = \frac{1}{\sqrt{2\pi}} \sum_k J_{jk} e^{ikt} \tag{3.25}$$

and similarly for all the other components. One has that the system (3.23) is equivalent to the system

$$ikJ'_{jk} := J_{jk}, \qquad ik\theta'_{jk} := \theta_{jk}, \quad j = 1, \ldots, r, \ k \neq 0, \tag{3.26}$$

$$\begin{bmatrix} ik & \tilde{\Omega}_j \\ -\tilde{\Omega}_j & ik \end{bmatrix} \begin{pmatrix} p'_{jk} \\ q'_{jk} \end{pmatrix} = \begin{pmatrix} p_{jk} \\ q_{jk} \end{pmatrix}, \quad j \geqslant r+1, \ k \in \mathbb{Z}. \tag{3.27}$$

Notice also that in terms of the Fourier variables the norm takes the form

$$\|w\|^2_{T,s,\sigma} = \sum_k (1+k^2) \left[\sum_{j=1}^r \left(|J_{jk}|^2 + |\theta_{jk}|^2\right) + \sum_{j \geqslant r+1} \left(|p_{jk}|^2 + |q_{jk}|^2\right)[j]^{2s} e^{2\sigma|k|} \right]. \tag{3.28}$$

It is immediate to solve the two equations (3.26). To solve (3.27) one can simply observe that the matrix defining such a system has eigenvalues and eigenvectors given by

$$\lambda_{kj} = i\left(k \pm \tilde{\Omega}_j(\epsilon)\right), \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}. \tag{3.29}$$

In particular the two eigenvectors are orthogonal to each other and independent of k and j. With this information it is easy to construct and estimate the solution of (3.27) getting

$$\left\|(p', q')\right\|_{T,s,\sigma} \leqslant \left\|(p, q)\right\|_{T,s+\tau,\sigma} \sup_{jk} \left[\frac{1}{j^\tau |i(k \pm \tilde{\Omega}_j(\epsilon))|} \right]. \tag{3.30}$$

If $\epsilon \in \mathcal{N}$ then the argument of the supremum is estimated by

$$\frac{1}{j^\tau |k \pm \tilde{\Omega}_j(\epsilon)|} \leqslant \frac{j^\tau}{j^\tau \gamma} = \frac{1}{\gamma}.$$

Adding the trivial estimate of J' and of θ' one gets the thesis. □

According to the previous lemma one can define L_ϵ^{-1} which is bounded as an operator from $R_{s+\tau,\sigma}$ to $R_{s,\sigma}$. It follows that one can rewrite the system (3.23) as a fixed point problem, namely

$$w = L_\epsilon^{-1} P\Theta\left(w + (a, \psi, 0, 0), \epsilon, \beta, \mu\right) \equiv \mu \mathcal{F}_\epsilon(w, \mu, a, \psi) \tag{3.31}$$

where $\mathcal{F}$, is a map which, for fixed $\epsilon \in \mathcal{N}$, is a smooth map from $R_{s,\sigma} \times (-\mu_*, \mu_*) \times \mathcal{U}'' \times \mathbb{T}^r$ to $R_{s,\sigma}$. We factorized a μ in front of $\mathcal{F}$, since it explicitly appears in the form of $P\Theta$ and eliminated β since $P\Theta$ is independent of it.

So one can apply the implicit function theorem in order to get the following corollary.

Corollary 3.1. *Fix $s > s_*$ and σ, then, for any $\epsilon \in \mathcal{N}$ there exists a function $w(\mu, a, \psi)$ which solves the system* (3.31)*. Moreover it is C^∞ as a map from $(-\mu_*, \mu_*) \times \mathcal{U}'' \times \mathbb{T}^r$ to $R_{s,\sigma}$, and fulfills the inequality*

$$\left\|w(\mu, a, \psi)\right\|_{T,s,\sigma} \leqslant \mu C \tag{3.32}$$

for all (μ, a, ψ) in the considered domain.

Then we substitute such a solution in (3.20)–(3.21) and we solve such a system. Since w does not depend on β, the solution of (3.20) is immediate. The solution of (3.21) is slightly more difficult since w does not depend smoothly on ϵ and moreover it is defined only for ϵ in the set $\mathcal{N}$. For this reason we proceed as in [2], i.e. we fix $\epsilon \in \mathcal{N}$ and we look for a vector $a = a_\epsilon(\mu, \psi)$ which fulfills such an equation.

We have the following lemma.

Lemma 3.4. *There exists a positive μ_* such that for any $|\mu| < \mu_*$ the following holds. For any $\epsilon \in \mathcal{A} \cap \mathcal{N} \cap B_\mu$ there exists a unique smooth function $\alpha_{\frac{\epsilon}{\mu}}(\mu, \psi)$ such that*

$$a = a_* + \mu \alpha_{\frac{\epsilon}{\mu}}(\mu, \psi) \tag{3.33}$$

solves Eq. (3.21)*.*

Remark 3.3. The function α is not smooth in ϵ/μ.

Proof. First notice that from (3.32) one has

$$\left\langle \frac{\partial \tilde{F}}{\partial I_j}\big((a, \mathbf{e}_1 t + \psi, 0, 0) + w(\mu, a, \psi, t)\big) \right\rangle = \left\langle \frac{\partial \tilde{F}}{\partial I_j}(a, \mathbf{e}_1 t + \psi, 0, 0) \right\rangle + \mu \mathcal{G}(a, \psi, \mu)$$

with $\mathcal{G}(a, \psi, \mu)$ a suitable smooth function. Thus the above expression is equal to

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \tilde{F}}{\partial I_j}(a, \mathbf{e}_1 t + \psi, 0, 0)\, dt + \mu \mathcal{G}(a, \psi, \mu) = \frac{\partial \langle \tilde{F} \rangle}{\partial a_j}(a) + \mu \mathcal{G}(a, \psi, \mu) \tag{3.34}$$

where of course

$$\langle \tilde{F} \rangle(a) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{F}(a, \mathbf{e}_1 t + \psi, 0, 0)\, dt.$$

Write $a = a_* + \mu\alpha$, thus we have to solve

$$0 = \frac{\epsilon}{\mu} + \frac{\partial \langle \tilde{F} \rangle}{\partial a_j}(a_* + \mu\alpha) + \mu \mathcal{G}(a_* + \mu\alpha, \psi, \mu) =: \mathcal{L}(\rho, \mu, \alpha, \psi)$$

for a fixed value of the parameters $\rho := \frac{\epsilon}{\mu}$. Due to the definition of a_* we have

$$\big\| \mathcal{L}(\rho, \mu, \alpha, \psi) \big\| \leqslant C\mu.$$

Moreover one has that $\frac{\partial \mathcal{L}}{\partial \alpha} \simeq \frac{\partial^2 \langle \tilde{F} \rangle}{\partial \alpha^2}$ is invertible with inverse having norm independent of μ. Then the implicit function theorem ensures the result.[1] $\square$

[1] Actually one needs a small variant of the standard implicit function theorem, since we do not know any solution of the equation $\mathcal{L} = 0$. However the standard proof based on the contraction mapping theorem allows one to prove that the implicit function exists also in this case.

End of the proof of Lemma 3.1. Thanks to Lemma 3.4 we have constructed the unique periodic function of the form (3.5) belonging to $\mathbf{e}_1 t + \mathcal{H}_{s,\sigma}$ which is μ close to $z_{0,\psi}$ and such that *the average of the angular component ϕ is equal to ψ*. In order to get Lemma 3.1 it is enough to show that there is a 1–1 correspondence between the average and the initial datum for the angular component. This is an immediate consequence of the trivial relation

$$\phi(0) = \psi + \theta(0), \tag{3.35}$$

and of the fact that $\theta(0)$ is a smooth function of ψ and has size smaller than μ. Thus from the implicit function theorem one can calculate the average ψ as a unique smooth function of $\phi(0)$. This concludes the proof. □

3.2. End of the proof of Theorem 2.1

Fix $|\mu| < \mu_*$ and $\epsilon \in \mathcal{A} \cap \mathcal{N} \cap B_\mu$. Consider the torus

$$\mathbb{T}_{\epsilon,\mu} := \bigcup_{\psi \in \mathbb{T}^r} \left\{ z^{\epsilon}_{\mu,\psi}(0) \right\} \tag{3.36}$$

where $z^{\epsilon}_{\mu,\psi}(t)$ is the periodic solution of $\tilde{H}$ constructed in Lemma 3.1. Moreover one has an important uniqueness property ensuring that there are no other periodic solutions of $\tilde{H}$ close to such a torus. We exploit now such a uniqueness property in order to prove the following lemma which is the last step for the proof of Theorem 2.1.

Lemma 3.5. *The torus $\mathbb{T}_{\epsilon,\mu}$ is invariant under the flow of the Hamiltonian vector fields of each of the functions $H^{(l)}_{\mu}$.*

Proof. Fix $z \in \mathbb{T}_{\epsilon,\mu}$, and a neighborhood $\mathcal{U}_0 \subset \mathbb{R}^r$ of zero. Consider the set

$$\mathcal{S} := \bigcup_{\varphi \in \mathcal{U}_0} \Phi^{\varphi}_{\mu}(z). \tag{3.37}$$

Since Φ^{φ}_{μ} commutes with the flow of $\tilde{H}$, all the points of $\mathcal{S}$ give rise to periodic orbits of $\tilde{H}$. Moreover $\mathcal{S}$ and $\mathbb{T}_{\epsilon,\mu}$ have at least z (and the periodic orbit it generates) in common. But there are no initial data outside $\mathbb{T}_{\epsilon,\mu}$ and close to it, which give rise to periodic solutions of $\tilde{H}$ (uniqueness property of the torus). It follows that $\mathcal{S} \subset \mathbb{T}_{\epsilon,\mu}$. So this property implies the thesis. □

4. Applications

4.1. Two-dimensional tori in a nonlinear wave equation

Consider the nonlinear wave equation

$$u_{tt} - u_{xx} + mu = f(u), \quad x \in \mathbb{T} \equiv \frac{\mathbb{R}}{2\pi\mathbb{Z}}, \tag{4.1}$$

where f is a smooth function having a zero of order higher than 1 at the origin. For simplicity we will consider explicitly only the case $f(u) = -u^3$. We will construct small amplitude invariant tori of dimension 2 exploiting the invariance of the equation under translations.

Eq. (4.1) is Hamiltonian with Hamiltonian function

$$K^{(1)}(U,u) = \int_{-\pi}^{\pi} \left[\frac{U^2 + u_x^2 + mu^2}{2} + \frac{u^4}{4} \right] dx \tag{4.2}$$

where $U \equiv \dot{u}$ is the momentum conjugated to u. Translations in $\mathbb{T}$ are generated by the Hamiltonian

$$K^{(2)}(U,u) = \int_{-\pi}^{\pi} U_x u \, dx \tag{4.3}$$

which commutes with $K^{(1)}$. To fit the scheme of the previous section we have to introduce suitable variables. First introduce the Fourier basis $\hat{e}_j(x)$ by

$$\hat{e}_j(x) = \begin{cases} \frac{\cos(jx)}{\sqrt{\pi}} & \text{if } j > 0, \\ \frac{\sin(-jx)}{\sqrt{\pi}} & \text{if } j < 0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } j = 0. \end{cases} \tag{4.4}$$

Notice that $\partial_x \hat{e}_j = -j\hat{e}_{-j}$ and introduce variables u_j, U_j by

$$u(x) = \sum_{j\in\mathbb{Z}} \frac{u_j}{\sqrt{\omega_j}} \hat{e}_j(x), \qquad U(x) = \sum_{j\in\mathbb{Z}} \sqrt{\omega_j}\, U_j \hat{e}_j(x), \tag{4.5}$$

where

$$\omega_j = \sqrt{j^2 + m}. \tag{4.6}$$

Then we introduce the variables p_j, q_j by

$$u_j = \frac{p_j + q_{-j}}{\sqrt{2}}, \qquad U_j = \frac{p_{-j} - q_j}{\sqrt{2}}, \quad j \neq 0, \tag{4.7}$$

and $p_0 = U_0$, $q_0 = u_0$. It is easy to check that these are canonical variables and that the Hamiltonians take the form

$$K^{(1)} = \sum_{j\in\mathbb{Z}} \omega_j \frac{p_j^2 + q_j^2}{2} + F, \tag{4.8}$$

$$K^{(2)} = \sum_{j\in\mathbb{Z}} j \frac{p_j^2 + q_j^2}{2} \tag{4.9}$$

where

$$F := \int_{-\pi}^{\pi} \frac{u^4}{4} \, dx \tag{4.10}$$

has to be thought as a function of the variables p, q.

Then we rescale the variables introducing the small parameter μ by defining $p=\sqrt{\mu}\tilde{p}$, $q=\sqrt{\mu}\tilde{q}$, we rewrite the Hamiltonians in terms of such variables, *and omit tildes*.

We fix two modes of the linear systems and we continue to the nonlinear system the family of invariant tori formed by the superposition of the linear oscillations involving only such two modes. Just to be determined (and because the computations turn out to be simple) we take the two modes with indexes 1 and -1.

Finally we introduce the action angle variables for the two considered modes by defining the actions

$$I_1=\frac{p_1^2+q_1^2}{2}, \qquad I_{-1}=\frac{p_{-1}^2+q_{-1}^2}{2},$$

and the corresponding angles. So the two Hamiltonians $K^{(1)}, K^{(2)}$ turn out to have the form (2.9). A simple computation shows that the two functions

$$H^{(1)}=\frac{K^{(1)}+\omega_1 K^{(2)}}{2\omega_1}, \qquad H^{(2)}=\frac{K^{(1)}-\omega_1 K^{(2)}}{2\omega_1} \tag{4.11}$$

have the structure (2.7), suitable for the application of Theorem 2.1. In particular the frequencies are

$$\Omega^{(1)}_{\pm j}=\frac{\omega_j\pm j\omega_1}{2\omega_1}, \qquad \Omega^{(2)}_{\pm j}=\frac{\omega_j\mp j\omega_1}{2\omega_1}, \quad j\geqslant 0,\ j\neq 1. \tag{4.12}$$

The nonlinearities are $F^{(1)}=F^{(2)}=F/2\omega_1$. We fix the domain $\mathcal{U}$ to be the open set $(0,+\infty)\times(0,+\infty)$. We fix the indexes of the phase space to be some arbitrary $\sigma>0$ and some $s>1/2$. Then, by the Sobolev embedding theorem the nonlinearity fulfills assumption (A.2) with $d=1$. We come to the nonresonance assumption (A.3). It turns out that it is convenient to choose $n_1=n_2=1$. We have the following

Lemma 4.1. *There exists an uncountable dense subset $\mathcal{S}$ of $(-1,\frac{4}{3})$, with zero measure, such that, if $m\in\mathcal{S}$ then property* (A.3) *holds with $\tau=1$ and a positive $\gamma=\gamma(m)$. Fix $m\in\mathcal{S}$, denote by $\mathcal{N}_0$ the intersection of the set $\mathcal{N}$ with a given neighborhood of the origin. Then $\mathcal{N}_0$ is uncountable*; *as $(\epsilon_1,\epsilon_2)\equiv\epsilon$ varies in $\mathcal{N}_0$, the quantity $\epsilon_1-\epsilon_2$ assumes uncountably many values, but also rational values.*

Remark 4.1. The quantity $\epsilon_1-\epsilon_2$ is important since it will be the ratio between the two frequencies of motion on the invariant torus.

Proof. One has, for $j\geqslant 0$,

$$\begin{aligned}\tilde{\Omega}_{\pm j}&:=(1+\epsilon_1)\Omega^{(1)}_{\pm j}+(1+\epsilon_2)\Omega^{(2)}_{\pm j}=\frac{\omega_j}{\omega_1}\left(1+\frac{\epsilon_1+\epsilon_2}{2}\right)\pm\frac{\epsilon_1-\epsilon_2}{2}j\\&=\left(\frac{1}{\omega_1}\left(1+\frac{\epsilon_1+\epsilon_2}{2}\right)\pm\frac{\epsilon_1-\epsilon_2}{2}\right)j+\frac{m}{\omega_j+j}.\end{aligned}$$

Consider first the case $\epsilon_1=\epsilon_2=0$. We construct a set $\mathcal{S}_0$ such that, if $m\in\mathcal{S}_0$ then $\tilde{\Omega}_j\equiv\frac{\omega_j}{\omega_1}$ fulfills

$$|k-\tilde{\Omega}_j|\geqslant\frac{\gamma}{|j|}, \quad \forall k,j, \text{ with } |k|,|j|\geqslant J_*,\ \gamma=\frac{1}{6} \tag{4.13}$$

and J_* a large number. We follow closely the construction of [6], proof of Lemma 3.1. Fix $\bar{m}\in(-1,4/3)$ and $\delta>0$; we construct uncountably many values of m which are δ close to $\bar{m}$ and such that (4.13) holds. Consider $\omega_1(m):=\sqrt{1+m}$, and $\nu(m):=[\omega_1(m)]^{-1}$, $\bar{\nu}:=\nu(\bar{m})$. Notice that there is 1–1 correspondence between m and ν. Consider the continued fraction expansion of $\bar{\nu}$. Fix some large integer Q and consider the set $\mathcal{S}_\nu$ of the ν's obtained by substituting the terms of index larger than Q in the continued fraction expansion of $\bar{\nu}$, with an infinite sequence of $0, 1$. If Q is large enough then $\mathcal{S}_\nu$ is contained in an arbitrarily small ball centered at $\bar{\nu}$. By standard continued fractions theory one has that if $\nu \in \mathcal{S}_\nu$, then

$$|k - \nu j| \geqslant \frac{\gamma'}{j}, \quad \forall j > J_*,\ \forall k \in \mathbb{Z},\ \gamma' = \frac{1}{3},$$

where J_* depends on Q. Denote by $\mathcal{S}_0$ the preimage of $\mathcal{S}_\nu$ under the map $m \mapsto \nu(m)$, and assume that Q is so large that $\mathcal{S}_0$ is contained in a ball of radius δ centered at $\bar{m}$. Thus, taking into account also the quantity $m/(\omega_j + j)$, we have that, for $m \in \mathcal{S}_0$,

$$|k - \tilde{\Omega}_j| \geqslant \frac{\gamma}{j}, \quad \forall j > J_*,\ \forall k.$$

Finally we take out of $\mathcal{S}_0$ the set of the m's such that $k - \tilde{\Omega}_j = 0$ for some $j \leqslant J_*$, $k \in \mathbb{Z}$. This is a finite set since ν is an analytic function of m. Thus the remaining set is contained in a ball of radius δ centered at $\bar{m}$, and it is uncountable. The union of such sets over all the points $\bar{m}$ is by definition the set $\mathcal{S}$.

We discuss now the case $\epsilon \neq 0$. Fix $m \in \mathcal{S}$, denote

$$\epsilon_+ := \frac{\epsilon_1 + \epsilon_2}{2}, \qquad \epsilon_- := \frac{\epsilon_1 - \epsilon_2}{2},$$
$$\nu_+ = \nu(1 + \epsilon_+) + \epsilon_-, \qquad \nu_- = \nu(1 + \epsilon_+) - \epsilon_-,$$

and notice that

$$\tilde{\Omega}_{\pm j} = j\nu_\pm + \frac{m}{\omega_j + j}, \quad j \geqslant 0.$$

The maps $(\epsilon_1, \epsilon_2) \mapsto (\epsilon_+, \epsilon_-) \mapsto (\nu_+, \nu_-)$ are 1–1. Consider the set $\mathcal{N}_1$ of the couples ϵ_1, ϵ_2 such that ν_+ and ν_- have a continued fraction expansion $[a_1, a_2, \ldots]$ with $a_j \leqslant 1$ for j large enough. Since the couples of (ν_+, ν_-) belonging to an arbitrary neighborhood of (ν, ν) are uncountable, the same property holds for $\mathcal{N}_1$. The result on the existence of rational values of $\epsilon_- = \epsilon_1 - \epsilon_2$ is obtained by fixing ϵ_+ in such a way that $\nu(1 + \epsilon_+)$ is of constant type, and by noticing that both ν_+ and ν_- remain of constant type when ϵ_- is a rational number. Thus in $\mathcal{N}_1$ one has

$$\bigl|k - \tilde{\Omega}_j(\epsilon)\bigr| \geqslant \frac{\gamma}{j}, \quad \forall j > J_*,\ \forall k.$$

To deal with the smaller values of j consider the (finite) set of functions

$$j\bigl(k - \tilde{\Omega}_j(\epsilon)\bigr), \quad j \leqslant J_*,\ k \leqslant K_*,$$

they are different from zero at $\epsilon = 0$, so there exists a neighborhood of 0 such that they remain different from zero. The set $\mathcal{N}$ is the intersection of $\mathcal{N}_1$ with such a neighborhood of zero. □

Finally we have to prove that the nondegeneracy assumption (A.4) holds. To this end we have to compute the function $F_{\mathbf{n}}$ and its average.

First of all we need to compute the flow over which we have to average. This is easily done by going back through the changes of variables we did. Thus one gets

$$u(a,\mathbf{n}t,0,0)=\sqrt{a_{-1}}\frac{\cos(x-t)}{\sqrt{\pi}}+\sqrt{a_1}\frac{\sin(x+t)}{\sqrt{\pi}}.$$

Inserting in F, in $F_{\mathbf{n}}$ and computing $\langle F_{\mathbf{n}}\rangle$ one gets that such a function is proportional (through a nonvanishing constant) to

$$a_1^2+4a_1a_{-1}+a_{-1}^2 \tag{4.14}$$

which is a nondegenerate quadratic form, and thus property (A.4) holds.

Thus we can apply the general theory and get quasiperiodic solutions in the nonlinear wave equation. In order to get a slightly more global description it is convenient to scale back the variables eliminating the parameter μ. We formulate the result for the nonlinear wave equation in terms of the original amplitudes $A=\mu a$.

Consider the set

$$\mathcal{A}_*:=\bigcup_{A\in(0,\infty)\times(0,\infty)}\left\{-\frac{\partial\langle F_{(1,1)}\rangle}{\partial a}(A)\right\} \tag{4.15}$$

and, for $\epsilon\in\mathcal{A}_*$, denote by $A_*(\epsilon)$ the unique solution of

$$\epsilon=-\frac{\partial\langle F_{(1,1)}\rangle}{\partial a}(A),$$

and by $\mathbb{T}_{\epsilon,*}$ the torus $p=q=0, I=A_*, \phi\in\mathbb{T}^2$.

Theorem 4.1. *Fix $m\in\mathcal{S}$. There exists a positive μ_* with the following property: for any $\epsilon\in\mathcal{N}\cap B_{\mu_*}\cap\mathcal{A}_*$ there exists a unique invariant torus $\mathbb{T}_\epsilon$ such that:*

1. *The flow on $\mathbb{T}_\epsilon$ has frequencies*

$$\left(\frac{\omega_1}{1+\frac{\epsilon_1+\epsilon_2}{2}},-\frac{\omega_1}{1+\frac{\epsilon_1+\epsilon_2}{2}}\frac{\epsilon_1-\epsilon_2}{2}\right). \tag{4.16}$$

2. *One has*

$$d(\mathbb{T}_\epsilon,\mathbb{T}_{\epsilon,*})\leqslant C\left\|A_*(\epsilon)\right\|^2. \tag{4.17}$$

Proof. We apply Theorem 2.1. To this end we must use the variables a and the parameter μ. For fixed ϵ define $\mu:=\|A_*(\epsilon)\|$, and fix the set $\mathcal{U}'$ to be the spherical shell $\|a\|=1$. Then using the notation of Section 2 one has $a_*(\epsilon/\mu)=A_*(\epsilon)/\mu$. Then remark that in this case one has

$$K^{(1)}=\frac{\omega_1}{1+\frac{\epsilon_1+\epsilon_2}{2}}\left(\tilde{H}-\frac{\epsilon_1-\epsilon_2}{2}K^{(2)}\right).$$

Thus the application of Theorem 2.1 and Corollary 2.2 immediately gives the result. □

Remark 4.2. Since the map that associates the frequencies to ϵ is 1–1 it follows that

$$\epsilon\neq\epsilon' \quad\Longrightarrow\quad \mathbb{T}_\epsilon\cap\mathbb{T}_{\epsilon'}=\emptyset.$$

Remark 4.3. The ratio between the two frequencies of motion on $\mathbb{T}_\epsilon$ is $(\epsilon_1-\epsilon_2)/2$. Since such a quantity varies in an uncountable set, the flow is dense on many of the tori $\mathbb{T}_\epsilon$. But there are also some tori on which $\epsilon_1-\epsilon_2$ is rational, thus we have also proved the persistence of some resonant tori, a phenomenon that does not occur when applying KAM theory.

4.2. A beam vibrating in 2 directions

As a second example we consider the system of coupled equations

$$\begin{aligned} u_{tt} + u_{xxxx} - mu &= -uf\left(u^2+v^2\right), \\ v_{tt} + v_{xxxx} - mv &= -vf\left(u^2+v^2\right), \end{aligned} \quad x \in \mathbb{T}. \tag{4.18}$$

This can be thought as a model of a beam which can oscillate in the y and in the z directions. u represents the displacement in the y direction and v the displacement in the z direction. In the case of negative m the terms mu and mv can be interpreted as centrifugal forces due to the fact that the beam is actually rotating with a constant velocity around its axis. To be determined here we will just consider the case $f(s) = s$.

(4.18) is a Hamiltonian system with Hamiltonian function

$$K^{(1)}(U,V,u,v) = \int_{-\pi}^{\pi} \left[\frac{U^2 + u_{xx}^2 + mu^2 + V^2 + v_{xx}^2 + mv^2}{2} + \frac{(u^2+v^2)^2}{4} \right] dx, \tag{4.19}$$

where U is the momentum conjugated to u and V the momentum conjugated to v. There are two symmetries, namely the translations in the torus and the rotations in the plane yz. They are generated by the two Hamiltonians

$$K^{(2)}(U,V,u,v) = \int_{-\pi}^{\pi} (U_x u + V_x v)\, dx, \qquad K^{(3)}(U,u) = \int_{-\pi}^{\pi} (Uv - Vu)\, dx. \tag{4.20}$$

We have to introduce suitable coordinates. First introduce the Fourier variables (U_j, V_j, u_j, v_j) as in the case of the nonlinear wave equation (with $\omega_j = \sqrt{j^4+m}$), then make the coordinate transformation (for $j \neq 0$)

$$u_j = \frac{P_{1,j} + Q_{1,-j}}{\sqrt{2}}, \qquad U_j = \frac{P_{1,-j} - Q_{1,j}}{\sqrt{2}}, \tag{4.21}$$

$$v_j = \frac{P_{2,j} + Q_{2,-j}}{\sqrt{2}}, \qquad V_j = \frac{P_{2,-j} - Q_{2,j}}{\sqrt{2}}, \tag{4.22}$$

which transforms the Hamiltonians to

$$K^{(1)} = \sum_{j \in \mathbb{Z}} \omega_j \frac{P_{1,j}^2 + Q_{1,j}^2 + P_{2,j}^2 + Q_{2,j}^2}{2} + F(u,v),$$

$$K^{(2)} = \sum_{j \in \mathbb{Z}} j \frac{P_{1,j}^2 + Q_{1,j}^2 + P_{2,j}^2 + Q_{2,j}^2}{2},$$

$$K^{(3)} = \sum_{j \in \mathbb{Z}} P_{1,j} Q_{2,j} - P_{2,j} Q_{1,j},$$

where we denoted

$$F(u,v) = \int_{-\pi}^{\pi} \frac{(u^2+v^2)^2}{4}\, dx. \tag{4.23}$$

Then for all j's make the further change of variables

$$Q_{1,j} = \frac{p_{1,j} + q_{2,j}}{\sqrt{2}}, \qquad P_{1,j} = \frac{p_{2,j} - q_{1,j}}{\sqrt{2}}, \tag{4.24}$$

$$Q_{2,j} = \frac{p_{2,j} + q_{1,j}}{\sqrt{2}}, \qquad P_{2,j} = \frac{p_{1,j} - q_{2,j}}{\sqrt{2}}, \tag{4.25}$$

which gives the Hamiltonians the form

$$K^{(1)} = \sum_{j\in\mathbb{Z}} \omega_j \left(\frac{p_{1,j}^2 + q_{1,j}^2}{2} + \frac{p_{2,j}^2 + q_{2,j}^2}{2} \right) + F(u,v), \tag{4.26}$$

$$K^{(2)} = \sum_{j\in\mathbb{Z}} j \left(\frac{p_{1,j}^2 + q_{1,j}^2}{2} + \frac{p_{2,j}^2 + q_{2,j}^2}{2} \right), \tag{4.27}$$

$$K^{(3)} = \sum_{j\in\mathbb{Z}} \frac{p_{1,j}^2 + q_{1,j}^2}{2} - \frac{p_{2,j}^2 + q_{2,j}^2}{2}. \tag{4.28}$$

Then we proceed as in the wave equation: we scale the variables, by $p = \sqrt{\mu}\tilde{p}, q = \sqrt{\mu}\tilde{q}$ and we omit tildes.

We fix now three modes that will be nonvanishing on the invariant torus of the linearized system. Just to give an example we fix the modes $(p_{1,j}, q_{1,j})$ with $j = 1, 2, 3$. We introduce action angle variables (I_j, ϕ_j), $j = 1, 2, 3$, for such modes and look for the linear combinations of the Hamiltonians having the form (2.7). They are given by

$$H^{(1)} = \frac{K^{(1)} + (\omega_2 - \omega_3)K^{(2)} + (2\omega_3 - 3\omega_2)K^{(3)}}{\omega_3 - 2\omega_2 + \omega_1}, \tag{4.29}$$

$$H^{(2)} = \frac{-2K^{(1)} + (\omega_3 - \omega_1)K^{(2)} + (3\omega_1 - \omega_3)K^{(3)}}{\omega_3 - 2\omega_2 + \omega_1}, \tag{4.30}$$

$$H^{(3)} = \frac{K^{(1)} + (\omega_1 - \omega_2)K^{(2)} + (\omega_2 - 2\omega_1)K^{(3)}}{\omega_3 - 2\omega_2 + \omega_1}. \tag{4.31}$$

Assumption (A.2) holds with $d = 2$. In order to apply Theorem 2.1 we choose $\mathbf{n} = \mathbf{e}_1 = (1, 0, 0)$. We first verify that the nonresonance condition (A.3) is fulfilled.

Lemma 4.2. *Fix $\tau > 1$. For any $\gamma > 0$ small enough there exists a subset $\mathcal{M}$ of $(-1, 4)$ whose complement has measure $O(\sqrt{\gamma})$ such that, if $m \in \mathcal{M}$ then, for any $R > 0$, the complementary of set $\mathcal{N} \cap B_R$ has measure estimated by $CR\sqrt{\gamma}$.*

Proof. Instead of studying (2.11) we multiply the quantity $k - \tilde{\Omega}_j$ by $\omega_3 - 2\omega_2 + \omega_1$ and study the so obtained quantity. Thus the quantities that have to be estimated are

$$\begin{aligned} f_{jk}(m,\epsilon) := {} & (1 + \epsilon_1 - 2\epsilon_2 + \epsilon_3)\omega_j \\ & + \left[(1 - \epsilon_2 - \epsilon_3)\omega_2 - (1 + \epsilon_1 + \epsilon_2)\omega_3 + (\epsilon_3 - \epsilon_2)\omega_1\right] j \\ & \pm \left[(3\epsilon_2 - 2\epsilon_3)\omega_1 - (1 + \epsilon_1 - \epsilon_2 + \epsilon_3)\omega_2 + (2 + 2\epsilon_1 - \epsilon_2)\omega_3\right] \\ & - k(\omega_3 - 2\omega_2 + \omega_1) \end{aligned}$$

where the sign $+$ holds for $\tilde{\Omega}_{1,j}$, while the sign $-$ holds for $\tilde{\Omega}_{2,j}$.

In the particular case $\epsilon = 0$ we thus have to estimate

$$k\omega_1 - (2k + j \pm 3)\omega_2 + (k + j \mp 2)\omega_3 - \omega_j, \tag{4.32}$$

where the first sign holds for $\tilde{\Omega}_{1,j} = \Omega^{(1)}_{1,j}$, while the second holds for $\tilde{\Omega}_{2,j} = \Omega^{(1)}_{2,j}$. Consider (4.32) as a function of m. It is easy to see that it vanishes identically only if (i) $(j,k) = (1,1)$ and the sign is $+$ (which corresponds to I_1), if (ii) $(j,k) = (2,0)$ and the sign is $+$ (which corresponds to I_2), if (iii) $(j,k) = (3,0)$ and the sign is $+$ (which corresponds to I_3). Thus all these cases are excluded. In all the other cases the function (4.32) is nontrivial. This remains true when ϵ is in a small neighborhood of zero.

Notice now that f_{jk} can vanish (or be small) only if

$$C_1 j^2 \leqslant |k| \leqslant C_2 j^2. \tag{4.33}$$

Thus we consider only such a case. First fix a large J_*, then exploiting the fact that $d\omega_j/dm = (2\omega_j)^{-1}$ one has

$$\left|\frac{\partial f_{jk}}{\partial m}\right| \geqslant C|k| \geqslant C' j^2. \tag{4.34}$$

It follows that there is at most a segment of length $C\gamma/|j|^{\tau-2}$ in the m space where f_{jk} is smaller than $\gamma/|j|^{\tau}$. Summing over k, taking into account (4.33) one gets that except for an interval of masses m of length $C\gamma/|j|^{\tau}$ one has

$$|f_{jk}| \geqslant \frac{\gamma}{|j|^{\tau}}, \quad \forall k \in \mathbb{Z}.$$

Summing over j one gets that, provided $\tau > 1$ one has

$$|f_{jk}| \geqslant \frac{\gamma}{|j|^{\tau}}, \quad \forall k \in \mathbb{Z},\ \forall |j| \geqslant J_*,$$

except for the m's in a set of measure $C\gamma$. And this is true *for any value of ϵ small enough*.

It remains to consider the set of j's smaller than J_*. But these are a finite number. Therefore excluding at most a further set of measure of order γ one gets that $\forall\epsilon$ fixed and close to the origin, f_{jk} fulfills the wonted estimate except for a set of m's having measure of order γ. Using Fubini's theorem one then concludes the proof. □

Finally we have to verify the nondegeneracy assumption. First notice that $F_{\mathbf{n}}$ is proportional to F, and that going through the change of coordinates one has

$$u(a, \mathbf{e}_1 t, 0, 0) = -\sqrt{a_1}\frac{\cos(x+t)}{\sqrt{\pi}} - \sqrt{a_2}\frac{\cos(2x)}{\sqrt{\pi}} - \sqrt{a_3}\frac{\cos(3x)}{\sqrt{\pi}}, \tag{4.35}$$

$$v(a, \mathbf{e}_1 t, 0, 0) = \sqrt{a_1}\frac{\sin(x+t)}{\sqrt{\pi}} + \sqrt{a_2}\frac{\sin(2x)}{\sqrt{\pi}} + \sqrt{a_3}\frac{\sin(3x)}{\sqrt{\pi}}. \tag{4.36}$$

Thus substituting in F and computing the integrals over x and t one obtains that $\langle F_{\mathbf{n}}\rangle$ is proportional to

$$a_1^2 + a_2^2 + a_3^2 + 4a_1a_2 + 4a_2a_3 + 4a_1a_3$$

which is a nondegenerate quadratic form. Therefore also assumption (A.4) holds.

Applying Theorem 2.1 one thus gets existence of quasiperiodic solutions with three frequencies. To get a precise result consider the set

$$\mathcal{A}_* := \bigcup_{A\in(0,\infty)^3}\left\{-\frac{\partial\langle F_{\mathbf{e}_1}\rangle}{\partial a}(A)\right\} \tag{4.37}$$

and, for $\epsilon\in\mathcal{A}_*$, denote by $A_*(\epsilon)$ the unique solution of

$$\epsilon = -\frac{\partial\langle F_{\mathbf{e}_1}\rangle}{\partial a}(A),$$

and by $\mathbb{T}_{\epsilon,*}$ the torus $p=q=0$, $I=A_*$, $\phi\in\mathbb{T}^3$. Thus the following theorem holds.

Theorem 4.2. *There exists a positive μ_* with the following property: for any $\epsilon\in\mathcal{N}\cap B_{\mu_*}\cap\mathcal{A}_*$ there exists a unique invariant torus $\mathbb{T}_\epsilon$ such that:*

1. *The flow on $\mathbb{T}_\epsilon$ has frequencies*

$$\begin{aligned}\Bigg(&\frac{\omega_3-2\omega_2+\omega_1}{1+\epsilon_1-2\epsilon_2+\epsilon_3}, -\frac{(1+\epsilon_1)\omega_2-(1+\epsilon_1-\epsilon_2)\omega_3-\epsilon_2\omega_3}{1+\epsilon_1-2\epsilon_2+\epsilon_3},\\ &-\frac{(2+2\epsilon_1-\epsilon_3)\omega_3-(3+3\epsilon_1-\epsilon_3)\omega_2+(3\epsilon_2-2\epsilon_3)\omega_1}{1+\epsilon_1-2\epsilon_2+\epsilon_3}\Bigg).\end{aligned} \tag{4.38}$$

2. *One has*

$$d(\mathbb{T}_\epsilon,\mathbb{T}_{\epsilon,*}) \leqslant C\left\|A_*(\epsilon)\right\|^2. \tag{4.39}$$

Remark 4.4. Since the set of $\epsilon\in\mathcal{N}\cap B_\mu\cap\mathcal{A}_*$ has positive measure, one has that for almost all allowed values of ϵ, the flow is ergodic on $\mathbb{T}_\epsilon$.

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