Creation of Kerr–de Sitter black hole in all dimensions

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Abstract

We discuss quantum creation scenario of Kerr–de Sitter black hole in all dimensions. We show that its relative creation probability is the exponential to the entropy of the black hole, using a topological argument. The action of the Euclidean regular instanton can be readily calculated in the same way.

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In gravitational physics, black holes and the universe are the two most attractive objects. Therefore, a quantum creation of black hole at the birth of the universe is one of the most interesting problems in the field. Extensive work has been done for the creation scenario for all kinds of black holes in a 4-dimensional spacetime. Most work is done for the maximal black hole creation [1], although the non-extreme sub-maximal black hole creation has also been discussed [2,3]. The case of non-rotating black holes in all dimensions can be simply treated, since its higher-dimensional metric is a quite straightforward generalization of the 4-dimensional one [2,4].

The Kerr solution was found about half century after the discovery of general relativity [5]. Two decades later Myers and Perry were able to get its higher-dimensional generalization in the flat background [6]. Only very recently the metrics of 5-dimensional and higher-dimensional Kerr black holes in the de Sitter background were found [7,8].

The creation scenario of general 4-dimensional Kerr–Newman black holes in the de Sitter, flat and anti-de Sitter backgrounds has been tackled, it turns out that the creation probability in the closed (open) background is the exponential to (the negative of) the entropy of the created hole [2]. The creation of
4-dimensional charged rotating extreme black holes from a regular seed instanton was discussed in [9].

In the $D$-dimensional Kerr–de Sitter spacetime there are $N \equiv [(D - 1)/2]$ rotation parameters $a_i$ for orthogonal planes, the rotations are associated with the azimuthal coordinates $\phi_i$ with periods $2\pi$ and latitudinal coordinates $\mu_i$, which lie in the interval $0 \leq \mu_i \leq 1$. One can set $n \equiv [D/2]$ and $\epsilon = n - N$. If $D$ is even, then $\epsilon = 1$, $n = N + 1$ and there is an extra $\mu_{N+1}$, for which $-1 \leq \mu_{N+1} \leq 1$. The sum of all the $\mu_i^2$ should be unity.

The Kerr–de Sitter metric with cosmological constant $\Lambda$ can be cast in the Boyer–Lindquist coordinate [8]

$$ds^2 = -W(1 - \lambda r^2) dt^2 + \frac{Udr^2}{V - 2M} + \sum_{i=1}^{n} \frac{a_i^2 \mu_i^2 d\phi_i}{1 + \lambda a_i^2} + \sum_{i=1}^{N} \frac{a_i^2 \mu_i^2 d\mu_i}{1 + \lambda a_i^2} + \frac{\lambda}{W(1 - \lambda r^2)} \left( \sum_{i=1}^{n} \frac{(r^2 + a_i^2) \mu_i d\mu_i}{1 + \lambda a_i^2} \right)^2,$$ \hspace{1cm} (1)

where $M$ is the mass parameter, $\lambda \equiv \frac{2\Lambda}{(D-1)(D-2)}$ and

$$W = \sum_{i=1}^{n} \frac{\mu_i^2}{1 + \lambda a_i^2},$$ \hspace{1cm} (2)

$$U = r^\epsilon \sum_{i=1}^{n} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{N} (r^2 + a_j^2)$$ \hspace{1cm} (3)

and

$$V = r^{-2+\epsilon} (1 - \lambda r^2) \prod_{i=1}^{N} (r^2 + a_i^2).$$ \hspace{1cm} (4)

For the creation of the above higher-dimensional Kerr black hole in the de Sitter background, one needs to find the seed instanton. The instanton can be obtained through the analytic continuation $\tau = it$ from metric (1). The horizons are located at the zeroes of $V - 2M = 0$. The instanton is constructed by some periodic identification of coordinate $\tau$ between two horizons $r_l (l = 1, 2)$, which are the two largest zeroes of $V - 2M = 0$ and we set $r_2 > r_1$.

To obtain a regular instanton $M$, one needs to identify $\tau$ by a period of $\beta_l = 2\pi/\kappa_l$ in order to avoid the conical singularity at horizon $r_l$, where $\kappa_l$ is its surface gravity [8]

$$\kappa_l = r_l (1 - \lambda r_l^2) \left( \sum_{i=1}^{N} \frac{1}{r_l^2 + a_i^2} + \frac{\epsilon}{2r_l^2} \right) - \frac{1}{r_l},$$ \hspace{1cm} (5)

To avoid the conical singularities for both the two horizons, their surface gravities must take the same value $\kappa$. This condition can be met only for a degenerate case of metric (1).

To avoid the irregularities caused by the differential rotation of two horizons, the following condition should also be satisfied [10]

$$\left( \Omega^2_1 - \Omega^2_j \right) 2\pi i \frac{\kappa_j}{\kappa} = 2m_j \pi,$$ \hspace{1cm} (6)

where $m_j$ are integers, and $\Omega^2_j$ are the angular velocities of the horizon $r_j$ [8]

$$\Omega^2_j = \frac{a_j (1 + \lambda a_j^2)}{r_l^2 + a_j^2}.$$ \hspace{1cm} (7)

It turns out that the smooth compact regular instanton is $S^{D-2}$ bundles over $S^2$. In particular, there are infinitely many for each odd $D \geq 5$ [8]. It is noted that in order to get an Euclidean instanton, the Lorentzian angular momentum parameters $a_j$ should take an imaginary value. However, we shall use an alternative prescription below, that is $a_j$ remains to be real.

We are interested in the creation of the black hole from a more general instanton, i.e., the constrained instanton, in which the above restrictions (6), (7) are relaxed. Of course, the creation of a maximal black hole is the special case of the consideration here. We choose an arbitrary period $\beta$, the identification we are using is $\tau \to \tau + \beta, r \to r, \mu_i \to \mu_i, \phi_i \to \phi_i + \eta_i$, where $\eta_i$ are arbitrary constants. The obtained metric is complex. Since this identification does not meet the regularity conditions for the regular instanton, there is at least one conical singularity associated with one of the horizons, and also other irregularities associated with the differential rotations there.

The quantum transition should occur at one equator, say the joint sections $\tau = 0$ and $\tau = \beta/2$. The
created Lorentzian spacetime metric can be obtained via the reversal analytic continuation at the equator. Strictly speaking, the seed instanton should be constructed by a south “hemisphere” with its time reversal, the north “hemisphere”. However, this consideration will not affect our calculation below. The relative creation probability of the black hole, at the WKB level, is [11]

\[ P \approx \exp(-I), \]

where \( I \) is the Euclidean action of the instanton. It is written as

\[ I = -\frac{1}{16\pi} \int M (D R - 2\Lambda) - \frac{1}{8\pi} \int_{\partial M} D^{-1} K, \]

(9)

where \( D R \) is the scalar curvature of the \( D \)-dimensional spacetime and \( D^{-1} K \) is the extrinsic curvature of its boundary. The second term includes contributions from the discontinuities of the extrinsic curvature and the conical singularity (as a degenerate form).

One could calculate the action directly, as for the Kerr–anti-de Sitter black hole case [12]. In the Kerr–anti-de Sitter case one has to perform the background subtraction to regularize the expression. Here, we use an alternative method, that is a topological argument, which is more transparent due to the topological origin of the entropy. The most convenient way is to divide \( M \) into three parts: \( M_1(r_1 \leq r \leq r_1 + \delta) \), \( M_2(r_2 - \delta \leq r \leq r_2) \) and \( M'(r_1 + \delta \leq r \leq r_2 - \delta) \), where \( \delta \) is a positive infinitesimal quantity. The total action is

\[ I = I_1 + I_2 + \int_{M'} (\pi^p q^q h_{pq} - N H_0 - N^p H^p) d^{D-1}x d\tau, \]

(10)

where \( I_i \) is the action for the submanifold \( M_i \). The third term of the action for \( M' \) has been cast into the canonical form in the \( 1 + (D - 1) \) decomposition. Here, \( \pi^p q^q \) is the canonical momentum conjugate to the metric \( h_{pq} \) of hypersurfaces \( \tau = \text{const} \). \( H_0 \) and \( H^p \) are the Einstein and momentum constraints, which should be zero for the instanton. The dot denotes the time derivative, it must vanish due to the \( U(1) \) symmetry associated with the Killing coordinate \( \tau \). Therefore, the third term of the action must be zero.

To calculate the first or second term of the action, one can apply the Gauss–Bonnet theorem to the 2-dimensional \((\tau, r)\) section of \( M_l \) under the condition that other coordinates are frozen,

\[ \frac{1}{4\pi} \int_{\hat{M}_l} 2 R + \frac{1}{2\pi} \int_{\partial \hat{M}_l} 1 K + \frac{\sigma}{2\pi} = \chi(l), \]

(11)

where \( \hat{M}_l \) represents the section, \( 2 R \) and \( 1 K \) are the scalar curvature of \( \hat{M}_l \) and extrinsic curvature of \( \partial \hat{M}_l \), respectively, \( \sigma \) is the deficit angle of the conical singularity at the horizon or the apex, and \( \chi(l) \) is the Euler characteristic of \( \hat{M}_l \), which is 1 here.

The first terms of both (9) for \( M_l \) and (11) are zero as \( \delta \) approaches zero. At the horizon one has \( D^{-1} K = 1 K \), and the deficit angle is included as a degenerate form in the second term of (9). Comparing (9) and (11), one obtains

\[ I_l = -\frac{1}{4} A_l, \]

(12)

where \( A_l \) is the horizon area. Its value is given [8]

\[ A_l = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)} \prod_{i=1}^N \frac{r_i^2 + a_i^2}{1 + \lambda a_i^2}. \]

(13)

One has to be cautious for the decomposition in (10), we have implicitly assumed that \( M_1 \) and \( M_2 \) are regular everywhere except the conical singularities, this is only possible when \( \phi_i - \Omega_i' \tau \) are fixed at a fixed point at the horizon \( r_1 \). However, the angular velocities at the two horizons are different in general. Therefore, there must be discrepancies of the coordinate angle \( \phi_i \) and discontinuity of extrinsic curvature across \( r = r_1 + \delta \) and \( r = r_2 - \delta \), for any identification parameters \( \eta_i \). As \( \delta \) approaches zero, all these irregularities will be located at the horizons, and the configuration of the wave function at the equator remains intact. This part of contribution should be cancelled by adding the Legendre terms, that is the differential rotating angles of the two horizons times their corresponding angular momenta, as we did earlier [2]. Using the canonical form, this problem has been taken care of automatically. Since the third term of the action is additive, it means that the Gibbons–Hawking boundary term has been implicitly included.

If one naively integrates the volume term of (9) for \( M' \), the result would not be zero. One can rephrase this phenomenon by saying that the angle is not the right representation for the metric of the equator in computing the creation probability. Instead, their conjugate
variables, the angular momenta are the right representation. This problem did not occur for the earlier research with other regular instantons, since then the relevant Legendre terms vanished anyway.

Therefore, the total action is one quarter of the sum of the two horizon areas. Since we have shown the constructed manifold satisfies the field equation everywhere except at the horizons, for the given equator configuration the only degree freedom left are $\beta$ and $\eta_i$. We have shown that the action is independent of these parameters. This means that the action is stationary with arbitrary variation with the restrictions at the equator. Therefore it is qualified as a constrained instanton and formula (8) is valid.

In gravitational thermodynamics [13] the Euclidean path integral can be interpreted as the partition function. Our right representation corresponds to the microcanonical ensemble. In this ensemble the entropy $S$ can be obtained from the partition function $Z$

$$S = \ln Z \approx -1,$$  \hspace{1cm} (14)

the approximation is at the WKB level. Therefore, the relative creation probability is the exponential to the entropy of the created hole. This is the result we wish to show.

Our method of computing the action using canonical form can also be applied to the Euclidean regular instanton [8], in which the two roots for the horizons coincide. For the selected parameters and the right angular velocity for the identification as required by (6), one can avoid all irregularities, and then the action should be the negative of half of the horizon area, as in the case of usual 4-dimensional Nariai instanton.

It is clear that the method used here is also applicable to all stationary metric with at least two horizons, in particular to the metric of the higher-dimensional charged Kerr–de Sitter black hole, which has been found for 5-dimensional case [14] and is expected to be found in all dimensions in the near future.

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References


