KNOTS IN SPATIAL EMBEDDINGS OF THE COMPLETE GRAPH ON FOUR VERTICES

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Let \( c_1, \ldots, c_7 \) denote the seven cycles in the complete graph, \( K_4 \), on four vertices. For a tame embedding of \( K_4 \) in Euclidean 3-space, let the ordered 7-tuple \( (k_1, \ldots, k_7) \) denote the associated list of knot types of \( (c_1, \ldots, c_7) \). The purpose of this paper is to show that any ordered 7-tuple of knot types can be realized as the associated list of knot types of \( (c_1, \ldots, c_7) \) with a tame embedding of \( K_4 \) in Euclidean 3-space.

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1. Introduction

We call a tame embedding of a graph in Euclidean 3-space \( R^3 \) a spatial embedding of the graph. Let \( K_n \) denote the complete graph on \( n \) vertices. Throughout this paper, all knots are oriented.

It is an interesting problem what types of spatial embeddings of a graph exist. Conway and Gordon [1] showed that every spatial embedding of \( K_7 \) contains a nontrivially knotted Hamiltonian cycle. This shows that every spatial embedding of \( K_n \) contains a nontrivially knotted cycle for \( n \geqslant 7 \) since \( K_n \) contains \( K_7 \) as a subgraph if \( n \geqslant 7 \). For an integer \( n \) greater than one, the \( \theta \)-curve of order \( n \) is the graph consisting of two vertices and \( n \) edges joining the vertices and it has \( \frac{1}{2} n(n - 1) \) cycles. Kinoshita [2] showed that any ordered \( \left( \frac{1}{2} n(n - 1) \right) \)-tuple \( (k_1, \ldots, k_{n(n-1)/2}) \) of knot types can be realized as the associated list of knot types of the cycles with a spatial embedding of the \( \theta \)-curve of order \( n \). Recently, Shimabara [4] studied the spatial embeddings of the complete bipartite graph \( K_{5,5} \) and several other graphs (see Sachs [3] also). In this paper we prove the following:

Theorem 1.1. Let \( c_1, \ldots, c_7 \) be the seven cycles in \( K_4 \). For any ordered 7-tuple \( (k_1, \ldots, k_7) \) of knot types, there is a spatial embedding of \( K_4 \) such that the associated list of knot types of \( (c_1, \ldots, c_7) \) is \( (k_1, \ldots, k_7) \).
Although $K_5$ and $K_6$ can be embedded in $\mathbb{R}^3$ so that all cycles are trivial knots, it is still unknown whether similar claims to Theorem 1.1 for $K_5$ and $K_6$ are true or not.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need a certain canonical diagram of a knot, where a *diagram* is the image of a regular projection together with an over/under information at each double point and such a double point is called a *crossing*. Let $\sigma_1$, $\sigma_2$, $\sigma_3$ and $\sigma_4$ denote the upper, left, lower and right sides of a rectangle respectively. By $\gamma_0$, we denote $\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4$ and we give $\gamma_0$ the counterclockwise orientation. Let $\gamma_1, \ldots, \gamma_u$ denote trivial circles such that the diagram of $\gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_u$ is given as in Fig. 1. We divide $\sigma_3$ into $2u+1$ subarcs $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_u, \beta_u$ and $\alpha_{u+1}$.

**Lemma 2.1.** Let $\gamma_0, \ldots, \gamma_u$ be as above. For any knot $K$ of the unknotting number at most $u$, there is a diagram of $K$ represented by $\gamma_0, \ldots, \gamma_u$ and mutually disjoint strips $S_1, \ldots, S_u$ in $\mathbb{R}^3$, satisfying the following conditions:

1. $\gamma_0 \cap S_i = \gamma_0 \cap \partial S_i = \beta_i$,
2. $\gamma_i$ meets $S_i$ in an arc $\delta_i$ of $\partial S_i$,
3. $\gamma_i$ does not meet $S_j$ if $i \neq j$,
4. $\partial S_i$ does not cross $\gamma_0, \ldots, \gamma_u$ other than $\alpha_i$ in the diagram, and
5. $K = (\gamma_0 \cup \cdots \cup \gamma_u \cup \partial S_i \cup \cdots \cup \partial S_u) - \text{Int}(\beta_1 \cup \cdots \cup \beta_u \cup \delta_1 \cup \cdots \cup \delta_u)$,

where $i, j = 1, \ldots, u$.

We call the diagram (5) of $K$ the *canonical diagram* of $K$. An example of our canonical diagram is illustrated in Fig. 2.
Proof of Lemma 2.1. Suzuki [5, Lemma 1.1] showed that there is a diagram represented by circles $\gamma_0, \ldots, \gamma_u$ and strips $S_1, \ldots, S_u$ like ours, but, in his diagram, $\gamma_1, \ldots, \gamma_u, \gamma_0 \cap \partial S_1, \ldots, \gamma_0 \cap \partial S_u$ are not arranged as ours. Although he said nothing about the orientation of $\gamma_0$, we may assume that $\gamma_0$ is counterclockwise oriented. For, if otherwise, the one obtained from the mirror image of the diagram by changing all crossings is also a diagram of the knot and the circle corresponding to $\gamma_0$ is counterclockwise oriented. Then we need to show that we can change the order of $\gamma_i$ and $\gamma_j$ and that of $\gamma_i$ and $\beta_j$ with respect to the orientation of $\gamma_0$ for $i \neq j, i, j = 1, \ldots, u$, keeping the conditions (1), (2), (3) and (5) satisfied. Allowing to intersect the strips $S_i$ and $S_j$ in an arc $\alpha$ contained in $\text{Int } S_i$, we can change the order of $\gamma_i$ and $\gamma_j$ and that of $\gamma_i$ and $\beta_j$. Then we deform $S_j$ so that $S_i \cap S_j = \emptyset$ by moving $\alpha$ on $S_i$ and across $\beta_j$, see Fig. 3. Repeating this, we have a diagram of $K$ satisfying (1), (2), (3) and (5) from the diagram of Suzuki [5]. In the diagram, $\partial S_i$ may cross

Fig. 3.
If $S_i$ crosses $\gamma_0 - \alpha_i$, we can deform $S_i$ isotopically so that $S_i$ does not cross $\gamma_0 - \alpha_i$, $\gamma_1, \ldots, \gamma_u$ and crosses at most $\alpha_i, S_1, \ldots, S_u$. 

The motivation behind Lemma 2.1 is to prove the following.

**Lemma 2.2.** For any nontrivial knot, there is a diagram of the knot such that the diagram has two nonempty sets $A$ and $B$ of crossings which satisfy the following; the sets $A$ and $B$ are mutually disjoint and for any set of $A$, $B$ and $A \cup B$, the knot is deformed into the trivial knot when we change all crossings of the set.

**Proof.** We show that the canonical diagram is the desired one. We may assume that each strip has even number of half twists, for we can add or subtract a half twist from a strip $S_i$, without changing the knot type of $K$, by changing both crossings of $\gamma_0$ with $\gamma_i$, see Fig. 4. Let $a_i$ be one of the two crossings of $\gamma_0$ with $\gamma_i$ for $i = 1, \ldots, u$. It is obvious that the crossing changes at $a_1, \ldots, a_u$ make $K$ trivial. Then, let $A$ be the set $\{a_1, \ldots, a_u\}$.

![Diagram](image)

We can also make $K$ trivial by crossing changes which deform so that $S_i$ crosses over $S_j$ and $\gamma_0$ for $j = i+1, \ldots, u$ if $S_i$ crosses $S_j$ and $\gamma_0$ respectively, and $S_i$ is unknotted and untwisted, see Figs. 2 and 5, where the knot of Fig. 2 and the knot at the left of Fig. 5 are of the same knot type. Then, let $B$ be the set of all such crossings. It is clear that $A$ and $B$ are mutually disjoint.

If we change the crossings of $B$ first, the diagram is deformed into a canonical diagram of the trivial knot since only over and under roles of crossings of the strips and $\gamma_0$ are changed. Then, if we change the crossings of $A$ next, the resultant diagram is a diagram of the trivial knot. Therefore $K$ is deformed into the trivial knot when we change the crossings of $A \cup B$. 

We construct the canonical diagram of the composite knot $k_1 \neq k_2$ of $k_1$ and $k_2$ as follows: Let $\sigma_j$ denote the corresponding side of the canonical diagram of $k_i$ to $\sigma_j$ of that of $K$ in Lemma 2.1 for $i = 1, 2$. Identifying $\sigma_{14}$ with $\sigma_{22}$ and deleting the interior of the identified side, we have a diagram of $k_1 \neq k_2$ which satisfies (1), (2), (3), (5) and may not satisfy (4). As in the proof of Lemma 2.1, we can deform the
diagram into the canonical diagram isotopically. The canonical diagram of the composite knot $k_1 \# \cdots \# k_n$ is constructed inductively as $(k_1 \# \cdots \# k_{n-1}) \# k_n$.

The following is immediate from Lemma 2.2 and the construction of the canonical diagram of a composite knot.

**Lemma 2.3.** In the canonical diagram of a composite knot $k_1 \# \cdots \# k_n$, there are sets of crossings $A_1, B_1, \ldots, A_n, B_n$ such that they are mutually disjoint and the diagram is deformed into that of $k_1 \# \cdots \# k_{i-1} \# k_i \# \cdots \# k_n$ when we change the crossings of any of $A_i, B_i$ and $A_i \cup B_i$ for $i = 1, \ldots, n$.

If two distinct edges of a graph are incident with a common vertex, then they are **adjacent at the vertex**. We say that three distinct edges $e_1, e_2$ and $e_3$ of a graph are **successive** if $e_1$ and $e_2$ are adjacent, $e_2$ and $e_3$ are adjacent and $e_1, e_2$ and $e_3$ are not incident with a common vertex.

**Lemma 2.4.** Let $G$ be a graph. Let $C = \{c_1, \ldots, c_n\}$ denote a set of cycles of $G$ which satisfies that each $c_i$ has successive three edges $e_{i1}, e_{i2}, e_{i3}$ such that $c_i$ is the unique cycle in $C$ containing $e_{i1}, e_{i2}, e_{i3}$. If there is a spatial embedding of $G$ such that all knot types of $c_1, \ldots, c_n$ of the embedding are trivial, then for any ordered $n$-tuple $(k_1, \ldots, k_n)$ of knot types, there is a spatial embedding such that the associated list of knot types of $(c_1, \ldots, c_n)$ is $(k_1, \ldots, k_n)$.

**Proof.** We denote a knot of the knot type $k_i$ by $k_i$ also. By renumbering if necessary, we may assume that $c_1, \ldots, c_m$ are all cycles of $C$ such that $e_{12} = \cdots = e_{m2}$, where $1 \leq m \leq n$. We denote the edge $e_{12} = \cdots = e_{m2}$ by $e$ and the vertices of $e$ by $v$ and $v'$. We may also assume that $e_{i1}$ and $e_{i2}$ are adjacent to $e$ at $v$ and $v'$ respectively, i.e., $e_{i1} \cap e = \{v\}$ and $e \cap e_{i3} = \{v'\}$, for $i = 1, \ldots, m$. 

For the embedding of the graph $G$ such that all knot types of $c_1, \ldots, c_n$ of the embedding are trivial, every edge contained in $c_1, \ldots, c_n$ is embedded locally unknotted, i.e., there is no $3$-ball $B^3$ such that $\partial B^3$ meets the embedded $G$ in two points of the edge and the pair of $B^3$ and the intersection of $B^3$ with the edge is a knotted ball pair. We replace the edge $e$ with an edge with knot type $k_1 \# \cdots \# k_m$; that is, the knot types of $c_1, \ldots, c_m$ are that of $k_1 \# \cdots \# k_m$, e.g., in Fig. 6, one edge is knotted in the composite knot of the trefoil knot and the figure-eight knot and the knot types of the cycles containing the edge are that of the composite knot. We assume that the knotted edge, denoted by $e$ again, is obtained from the canonical diagram of $k_1 \# \cdots \# k_m$ by deleting the interior of the arc of $k_1$ and the vertex $v$ is the point $\sigma_{11} \cap \sigma_{12}$ of $k_1$.

Let $A_1, B_1, \ldots, A_m, B_m$ denote the set of crossings of the canonical diagram satisfying Lemma 2.3, where $A_i = B_i = \emptyset$ if $k_i$ is the trivial knot. We denote the crossings of $A_i$ by $a_{i1}, \ldots, a_{iu_i}$. We denote the crossings of $B_i$ by $b_{i1}, \ldots, b_{iu_i}$, where the second index $j$ of $b_{ij}$ is given so that we meet $b_{1j}, \ldots, b_{uj}$ in this order when we walk along the edge $e$ from $V'$ to $v$. We note that a crossing of an edge with itself can be replaced by crossings of the edge with the adjacent edges at a vertex of the edge, see Fig. 7. We replace the crossing $a_{ij}$ of $e$ with itself to the crossings of $e$ with all adjacent edges at the vertex $v$ and change the crossings except that of $e$ with $e_{i1}$ for $(i, j) = (1, 1), \ldots, (1, u_1), (2, 1), \ldots, (2, u_2), \ldots, (m, 1), \ldots, (m, u_m)$. We
replace the crossing $b_j$ of $e$ with itself to the crossings of $e$ with all adjacent edges at the vertex $v'$ and change the crossings except that of $e$ with $e_{i3}$ for $(i,j) = (1,1), \ldots, (1,x_1), (2,1), \ldots, (2,x_2), \ldots, (m,1), \ldots, (m,x_m)$. Then, the knot type of $c_i$ is $k_i$ since we have changed the crossings of $A_j$, $B_j$ or $A_j \cup B_j$ for $j = 1, \ldots, i-1, i+1, \ldots, m$ of $c_i$ and by Lemma 2.3. The knot type of any cycle of $c_{m+1}, \ldots, c_n$ is unchanged since, if a cycle of $c_{m+1}, \ldots, c_n$ contains $e$, we have changed the crossings of $A_j$, $B_j$ or $A_j \cup B_j$ for $j = 1, \ldots, m$ of the cycle and by Lemma 2.3. Although the knot types of other cycles containing $e_{i1}, e_{i2}$ and $e_{i3}$ may be changed, such cycles are not in $C$. In Fig. 8, an example is illustrated, where $G = K_4$, the cycle consisting of the edges $e_1, e_2, e_3$ is knotted in the trefoil knot, the cycle consisting of the edges $e_1, e_2, e_4, e_5$ is knotted in the figure-eight knot and the other cycles are trivial. Repeating the same constructions on edges $e_{m+1,2}, \ldots, e_{n,2}$, we have the embedding such that the knot type of $c_i$ is $k_i$ for each $i = m+1, \ldots, n$ without changing the knot types of the other cycles of $C$. Therefore we have the desired embedding.

Fig. 8.

Now we can prove Theorem 1.1 immediately. In $K_4$, every cycle has successive three edges such that the cycle is the unique one containing the three edges. We can embed $K_4$ in Euclidean 3-space so that the knot type of every cycles of $K_4$ is trivial since $K_4$ is a planar graph. This completes the proof of Theorem 1.1 by Lemma 2.4.

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References