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A Characterization of the Ree Groups ${}^2F_4(q)$

DAVID PARROTT[1]

The Ohio State University, Columbus, Ohio 43210

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1. INTRODUCTION

The Ree groups ${}^2F_4(q)$ are the fixed points of a certain automorphism of the Chevalley groups of type F_4 over a finite field $K = GF(q)$, where $q = 2^{2n+1}$, $n \geqslant 0$. Ree [7] showed that the groups ${}^2F_4(q)$ are simple if $q > 2$, while Tits [12] showed that ${}^2F_4(2)$ is not simple but possesses a simple subgroup $\mathscr{T}$ of index 2.

In this paper, we give a characterization of the Ree groups in terms of the centralizers of involutions in the center of a Sylow 2-subgroup. Namely, if $H(q)$ denotes the centralizer of each involution in the centre of a Sylow 2-subgroup of ${}^2F_4(q)$, we have the following.

THEOREM. *Let G be a finite group which possesses a subgroup $H \cong H(q)$ so that for every involution $z \in Z(H)$ we have $H = C_G(z)$. Then one of the following possibilities holds:*

(i) $G \cong {}^2F_4(q)$, $q = 2^{2n+1}$, $n \geqslant 0$.

(ii) $q = 2$ *and* $G = H \cdot O(G)$.

(iii) $q > 2$ *and* $Z(H) \lhd G$ *(with* $|G : H| \mid q - 1$*)*.

The notation of this paper will follow [3], and we will follow [7] in regard to the structure of ${}^2F_4(q)$. In particular, if X_1, X_2 are subsets of a finite group X, $X_1 \sim_X X_2$ means $X_1 = x^{-1}X_2x = X_2^x$ for some $x \in X$, while $L_n(X)$ is defined by $L_1(X) = X$ and $L_n(X) = [X, L_{n-1}(X)]$ for each $n \geqslant 2$.

2. THE STRUCTURE OF $H \cong H(q)$

Since we will only consider the structure of $H(q)$ and not that of ${}^2F_4(q)$, we will identify H with $H(q)$.

Let $K = GF(q)$ be a finite field of characteristic 2 and order $q = 2^{2n+1}$, $n \geqslant 0$. Further, let K^* denote the multiplicative group of K. As is well

[1] Department of Pure Mathematics, The University of Adelaide, South Australia 5001.

known, K admits and automorphism θ such that $2\theta^2 = 1$. In [7], Ree gives generators $\alpha_i(t)$; $i = 1,\ldots, 12$, $t \in K$, for a Sylow 2-subgroup $T = T(q)$ of ${}^2F_4(q)$. He also shows $Z(T) = \langle \alpha_{12}(t) \mid t \in K \rangle$ is elementary of order q. Using the definition of the $\alpha_i(t)$ and the commutator relations for the Chevalley groups of type F_4 (see [7, p. 404]) all commutators $[\alpha_i(t), \alpha_j(u)]$ can be computed. From Ree's work, it is straightforward to show that $H = C_{{}^2F_4(q)}(\alpha_{12}(t))$ is independent of our choice of $t \in K^*$.

Further, in the notation of [7],

$$\begin{aligned} H &= \langle T, \omega(w(1, -1, \infty)) = \omega_{34'} \cdot \omega_{34}, h(\chi_{34', t^{1-2\theta}} \cdot \chi_{34, t^{-1}}) \mid t \in K \rangle \\ &= \langle T, \omega_1, h(t) \mid t \in K \rangle. \end{aligned}$$

Below we list relations between the generators of H:

1. For all $t, u \in K$ we have:

(a) $\alpha_i(t)\,\alpha_i(u) = \alpha_i(t + u) = \alpha_i(u)\,\alpha_i(t)$, $i = 2, 3, 7, 8, 9, 10, 11, 12$

$$\begin{aligned} \alpha_1(t)^2 &= \alpha_2(t^{2\theta+1}),\ [\alpha_1(t), \alpha_1(u)] = \alpha_2(t^{2\theta}u + tu^{2\theta}); \\ \alpha_4(t)^2 &= \alpha_8(t^{2\theta+1}),\ [\alpha_4(t), \alpha_4(u)] = \alpha_8(t^{2\theta}u + tu^{2\theta}); \\ \alpha_5(t)^2 &= \alpha_{12}(t^{2\theta+1}),\ [\alpha_5(t), \alpha_5(u)] = \alpha_{12}(t^{2\theta}u + tu^{2\theta}); \\ \alpha_6(t)^2 &= \alpha_{11}(t^{2\theta+1}),\ [\alpha_6(t), \alpha_6(u)] = \alpha_{11}(t^{2\theta}u + tu^{2\theta}). \end{aligned}$$

(b)

$$\begin{aligned} [\alpha_{11}(t), \alpha_3(u)] &= [\alpha_{10}(t), \alpha_4(u)] = [\alpha_9(t), \alpha_6(u)] \\ &= [\alpha_8(t), \alpha_7(u)] = \alpha_{12}(tu); \\ [\alpha_{10}(t), \alpha_1(u)] &= [\alpha_9(t), \alpha_2(u)] = \alpha_{11}(tu); \\ [\alpha_9(t), \alpha_1(u)] &= \alpha_{10}(tu^{2\theta})\,\alpha_{11}(tu^{2\theta+1})\,\alpha_{12}(t^{2\theta}u); \\ [\alpha_8(t), \alpha_2(u)] &= \alpha_{10}(tu)\,\alpha_{11}(u^{2\theta}t)\,\alpha_{12}(t^{2\theta}u), \\ [\alpha_8(t), \alpha_1(u)] &= \alpha_9(tu)\,\alpha_{11}(u^{2\theta+2}t)\,\alpha_{12}(t^{2\theta}u^{2\theta+1}); \\ [\alpha_5(t), \alpha_7(u)] &= \alpha_{11}(tu),\ [\alpha_5(t), \alpha_6(u)] = \alpha_{10}(tu), \\ [\alpha_5(t), \alpha_4(u)] &= \alpha_9(tu),\ [\alpha_5(t), \alpha_3(u)] = \alpha_8(tu); \\ [\alpha_7(t), \alpha_4(u)] &= \alpha_{10}(u^{2\theta}t)\,\alpha_{11}(t^{2\theta}u)\,\alpha_{12}(u^{2\theta+1}t) \\ [\alpha_7(t), \alpha_3(u)] &= \alpha_9(tu^{2\theta})\,\alpha_{10}(t^{2\theta}u); \\ [\alpha_6(t), \alpha_3(u)] &= \alpha_8(tu^{2\theta})\,\alpha_9(t^{2\theta}u)\,\alpha_{12}(t^{2\theta+1}u) \\ [\alpha_6(t), \alpha_1(u)] &= \alpha_7(tu); \\ [\alpha_4(t), \alpha_2(u)] &= \alpha_7(tu)\,\alpha_{11}(t^{2\theta+1}u^{2\theta})\,\alpha_{12}(t^{2\theta+2}u) \\ [\alpha_4(t), \alpha_1(u)] &= \alpha_5(t^{2\theta}u)\,\alpha_6(tu^{2\theta})\,\alpha_7(tu^{2\theta})\,\alpha_9(t^{2\theta+1}u) \\ &\quad \alpha_{11}(t^{2\theta+1}u^{2\theta+2})\,\alpha_{12}(t^{2\theta+2}u^{2\theta+1}); \\ [\alpha_3(t), \alpha_2(u)] &= \alpha_5(t^{2\theta}u)\,\alpha_6(tu)\,\alpha_7(tu^{2\theta})\,\alpha_8(t^{2\theta+1}u)\,\alpha_9(t^{2\theta+1}u^{2\theta}) \\ &\quad \alpha_{10}(t^{2\theta+1}u^2)\,\alpha_{12}(t^{2\theta+2}u^{2\theta+1}), \\ [\alpha_3(t), \alpha_1(u)] &= \alpha_4(tu)\,\alpha_5(t^{2\theta}u^{2\theta+1})\,\alpha_7(tu^{2\theta+2})\,\alpha_8(t^{2\theta+1}u^{2\theta+1}) \\ &\quad \alpha_9(t^{2\theta+1}u^{2\theta+2})\,\alpha_{10}(t^{2\theta+1}u^{4\theta+2})\,\alpha_{11}(t^{2\theta+1}u^{4\theta+3}) \\ &\quad \alpha_{12}(t^{2\theta+2}u^{4\theta+3}). \end{aligned}$$

All other commutators $[\alpha_i(t), \alpha_j(u)] = 1$, $i, j = 1,\ldots, 12$.

2. Put $h(u) = h(\chi_{34'}, {}_{u^{1-2\theta}} \cdot \chi_{34,u^{-1}})$ for each $u \in K^*$. Then

$$\langle h(u) \mid u \in K^* \rangle \simeq K^*,$$

and for each $t \in K$, $u \in K^*$ we have the following action of $h(u)$ on T:

$\alpha_i(t)$	$\alpha_{12}(t)$	$\alpha_{11}(t)$	$\alpha_{10}(t)$	$\alpha_9(t)$	$\alpha_8(t)$	$\alpha_7(t)$
$\alpha_i(t)^{h(u)}$	$\alpha_{12}(t)$	$\alpha_{11}(u^{-2\theta}t)$	$\alpha_{10}(u^{2\theta-2}t)$	$\alpha_9(u^{2-2\theta}t)$	$\alpha_8(u^{2\theta}t)$	$\alpha_7(u^{-2\theta}t)$
$\alpha_i(t)$	$\alpha_6(t)$	$\alpha_5(t)$	$\alpha_4(t)$	$\alpha_3(t)$	$\alpha_2(t)$	$\alpha_1(t)$
$\alpha_i(t)^{h(u)}$	$\alpha_6(u^{2\theta-2}t)$	$\alpha_5(t)$	$\alpha_4(u^{2-2\theta}t)$	$\alpha_3(u^{2\theta}t)$	$\alpha_2(u^{-2}t)$	$\alpha_1(u^{2-4\theta}t)$

3. Put $\omega_1 = \omega(w(1, -1, \infty))$; then $\omega_1^2 = 1$ and for each $t \in K$,

$$\alpha_{12}(t)^{\omega_1} = \alpha_{12}(t),\ \alpha_{11}(t)^{\omega_1} = \alpha_8(t),\ \alpha_{10}(t)^{\omega_1} = \alpha_9(t),$$
$$\alpha_5(t)^{\omega_1} = \alpha_5(t),\ \alpha_7(t)^{\omega_1} = \alpha_3(t),\ \alpha_6(t)^{\omega_1} = \alpha_4(t).$$

Further, for each $u \in K^*$, $h(u)^{\omega_1} = h(u)^{-1}$.

4. $S = \langle \alpha_1(t), \omega_1, h(u) \mid t \in K, u \in K^* \rangle \simeq Sz(q)$, the Suzuki simple group (described in [8]). (This is probably most easily seen by showing that the Sylow 2-subgroups of S are T.I. sets and then using Suzuki's result [9].)

From these relations, we can describe the structure of H using the following notation:

$$\begin{aligned} U_i &= \langle \alpha_i(t), \alpha_{i+1}(t), \dots, \alpha_{12}(t) \mid t \in K \rangle; \\ V_i &= \langle \alpha_i(t) \mid t \in K \rangle, \qquad i = 1, \dots, 12; \\ J &= U_3 = O_2(H); \\ Z &= Z(H) = Z(T) = U_{12} = V_{12}; \\ E &= U_8 = J'; \\ Q &= \langle h(u) \mid u \in K^* \rangle. \end{aligned}$$

(Note that $T = U_1$ is a Sylow 2-subgroup of H.)

We have that H is a faithful split extension of the 2-group J of class 3 and order q^{10} by the group $S \simeq Sz(q)$ of order $q^2(q-1)(q+1)$. A Sylow 2-subgroup T of H has order q^{12}, $N_H(T) = T \cdot Q$, where Q is cyclic of order $q - 1$ and $Z = Z(T) = Z(H)$ is elementary of order q. The subgroup $E = J'$ is elementary of order q^5 with $C_H(E) = \langle E, V_5 \rangle$ of order q^6 and $\Omega_1(C(E)) = E$.

Also $Z = L_3(J) = [J, J'], U_i \lhd T$ and $V_i^Q = V_i$ for each $i = 1, \ldots, 12$. Note that V_i is elementary abelian for $i = 2, 3, 7, 8, 9, 10, 11, 12$, but $V_1 \cong V_4 \cong V_5 \cong V_6$ are "Suzuki 2-groups" of order q^2.

Some Properties of $S \cong Sz(q)$ (see [8])

The groups $Sz(q)$ are simple if $q > 2$, while $Sz(2)$ is a Frobenius group of order 20. A Sylow 2-subgroup V_1 of S is non-Abelian of order q^2, $\Omega_1(V_1) = V_2 = Z(V_1) = V_1'$ is elementary of order q with $N_S(V_1) = Q \cdot V_1$. For each involution $\alpha_2(t) \in V_2$, $C_S(\alpha_2(t)) = V_1$. The other conjugacy classes of maximal local subgroups of S are Frobenius groups of order $4(q + \sqrt{2q} + 1), 4(q - \sqrt{2q} + 1), 2(q - 1)$ (the latter having $N_S(Q) = Q\langle\omega_1\rangle$ as a representative). If $S_1 \subseteq S$ then either S_1 is conjugate to a subgroup of one of the maximal subgroups above or $S_1 \cong Sz(q_1)$, where $q_1 \mid q$. Finally, each outer automorphism of S is induced by a field automorphism of the underlying field K.

With regard to the action of S on J, we have $C_J(S) = V_5$ (of order q^2) while if $q > 2$ S acts indecomposably on both E and J/E.

The Conjugacy Classes of Involutions in H

In $E - Z$ there are two classes of involutions (in H) with representatives $\alpha_{11}(1), \alpha_{10}(1)$. Put $C = C_H(\alpha_{11}(1)) = C_T(\alpha_{11}(1))$ and $D = C_H(\alpha_{10}(1)) = C_T(\alpha_{10}(1))$. We have $C = \langle V_i \mid i \neq 3\rangle$ is of order q^{11} and class 5 with $Z(C) = L_5(C) = U_{11}$ and put $F = L_3(C) = \langle V_{12}, V_{11}, V_{10}, V_9, V_7\rangle$ so F is elementary Abelian of order q^5. Note that $F \lhd T$. Further, there are precisely $2q - 1$ cosets of E in J which contain an element of order four whose square is $\alpha_{11}(1)$. Similarly, $D = \langle V_i \mid i \neq 1, 4\rangle$ is of order q^{10}, $Z(D) = Z \times V_{10}$ and $L_5(D) = Z$ if $q > 2$ while $L_4(D) = Z$ if $q = 2$. Finally, if $e \in E - U_{10}$ we have $C_T(e) = C_J(e)$ (of order q^9) while $e \in E \cap F - U_{11} = U_9 - U_{11}$ implies $e \sim_H \alpha_{10}(1)$ and $e \in U_{11} - Z$ implies $e \sim_H \alpha_{11}(1)$.

In $J - E$ there is one class of involutions with representative $\alpha_7(1)$ and there are precisely $(q^2 + 1)(q - 1) + 1 = q^3 - q^2 + q$ cosets of E in J which contain involutions. We have that $C_H(\alpha_7(1)) = C_T(\alpha_7(1)) = \langle V_i \mid i = 1, 2, 6, 7, 9, 10, 11, 12\rangle$ is of order q^8 and with center $\langle Z, V_{11}, V_7\rangle$ elementary of order q^3. Also, $\Omega_1(C_T(\alpha_7(1)))$ has index q in $C_T(\alpha_7(1))$ so that $C_T(\alpha_7(1)) = \Omega_1(C_T(\alpha_7(1)) \cdot V_1$.

Finally in $T - J$ there are q classes of involutions with representatives $\alpha_2(1)\, \alpha_{12}(t)$, $t \in K$. For each $t \in K$, $C_H(\alpha_2(1)\, \alpha_{12}(t)) = C_T(\alpha_2(1)) = C_H(\alpha_2(1)) = \langle V_i \mid i = 1, 2, 5, 6, 7, 10, 11, 12\rangle$. Further, $Z(C_T(\alpha_2(1))) = \langle Z, V_{11}, V_2\rangle$ is elementary Abelian of order q^3 while $\Omega_1(C_T(\alpha_2(1))) = \langle V_{12}, V_{11}, V_{10}, V_7, V_2\rangle$ is elementary of order q^5.

To summarize, we list representatives of the conjugate classes of involutions in $H - Z$ and some properties of their centralizers:

Involution x	Location	$\lvert C_H(x) \rvert$	$Z(C_H(x))$
$\alpha_{11}(1)$	$E - Z$	q^{11}	$Z \times V_{11} = U_{11}$
$\alpha_{10}(1)$	$E - Z$	q^{10}	$Z \times V_{10}$
$\alpha_7(1)$	$J - E$	q^8	$U_{11} \times V_7$
$\alpha_2(1)\, \alpha_{12}(t)$, $t \in K$	$H - J$	q^8	$U_{11} \times V_2$

3. A Preliminary Result

The idea used in the proof of Proposition 2 is due to Suzuki [10]; the first result is well known (see [10] also).

Proposition 1. *Let w, v be involutions in a finite group X with $\langle wv \rangle$ of even order. If i is the unique involution in $\langle wv \rangle$ then $\langle w, v \rangle \subseteq C_X(i)$ and either*

$$wi \underset{X}{\sim} w \quad \text{and} \quad vi \underset{X}{\sim} v \qquad \text{or} \quad wi \underset{X}{\sim} v \quad \text{and} \quad vi \underset{X}{\sim} w.$$

Proposition 2. *Let Y be a 2-subgroup of a finite group X. Suppose that $Y \subset W \subseteq X$ so that for every $x \in X$ and $y \in Y$ with $y^x \in W$, we have $y^x \in Y$. Further, suppose there is an involution $w \in W - Y$ with $C(w_1) \subseteq W$ for all $w_1 \in \langle w, Y \rangle^{\#}$. Then $Y \lhd X$.*

Proof. Note that $Y \lhd W$, whence $\langle w, Y \rangle$ is a group of order $2 \mid Y \mid$. Suppose $Y \ntriangleleft X$, so $Y \neq 1$. There exists $x \in X$ with $Y^x \neq Y$. If $1 \neq a \in Y^x \cap W$, $a \in Y^x \cap Y$ so that a centralizes $Z(Y^x) \neq 1$ (as Y is a 2-group), whence $Z(Y^x) \subseteq W$ by assumption and so $Z(Y^x) \subseteq Y$. But then if $1 \neq b \in Z(Y^x)$, $C_X(b) \subseteq W$ so $Y^x \subseteq W$, whence $Y^x = Y$. Thus, $Y^x \cap W = 1$.

Let y be an involution in Y and put $v = y^x$. Then $v \not\sim_X w$ by assumption so $\langle vw \rangle$ has even order. Let i be unique involution in $\langle vw \rangle$. Note that $i \in C_X(w) \subseteq W$. By Proposition 1, either $wi \sim_X v$ or $vi \sim_X v$.

If $wi \sim_X v$, as $wi \in W$, $wi \in Y$, whence $i \in Yw$. By assumption $C_X(i) \subseteq W$ so $v \in W$ which contradicts $v \in Y^x$ and $Y^x \cap W = 1$.

We may assume, therefore, that $vi \sim_X v$. Hence, $vi \in Y^x$ (as $i \in C_X(v) \subseteq W^x$) so $i \in Y^x$, whence $i \in W \cap Y^x = 1$, a contradiction. The proposition is proved.

4. The Cases (ii) and (iii) of the Theorem

For the rest of the paper, we suppose G, H satisfy the hypotheses of the theorem. We use the same notation as in Section 2; as well, $C(X)$ and $N(X)$ will denote $C_G(X)$, $N_G(X)$ respectively, for any subset X of G.

Lemma 1. *A Sylow 2-subgroup T of H is a Sylow 2-subgroup of G.*

Proof. If T is not a Sylow 2-subgroup of G, by Sylow's theorem there exists a 2-group T_1 with $| T_1 : T | = 2$. Let $y \in T_1 - T$, so $y \notin H$. As $Z(T) = Z$ char $T \lhd T_1$, $y \in N(Z)$, whence $C_Z(y) \neq 1$, contradicting our assumption.

Lemma 2. *Let x be an element of order p, p an odd prime, with $x \in N(Z)$. If $C(x) \cap N(Z)$ covers H/J then $C_J(x) = 1$.*

Proof. By assumption, $C(Z) = H$. Clearly $J = O_2(H) \lhd N(Z)$ and since $H/J \simeq Sz(q)$, if x is as in the statement of the lemma, $x \notin H$. If $E_0 = C_n(x) \neq 1$, as $C_E(x) \lhd C_H(x)$ we have $| E_0 | \geqslant q^4$ and so $E_0 \times Z = E$. It follows that E_0 contains an element $e \sim_H \alpha_{11}(1)$ and so x permutes the $2q - 1$ cosets of E in J which possess an element of order 4 whose square is e. As $| N(Z) : H | \mid q - 1, p \mid q - 1$ and so $C_{J/E}(x) \neq 1$.

On the other hand, if $C_{J/E_0}(x) \neq 1$, as $C_J(x)/E_0 \lhd C(x)/E_0$, $| C_J(x)/E_0 | \geqslant q^4$. Thus $C_J(x)$ contains cosets of E_0 which contain only elements of order four, and so $E_0 \neq 1$.

We have proved that if $C_J(x) \neq 1$ then $| C_J(x)| \geqslant q^8$ and $E_0 \times Z = E$. However, $C_J(E_0) = C_J(E)$, so $[C_J(x), E_0] \neq 1$. But $[C_J(x), E_0] \subseteq [J, E] = Z$, and, hence, $Z \cap C_J(x) \neq 1$ which is a contradiction. The lemma is proved.

In the rest of this section, we will show that if Z is weakly closed in H with respect to G, then G satisfies either conclusion (ii) or (iii) of the Theorem.

Lemma 3. *If z is an involution in Z and $\langle z \rangle$ is weakly closed in H with respect to G then $G = H \cdot O(G)$, where $O(G) = 1$ if $q > 2$. In particular if $q = 2$ and Z is weakly closed in H then $G = H \cdot O(G)$.*

Proof. The first statement follows immediately from Glauberman's theorem [2]. If $q = 2$, $Z = \langle z \rangle$ is of order 2 and the last statement follows. If $q > 2$ and $O(G) \neq 1$, since Z contains a four group there exists an involution $z \in Z$ with $C(z) \cap O(G) \neq 1$. But then $H \cap O(G) \neq 1$ which contradicts the structure of H. Hence, if $q > 2$, $O(G) = 1$ as required.

The proof of the next lemma uses an idea of Suzuki [10].

Lemma 4. *If $q > 2$ and Z is weakly closed in H with respect to G then $Z \lhd G$, and so $| G : H | \mid q - 1$.*

Proof. Suppose $q > 2$, Z is weakly closed in H but $Z \ntriangleleft G$. By Lemma 3, for each involution $z \in Z$ there exists $g \in G$ with $z \neq z^g \in H$ (otherwise $G = H \subseteq N(Z)$). As $Z \subseteq C(z^g) = H^g$, it follows that $Z^{g^{-1}} \subseteq H$ so $Z^{g^{-1}} = Z$. Thus, $N(Z) \supset H$ and if $z^g \in H$ for any $g \in G$, $z \in Z$, we have $z^g \in Z$. If we show that $C(\alpha_2(1)\,\alpha_{12}(t)) \subseteq N(Z)$ for each $t \in K$, the assumptions of Proposition 2 will be satisfied (with $X = G$, $Y = Z$, $W = N(Z)$ and $w = \alpha_2(1)$) and we can conclude that $Z \triangleleft G$.

Let $y = \alpha_2(1)\,\alpha_{12}(t)$ for any $t \in K, P = C_H(y)$ and $A = C_G(y)$. Note that P is a normal Sylow 2-subgroup of $C_{N(Z)}(y) = A \cap N(Z)$. Under the assumptions of the lemma, we want to prove that $A \subseteq N(Z)$. We argue by way of contradiction and assume $Z \ntriangleleft A$.

Since Z is weakly closed in H, Z is weakly closed in P and so P (of order q^8) is a Sylow 2-subgroup of A. By assumption A contains more than one Sylow 2-subgroup. Among all Sylow 2-subgroups of A, choose a Sylow 2-subgroup P_1 of A so that $|P \cap P_1|$ is maximal. Put $I = P_1 \cap P$ and note that $1 \neq \langle y \rangle \subseteq I$.

Suppose $1 \neq z \in Z \cap I$. Let $a \in A$ with $P^a = P_1$ so $z = z_0{}^a$ for some $z_0 \in Z$. Thus, $z \in Z(P)^a = Z(P_1)$ so $P_1 \subseteq H$, contradicting $H \cap A = P$. We have shown that $I \cap Z = 1$.

Next we prove that $IC_A(I)/I$ is a T.I. group in the sense of Suzuki [9]. Put $R = IC_A(I)$. Clearly $ZI \subseteq R \cap P$ and so Z is weakly closed in $R \cap P$, whence $R \cap P$ is a Sylow 2-subgroup of R. Clearly $Z^a \subseteq R$ (but $Z^a \nsubseteq I$), whence $R \cap P$ is not the only Sylow 2-subgroup of R; i.e., $R \cap P \ntriangleleft R$. Let R_1 be a Sylow 2-subgroup of R with $R_1 \cap (R \cap P) = R_1 \cap P \supset I$. Then if A_1 is a Sylow 2-subgroup of A with $A_1 \supseteq R_1$, we have $A_1 \cap P \supset I$, i.e.,

$$|A_1 \cap P| > |P_1 \cap P|.$$

By the maximality of $|P_1 \cap P|$, we must have $A_1 = P$ which means $R_1(P \cap R) \subseteq P$ so $R_1(P \cap R)$ is a 2-group. This forces $R_1 = P \cap R$ as $P \cap R$ is a Sylow 2-subgroup of R. The distinct Sylow 2-subgroups of R/I therefore have trivial intersection with each other; i.e., R/I is a T.I. group. As ZI/I is elementary of order $q > 2$, Suzuki's result ([9], Theorem 2) yields that R/I possesses a normal series $R/I \trianglerighteq R_2/I \trianglerighteq R_3/I \trianglerighteq I/I$, where $|R : R_2|$, $|R_3 : I|$ are odd and $R_2/R_3 \simeq L_2(q)$, $U_3(q)$ or $Sz(q)$.

In any case, all involutions in $P \cap R/I$ are conjugate in R/I so $\Omega_1(P \cap R/I) = ZI/I$ and $|N_R(Z) : P \cap R| = q - 1$. From the structure of H it follows that $R_3 = I$, while $Z(P) \subseteq R$ so $|Z(P) \cap I| = q^2$ (as $Z(P) = U_{11} \times V_2$ is elementary of order q^3). Let x be an element of order p, p an odd prime, $p \mid q - 1$ with $x \in N_R(Z)$. Then $x \in C(I)$ so $C_E(x)$ has order $\geqslant q$ and $C(x)$ covers $\Omega_1(T/J)$. Since $H/J \simeq Sz(q)$ and $x \in N(Z) - H$, $C(x)$ covers H/J (as $C(x)$ covers $\Omega_1(T/J)$). By Lemma 2, $C_J(x) = 1$ which contradicts $|C_E(x)| \geqslant q$. We have shown that $C(y) \subseteq N(Z)$ as required.

5. Fusion of Involutions

For the rest of the paper, we assume that Z is not weakly closed in H with respect to G. In order to complete the proof of the theorem we have to show, therefore, that $G \simeq {}^2F_4(q)$ under this assumption.

Lemma 5. *We have that* $D = C_H(\alpha_{10}(1))$ *is a Sylow 2-subgroup of* $C_G(\alpha_{10}(1))$ *and so* $\alpha_{10}(1) \not\sim_G z$ *for all* $z \in Z$.

Proof. From Sec. 2, $L_i(D) = Z$, $i = 4$ or 5, and so Z char D. The result now follows from Sylow's theorem and our assumption that $C(z) = H$ for all $z \in Z^{\#}$.

Lemma 6. *If* $C_H(\alpha_7(1))$ *is not a Sylow 2-subgroup of* $C_G(\alpha_7(1))$ *then* $\alpha_{11}(1) \sim_G z$ *for some* $z \in Z^{\#}$.

Proof. Suppose $C_H(\alpha_7(1)) = C_T(\alpha_7(1))$ is not a Sylow 2-subgroup of $C_G(\alpha_7(1))$. By Sylow's theorem there exists a 2-element $b \in N(C_H(\alpha_7)))$ with $b^2 \in C_H(\alpha_7(1))$ but $b \in C(\alpha_7(1)) - C_H(\alpha_7(1))$. Clearly b normalizes the subgroup $U_{11} \times V_7 = Z(C_H(\alpha_7(1)))$. Recall that all involutions in $U_{11} - Z$ are conjugate to $\alpha_{11}(1)$ in H and all involutions in $U_{11} \times V_7 - U_{11}$ are conjugate to $\alpha_7(1)$ in H. If the lemma is false, $Z^b \cap U_{11} = 1$ (note that $Z^b \cap Z = 1$; otherwise, $b \in N(Z^b \cap Z)$ and $C_Z(b) \neq 1$). Thus, b normalizes $Z \cup (U_{11} \times V_7 - U_{11})$ and also $U_{11} - Z$, whence b normalizes

$$\langle U_{11} - Z \rangle = U_{11}$$

which is impossible.

Lemma 7. *If some involution* $z \in Z$ *is conjugate to an involution* $y \in T - J$ *(in G) then* $\alpha_{11}(1) \sim_G z'$ *for some* $z' \in Z^{\#}$.

Proof. Without loss we take $y = \alpha_2(1)\,\alpha_{12}(t)$ for some $t \in K$. Then $C_H(y) = C_T(y)$ has order q^8 and $W = \Omega_1(C_H(y)) = U_{10} \times V_7 \times V_2$ is elementary of order q^5. Under the assumption $y \sim_G z$ for some $z \in Z$, there exists a 2-element $v \in C(y) \cap N_H(C(y))$ with $v^2 \in C_H(y)$ but $v \notin C_H(y)$. Clearly $v \in N(W) - H$. If v normalizes $U_{10} \times V_7 - U_{10}$ then v normalizes $\langle U_{10} \times V_7 - U_{10}\rangle = U_{10} \times V_7 = C_J(y) \cap W$. Thus, $v \in N(U_{10})$. Now $Z^v \cap Z = 1$ (as above) and $Z^v \subseteq U_{11} - Z$ by Lemma 5 and the fact that all involutions in $U_{10} - U_{11}$ are conjugate to $\alpha_{10}(1)$ in H. In this case $\alpha_{11}(1) \sim_G z$ for all $z \in Z^{\#}$.

On the other hand, if v does not normalize $U_{10} \times V_7 - U_{10}$, $\alpha_7(1)$ must be conjugate to an involution in U_{10} and/or an involution in $W - W \cap J = W - (U_{10} \times V_7)$. This forces that $C_H(\alpha_7(1))$ is not a Sylow 2-subgroup of $C(\alpha_7(1))$ for if $\alpha_7(1) \sim_G e$, $e \in U_{10}$ then $|C_H(e)| \geqslant q^{10}$, while if $\alpha_7(1) \sim_G h$,

$h \in W - W \cap J$ then $| C_H(h)| = | C_H(\alpha_7(1))| = q^8$ but $C_H(\alpha_7(1)) \nsubseteq C_H(h)$ (see Section 2). Lemma 6 gives the desired result in this case.

LEMMA 8. *We have that* $\alpha_{11}(1) \sim \alpha_{12}(t)$ *for all* $t \in K^*$. *In fact, all involutions in* U_{11} *are conjugate in* $M = N(U_{11})$.

Proof. Under the assumption that Z is not weakly closed in H with respect to G, there exists $z \in Z^{\#}$ with z conjugate to (at least) one of $\alpha_{11}(1)$, $\alpha_7(1)$, $\alpha_2(1)\alpha_{12}(t)$, $t \in K$ (using Lemma 5). It follows immediately from Lemmas 6 and 7 that $\alpha_{11}(1) \sim_G z'$ for some $z' \in Z$. Thus $C = C(\alpha_{11}(1))$ is not a Sylow 2-subgroup of $C(\alpha_{11}(1))$. Choose $c \in C(\alpha_{11}(1)) \cap N(C) - C$ with c a 2-element such that $c^2 \in C$. Then $Z^c \cap Z = 1$ and so $Z^c \subseteq U_{11} - Z$ as $Z(C) = U_{11}$ must be normalized by c. The lemma now follows from the fact that all involutions in $U_{11} - Z$ are conjugate in $N_H(T) = N_H(C) = N_H(U_{11})$.

LEMMA 9. *When* $q = 2$, $N(Z) = H$ *while if* $q > 2$, $| N(Z): H | = q - 1$. *In particular,* $N(Z) = J \cdot L$ *with* $J \cap L = 1, L = Q_2 \times S_2$ *where* Q_2 *is cyclic of order* $q - 1$ *and* $S_2 \cong Sz(q)$. *Further,* $C(Q_2) \cap N(Z) = L$ *and* $B = N_G(T) = T \cdot Q_0$ *is of order* $q^{12}(q-1)^2$, *where* Q_0 *is the direct product of two cyclic groups of order* $q - 1$.

Proof. By Lemma 8, all involutions of Z are conjugate in G. Thus, all involutions of Z are conjugate in $B = N_G(T) \subseteq N_G(Z)$ by a result of Burnside [3, p. 240]. Since $C_G(z) = H$ for all $z \in Z^{\#}$, $| N(Z) : H | = q - 1$ ($= 1$ when $q = 2$).

We see that U_{11} is the only normal elementary Abelian subgroup of T of order q^2 which implies that $U_{11} \lhd B$. By the Frattini argument, $N(T) = B$ covers $N(Z)/H$ and so $| B | = q^{12}(q-1)^2$ as obviously $B \subseteq N(Z)$. It is clear that B is soluble, so we may choose a complement Q_0 of T with $Q_0 \cap H = Q$.

Since Q_0 normalizes $[U_{11}, Q] = V_{11}$ there exists $t \in K^*$ such that $C(\alpha_{11}(t))$ covers Q_0/Q. It follows that $Q_1 = C_{Q_0}(\alpha_{11}(t))$ is cyclic of order $q - 1$ as $C_G(\alpha_{11}(t)) \cong H$. From Lemma 8, we see that $\alpha_{11}(t) = z^x$ for some $x \in M = N(U_{11})$ and $z \in Z^{\#}$. Thus, T^x is the normal Sylow 2-subgroup of $C_M(\alpha_{11}(t))$. In particular, Q_1 normalizes T^x and, therefore, centralizes $Z(T^x)$. Now $U_{11} = C_{U_{11}}(Q_1) \times [U_{11}, Q_1] = Z(T^x) \times Z$ which implies $C(Q_1) \cap U_{11} = V_{11}$ as Q_1 normalizes V_{11}. Hence, $Q_1 \lhd Q_0$ as Q_0 normalizes V_{11}, and so $Q_0 = Q_1 \times Q$.

We have that Q_1 normalizes $H/J \cong Sz(q)$ and centralizes QJ/J, a Hall subgroup of H/J. It follows that $[Q_1, H] \subseteq J$ and so $N(Z)/J = H/J \times Q_2J/J$, where Q_2 is a complement of Q in Q_0. The Frattini argument yields that $N(Q_2)$

covers H/J, whence $C(Q_2)$ covers H/J. From Lemma 2, we see that $C_J(Q_2) = 1$, whence $C(Q_2) \cap N(Z)/Q_2 \simeq Sz(q)$. Clearly

$$C_H(Q_2) = S_2 \lhd C(Q_2) \cap N(Z)$$

and so $C_{N(Z)}(Q_2) = Q_2 \times S_2$, where $S_2 \simeq Sz(q)$.

LEMMA 10. *We have the following fusion of involutions in G:*

$$\alpha_{11}(1) \sim \alpha_{12}(t) \sim \alpha_2(1)\,\alpha_{12}(t_0), \textit{ for all } t \in K^* \textit{ and some}$$

fixed $t_0 \in K$;

$$\alpha_{10}(1) \sim \alpha_7(1) \sim \alpha_2(1)\,\alpha_{12}(u), \textit{ for all } u \in K - \{t_0\}.$$

Proof. Recall from Section 2 that $\langle \omega_1, V_5 \rangle$ is a Sylow 2-subgroup of $N_H(Q)$. Further, $\Omega_1(\langle \omega_1, V_5 \rangle) = \langle \omega_1, Z \rangle$ is elementary of order $2q$. Now Q_2 normalizes $N_H(Q)$ and so Q_2 centralizes an involution $\omega_1 \alpha_{12}(t')$ for some $t' \in K$, and acts transitively on the other $q - 1$ involutions in $\langle \omega_1, Z \rangle - Z$. Since $\omega_1 \sim_S \alpha_2(1)$, $\langle \alpha_2(1), Z \rangle - Z$ also possesses two classes of involutions in $N(Z)$; i.e., for some fixed $t_0 \in K$, $\alpha_2(1)\,\alpha_{12}(t_0) \not\sim_{N(Z)} \alpha_2(1)\,\alpha_{12}(t)$ for all $t \in K - \{t_0\}$, but the $q - 1$ involutions $\alpha_2(1)\,\alpha_{12}(t)$, $t \in K - \{t_0\}$ are conjugate in $N(Z)$.

A computation yields $\Omega_1(C) = \langle F, \alpha_2(t), \alpha_8(t), \alpha_6(t)\,\alpha_5(t^{2\theta}) | \, t \in K \rangle$ is of order q^8 (where $F = L_3(C) = U_9 \times V_7$, so $F \lhd M$), $Z(\Omega_1(C)) = U_{10}$ and $\Omega_1(C \cap J) = \langle E, F \rangle$ is of order q^6. In $N_M(Z)$ there are precisely 5 conjugate classes of involutions in F with representatives:

$$\left.\begin{array}{l} \alpha_{12}(1) \in Z \\ \alpha_{11}(1) \in U_{11} - Z \end{array}\right\} \alpha_{12}(1) \underset{M}{\sim} \alpha_{11}(1)$$

$$\left.\begin{array}{l} \alpha_{10}(1) \in U_{10} - U_{11} \\ \alpha_9(1) \in U_9 - U_{10} \end{array}\right\} \alpha_{10}(1) \underset{H}{\sim} \alpha_9(1)$$

$$\alpha_7(1) \in F - U_9 .$$

Also in $N_M(Z)$ there are precisely two conjugacy classes of (nontrivial) cosets of F in $\Omega_1(C)$ which contain involutions. They have representatives $\alpha_8(1)F$ and $\alpha_2(1)F$. The coset $\alpha_8(1)F$ contains q^3 involutions conjugate to $\alpha_{11}(1)$ in H with representative $\alpha_8(1)$ and $(q - 1)\,q^3$ involutions conjugate to $\alpha_{10}(1)$ in H with representative $\alpha_8(1)\,\alpha_9(1)$. The coset $\alpha_2(1)F$ contains q^3 involutions conjugate to $\alpha_2(1)\,\alpha_{12}(t_0)$ and $q^3(q - 1)$ involutions conjugate to $\alpha_2(1)\,\alpha_{12}(t_1)$, where $t_1 \in K - \{t_0\}$.

All that remains to be shown is that $\alpha_9(1) \sim_M \alpha_7(1)$ and $\alpha_2(1)F \sim_M \alpha_8(1)F$. This follows because:

(a) $Z \ntriangleleft M$ but $U_{10} \lhd M$,

(b) for any $e \in U_9 - U_{10}$, $(C(e) \cap \Omega_1(C))' = Z$,

(c) for any involution $e \in \{\alpha_8(t)F \mid t \in K^*\}$ (i.e., $e \in E - F$),

$$(C(e) \cap \Omega_1(C))' = Z.$$

LEMMA 12. *We have* $M/C \cong GL(2, q)$ *and*

$$C_M(\alpha_{10}(1))/C_C(\alpha_{10}(1)) \cong SL(2, q).$$

Further, all involutions in $M - C$ *are conjugate to* $\alpha_{10}(1)$ *in* G.

Proof. We know that $T = CV_3$ is a Sylow 2-subgroup of M and so T/C, a Sylow 2-subgroup of M/C, is elementary Abelian of order q. A simple computation shows that all involutions in $T - C$ lie in $J - C$. Therefore, all involutions in $T - C$ are conjugate to $\alpha_7(1)$ in H and to $\alpha_{10}(1)$ in G (by Lemma 11).

All involutions in U_{11} are conjugate in M, so

$$| M | = | H \cap M | \cdot (q^2 - 1) = q^{12} \cdot (q - 1)^2$$

and

$$| M/C | = q \cdot (q - 1)^2 (q + 1).$$

If $x \in T - C$, $C(x) \cap U_{11} = Z$, which means that if T_1 is any other Sylow 2-subgroup of M besides T, $T_1 \cap T = C$ (using the fact that $N_M(Z) = N_M(T)$). Thus, M/C is a T.I. group with Abelian Sylow 2-subgroups.

If $q = 2$, $| M/C | = 6$, whence $M/C \cong SL(2, 2) = GL(2, 2)$, and as $\alpha_{10}(1)$ has precisely four conjugates in M, $C_M(\alpha_{10}(1))$ covers M/C. The lemma is proved in this case.

In the case $q > 2$, Suzuki's result [9] yields that M possesses normal subgroups L_1, L_2 with $M \rhd L_1 \rhd L_2 \lhd C$, $| M : L_1 |$, $| L_2 : C |$ odd and $L_2/L_1 \cong SL(2, q)$. However, as $Q_0 = Q \times Q_1$, $M/L_2 \cong SL(2, q)$ and so by Gaschutz' theorem [5, Satz I.17.4], $M/C \cong GL(2, q)$. Finally, $\alpha_{10}(1)$ has $q^3 - q^2 = q^2(q - 1)$ conjugates in M and $| C : C_C(\alpha_{10}(1))| = q^2$ so

$$| C_M(\alpha_{10}(1)) : C_C(\alpha_{10}(1))| = q(q - 1)(q + 1).$$

The structure of $GL(2, q)$ yields immediately that

$$C_M(\alpha_{10}(1))/C_C(\alpha_{10}(1)) \cong SL(2, q).$$

LEMMA 13. *We have* $C(\alpha_{10}(1)) \subseteq M$.

Proof. Put $X = C_G(\alpha_{10}(1))$, $Y = C_M(\alpha_{10}(1))$ and recall that $D = C_H(\alpha_{10}(1))$ is a Sylow 2-subgroup of X.

Suppose that $w^x \in Y$ for some $w \in U_{11}$ and $x \in X$. We claim that $w^x \in U_{11}$. If not, $w^x \in Y \cap C - F$, and so w^x is conjugate to an involution in $\alpha_8(1)F$. However, for any $f \in C_F(\alpha_8(1))$, an easy computation yields $L_3(C_D(\alpha_8(1)f)) = Z$. Thus, $C_D(\alpha_8(1)f)$ is a Sylow 2-subgroup of $C_X(\alpha_8(1)f)$, whence $\alpha_8(1)f$ can not be conjugate to any involution in U_{11} (as $C_D(\alpha_8(1)f) \subset D$). We have $w^x \in U_{11}$ as required.

For any $w \in U_{11}^{\#}$, $C_X(w) \subseteq Y$. This is clear because all involutions in U_{11} are conjugate in Y and for any $z \in Z^{\#}$, $C_X(z) = C_H(\alpha_{10}(1)) = D \subseteq Y$.

Finally we show that for any $w \in U_{11}$, $C_X(\alpha_3(1)w) \subseteq Y$. It is enough to consider $w \in Z$ since $C(\alpha_3(1)) \cap U_{11} = \mho^1(\langle \alpha_3(1), U_{11} \rangle) = Z$. Put $y = \alpha_3(1)z$ for any $z \in Z$. Since $Y/Y \cap C \simeq SL(2, q)$,

$$C_Y(y) = C_D(y) = \langle Z, V_{10}, V_9, V_8, V_3 \rangle$$

is elementary Abelian of order q^5. From the fact that U_{11} is weakly closed in Y with respect to X, it follows that Z is weakly closed in $C_Y(y)$ with respect to X. Thus, $N_X(C_Y(y)) \subseteq N_X(Z) \cap C_Y(y) = C_Y(y)$ which implies (by Sylow) that $C_Y(y)$ is a self-normalizing (Abelian) Sylow 2-subgroup of $C_X(y)$. An application of Burnside's transfer theorem yields that $C_X(y)$ possesses a normal 2-complement R. We act on R with the four group $\langle \alpha_{12}(1), \alpha_8(1) \rangle$, all of whose involutions are conjugate to $\alpha_{12}(1)$. Clearly $C_R(\alpha_{12}(1)) = 1$, so if $R \neq 1$ we may assume without loss that $C_R(\alpha_8(1)) \neq 1$. Thus, $\alpha_{12}(1)$ inverts an element of odd order in $C_G(\alpha_8(1))$. However,

$$| H \cap C_G(\alpha_8(1)) | = q^{11}$$

which forces $\alpha_{12}(1) \in O_2(C(\alpha_8(1)))$ as $C(\alpha_8(1)) \sim_G H$. This gives a contradiction to $R \neq 1$ and we conclude that $C_X(y) \subseteq Y$.

The conditions of Proposition 2 are satisfied (with $Y = U_{11}$, $w = \alpha_3(1)$, $W = Y$ and $X = X$) and we conclude, therefore, that $U_{11} \lhd X$; i.e., $C(\alpha_{10}(1)) \subseteq M$.

6. The Identification of G with ${}^2F_4(q)$

We divide this section into two parts depending on whether $q > 2$ or $q = 2$.

Case I. $q > 2$

Put $N = N(Q_0)$ and recall $B = N(T) = T \cdot Q_0$. From the proof of Lemma 10, $C(Q_2) \cap N_H(Q) = Q\langle \omega_1 \alpha_{12}(t') \rangle$ for some $t' \in K$. We put $\omega = \omega_1 \alpha_{12}(t')$ and have $N_{N(z)}(Q_0) = Q_0 \langle \omega \rangle$. Further, $C_C(Q_0) = 1$, so

Lemma 12 shows that $N_M(Q_0) = Q_0\langle v\rangle$ where v is an involution in $M - T$. We can now state the following.

LEMMA 14. *The group* Q_0 *is self-centralizing in* G *and* $N = N(Q_0) = Q_0\langle v, \omega\rangle$, *where* $\langle v, \omega\rangle \simeq D_{16}$, *the dihedral group of order* 16.

Proof. We show first that $C(Q_0) = Q_0$. From Lemma 9, $C_{N(Z)}(Q_2) = Q_2 \times S_2$, where $S_2 \simeq Sz(q)$. The proof of Lemma 10 shows that $\omega \sim_G \alpha_{12}(1)$ and so $Q_2 \sim_G Q$ as $\omega \in C(Q_2)$. Thus, $C(Q) \supset C_{N(Z)}(Q) = Q_0 V_5$. Since $C(z) \cap C(Q) = Q \times V_5$ for all $z \in Z^{\#}$, $C(Q)$ must be a T.I. group. Applying Suzuki's result [9] and using the fact that $N(Z) \cap C(Q) = Q_0 V_5$ we get $C(Q)/Q \simeq Sz(q)$. Hence, $C(Q) = Q \times S_1$, $S_1 \simeq Sz(q)$ as $Q \sim_G Q_2$. The structure of $S_1 \simeq Sz(q)$ yields immediately that $C(Q_0) = Q_0$.

Our remarks above give us $N = N(Q_0) \supseteq Q_0\langle v, \omega\rangle$. Lemma 12 implies $v \sim_G \alpha_{10}(1)$ and as $\omega \sim_G \alpha_{12}(1)$, $\langle v\omega\rangle$ has even order and therefore possesses a unique involution i. Further, $C_{Q_0}(c) \simeq [v, Q_0] \simeq Q$ and we know i normalizes $C_{Q_0}(v)$. If i centralizes $C_{Q_0}(v)$, $v \sim_G i \sim_G \alpha_{10}(1)$ as $C_H(Q) = Q \times V_5$. On the other hand, if $[i, C_{Q_0}(v)] \neq 1$, i inverts an element of odd order in $C_G(v)$. In particular $i \notin O_2(C(v))$ which implies, by Lemmas 12, 13, that $i \sim_G \alpha_{10}(1)$. Since ω normalizes $C_{Q_0}(i)$, the same argument yields that $C_{Q_0}(i) = 1$, as we have proved $i \sim_G \alpha_{10}(1)$.

As $C(Q_0) = Q_0$ and $C_{Q_0}(i) = 1$, $N(Q_0) = N = Q_0 \cdot C_N(i)$. Since $C_{Q_0}(\omega) = Q_2 \lhd C_N(\omega)$, $Q \sim_G Q_2$ and $N(Q) = (Q \times S_1)\langle\omega\rangle$, we have that $C_N(\omega) = \langle i, \omega\rangle \cdot Q_2$. Now $i \sim_G \alpha_{10}(1)$ so by Lemma 12, 13 $\omega \in O_2(C(i))$, whence $O_2(C_N(i))$ is a 2-group of maximal class. Thus, $C_N(i)$ is a 2-group of maximal class and as $v \not\sim \omega$, $C_N(i)$ is dihedral. Further, $\Omega_1(T)$ does not contain elements of order 16, so $C_N(i)$ is dihedral of order 8, 16. The proof of the Lemma is completed by showing $\langle v, \omega\rangle \simeq D_{16}$.

It is enough to show that $(v\omega)^4 \neq 1$. Since $v \in M - N(Z)$, $\alpha_{12}(1)^v = \alpha_{11}(t)\,\alpha_{12}(u)$ for some $t, u \in K$ while $(\alpha_8(t)\,\alpha_{12}(u))^v = \alpha_2(t_1)\,j$ for some $t_1 \in K$, $j \in J$. Thus, $(\alpha_2(t_1)\,j)^{(v\omega)^3 v} = \alpha_2(t_1 j)$ which implies $(v\omega)^4 \neq 1$ as $\omega \in H - T$ while $C_H(\alpha_2(t_1)\,j) \subseteq T$. The lemma is proved.

Next we derive some results about the action of v on C. By definition, $v \in N \cap M - N(Z)$. Thus, $Q^v \times Q = Q_0$, and as Q_0 normalizes $C(Q^v) \cap U_{11}$, $C(Q^v) \cap U_{11} = V_{11}$, i.e., $Z^v = V_{11}$. By Lemma 2, $C_T(Q^v) \subseteq J$, so $V_5{}^v = \langle \alpha_6(t^{2\theta-2})\,\alpha_7(t^{-2\theta}u)\,\alpha(t, u) \mid t \in K$, u is some fixed element of K and $\alpha(t, u) \in E$ (here $\alpha(t, u)$ depends on t, u)$\rangle$. (This follows because $V_5 = C_C(Q)$ so $V_5{}^v = C_C(Q^v)$ and $V_5{}^v$ is Q-invariant.) Now

$$\begin{aligned}[V_5, V_5{}^v] &= \langle [\alpha_6(t^{2\theta-2})\,\alpha_7(t^{-2\theta}u)\,\alpha(t, u), \alpha_5(v)] \mid t, v \in K\rangle \\ &= \langle [\alpha_6(t^{2\theta-2})\,\alpha_7(t^{-2\theta}u), \alpha_5(v)] \mid t, v \in K\rangle \\ &= \begin{cases} \langle \alpha_{10}(t), \alpha_{11}(t) \mid t \in K\rangle & \text{if } u \neq 0, \\ \langle \alpha_{10}(t) \mid t \in K\rangle & \text{if } u = 0. \end{cases}\end{aligned}$$

However, $[V_5^v, V_5]$ is v-invariant which implies $u = 0$ and

$$V_5^v = \langle \alpha_6(t)\, \alpha(t) |\, t \in K, \alpha(t) \in E \rangle = V_6^*.$$

Further, $V_{10}^v = V_{10}$, since $u = 0$.

From the proof of Lemma 10, $V_9^v = \langle \alpha_7(t)\, \alpha_0(t) |\, t \in K, \alpha_0(t) \in E \cap F \rangle = V_7^*$. Put $(V_6^*)^\omega = \langle \alpha_4(t)\, \alpha(t)^\omega \mid t \in K, \alpha(t)^\omega \in E \rangle = V_4^*$. Since

$$[V_4^*, U_{10}] = Z, (V_4^*)^v = V_1^*$$

must cover T/J. Put $(V_7^*)^\omega = \langle \alpha_3(t)\, \alpha_0(t)^\omega \mid \alpha_0(t)^\omega \in E, t \in K \rangle = V_3^*$ and note that $T = CV_3^*$. We have the following table for the action of v, ω on subgroups of T:

Subgroup X	V_{10}	V_9	V_7^*	V_5	V_6^*	V_4^*	C	J
X^v	V_{10}	V_7^*	V_9	V_6^*	V_5	V_1^*	C	
X^ω	V_9	V_{10}	V_3^*	V_5	V_4^*	V_6^*		J

For each $w \in \langle \omega, v \rangle$ define $l(w)$ to be the minimal length of w as a word in ω, v. (Clearly $l(w) \leqslant 8$ as $(\omega v)^8 = 1$). Using the table and the fact that $B = (JQ_0)\, V_1^* = V_1^*(JQ_0) = V_3^*(CQ_0) = (CQ_0)\, V_3^*$ we have: if $l(vw) \geqslant l(w)$ then $vBw \subseteq BvwB$ and if $l(\omega w) \geqslant l(w)$ then $\omega Bw \subseteq B\omega wB$, for all $w \in \langle \omega, v \rangle$.

Since $N(Z)/Q_2 \cdot J \simeq Sz(q)$, $Q_2 \cdot J \subseteq B$ and $B/Q_2 \cdot J$ is a Sylow 2-normalizer of $N(Z)/Q_2 \cdot J$, $N(Z) = B \cup B\omega B$ (i.e., $Sz(q)$ acts doubly transitively on its Sylow 2-subgroups; see [8]). Similarly, we have $M = B \cup BvB$. For if $M_1 \subseteq B$ so that $M_1/C = O(M/C)$ then $M/M_1 \simeq SL(2, q)$ and B/M_1 is a Sylow 2-normalizer of M/M_1 (and again, $SL(2, q)$ acts doubly transitively on its Sylow 2-subgroups; see [8] or [3, p. 41]).

By a result of Tits [11], the above two facts imply that

(a) $sBw \subseteq BwB \cup BswB$ for each $s \in \{\omega, v\}$ and $w \in \langle \omega, v \rangle$. Clearly, we also have

(b) $sBs \neq B$ for $s \in \{\omega, v\}$.

Next we show G is simple. Suppose $1 \neq G_2 \lhd G$. As $|Z| = q > 2$, G_2 must be even as $O(H) = 1$. Thus at least one of $\alpha_{10}(1), \alpha_{12}(1) \in G_2$. However, $[\alpha_7(1), \alpha_8(1)\, \alpha_9(1)] = \alpha_{12}(1)$ and

$$[\alpha_2(1)\, \alpha_{12}(t_0), \alpha_8(1)] = \alpha_{10}(1)\, \alpha_{11}(1)\, \alpha_{12}(1) \sim_H \alpha_{10}(1),$$

so G_2 contains both $\alpha_{10}(1)$ and $\alpha_{12}(1)$. In particular, $H \subseteq G_2$ and $M' \subseteq G_2$, where $|M : M'| = q - 1$ and $M'/C \simeq SL(2, q)$. But $Q \subseteq M - M'$ and $Q \subseteq H$ so $G_2 \supseteq \langle N(Z), M\rangle$ whence $G_2 = G$ by the Frattini argument applied to the Sylow 2-subgroup T. We have

(c) G is a simple group.

Put $G_1 = \langle B, N\rangle = \langle T, Q_0, v, \omega\rangle = \langle N(Z), M\rangle$. From Section 5 it is clear that for any pair of involutions $x, y \in T$ with $x \sim_G y$ then $x \sim_{G_1} y$. Since $H \subseteq G_1$ and $C(\alpha_{10}(1)) \subseteq M \subseteq G_1$ it follows that $C(x) \subseteq G_1$ for each involution $x \in T$. Clearly T is a Sylow 2-subgroup of G_1 so G_1 contains the centralizer of each of its involutions. If $G_1 \subset G$, G_1 would be "strongly embedded in G" and G would have one class of involutions, see [3, p. 306]. This is not the case so we get

(d) $G = \langle B, N\rangle$.

As the statements (a) and (b) above remain valid if we replace each element $x \in \langle \omega, v\rangle$ by $\bar{x} = x(B \cap N)$, Lemma 14, (a), (b), (c), (d) imply the following.

LEMMA 15. *The group G is a finite simple group with a (B, N)-pair of rank 2 such that $N = N(Q_0)$ and $B = Q_0T = (B \cap N)\, T$.*

We remark that the term "rank 2" refers to the fact that $N/B \cap N = N/Q_0$ is generated by two involutions.

It follows immediately from Lemma 15 and Theorem B of Fong and Seitz [1] that $G \simeq {}^2F_4(q)$.

Case II. $q = 2$

LEMMA 16. *Let $T^* = \langle M' \cap T, \alpha_3(1)\rangle$, a subgroup of index 2 in T. Then for any element $x \in T^*$, x of order four, we have that $x \not\sim_G \alpha_5(1)$.*

Proof. A simple computation gives $C' = \Omega_1(C) = \langle U_7, \alpha_2(1), \alpha_5(1)\, \alpha_6(1)\rangle$ is of order 2^8. Further, $[\alpha_3(1), \alpha_1(1)] = \alpha_4(1)\, \alpha_5(1)\, c'$, $c' \in C'$, and so $\alpha_4(1)\, \alpha_5(1) \in M'$. The factor group M/C is a faithful extension of an elementary group of order 8 by $SL(2, 2) \simeq D_6$. If P is a Sylow 3-subgroup of M, P fixes the coset $\alpha_5(1)\, C'$ (because $\langle \alpha_5(1), C'\rangle/F$ is the only elementary subgroup of C/F of order 16). Thus, $|M : M'| = 4$ with $\alpha_5(1) \notin M'$. Put

$$T^* = \langle M' \cap T, \alpha_3(1)\rangle$$

so $|T : T^*| = 2$ and $\alpha_5(1) \notin T^*$.

Let x be an element of order four in T^*. Suppose $x \sim_G \alpha_5(1)$, so $x^2 \sim_G \alpha_{12}(1)$. Since all cosets in $M' \cap T/C'$ are conjugate to $\alpha_4(1)\, \alpha_5(1)\, C'$ and if y is an element of order four in $\alpha_4(1)\, \alpha_5(1)\, C'$ then $y^2 \sim_G \alpha_{10}(1)$, we have $x \notin M' \cap T - C'$. We next consider $x \in T^* - C \cap T^*$. If $x \in J$ and xE contains involutions, $x^2 = \alpha_{12}(1)$ so $x \not\sim_T \alpha_5(1)$, whence $x \not\sim_G \alpha_5(1)$ (as $\alpha_5(1)\, E \lhd H$). On the other hand, if xE does not contain involutions (and $x \in J$), $x^2 \sim_G \alpha_{10}(1)$ as $xE \sim_H \alpha_5(1)\, \alpha_6(1)\, E$. In the case $x \notin J$, as $xE = \alpha_3(1)\, cE$ for some $c \in C \cap T^* - J$, a simple computation shows $(xE)^2 = x^2E = jE$ for some $j \in J$, whence $x^2 \sim_H \alpha_7(1) \sim_G \alpha_{10}(1)$.

Finally we consider the case when $x \in C' - F$. If xF does not contain involutions, $x^2 \sim_G \alpha_{10}(1)$. When xF does contain involutions, $xF \sim_M \alpha_8(1)F$. But all elements of order four in $\alpha_8(1)F$ lie in $\alpha_7(1)\, E$ and so have square $\alpha_{12}(1)$. Thus, $x \not\sim_G \alpha_5(1)$ in this case as $\alpha_7(1)\, E \not\sim_H \alpha_5(1)\, E$. The lemma is proved.

LEMMA 17. *The group G possesses a normal subgroup G^* of index 2 with $G^* \cong \mathcal{T}$, the Tits simple group.*

Proof. It is easily seen that $T - T^*$ possesses no involutions and so $\alpha_5(1)$ is an element of minimal order in $T - T^*$. Thus, Lemma 16 and Harada's transfer lemma [4, Lemma 16] yield that G possesses a subgroup G^* of index 2 with $\alpha_5(1) \in G - G^*$. Now $C_{G^*}(\alpha_{12}(1)) = H \cap G^* = H^*$ is a faithful extension of a 2-group $J^* = J \cap H^*$ of order 2^9 and class 3 by a Frobenius group of order 20. Further, if P^* is a Sylow 5-subgroup of H^*,

$$C_{J^*}(P^*) = Z(H^*) = \langle \alpha_{12}(1) \rangle.$$

Finally, $\langle \alpha_{12}(1) \rangle$ is not weakly closed in T^* with respect to G^* which implies $G^* \neq H^*O(G^*)$. It follows immediately from the author's result [6] that $G^* \cong \mathcal{T}$.

We conclude that $G \cong {}^2F_4(2)$ by using an unpublished result of J. Tits that Aut $\mathcal{T} \cong {}^2F_4(2)$, as clearly $G \subseteq$ Aut G^*. This completes the proof of the theorem.

REFERENCES

1. P. FONG AND G. SEITZ, Groups with a (B, N)-pair of Rank 2, to appear.
2. G. GLAUBERMAN, Central elements in core-free groups, *J. Algebra* 4 (1966), 403–420.
3. D. GORENSTEIN, "Finite Groups," Harper and Row, New York, 1968.
4. K. HARADA, Finite simple groups with short chains of subgroups, *J. Math. Soc. Japan* **20** (1968), 655–672.
5. B. HUPPERT, "Endliche Gruppen," Vol. I, Springer-Verlag, Berlin/Heidelberg/New York, 1967.
6. D. PARROTT, A characterization of the Tits' simple group, *Canad. J. Math.* **24** (1972), 672–685.
7. R. REE, A family of simple groups associated with the simple Lie algebra of type (F_4), *Amer. J. Math.* **83** (1961), 401–420.

8. M. SUZUKI, On a class of doubly transitive groups, I, *Ann. Math.* **75** (1962), 105–145.
9. M. SUZUKI, Finite groups of even order in which Sylow 2-subgroups are independent, *Ann. Math.* **80** (1964), 58–77.
10. M. SUZUKI, Characterizations of linear groups, *Bull. Amer. Math. Soc.* **75** (1969), 1043–1091.
11. J. TITS, Théorème de Bruhat et sous-groupes paraboliques, *C. R. Acad. Sci. Paris* **254** (1962), 2910–2912.
12. J. TITS, Algebraic and abstract simple groups, *Ann. Math.* **80** (1964), 313–329.