Line-primitive Automorphism Groups of Finite Linear Spaces

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If $G$ is a line-primitive automorphism group of a $2-(v, k, 1)$ design, then $G$ is almost simple, unless the design is a projective plane with a prime number of points and $G$ acts on the point set as a regular group or as a Frobenius group of dividing $vk$ or $v(k - 1)$. If $k < 30$, then $G$ is point-primitive.

1. Introduction

A (non-trivial) linear space $S$ is an incidence structure of points and lines such that any two points are incident with exactly one line, any point being incident with at least two lines and any line with at least two points. As usual, we shall identify each line with the set of points incident with it. A flag is an incident point-line pair. We always assume $S$ to be finite, i.e. to have a finite number of points.

Let $G \leq \text{Aut} \ S$ be an automorphism group of $S$. The following is well known.

**Lemma 1.** (Block [1]). If $G$ is line-transitive, then $G$ is point-transitive.

**Lemma 2.** (Higman–McLaughlin [10]). If $G$ is flag-transitive, then $G$ is point-primitive.

This implication does not hold any longer if flag-transitivity is replaced by the weaker assumption of line-transitivity, but the number of points of any counterexample is necessarily small with respect to the line-size, as shown in [7]. A class of counterexamples is provided by the groups acting regularly on the points and lines of a projective plane whose number of points is not a prime. Line-primitivity is fairly well under control in finite projective planes.

**Lemma 3.** (Kantor [13]). In a finite projective plane $S$ of order $n$, $G$ is line-primitive if and only if $G$ is point-primitive. Moreover, if $G$ is line-primitive, then:

(i) $S$ is Desarguesian and $\text{PSL}(3, n) \leq G \leq \text{PFL}(3, n)$; or
(ii) $n^2 + n + 1$ is prime and $G$ is a regular group or a Frobenius group whose order divides $(n^2 + n + 1)(n + 1)$ or $(n^2 + n + 1)n$.

Other examples of pairs $(S, G)$ where $G$ is line-primitive (and point-primitive) on $S$ are the following:

1. the linear space on $v$ points all of whose lines have size 2, together with any subgroup of $\text{Sym}(v)$ which is primitive on the pairs of points;
2. the $d$-dimensional projective space of order $q$ ($d \geq 3$), together with any $G \geq \text{PSL}(d + 1, q)$;
3. the Hermitian unital of order $q \geq 3$, together with any $G \geq \text{PSU}(3, q)$;
4. the Ree unital of order $q \geq 27$, together with any $G \geq ^2G_2(q)$.

Note that, in the spaces (2), (3) and (4), any line-transitive group is necessarily line-primitive, with the only exception where $S$ is $\text{PG}(4, 2)$ and $G$ is a Frobenius group of order 31·5. This is not necessarily true in the spaces of case (1), where the classes of any non-trivial partition of the line-set are the lines of a new linear space on $v$ points, so that the existence
of a line-transitive but line-imprimitive group is equivalent to the existence of a line space on \( v \) points with line-size \( > 2 \) admitting a 2-homogeneous automorphism group (such spaces are classified in [11] and [8]).

We remark also that the Hermitian unital of order 2 and the Ree unital of order 3 are not included in the list because they are members of two families of linear spaces whose full automorphism group, although flag-transitive, is line-imprimitive, namely the affine spaces and the linear spaces \( W(2^n) \) whose points are the lines disjoint from a complete conic in \( PG(2, 2^n) \), whose lines are the points not belonging to that conic, the incidence being the obvious one (see for example [2, section 2.6]).

Those remarks support the conjecture that in any finite linear space \( S \), if \( G \) is line-primitive then \( G \) is point-primitive. We will prove that if a counterexample \((S, G)\) exists, then \( S \) has line-size \( k \geq 30 \) and \( G \) is an almost simple group. A more ambitious goal would be to classify all pairs \((S, G)\) where \( S \) is a finite linear space and \( G \) acts line-primitively on \( S \). We will reduce this to an investigation of all finite simple groups, a result similar to the following theorem (Buekenhout, Delandtsheer and Doyen [2]): if \( G \) is a flag-transitive automorphism group of a finite linear space \( S \), then either \( G \) is almost simple or \( G \) is of affine type. This result was the starting point of a programme of classification of all flag-transitive pairs \((S, G)\) by the same team, together with P. Kleidman, M. Liebeck and J. Saxl.

We close this introduction by stating the two main results of the present paper.

**Theorem 1.** Let \( S \) be a finite linear space other than a projective plane and let \( G \) be an automorphism group of \( S \). If \( G \) acts primitively on the \( b \) lines of \( S \), then \( G \) is almost simple, i.e. there is a non-abelian simple group \( T \) of order \( > b \) such that \( T \cong G \cong \text{Aut} T \).

**Theorem 2.** Let \( S \) be a finite linear space and let \( G \) be an automorphism group of \( S \). If \( G \) acts primitively on the lines of \( S \) and if \( S \) has line-size \( < 30 \), then \( G \) acts primitively on the points of \( S \).

The author would like to thank the referee for pointing out an error in the preliminary version of the proof of Theorem 1.

### 2. LINE-PRIMITIVE GROUPS ARE ALMOST SIMPLE

Let \( S \) be a finite linear space other than a projective plane and let \( G \) be a line-primitive automorphism group of \( S \). Hence \( S \) has constant line-size \( k \) and is a 2-(\( v, k, 1 \)) design with \( v \) points and \( b \) lines. Since \( S \) is not a projective plane, \( b > v \). Together with the fact that \( b = v(v - 1)/k(k - 1) \), this implies:

**Lemma 4.** There is a prime number dividing \( b \) but not \( v \).

Let \( \mathcal{P} \) be the point-set and \( \mathcal{L} \) the line-set of \( S \). We define \( \mathcal{C} \) as follows: if \( G \) is primitive on \( \mathcal{P} \), then \( \mathcal{C} = \mathcal{P} \) (or equivalently \( \mathcal{C} \) is the set of classes of the trivial partition of \( \mathcal{P} \) into singletons). If \( G \) is not primitive on \( \mathcal{P} \), then \( \mathcal{C} \) is the set of classes of some non-trivial partition of \( \mathcal{P} \) which is preserved by \( G \) and on which \( G \) acts primitively. Let \( c \) be the cardinality of \( \mathcal{C} \) and \( s \) the common size of the classes in \( \mathcal{C} \), so that \( v = cs \).

**Lemma 5.** \( G \) acts faithfully on \( \mathcal{C} \).

**Proof.** Let \( K \) be the kernel of the action of \( G \) on \( \mathcal{C} \). Since \( K \) is normal in \( G \) and \( G \) is primitive on \( \mathcal{L} \), the group \( K \) acts either trivially or transitively on \( \mathcal{L} \). By Lemma 1, if \( K \) is transitive on \( \mathcal{L} \), then \( K \) is transitive on \( \mathcal{P} \), and so on \( \mathcal{C} \), a contradiction. Hence \( K \) stabilizes every line of \( S \) and so every point of \( S \).
COROLLARY 5. \( G \leq \text{Sym}(c) \) and \( s| \langle c - 1 \rangle! \)

PROOF. This follows immediately from Lemmas 5 and 1, which yield \( sc = v| \langle G \rangle| e! \).

PROOF OF THEOREM 1. Keeping in mind that \( G \) acts faithfully on \( \mathcal{P} \), \( \mathcal{L} \) and \( \mathcal{C} \) (Lemma 5) and that \( G \) acts primitively on \( \mathcal{C} \) and \( \mathcal{L} \), we shall use the O'Nan-Scott theorem as stated in [14]. Let \( N \) be the socle of \( G \), so that \( N \cong T^d \) for some simple group \( T \). Since \( G \) acts primitively on \( \mathcal{L} \), its socle \( N \) acts transitively on \( \mathcal{L} \), and so \( b \) divides \( |T|^d \). Lemma 4 forces the existence of a prime number dividing \( b \) (and hence dividing \( |T| \)) but not dividing \( v \) (and hence not dividing \( c \)), so that \( c \) cannot be a power of \( |T| \). This, together with the O'Nan-Scott theorem, implies that there are only two possible types of primitive action of \( G \) on \( \mathcal{C} \), namely:

(i) \( G \) is almost simple; or

(ii) \( d = 2, c = \gamma^d \) and there is a group \( H \) acting primitively on a set \( F \) of cardinality \( \gamma \) such that \( T \leq H \leq \text{Aut} T \) and \( G \leq H \wr \text{Sym}(d) \) acting on \( \mathcal{C} = \Gamma^d \) in its natural product action.

In case (i), the O'Nan-Scott theorem forces the socle \( T \) of the line-primitive group \( G \) to have order \( > b \). Consider now case (ii). Let \( N = T_1 \times \cdots \times T_d \), write \( T_i \) instead of \( 1 \times \cdots \times 1 \) and \( T_i^* \) instead of \( 1 \times T_2 \times \cdots \times T_d \), etc. Since \( T_i^* \) is intransitive on \( \mathcal{C} \), it is intransitive on \( \mathcal{P} \) and hence, by Lemma 1, it is also intransitive on \( \mathcal{L} \). Then the O'Nan-Scott theorem for the primitive action of \( G \) on \( \mathcal{L} \) indicates that \( N \) is the stabilizer of \( \mathcal{L} \) on \( \mathcal{C} \) (resp. on \( \mathcal{L} \)), and call \( \mathcal{C} \)- (resp \( \mathcal{L} \)-) fibres of direction \( i \) the orbits of \( T_i \) on \( \mathcal{C} \) (resp. on \( \mathcal{L} \)), and call \( \mathcal{C} \)- (resp \( \mathcal{L} \)-) hyperplanes of direction \( i^* \) the orbits of \( T_i^* \) on \( \mathcal{C} \) (resp. on \( \mathcal{L} \)). Note that every intersection of \( d \) \( \mathcal{L} \)-hyperplanes of pairwise distinct directions is a line \( L \) and that, conversely, for every line \( L, L = L_1 \cap \cdots \cap L_{d^l} \), where \( L_i^l \) denotes the orbit of \( L \) under \( T_i^l \). Also the stabilizer \( (T_i)_L \) of \( L \) in \( T_i \) is the same as the element-wise stabilizer \( (T_i)_L \cap T_i^l \) of \( L_i^l \) and \( N_L = (T_1)_L \times \cdots \times (T_d)_L = (T_1)_L \cap T_i^l \times \cdots \times (T_d)_L \cap T_i^d \).

We now divide the proof into several steps.

Step 1. \( G \) does not act primitively on \( \mathcal{P} \).

PROOF. Suppose the contrary, that is \( \mathcal{P} = \mathcal{C} \). Let \( F \) be one of the \( \gamma^{d-1} \mathcal{C} \)-fibres of direction \( 1 \). Since \( |F| = \gamma \geq 2 \), there is a line \( L \) containing at least two points of \( F \), so that the set of points \( L \cap F \) uniquely determines both the line \( L \) and the \( \mathcal{C} \)-fibre \( F \). Denoting by \( L^N \) the orbit of \( L \) under \( N \), we obtain

\[
\beta^d = |L^N| \leq |(L \cap F)^N| = |F^N| \cdot |(L \cap F)^N|.
\]

Since \( N \) acts transitively on the \( \gamma^{d-1} \mathcal{C} \)-fibres of direction \( 1 \), the length of \( F^N \) is \( \gamma^{d-1} \). Since \( T_i \) acts on \( F \) in the same way as \( N_F \) does, and since any line having at least two points in \( F \) is uniquely determined by its intersection with \( F \),

\[
|(L \cap F)^N| = |L^T| = \beta.
\]

Hence (1) becomes

\[
\beta^d \leq \gamma^{d-1} \beta,
\]

so that \( \beta \leq \gamma \), contradicting \( b > v \).

Step 2. For any two distinct lines \( L \) and \( L' \), \( N_L \neq N_{L'} \neq 1 \).

PROOF. Notice first that all line-stabilizers in \( N \) have the same order. By Feit-Thompson's theorem, \( N \) contains an involution \( \sigma \). For any two points \( x \) and \( y \) interchanged by \( \sigma \), the line \( L = xy \) is stabilized by \( \sigma \), so that \( \sigma \in N_L \neq 1 \).
Now define a partition of \( L \) by putting two lines in the same class iff their stabilizers in \( N \) coincide. Since \( N_L \neq 1 \), this partition has at least two classes. On the other hand, \( g \) preserves this partition and acts primitively on \( L \), so that all classes of the partition must be singletons and Step 2 is proved.

**Step 3.** Let \( \mathbb{L}_1(L) \) be the set of all lines fixed by \((T_1)_L\). Then \( \mathbb{L}_1(L) = L^{T_1} \).

**Proof.** Trivially, \( L^{T_1} \subseteq \mathbb{L}_1(L) \). If there is a line \( L' \) in \( \mathbb{L}_1(L) \setminus L^{T_1} \), then \((T_1)_L = (T_1)_L'\), so that

\[
N_L = (T_1)(L^{T_1}) \times \cdots \times (T_d)(L^{T_2}) = (T_1)(L^{T_1}) \times (T_2)(L^{T_2}) \times \cdots \times (T_d)(L^{T_2}) = N_L' ,
\]

where the line \( L'' = L^{T_1} \cap L^{T_2} \cap \cdots \cap L^{T_d} \) is distinct from \( L \) since the hyperplanes \( L^{T_1} \) and \( L^{T_2} \) are disjoint. This contradicts Step 2.

**Step 4.** For every point \( x \in \mathbb{P} \) and every \( i \in \{1, \ldots , d\} \), there is a line \( L_i \) through \( x \) and an involution \( \sigma_i \in T_i \) fixing \( x \) but not \( L_i \).

**Proof.** By Feit–Thompson’s theorem, \( T_i \) contains an involution \( \sigma_i \). If \( \sigma_i \) fixes no point, then, as noticed by Camina and Siemons [4], this involution would define a partition of \( \mathbb{P} \) into pairwise disjoint lines, (all stabilized by \( \sigma_i \)), so that \( k \mid \nu \). By results of Camina and Gagen [3] and Higman and MacLaughlin [10] (see Lemma 6 in section 3), this forces \( G \) to act primitively on \( \mathbb{P} \), contradicting Step 1. Hence \( \sigma_i \) fixes at least one point \( x \). Because \( \sigma_i \) commutes with \( T_i^* \), the orbit \( x^{T_i^*} \) of \( x \) under \( T_i^* \) is fixed pointwise by \( \sigma_i \). Since \( |x^{T_i^*}| \geq \nu^{d-1} \geq 2 \), \( \sigma_i \) fixes at least two points \( x \) and \( y \). If all lines through any fixed point of \( \sigma_i \) are fixed, then every point \( z \) not on the line \( xy \) is the intersection of the two fixed lines \( xz \) and \( yz \), and so \( z \) is fixed. This forces all points of \( \mathbb{P} \) to be fixed, contradicting the fact that \( \sigma_i \) is an involution. Hence there is at least one line \( L_i \), which passes through a fixed point of \( \sigma_i \) and is not fixed by \( \sigma_i \). The assertion of Step 4 follows from the transitivity of \( N \) and \( \mathbb{P} \) and the fact that \( T_i \triangleleft N \).

**Step 5.** For every point \( x \in \mathbb{P} \), there is a line \( L \) such that \( N_L \leq N_x \).

**Proof.** In what follows, the indices \( i \) and \( i+1 \) are in \( \{1, \ldots , d\} \) and are computed modulo \( d \). Let \( L_{i+1} \) be as in Step 4, that is a line through \( x \) such that there is an involution \( \sigma_{i+1} \) in \( T_{i+1} \) which fixes \( x \) but not \( L_{i+1} \). We know that the stabilizer of \( L_{i+1} \) in \( T_i \) is the same as the elementwise stabilizer in \( T_i \) of the \( L \)-hyperplane \( L^{T_{i+1}}_{i+1} \), which contains the \( L \)-fibre \( L^{T_{i+1}}_i \). Therefore the elementwise stabilizer \((T_i)(L^{T_{i+1}}_{i+1})\) fixes \( \sigma_{i+1}(L_{i+1}) \) as well as \( L_{i+1} \), and so fixes \( x = L_{i+1} \cap \sigma_{i+1}(L_{i+1}) \). Hence

\[
(T_i)(L^{T_{i+1}}_i) \subseteq (T_i)_x ,
\]

Consequently,

\[
(T_i)(L^{T_i}_i) \times (T_2)(L^{T_2}_i) \times \cdots \times (T_d)(L^{T_d}_i) \subseteq (T_i)_x \times \cdots \times (T_d)_x \leq N_x .
\]

The l.h.s. is the stabilizer in \( N \) of the intersection of the \( d \) \( L \)-hyperplanes \( L^{T_{i+1}}_i \), which is a line \( L \). So Step 5 is proved.

**Step 6** (final contradiction). Let \( F(L) \) be the set of all points fixed by \( N_L \). By Step 5, \( F(L) \) is non-empty. Let

\[
A = \bigcup_{x \in F(L)} x^{T_i} ,
\]
The canonical projection of \( A \) to \( C \) (mapping every point onto the class it belongs to) is a union of \( m \) \( C \)-hyperplanes of direction \( 1^* \); let \( A = A_1 \cup \cdots \cup A_m \), where \( A_1, \ldots, A_m \) are the pre-images of those \( m \) \( C \)-hyperplanes. Trivially, \( T^*_1 \) acts on \( A \) and leaves each \( A_j \) invariant. Since any two points \( x \) and \( y \) in \( A \) are fixed by \((T^*_1)_L\), the line \( xy \) belongs to \( L^T \) by Step 3. Hence \( T^*_i \) acts transitively on the lines of the linear space \( S|_A \) induced by \( S \) on the point-set \( A \) (the points of \( S|_A \) are those in \( A \), while the lines of \( S|_A \) are the intersections of cardinality \( \geq 2 \) of the lines of \( S \) with \( A \)). Therefore the cardinality of the intersection with \( A_j \) of a line of \( S|_A \) is a number \( k_j \), depending only on \( j \). Since \( |A_j| = a_j \geq \gamma^{d-1} \geq 2 \) and since \((T^*_1)_L\) acts line-transitively on \( S|_A \), we know that \( k_j \geq 2 \) for \( j = 1, \ldots, m \).

Suppose that \( m \geq 2 \). Let \( x \in A_1 \), \( y \in A_2 \) and count the numbers \( r_x \) and \( r_y \) of lines of \( S|_A \) through \( x \) and \( y \) respectively:

\[
r_x = (a_1 - 1)/(k_1 - 1) = a_2/k_2, \\
r_y = (a_2 - 1)/(k_2 - 1) = a_1/k_1,
\]

a contradiction.

Thus \( m = 1 \); that is, the canonical projection of \( A \) to \( C \) is precisely one \( C \)-hyperplane of direction \( 1^* \). Hence the canonical projection of \( F(L) \) to \( C \) is contained in this \( C \)-hyperplane of direction \( 1^* \). Since \( G \) acts transitively on the directions \( i^* \), the same holds for every direction \( i^* \) \((i = 1, \ldots, d)\). Thus all points of \( F(L) \) are in a unique class \( C \) (namely the intersection of those \( d \) \( C \)-hyperplanes). Hence \( G_L \), which normalizes \( N_L \), stabilizes \( F(L) \), thus also \( C \), so that \( G_L \leq G_C \). Since \( b > c \), this contradicts the maximality of \( G_L \) in \( G \).

3. LINE-PRIMITIVE BUT POINT-IMPRIMITIVE GROUPS

Let \( S \) be a finite linear space and let \( G \) be a line-primitive but point-imprimitive automorphism group of \( S \). As before, \( S \) is a \( 2-(v, k, 1) \) design which is not a projective plane because of Lemma 3. By Lemma 2, \( G \) cannot be flag-transitive. On the other hand, Camina and Gagen have proved in [3] that in any \( 2-(v, k, 1) \) design where \( k > v \), all line-transitive groups are flag-transitive. Since the only linear spaces with \( v \leq k^2 \) are the affine and projective planes, we deduce:

**Lemma 6.** \( k \neq v \) and \( v > k^2 \).

Actually, Higman and McLaughlin derived Lemma 2 from the following combinatorial fact, which is implicit in their proof:

**Lemma 7.** The point-set \( \mathcal{P} \) of a \( 2-(v, k, 1) \) design cannot be partitioned into classes of equal size \( > 1 \) in such a way that, for some integer \( e \), each line intersects each class in \( 0 \) or \( e \) points.

Let \( \mathcal{C} = \{C_1, \ldots, C_c\} \) be a non-trivial partition of \( \mathcal{P} \) into \( c \) classes of size \( s \), which is preserved by \( G \) and on which \( G \) acts primitively. For any non-negative integer \( i \) and any line \( L \), we denote by \( X_i(L) \) the set of all classes of \( \mathcal{C} \) intersecting \( L \) in exactly \( i \) points. By the line-transitivity of \( G \), the number \( x_i \) of classes in \( X_i(L) \) is independent of \( L \). Let \( I \) be the set of all integers \( i \) for which \( x_i \) is non-zero, and let \( I_0 = I \setminus \{0\} \). With this notation, Lemma 7 yields:

**Lemma 8.** \( |I_0| \geq 2 \).

**Lemma 9.** For any \( i \in I \) and for any line \( L \), the stabilizer of \( L \) in \( G \) is the same as the stabilizer of \( X_i(L) \) in \( G \).
PROOF. Note first that this statement makes sense because $G$ acts faithfully both on $L$ and on $C$. Since the stabilizer $G_L$ of $L$ necessarily stabilizes $X_i(L)$, the lemma follows from the maximality of $G_L$ in $G$, the transitivity of $G$ on $C$ and the fact that $X_i(L)$ is a proper subset of $C$.

Recall that a permutation group on $\Omega$ is said to be $\lambda$-homogeneous if it acts transitively on the set of all $\lambda$-subsets of $\Omega$.

**Corollary 9.** For any $i \in I$:
(i) the mapping $X_i: L \rightarrow X_i(L)$ is injective;
(ii) $b \leq \binom{c}{x_i}$;
(iii) $x_i \geq 2$;
(iv) if $G$ acts $x_i$-homogeneously on $C$, then $b = \binom{c}{x_i}$.

**Proof.** (i) follows from Lemma 9 and the maximality of $G_L$; (ii) follows from (i); (iii) follows from (ii) together with $b > v > c$; (iv) follows from (i).

An unordered pair of points of $S$ is said to be inner if both points are contained in the same class of $C$. We denote by $n$ the number of inner pairs of points on a given line.

**Lemma 10.**

$$c = \sum_{i \in I} x_i \geq 4;$$
$$k = \sum_{i \in I} ix_i \geq 6;$$
$$n = \sum_{i \in I} \binom{i}{2} x_i \geq 2.$$

**Proof.** Easy counting. The lower bounds follow from Lemma 8 and Corollary 9 (iii).

**Lemma 11** (Delandtsheer-Doyen [7]). There is a positive integer $m$ such that

$$s = \left(\binom{k}{2} - n\right)|m$$

and

$$c = \left(\binom{k}{2} - m\right)|n.$$

Actually, Lemma 11 was proved under the weaker hypothesis that $G$ is line-transitive.

**Corollary 11.1** $\nu \leq \left(\binom{k}{2} - 2\right)\left(\binom{k}{2} - 1\right)/2$.

**Proof.** Immediate consequence of Lemmas 10 and 11.

**Corollary 11.2** $m < \left(\binom{k}{2}\right)/(2n + 1)$ and $n < \left(\binom{k}{2}\right)/(2m + 1)$. Moreover, either $2 \leq n \leq m$ or $1 \leq m < n$.

**Proof.** The second part of the statement follows from Lemma 10. In order to prove the first part, note that

$$\nu = \left(\binom{k}{2} - m\right)\left(\binom{k}{2} - n\right)/nn.$$
Since \( \binom{k}{2} < v/2 \) in any 2-(v, k, 1) design, we obtain

\[
v < v \left( \binom{k}{2} - m \right) 2nm
\]

and so

\[
(2n + 1)m < \binom{k}{2}.
\]

We obtain the other inequality in the same way.

Denote by \( f \) the number of flag-orbits of \( G \). By Lemma 2, \( f \geq 2 \).

**Lemma 12** (Delandtsheer [6]). \( G \) has line-rank at least \( f^2 + 4 \geq 8 \).

**Lemma 13.** There are pairs of lines which cannot be interchanged by \( G \).

**Proof.** Let \( (x, L) \) be a flag and let \( C \) be the class containing \( x \). By Lemma 8, there is an element \( j \) of \( I_0 \) such that \( j \neq i = |L \cap C| \). Since \( G \) is transitive on \( \mathcal{P} \), there is a line \( L' \) through \( x \) intersecting \( C \) in \( j \) points. An element of \( G \) interchanging \( L \) and \( L' \) should fix \( x \), hence also \( C \), contradicting \( i \neq j \).

**Corollary 13.** (i) if \( G \ni \text{Alt}(\omega) \) acts on \( \Omega \) with \( |\Omega| = \omega \geq 5 \), then \( G_L \) acts primitively on \( \Omega \);

(ii) \( G \not\ni \text{Alt}(c) \).

**Proof.** (i) follows from Lemma 13 since \( \text{Alt}(\omega) \) contains an element interchanging any two given subsets of \( \Omega \) having the same size, as well as any two given partitions of \( \Omega \) into the same number of equicardinal classes. In order to prove (ii), suppose to the contrary that the socle of \( G \) is \( \text{Alt}(c) \), so that \( G = \text{Alt}(c) \) or \( \text{Sym}(c) \) acting faithfully on \( C \) (whose size is \( c \)). By Lemma 8, the set \( I_0 \) contains two distinct elements \( i \) and \( j \). Then the stabilizer \( G_c \) of \( L \) preserves the subsets \( X_i(L) \) and \( X_j(L) \) on \( C \), contradicting (i).

**Proof of Theorem 2.** Given any \( k < 30 \), we look for all pairs \( (n, m) \) satisfying Corollary 11.2 and such that \( s \) and \( c \) given by Lemma 11 are integers. Then, setting \( v = sc \), we keep only those pairs \( (n, m) \) for which the numbers \( r = (v - 1)/(k - 1) \) and \( b = \nu(v - 1)/k(k - 1) \) are integers and Corollary 5 and Lemma 6 are satisfied. In this way, we are left with 56 5-tuples \( (k, s, c, n, m) \). Table 1 gives the values of \( k, s, c \) and \( n \), followed by symbols (explained later) which refer to the arguments used to rule out the corresponding 5-tuple. Remember that, by Lemma 5 and Corollary 13, \( G \) is a subgroup of \( \text{Sym}(c) \) which does not contain \( \text{Alt}(c) \).

Let us now explain the symbols appearing in Table 1.

**Lemma M** (Manning [18, Theorem 14.1]) If \( G \) is a primitive group of degree \( c \) not containing \( \text{Alt}(c) \), then the index of \( G \) in \( \text{Sym}(c) \) is divisible by \( \pi = \prod_{q=1}^{c} \tau_q \), where \( \tau_q \) denotes the product of all prime numbers from the following intervals:

\[
\begin{align*}
q &= 1, & 2 \leq p < c - 2; \\
q &= 2, 3, 4, & q + 1 < p < (c - q)/q; \\
q &= 5, & 5 < p < (c - 6)/q; \\
q &= 6, & 5 < p < (c - 10)/q; \\
q &\geq 7, & 2q - 2 < p < (c - 4q + 4)/q.
\end{align*}
\]
The symbol $Mp$ (where $p$ is a prime) means that the product of the highest power of $p$ dividing $v$ or $b$ by the highest power of $p$ dividing $\pi$ in Lemma $M$ does not divide $c!$, a contradiction.

**Lemma W.** (Manning and Weiss [18, Theorem 13.10]). Let $p$ be a prime and $G$ a primitive group of degree $c = qp + h$, which contains an element of order $p$ and degree $qp$ but does not contain $\text{Alt}(c)$. Then

from \quad q = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \geq 8

and \quad p \geq 2 \quad 5 \quad 5 \quad 7 \quad 7 \quad 11 \quad 11 \quad 2q - 1

it follows that \quad h \leq 2 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 6 \quad 8 \quad 4q - 4.

The symbol $Wp$ means that Lemma W provides a contradiction. The next lemma is an easy consequence of a theorem of C. Praeger [16].

**Lemma P.** If $9 \neq c < p^2$, where $p$ is a prime, then $p^2 \nmid |G|$. The symbol $Pp$ indicates that $p^2 \mid |G|$, contradicting Lemma P. The lemma below follows from Theorem 1 and a result of Guralnick [9].

**Lemma G.** If $c$ is a prime power distinct from 11, 23 and 27, then $c = (q^l - 1)/(q - 1)$ and $G \geq \text{PSL}_l(q)$ acting on the points of $\text{PG}(l - 1, q)$.

The symbols $G1, G2$ and $G3$ all mean that lemma G applies. $G1$ indicates that there is no prime power $q$ such that $c = (q^l - 1)/(q - 1)$, $G2$ indicates that $G$ is 2-transitive, and $G3$ indicates that $l \geq 3$ and the group $G$ has exactly two orbits on the 3-subsets of $C$, one of which has length $c(c - 1)(q - 1)/6$.

Finally, $Li$ or $Ci$ refer to the $i$th lemma or corollary, while the symbol OA means that other arguments are used (to be detailed later).

Let us take a few examples. In the case (12, 32, 32, 2; G2, C9) of Table 1, we know by Lemma G that the group $G$ acts 2-transitively on the 32 classes of $C$. On the other hand, $n = 2$ forces $x_2 = 2$ and Corollary 9 yields $b = 32\cdot31/2$, contradicting $b = 32^2 \cdot (32^2 - 1)/12\cdot11$.

In the case (28, 25, 121, 3; G3, L9), Lemma G implies that the group $G$ normalizes $\text{PSL}_3(3)$ and acts on $C$ as it does on the 121 points of $\text{PG}(4, 3)$. Hence $G$ has two orbits on...
the 3-subsets of \( C \), of lengths 7260 and 280720. But \( n = 3 \) forces \( x_2 = 3 \) and Lemma 9 enables us to identify the set of lines with one orbit of 3-subsets of \( C \), contradicting \( b = 12 \, 100 \).

In the case \((23, 25, 81, 3; G1)\), Lemma \( G \) yields a contradiction because 81 is not the number of points of a Desarguesian projective space.

The case \((24, 45, 45, G; OA)\) is ruled out by noticing that 11 divides \( b \) (hence also \( |G| \)) and by checking, for example in [16], that the only primitive groups of degree 45 whose order is a multiple of 11 are \( \text{Alt}(45) \) and \( \text{Sym}(45) \).

The case \((29, 25, 65, 6; OA)\) is the hardest one, because we have no list of all primitive groups of degree 65, and the lemmas \( M \) and \( W \) do not help very much. The socle \( T \) of \( G \) has transitive actions of degrees \( c = 65 \) and \( b = 3250 \). By Lemma \( W \), the order of \( G \) is not divisible by 11, 17, 19, 29 or 37. Using [5] and [15] it is easy but tedious to check that no simple group satisfies these conditions.

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