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Convexificators and strong Kuhn–Tucker conditions

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ABSTRACT

This study is devoted to constraint qualifications and strong Kuhn–Tucker necessary optimality conditions for nonsmooth multiobjective optimization problems. The main tool of the study is the concept of convexificators. Mangasarian–Fromovitz type constraint qualification and several other qualifications are proposed and their relationships are investigated. In addition, sufficient optimality conditions are studied.

1. Introduction

In nonlinear programming with scalar-valued objective function, constraint qualifications play an important role in deriving the Lagrange multiplier rules. For instance, constraint qualifications assure the existence of positive Lagrange multipliers associated with the objective function, and this implies that the Fritz-John conditions and the Kuhn–Tucker conditions are equivalent to each other. In multiobjective optimization problems, many authors have derived the necessary conditions under the same constraint qualifications as are used in nonlinear programming with scalar-valued objective function. In these approaches, however, we could not obtain positive Lagrange multipliers associated with the vector-valued objective function, namely, some of the multipliers may be equal to zero. This means that the components of the vector-valued objective function have no role in the necessary conditions for efficiency. In order to avoid the case where some of the Lagrange multipliers associated with the objective function vanish for a multiobjective optimization problem, several approaches have been developed in recent years, and strong Kuhn–Tucker (K–T) necessary optimality conditions have been obtained. We say that strong K–T conditions hold when the Lagrange multipliers are positive for all components of the objective function.

Maeda [1] considered differentiable multiobjective optimization problems and gave strong K–T necessary conditions for a Pareto minimum of a function over a feasible set defined by inequality constraints under a regularity condition. Later, Preda and Chitescu [2] extended the results obtained by Maeda for the continuously differentiable case of the optimization problem to the directionally differentiable case.

Recently, the notion of convexificators has been used to extend various results in nonsmooth analysis and optimization (see [3–5]). Convexificators can be viewed as a weaker version of the notion of subdifferentials and thus are more amenable to analysis and applications. Indeed, convexificators are in general closed sets, unlike the well-known subdifferentials which are convex and compact objects. For a locally Lipschitz function, most known subdifferentials are convexificators and these known subdifferentials may contain the convex hull of a convexificator; see, for instance, [3,6,7]. Therefore, from the

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viewpoint of optimization and applications, the descriptions of the optimality conditions in terms of convexificators provide sharper results. For nonsmooth optimization problems, various results concerning Fritz-John type and Kuhn–Tucker type necessary optimality conditions that use convexificators have been developed in [4,7–9]. In the framework of the locally Lipschitz case, Li and Zhang [10] consider only inequality constraints and provide strong Kuhn–Tucker necessary optimality conditions that are expressed in terms of upper convexificators, by imposing some convexity assumptions on the derivatives.

In this paper, by using the idea of convexificators, we study strong Kuhn–Tucker necessary optimality conditions for a nonsmooth multiobjective optimization problem with inequality constraints and an arbitrary set constraint. We propose first a nonsmooth analogue of the generalized Mangasarian–Fromovitz constraint qualification by using the convexificators and for efficient solutions we derive stronger K–T type necessary conditions that are expressed in terms of the convexificators. Moreover, we propose some other constraint qualifications and we explore the relationships among them. Then, we give examples to show that the regularity conditions imposed on the upper convexificators are necessary for our qualifications to guarantee nonemptiness of the Kuhn–Tucker multipliers set.

In this paper, we consider locally Lipschitz functions and we do not assume that the directional derivatives of the objective function and inequality constraints are sublinear in the second variable.

The outline of the paper is as follows. In Section 2, notation, definitions, and preliminaries are given. In Section 3, the nonsmooth version of the Mangasarian–Fromovitz constraint qualification is given and the strong Kuhn–Tucker optimality conditions are obtained. Moreover, other constraint qualifications are given and the relationships among these qualifications are discussed.

2. Preliminaries

Throughout this paper, $\mathbb{R}^\ell$ is the usual ℓ-dimensional Euclidean space. Let $x = (x_1, \dots, x_\ell)$ and $y = (y_1, \dots, y_\ell)$ be two vectors in $\mathbb{R}^\ell$. Then,

$$x \leqq y \iff x_i \le y_i, \quad i = 1, \dots, \ell,$$
$$x \le y \iff x_i \le y_i, \quad i = 1, \dots, \ell, \text{ and } x \neq y,$$
$$x < y \iff x_i < y_i, \quad i = 1, \dots, \ell.$$

Let S be a nonempty subset of $\mathbb{R}^\ell$. The convex hull of S, the closure of S and the convex cone (containing the origin of $\mathbb{R}^\ell$) generated by S are denoted by $\operatorname{co} S$, $\operatorname{cl} S$ and $\operatorname{cone} S$, respectively. The negative polar cone S^- and the strictly negative polar cone S^s are defined respectively by

$$S^- = \{v \in \mathbb{R}^\ell \mid \langle x, v\rangle \le 0\, \forall x \in S\},$$
$$S^s = \{v \in \mathbb{R}^\ell \mid \langle x, v\rangle < 0 \forall x \in S\}.$$

The contingent cone $T(S, x)$ and the normal cone $N(S, x)$ at $x \in \operatorname{cl} S$ are defined respectively by

$$T(S, x) = \left\{v \in \mathbb{R}^\ell | \exists t_n \downarrow 0 \text{ and } v_n \to v \text{ such that } x + t_n v_n \in S \forall n\right\},$$
$$N(S, x) = T(S, x)^- = \left\{\xi \in \mathbb{R}^\ell | \langle \xi, v\rangle \le 0 \forall v \in T(S, x)\right\}.$$

We recall the following definitions from [6].

Let $f : \mathbb{R}^\ell \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, $x \in \mathbb{R}^\ell$, and let $f(x)$ be finite. The lower and upper Dini derivatives of f at x in the direction $v \in \mathbb{R}^\ell$ are defined, respectively, by

$$f^-(x; v) := \liminf_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t},$$
$$f^+(x; v) := \limsup_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

The function $f : \mathbb{R}^\ell \to \overline{\mathbb{R}}$ is said to have an upper convexificator $\partial^* f(x) \subset \mathbb{R}^\ell$ at $x \in \mathbb{R}^\ell$ if $\partial^* f(x)$ is closed and for each $v \in \mathbb{R}^\ell$,

$$f^-(x; v) \le \sup_{\xi \in \partial^* f(x)} \langle \xi, v\rangle.$$

The function $f : \mathbb{R}^\ell \to \overline{\mathbb{R}}$ is said to have a lower convexificator $\partial_* f(x) \subset \mathbb{R}^\ell$ at $x \in \mathbb{R}^\ell$ if $\partial_* f(x)$ is closed and for each $v \in \mathbb{R}^\ell$,

$$f^+(x; v) \ge \inf_{\xi \in \partial_* f(x)} \langle \xi, v\rangle.$$

The function $f : \mathbb{R}^\ell \to \overline{\mathbb{R}}$ is said to have an upper regular convexificator $\partial^* f(x) \subset \mathbb{R}^\ell$ at $x \in \mathbb{R}^\ell$ if $\partial^* f(x)$ is closed and for each $v \in \mathbb{R}^\ell$,

$$f^+(x; v) = \sup_{\xi \in \partial^* f(x)} \langle \xi, v\rangle.$$

The function $f:\mathbb{R}^\ell \to \overline{\mathbb{R}}$ is said to have a lower regular convexificator $\partial_* f(x) \subset \mathbb{R}^\ell$ at $x \in \mathbb{R}^\ell$ if $\partial_* f(x)$ is closed and for each $v \in \mathbb{R}^\ell$,

$$f^-(x;v) = \inf_{\xi\in\partial_* f(x)} \langle \xi, v\rangle.$$

The notion of convexificators in [4] has been extended and used to unify and strengthen various results in nonsmooth analysis and optimization. Along the lines of [11], we give now the definition of upper semi-regular convexificators which will be useful in what follows.

The function $f:\mathbb{R}^\ell \to \overline{\mathbb{R}}$ is said to have an upper semi-regular convexificator $\partial^* f(x) \subset \mathbb{R}^\ell$ at $x \in \mathbb{R}^\ell$ if $\partial^* f(x)$ is closed and for each $v \in \mathbb{R}^\ell$,

$$f^+(x;v) \le \sup_{\xi\in\partial^* f(x)} \langle \xi, v\rangle.$$

The function $f:\mathbb{R}^\ell \to \overline{\mathbb{R}}$ is said to have a lower semi-regular convexificator $\partial_* f(x) \subset \mathbb{R}^\ell$ at $x \in \mathbb{R}^\ell$ if $\partial_* f(x)$ is closed and for each $v \in \mathbb{R}^\ell$,

$$f^-(x;v) \ge \inf_{\xi\in\partial_* f(x)} \langle \xi, v\rangle.$$

Obviously, an upper (lower) regular convexificator of f at a point is an upper (lower) semi-regular convexificator of f at the point and every upper (lower) semi-regular convexificator is an upper (lower) convexificator.

Moreover, for a locally Lipschitz function $f:\mathbb{R}^\ell \to \mathbb{R}$ the Clarke subdifferential $\partial_C f(x)$ [12], Michel-Penot subdifferential $\partial^\diamond f(x)$ [13], Mordukhovich subdifferential $\partial_M f(x)$ [14] and Trieman subdifferential $\partial_T f(x)$ [15] are examples of upper semi-regular convexificators. It has been shown [7, Example 2.1] that the convex hull of a convexificator of a locally Lipschitz function may be strictly contained in these subdifferentials.

Now, we recall from [11] two classes of generalized convex functions, called ∂^*-pseudoconvex and ∂^*-quasiconvex functions, which will be used in the sequel.

Let $f:\mathbb{R}^\ell \to \overline{\mathbb{R}}$ be an extended real-valued function and let f have an upper semi-regular convexificator at $x \in \mathbb{R}^\ell$;

(i) f is said to be ∂^*-pseudoconvex at x if for every $y \in \mathbb{R}^\ell$ and $x \neq y$

$$f(y) < f(x) \quad \text{implies } \langle \xi, y-x\rangle < 0,\ \forall \xi \in \partial^* f(x).$$

(ii) f is said to be ∂^*-quasiconvex at x if for every $y \in \mathbb{R}^\ell$

$$f(y) \le f(x) \quad \text{implies } \langle \xi, y-x\rangle \le 0,\ \forall \xi \in \partial^* f(x).$$

3. Strong Kuhn–Tucker conditions

In this section strong Kuhn–Tucker necessary conditions are given for a point to be a locally efficient solution. In order to obtain the positivity of the multipliers associated with the objective function a generalized constraint qualification will be assumed.

We consider the following multiobjective programming problem:

$$\begin{aligned}&(\mathbf{P})\quad \min f(x) = (f_1(x),\ldots,f_m(x))\\ &\text{s.t. } g(x) = (g_1(x),\ldots,g_n(x)) \leqq 0\\ &x\in Q\end{aligned}$$

where $f_i:\mathbb{R}^\ell \to \overline{\mathbb{R}}$ and $g_j:\mathbb{R}^\ell \to \overline{\mathbb{R}}$ are real-valued functions for $i \in I = \{1,\ldots,m\}$ and $j \in J = \{1,\ldots,n\}$ and where Q is a convex subset of $\mathbb{R}^\ell$.

For convenience we introduce some notation which will be used in the sequel.

$$J(\bar{x}) = \Big\{k\in J | g_k(\bar{x}) = 0\Big\},$$

$$F = \bigcup_{i=1}^{m} \operatorname{co}\partial^* f_i(\bar{x}),$$

$$F^i = \bigcup_{j\in I\setminus\{i\}} \operatorname{co}\partial^* f_j(\bar{x}),$$

$$G = \bigcup_{k\in J(\bar{x})} \operatorname{co}\partial^* g_k(\bar{x}),$$

$$S = \Big\{x\in\mathbb{R}^\ell | g(x) \leqq 0, x\in Q\Big\},$$

$$S^i = \Big\{x\in\mathbb{R}^\ell | f_j(x) \le f_j(\bar{x}), \forall j\in I\setminus\{i\}, g(x)\leqq 0, x\in Q\Big\},$$

$$S^\circ = \Big\{x\in\mathbb{R}^\ell | f_j(x) \le f_j(\bar{x}), \forall j\in I, g(x)\leqq 0, x\in Q\Big\}.$$

A point $\bar{x} \in S$ is said to be a locally efficient solution for (P) if there exists no $x \in S$ near $\bar{x}$ such that $f(x) \leq f(\bar{x})$.

A point $\bar{x} \in S$ is said to be a locally weakly efficient solution for (P) if there exists no $x \in S$ near $\bar{x}$ such that $f(x) < f(\bar{x})$.

Definition 1. We say that the generalized Mangasarian–Fromovitz constraint qualification is satisfied at $\bar{x}$ if for every $i \in I$,

(CQ1) $(F^i)^s \bigcap G^s \bigcap T(Q, \bar{x}) \neq \emptyset.$

Now, we are ready to prove our result of strong Kuhn–Tucker type necessary conditions in terms of upper semi-regular convexificators.

Theorem 1. *Let $\bar{x}$ be a locally weakly efficient solution for (P). Suppose that f_i and g_j are locally Lipschitz functions at $\bar{x}$, and admit bounded upper semi-regular convexificators $\partial^* f_i(\bar{x})$ and $\partial^* g_j(\bar{x})$ for all $i \in I$ and $j \in J$. If (CQ1) holds at $\bar{x}$, then there exists $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_+$ such that*

$$0 \in \sum_{i=1}^{m} \lambda_i \mathrm{co}\, \partial^* f_i(\bar{x}) + \sum_{j=1}^{n} \mu_j \mathrm{co}\, \partial^* g_j(\bar{x}) + N(Q, \bar{x}), \tag{1}$$

$$\mu_j g_j(\bar{x}) = 0, \quad j = 1, \dots, n.$$

Proof. We claim that for every index $i_0 \in I$ there exists $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \geqq 0$ which satisfies (1) and $\lambda_{i_0} > 0$. We proceed by contradiction. Therefore there exists $i_0 \in I$ such that for every $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \geqq 0$,

$$0 \notin \mathrm{co}\, \partial^* f_{i_0}(\bar{x}) + \sum_{i \in I \setminus \{i_0\}} \lambda_i \mathrm{co}\, \partial^* f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \mu_j \mathrm{co}\, \partial^* g_j(\bar{x}) + N(Q, \bar{x}).$$

We assert that

$$0 \notin \mathrm{co}\, (F \cup G) + N(Q, \bar{x}). \tag{2}$$

Indeed, if the assertion (2) is not true, then there exist $\lambda_i \geq 0$, $i \in I$ and $\mu_j \geq 0$, $j \in J(\bar{x})$, not all zero, and $\xi_i \in \mathrm{co}\, \partial^* f_i(\bar{x})$, $\zeta_j \in \mathrm{co}\, \partial^* g_j(\bar{x})$ and $\eta \in N(Q, \bar{x})$ such that

$$\sum_{i=1}^{m} \lambda_i \xi_i + \sum_{j \in J(\bar{x})} \mu_j \zeta_j + \eta = 0. \tag{3}$$

By assumption we have $\lambda_{i_0} = 0$. Let $v \in \mathbb{R}^\ell$ be the vector which is satisfied in (CQ1) for the index i_0. Then we obtain

$$0 = \sum_{i \in I \setminus \{i_0\}} \lambda_i \langle \xi_i, v \rangle + \sum_{j \in J(\bar{x})} \mu_j \langle \zeta_j, v \rangle + \langle \eta, v \rangle < 0,$$

which is a contradiction. Then, the assertion (2) is true or equivalently

$$\mathrm{co}\, (F \cup G) \bigcap -N(Q, \bar{x}) = \emptyset.$$

Since $\mathrm{co}\, (F \cup G)$ is a compact and convex set and $N(Q, \bar{x})$ is a closed and convex set it follows that by the separation theorem there exist $v \in \mathbb{R}^\ell$ and $\alpha \in \mathbb{R}$ such that

$$\langle \xi, v \rangle < \alpha < \langle \eta, v \rangle, \quad \forall \xi \in \mathrm{co}\, (F \cup G)\,, \ \forall \eta \in -N(Q, \bar{x}).$$

By the definition of normal cone we obtain

$$v \in (N(Q, \bar{x}))^- = T(Q, \bar{x}). \tag{4}$$

Since $N(Q, \bar{x})$ is a normal cone, then we conclude that $\alpha < 0$ which implies

$$v \in (F \cup G)^s\,. \tag{5}$$

For each $i \in I$ and $j \in J(\bar{x})$, since f_i and g_j admit an upper semi-regular convexificator it follows from (5) that

$$f_i^+(\bar{x}; v) < 0, \quad i \in I,$$

$$g_j^+(\bar{x}; v) < 0, \quad j \in J(\bar{x}).$$

Using (4), we conclude that there exist $t_n \downarrow 0$ and $v_n \to v$ such that

$$\bar{x} + t_n v_n \in Q, \quad \forall n \in \mathbb{N}. \tag{6}$$

Now suppose that $\varepsilon > 0$ is such that

$$f_i^+(\bar{x}; v) < -\varepsilon, \quad i \in I, \tag{7}$$

$$g_j^+(\bar{x}; v) < -\varepsilon, \quad j \in J(\bar{x}). \tag{8}$$

It follows from (8) that, for all sufficiently large n,

$$\frac{g_j(\bar{x} + t_n v) - g_j(\bar{x})}{t_n} < -\varepsilon. \tag{9}$$

Let ℓ_j be the Lipschitzian constant for g_j near $\bar{x}$; then

$$g_j(\bar{x} + t_n v) \leq g_j(\bar{x} + t_n v_n) + \ell_j t_n \|v - v_n\|.$$

Therefore

$$\frac{g_j(\bar{x} + t_n v_n) - g_j(\bar{x})}{t_n} \leq \frac{g_j(\bar{x} + t_n v) - g_j(\bar{x})}{t_n} + \ell_j \|v - v_n\|. \tag{10}$$

Since $v_n \to v$, (9) and (10) are satisfied; thus for n large enough we have

$$g_j(\bar{x} + t_n v_n) < g_j(\bar{x}). \tag{11}$$

On the other hand, from (6) and the continuity of constraint functions we obtain that $\bar{x} + t_n v_n$ is a feasible point for (P) for all sufficiently large n.

Similarly, in view of (7) for all sufficiently large n

$$f_i(\bar{x} + t_n v_n) < f_i(\bar{x}), \quad \forall\, i \in I,$$

which is a contradiction with locally weakly efficiency of $\bar{x}$.

Since i_0 was chosen arbitrarily, it follows that for every index $i \in I$ there exists $(\boldsymbol{\lambda}^i, \boldsymbol{\mu}^i) \geqq 0$ such that (1) is satisfied and $\lambda_i^i > 0$. Thus, by adding on these indices, the proof is complete. □

It is worth noting that Theorem 1 is not valid if the convex hull in (1) is removed. This fact is demonstrated by the following example.

Example 1. Consider the following nonsmooth scalar optimization problem:

$$\begin{aligned} (\mathbf{P_1})\ \ & \min f(x, y) = |x| + y^2 \\ & \text{s.t.} \quad g(x, y) = -|y| \leq 0 \\ & \qquad\ \ Q = \mathbb{R}^2. \end{aligned}$$

It is obvious that $(x, y) = (0, 0)$ is the global minimum for (P_1) and we have

$$f^+((0, 0); (v_1, v_2)) = |v_1|,$$
$$g^+((0, 0); (v_1, v_2)) = -|v_2|.$$

Hence, the objective and constraint functions admit bounded upper semi-regular convexificators as follows:

$$\partial^* f(0, 0) = \{(-1, 0), (1, 0)\},$$
$$\partial^* g(0, 0) = \{(0, -1), (0, -1/2)\}.$$

It is easy to verify that

$$\left(\text{co}\, \partial^* g(0, 0)\right)^s \cap T(Q, (0, 0)) \neq \emptyset$$

and it is trivial that there exists no $\mu \geq 0$ such that

$$0 \in \partial^* f(0, 0) + \mu \partial^* g(0, 0) + N(Q, (0, 0)).$$

In Theorem 1 we cannot replace the upper semi-regular convexificator with an upper convexificator. Let us illustrate this with the following example.

Example 2. Consider the following nonsmooth optimization problem:

$$(\mathbf{P_2})\ \min f(x) = \begin{cases} |x| \sin \dfrac{1}{x} & x \neq 0, \\ 0 & x = 0, \end{cases}$$
$$\text{s.t.}\quad g(x) = -x \le 0,$$
$$Q = \left\{ \frac{1}{2n\pi + \pi/2} : n \in \mathbb{N} \right\} \cup \{0\}.$$

Then $\bar{x} = 0$ is the global minimum for (P_2). Moreover,

$$f^+(0; v) = |v|,$$
$$f^-(0; v) = -|v|,$$
$$g^+(0; v) = -1.$$

Observe that the upper convexificators for f and g at $\bar{x}$ are given as follows:

$$\partial^* f(0) = \{-1, -1/2\},$$
$$\partial^* g(0) = \{-1\},$$

So

$$\left(\mathrm{co}\, \partial^* g(0)\right)^s \cap T(Q, 0) \neq \emptyset.$$

Thus, there is no $\mu \ge 0$ such that

$$0 \in \mathrm{co}\, \partial^* f(0) + \mu \mathrm{co}\, \partial^* g(0) + N(Q, 0).$$

In what follows we are going to derive sufficient optimality conditions for (P) under the generalized convexity assumptions.

Theorem 2. *Let $\bar{x}$ be a feasible solution for (P) and Q be a convex set. Suppose that f_i are ∂^*-pseudoconvex at $\bar{x}$ for all $i \in I$ and g_j are ∂^*-quasiconvex at $\bar{x}$ for all $j \in J(\bar{x})$. If there exist $\boldsymbol{\lambda} \ge 0$ (resp. $\boldsymbol{\lambda} > 0$) and $\boldsymbol{\mu} \geqq 0$ such that*

(i) $0 \in \sum_{i=1}^m \lambda_i \mathrm{co}\, \partial^* f_i(\bar{x}) + \sum_{j=1}^n \mu_j \mathrm{co}\, \partial^* g_j(\bar{x}) + N(Q, \bar{x})$, *and*
(ii) $\mu_j g_j(\bar{x}) = 0, j = 1, \ldots, n$

then $\bar{x}$ is a globally weakly efficient (resp. efficient) solution for (P).

Proof. Suppose that $\bar{x}$ is not the globally weakly efficient (resp. efficient) solution for (P). Then there exists a feasible solution x_0 such that

$$f(x_0) < f(\bar{x}) \left(\text{resp.}\, f(x_0) \le f(\bar{x})\right).$$

Since all components of the objective function are ∂^*-pseudoconvex then

$$\langle \xi_i, x_0 - \bar{x} \rangle < 0, \quad \forall \xi_i \in \partial^* f_i(\bar{x}),\ \forall i \in I\ (\text{resp.}\ \exists i \in I). \tag{12}$$

On the other hand, for the feasible point x_0 we have

$$\mu_j g_j(x_0) \le 0 = \mu_j g_j(\bar{x}).$$

By the ∂^*-quasiconvexity of $g_j, j \in J(\bar{x})$, we obtain

$$\langle \eta_j, x_0 - \bar{x} \rangle \le 0, \quad \forall \eta_j \in \partial^* g_j(\bar{x}),\ \forall j \in J(\bar{x}). \tag{13}$$

Therefore for every $\xi_i \in \mathrm{co}\, \partial^* f_i(\bar{x})$, $\eta_j \in \mathrm{co}\, \partial^* g_j(\bar{x})$ and $v \in N(Q, \bar{x})$ we have

$$0 > \sum_{i=1}^m \lambda_i \langle \xi_i, x_0 - \bar{x} \rangle + \sum_{j=1}^n \mu_j \langle \eta_j, x_0 - \bar{x} \rangle$$
$$\ge \sum_{i=1}^m \lambda_i \langle \xi_i, x_0 - \bar{x} \rangle + \sum_{j=1}^n \mu_j \langle \eta_j, x_0 - \bar{x} \rangle + \langle v, x_0 - \bar{x} \rangle.$$

This is a contradiction and the proof is complete. □

Let us now present some constraint qualifications and investigate their relationships. We show that some of these qualifications ensure (CQ1). Consequently, they validate also the stronger Kuhn–Tucker necessary conditions for the efficiency of problem (P). Let $x \in Q$ be a feasible solution of problem (P).

We begin with the following two constraint qualifications which are the nonsmooth version of the Abadie constraint qualification.

(CQ2) $F^- \bigcap G^- \bigcap T(Q,\bar{x}) \subset \bigcap_{i=1}^m T(S^i,\bar{x})$.

(CQ3) $F^- \bigcap G^- \bigcap T(Q,\bar{x}) \subset T(S^\circ,\bar{x})$.

The next two constraint qualifications can be considered as the nonsmooth types of the Cottle constraint qualification.

(CQ4) $G^s \bigcap T(Q,\bar{x}) \neq \emptyset$.

(CQ5) For all $i \in I$,

$$(F^i)^s \bigcap G^- \bigcap T(Q,\bar{x}) \neq \emptyset, \quad \text{and} \quad G^s \bigcap T(Q,\bar{x}) \neq \emptyset.$$

The following constraint qualification is a generalized version of the constraint qualification which is known as the basic constraint qualification.

For every $i \in I$ and for every $\lambda_j \geq 0, j \in I \setminus \{i\}$, and $\mu_k \geq 0$, $k \in J$, not all zero, we have:

(CQ6) $0 \notin \sum_{j\in I\setminus\{i\}} \lambda_j \mathrm{co}\, \partial^* f_j(\bar{x}) + \sum_{k=1}^m \mu_k \mathrm{co}\, \partial^* g_k(\bar{x}) + N(Q,\bar{x})$.

Now, we present the relationships between the qualifications introduced in this section.

Proposition 1. *The following relations between constraint qualifications hold:*

(1) (CQ3) *implies* (CQ2) ,
(2) (CQ1) *implies* (CQ4) ,
(3) (CQ1) *implies* (CQ2) ,
(4) (CQ1) *holds if and only if* (CQ5) *holds,*
(5) (CQ1) *holds if and only if* (CQ6) *holds.*

Proof. (1) Since

$$T(S^\circ,\bar{x}) \subset \bigcap_{i=1}^m T(S^i,\bar{x}),$$

obviously (CQ5) implies (CQ4).

(2) The proof is trivial.

(3) According to (CQ1) for $i \in I$,

$$(F^i)^s \bigcap G^s \bigcap T(Q,\bar{x}) \neq \emptyset.$$

Now let

$$v \in (F^i)^s \bigcap G^s \bigcap T(Q,\bar{x}).$$

By an easy calculation we have

$$v \in T(S^i,\bar{x})$$

and we conclude that

$$\begin{aligned} F^- \bigcap G^- \bigcap T(Q,\bar{x}) &\subset (F^i)^- \bigcap G^- \bigcap T(Q,\bar{x}) \\ &= \mathrm{cl}\left((F^i)^s\right) \bigcap \mathrm{cl}\, G^s \bigcap T(Q,\bar{x}) \\ &= \mathrm{cl}\left((F^i)^s \bigcap G^s \bigcap T(Q,\bar{x})\right) \\ &\subset \mathrm{cl}\, T(S^i,\bar{x}) \\ &= T(S^i,\bar{x}). \end{aligned}$$

(4) The proof is trivial.

(5) Obviously, (CQ1) implies (CQ6). Now, let (CQ6) hold. Therefore

$$0 \notin \mathrm{co}\left(F^i \cup G\right) + N(Q,\bar{x}). \tag{14}$$

On the other hand, $\mathrm{co}\left(F^i \cup G\right)$ is a compact and convex set and $N(Q,\bar{x})$ is a closed set. Thus the right hand side of (14) is a closed and convex set in $\mathbb{R}^\ell$. By the separation theorem there exists $v \in \mathbb{R}^\ell$ such that

$$\langle \varrho, v\rangle < 0, \quad \forall\, \varrho \in \mathrm{co}\left(F^i \cup G\right) + N(Q,\bar{x}).$$

Thus,

$$\begin{aligned} &\langle \xi_j, v\rangle < 0, \quad \forall \xi_j \in \mathrm{co}\, \partial^* f_j(\bar{x}),\ \forall j \in I \setminus \{i\}, \\ &\langle \zeta_k, v\rangle < 0, \quad \forall \zeta_k \in \mathrm{co}\, \partial^* g_k(\bar{x}),\ \forall k \in J(\bar{x}), \\ &\langle \eta, v\rangle \leq 0, \quad \forall \eta \in N(Q,\bar{x}). \end{aligned}$$

Hence, (CQ1) is satisfied. □

In conclusion of this section we summarize the relations among the presented constraint qualifications in Fig. 1.

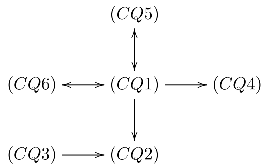

Fig. 1. Relations between various constraint qualifications.

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