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J. Math. Anal. Appl. 281 (2003) 501–515
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A formula for the derivatives of holomorphic functions in $\mathbb{C}^2$ in terms of certain integrals taken on boundaries of analytic varieties

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Received 26 June 2002
Submitted by T. Akahori

Abstract

We derive a Cauchy–Fantappiè type formula which expresses the derivatives of holomorphic functions at a point on a given analytic variety, in terms of the values of the function in an arbitrarily small neighborhood of the curve which bounds the variety. The formula involves derivatives of functions defined by integrals taken on boundaries of nearby analytic varieties. We also apply these formulas to questions related to analytic functionals.

Keywords: Derivatives of holomorphic functions; Analytic varieties; Residue process; Cauchy–Fantappiè formula; Bochner–Martinelli kernel; Analytic functionals

1. Introduction

Let $D \subset \mathbb{C}^2$ be a bounded open set with smooth boundary and $\gamma : (\partial D) \times D \to \mathbb{C}^2$ a smooth map with $\gamma(\zeta, z) = (\gamma_1(\zeta, z), \gamma_2(\zeta, z))$ defined for $(\zeta, z) \in (\partial D) \times D$ so that the quantity

$$\Gamma(\zeta, z) = (\zeta_1 - z_1)\gamma_1(\zeta, z) + (\zeta_2 - z_2)\gamma_2(\zeta, z) \neq 0 \quad \text{for every } (\zeta, z) \in (\partial D) \times D.$$

Then for a function $f \in \mathcal{O}(\bar{D})$ (i.e., holomorphic in a neighborhood of $\bar{D}$),

$$f(z) = \int_{\zeta \in \partial D} f(\zeta) K(\zeta, z) \wedge d\zeta_1 \wedge d\zeta_2 \quad \text{when } z \in D, \tag{1}$$

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doi:10.1016/S0022-247X(03)00134-3

where

$$K(\zeta,z) = -\frac{1}{(2\pi i)^2}\frac{1}{[\Gamma(\zeta,z)]^2}\det\begin{pmatrix}\gamma_1(\zeta,z) & \bar\partial_\zeta\gamma_1(\zeta,z)\\ \gamma_2(\zeta,z) & \bar\partial_\zeta\gamma_2(\zeta,z)\end{pmatrix}.$$

(To compute $\bar\partial_\zeta\gamma_1$ and $\bar\partial_\zeta\gamma_2$, we may use any extension of $\gamma(\zeta,z)$ to a neighborhood of ∂D, as a function of ζ, having restricted z to a compact subset of D.) This is the Cauchy–Fantappiè formula in the domain D, associated to the map γ (see [4]).

Let us also consider a function $\phi(z_1,z_2)$, holomorphic in a neighborhood U of $\bar D$, and let us set $V=\{(z_1,z_2)\in U\colon\ \phi(z_1,z_2)=0\}$ and $M=V\cap D$. We assume that $d\phi\neq 0$ at the points of $\partial M = V\cap(\partial V)$, so that V is smooth near ∂M, and that V meets ∂D transversally, so that ∂M is a smooth curve. In this setting the following formula holds:

$$f(z)=\int_{\zeta\in\partial M} f(\zeta)\Omega(\zeta,z)\beta(\zeta)\quad \text{for } z\in M, \tag{2}$$

where

$$\Omega(\zeta,z)=\frac{1}{2\pi i}\frac{1}{\Gamma(\zeta,z)}\det\begin{pmatrix}\gamma_1(\zeta,z) & \phi_1(\zeta,z)\\ \gamma_2(\zeta,z) & \phi_2(\zeta,z)\end{pmatrix},$$

$$\beta(\zeta_1,\zeta_2)=\frac{\overline{\partial\phi/\partial\zeta_2}\,d\zeta_1-\overline{\partial\phi/\partial\zeta_1}\,d\zeta_2}{|\partial\phi/\partial\zeta_1|^2+|\partial\phi/\partial\zeta_2|^2},$$

and $\phi_1(\zeta,z),\phi_2(\zeta,z)$ are holomorphic functions (with respect to (ζ,z) in a neighborhood of $\bar D\times\bar D$), with the property

$$\phi(\zeta_1,\zeta_2)-\phi(z_1,z_2)=(\zeta_1-z_1)\phi_1(\zeta,z)+(\zeta_2-z_2)\phi_2(\zeta,z). \tag{3}$$

This is Cauchy–Fantappiè type formula in M (see [1,5]).

With the integral formula (2), we express the values of the functions f, at the points of M, in terms of the values of the function on ∂M. Since (2) holds only for $(z_1,z_2)\in M$, we cannot differentiate it (with respect to z_1,z_2), if we want to obtain a similar formula for the derivatives of the functions f. Only certain combinations of derivatives of f can be expressed in such a way. For example, if the point $z\in M$ is a regular point, then the holomorphic vector field

$$\frac{\partial\phi}{\partial z_1}(z)\frac{\partial}{\partial z_2}-\frac{\partial\phi}{\partial z_2}(z)\frac{\partial}{\partial z_1}$$

is tangential to M at z, and therefore

$$\begin{aligned}&\frac{\partial\phi}{\partial z_1}(z)\frac{\partial f}{\partial z_2}(z)-\frac{\partial\phi}{\partial z_2}(z)\frac{\partial f}{\partial z_1}(z)\\ &\quad=\int_{\zeta\in\partial M} f(\zeta)\left(\frac{\partial\phi}{\partial z_1}(z)\frac{\partial\Omega(\zeta,z)}{\partial z_2}-\frac{\partial\phi}{\partial z_2}(z)\frac{\partial\Omega(\zeta,z)}{\partial z_1}\right)\beta(\zeta).\end{aligned} \tag{4}$$

However, we may apply to (1), any derivative of the form $\mathfrak{D}^s=\partial^{s_1+s_2}/\partial z_1^{s_1}\partial z_2^{s_2}$. The result is the formula

$$\mathfrak{D}^s f(z)=\int_{\zeta\in\partial D} f(\zeta)\mathfrak{D}^s K(\zeta,z)\wedge d\zeta_1\wedge d\zeta_2\quad\text{for } z\in D. \tag{5}$$

Now what we will do in this paper, is to show that the integral of (5) (in the case $z \in M$) can be transformed to a certain combination of derivatives, evaluated at $\tau = 0$, of functions defined by line integrals, which are taken on the curves $(\partial D) \cap \{\phi = \tau\}$ (where $\tau \in \mathbb{C}$ with small $|\tau|$). Thus, in particular, we express the derivatives $\mathfrak{D}^s f(z)$ in terms of the values of f in arbitrarily small neighborhoods of the curve ∂M.

The construction is quite explicit and is based on a residue process. Similar residue processes were first used by Stout in [5] and subsequently they were generalized in [1] and [2]. (It is the type of residue process, which leads from formula (1) to (2); see [1].) To make the presentation clear, we will start this reduction of the integrals with the case of the first-order derivatives. But first, let us describe the residue process.

2. The residue process

Recall that for a continuous function $\Psi : \mathbb{C} \to \mathbb{C}$,

$$\lim_{\varepsilon \to 0} \int\limits_{|\tau|=\varepsilon} \Psi(\tau) \frac{d\tau}{\tau} = 2\pi i \Psi(0).$$

This follows from the inequality

$$\left| \int\limits_{|\tau|=\varepsilon} \big(\Psi(\tau) - \Psi(0)\big) \frac{d\tau}{\tau} \right| \leqslant \frac{1}{\varepsilon} \sup_{|\tau|=\varepsilon} \big|\Psi(\tau) - \Psi(0)\big| \cdot 2\pi\varepsilon = 2\pi \sup_{|\tau|=\varepsilon} \big|\Psi(\tau) - \Psi(0)\big|.$$

If the function Ψ is C^1 then integration by parts shows that

$$\begin{aligned}\int\limits_{|\tau|=\varepsilon} \Psi(\tau) \frac{d\tau}{\tau^2} &= -\int\limits_{|\tau|=\varepsilon} \Psi(\tau)\, d\left(\frac{1}{\tau}\right) = \int\limits_{|\tau|=\varepsilon} \frac{1}{\tau}\, d\Psi(\tau) \\ &= \int\limits_{|\tau|=\varepsilon} \frac{1}{\tau}\left(\frac{\partial \Psi}{\partial \tau}\, d\tau + \frac{\partial \Psi}{\partial \bar\tau}\, d\bar\tau\right) = \int\limits_{|\tau|=\varepsilon} \frac{\partial \Psi}{\partial \tau} \frac{d\tau}{\tau} + \int\limits_{|\tau|=\varepsilon} \frac{\partial \Psi}{\partial \bar\tau} \frac{d\bar\tau}{\tau}.\end{aligned}$$

We claim that

$$\lim_{\varepsilon \to 0} \int\limits_{|\tau|=\varepsilon} \frac{\partial \Psi}{\partial \bar\tau} \frac{d\bar\tau}{\tau} = 0.$$

To prove this, notice that on the circle $|\tau| = \varepsilon$, $1/\tau = \bar\tau/\varepsilon^2$, and therefore

$$\int\limits_{|\tau|=\varepsilon} \frac{\partial \Psi}{\partial \bar\tau} \frac{d\bar\tau}{\tau} = \frac{1}{\varepsilon^2} \int\limits_{|\tau|=\varepsilon} \frac{\partial \Psi}{\partial \bar\tau} \bar\tau\, d\bar\tau = \frac{1}{\varepsilon^2} \int\limits_{|\tau|=\varepsilon} \left(\frac{\partial \Psi}{\partial \bar\tau} - \frac{\partial \Psi}{\partial \bar\tau}(0)\right) \bar\tau\, d\bar\tau,$$

where we have also used the fact that $\int_{|\tau|=\varepsilon} \bar\tau\, d\bar\tau = 0$.

Now the claim follows from the inequality

$$\left| \frac{1}{\varepsilon^2} \int\limits_{|\tau|=\varepsilon} \left(\frac{\partial \Psi}{\partial \bar\tau} - \frac{\partial \Psi}{\partial \bar\tau}(0)\right) \bar\tau\, d\bar\tau \right| \leqslant 2\pi \sup_{|\tau|=\varepsilon} \left| \frac{\partial \Psi}{\partial \bar\tau} - \frac{\partial \Psi}{\partial \bar\tau}(0) \right|.$$

Since

$$\int\limits_{|\tau|=\varepsilon} \frac{\partial\Psi}{\partial\tau}\frac{d\tau}{\tau} \to 2\pi i\frac{\partial\Psi}{\partial\tau}(0),$$

it follows that

$$\lim_{\varepsilon\to 0}\int\limits_{|\tau|=\varepsilon} \Psi(\tau)\,\frac{d\tau}{\tau^2} = 2\pi i\frac{\partial\Psi}{\partial\tau}(0).$$

More generally, the following lemma holds.

Lemma 1. *For a C^m function $\Psi:\mathbb{C}\to\mathbb{C}$,*

$$\lim_{\varepsilon\to 0}\int\limits_{|\tau|=\varepsilon} \Psi(\tau)\,\frac{d\tau}{\tau^{m+1}} = \frac{2\pi i}{m!}\frac{\partial^m\Psi}{\partial\tau^m}(0).$$

Proof. By Taylor's theorem,

$$\begin{aligned}\Psi(\tau) = {}&\Psi(0) + \frac{\partial\Psi}{\partial\tau}(0)\tau + \frac{\partial\Psi}{\partial\bar\tau}(0)\bar\tau\\ &+ \frac{1}{2}\left(\frac{\partial^2\Psi}{\partial\tau^2}(0)\tau^2 + 2\frac{\partial^2\Psi}{\partial\tau\partial\bar\tau}(0)\tau\bar\tau + \frac{\partial^2\Psi}{\partial\bar\tau^2}(0)\bar\tau^2\right) + \cdots\\ &+ \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{\partial^m\Psi}{\partial\tau^k\partial\bar\tau^{m-k}}(0)\tau^k\bar\tau^{m-k} + \frac{1}{(m-1)!}\sum_{k=0}^{m}\binom{m}{k}\\ &\times\left(\int\limits_{s=0}^{1}\left[\frac{\partial^m\Psi}{\partial\tau^k\partial\bar\tau^{m-k}}(s\tau) - \frac{\partial^m\Psi}{\partial\tau^k\partial\bar\tau^{m-k}}(0)\right](1-s)^m\,ds\right)\tau^k\bar\tau^{m-k}.\end{aligned}$$

When we substitute this expansion of $\Psi(\tau)$ in the integral $\int_{|\tau|=\varepsilon}(\Psi(\tau)/\tau^{m+1})\,d\tau$, we run into integrals of the form

$$\int\limits_{|\tau|=\varepsilon} \tau^k\bar\tau^{l-k}\,\frac{d\tau}{\tau^{m+1}} \quad (0\leqslant l\leqslant m,\ 0\leqslant k\leqslant l).$$

Since on the circle $|\tau|=\varepsilon$, $\bar\tau=\varepsilon^2/\tau$, these integrals become

$$\int\limits_{|\tau|=\varepsilon} \tau^k\bar\tau^{l-k}\,\frac{d\tau}{\tau^{m+1}} = \varepsilon^{2(l-k)}\int\limits_{|\tau|=\varepsilon} \frac{d\tau}{\tau^{m+l-2k+1}}.$$

It follows that the only case that the limit

$$\lim_{\varepsilon\to 0}\int\limits_{|\tau|=\varepsilon} \tau^k\bar\tau^{l-k}\,\frac{d\tau}{\tau^{m+1}}$$

can be $\neq 0$ is when $k=l=m$, in which case each integral, and therefore this limit, is $2\pi i$.

Since the coefficient of $\tau^k\bar{\tau}^{l-k}$ in Taylor's expansion, in the case $k=l=m$, is

$$\frac{1}{m!}\frac{\partial^m\Psi}{\partial\tau^m}(0),$$

the formula of the lemma will follow if we show that

$$\int\limits_{|\tau|=\varepsilon}\left(\int\limits_{s=0}^{1}\left[\frac{\partial^m\Psi}{\partial\tau^k\partial\bar{\tau}^{m-k}}(s\tau)-\frac{\partial^m\Psi}{\partial\tau^k\partial\bar{\tau}^{m-k}}(0)\right](1-s)^m\,ds\right)\tau^k\bar{\tau}^{m-k}\,\frac{d\tau}{\tau^{m+1}}\to 0$$

as $\varepsilon\to 0$.

But this follows from the fact that the above integral, in absolute value, is

$$\leqslant\frac{2\pi}{m}\sup_{|\tau|\leqslant\varepsilon}\left|\frac{\partial^m\Psi}{\partial\tau^k\bar{\tau}^{m-k}}(\tau)-\frac{\partial^m\Psi}{\partial\tau^k\partial\bar{\tau}^{m-k}}(0)\right|.$$

This completes the proof of the lemma. □

Now we can prove the following theorem.

Theorem 1. *Suppose that the function $\Theta(\zeta)$ and the differential form $\Xi(\zeta)$, defined and being smooth for ζ in a neighborhood of ∂D, satisfy the equation*

$$\Xi(\zeta)=\frac{1}{(\phi(\zeta))^{m+1}}\bar{\partial}_\zeta\big[\Theta(\zeta)\big]\quad \textit{when } \phi(\zeta)\neq 0.$$

Then, for $f\in\mathcal{O}(\bar{D})$,

$$\int\limits_{\zeta\in\partial D} f(\zeta)\Xi(\zeta)\wedge d\zeta_1\wedge d\zeta_2=\frac{2\pi i}{m!}\frac{\partial^m}{\partial\tau^m}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Theta(\zeta)\beta(\zeta)\right).$$

More generally, if the functions $\Theta_j(\zeta)$, $j=0,\dots,m$, and the differential form $\Xi(\zeta)$ are defined and smooth for ζ in a neighborhood of ∂D, and satisfy the equation

$$\Xi(\zeta)=\sum_{j=0}^{m}\frac{1}{(\phi(\zeta))^{j+1}}\bar{\partial}_\zeta\big[\Theta_j(\zeta)\big]\quad \textit{when } \phi(\zeta)\neq 0,$$

then

$$\int\limits_{\zeta\in\partial D} f(\zeta)\Xi(\zeta)\wedge d\zeta_1\wedge d\zeta_2=2\pi i\sum_{j=0}^{m}\frac{1}{j!}\frac{\partial^j}{\partial\tau^j}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Theta_j(\zeta)\beta(\zeta)\right).$$

Proof. It is easy to check that

$$d\zeta_1\wedge d\zeta_2=\beta(\zeta_1,\zeta_2)\wedge\left(\frac{\partial\phi}{\partial\zeta_1}d\zeta_1+\frac{\partial\phi}{\partial\zeta_2}d\zeta_2\right). \tag{6}$$

In carrying out the residue process, we will be working on the surface ∂D. Deforming slightly this surface (away from the curve ∂M), if necessary, we may assume that $d\phi\neq 0$

at the points of ∂D. Then (6) will hold in a neighborhood of ∂D. (Notice that the value of the integral $\int_{\zeta\in\partial D} f(\zeta)\Xi(\zeta)\wedge d\zeta_1\wedge d\zeta_2$ remains unchanged under small deformations of ∂D, since the differential form $f(\zeta)\Xi(\zeta)\wedge d\zeta_1\wedge d\zeta_2$ is d-closed.) Then

$$\Xi(\zeta)\wedge d\zeta_1\wedge d\zeta_2=\bar\partial_\zeta\left[\frac{1}{(\phi(\zeta))^{m+1}}\Theta(\zeta)\beta(\zeta)\wedge d\phi(\zeta)\right],$$

and therefore

$$f(\zeta)\Xi(\zeta)\wedge d\zeta_1\wedge d\zeta_2=d_\zeta\left[f(\zeta)\Theta(\zeta)\beta(\zeta)\wedge\frac{d\phi(\zeta)}{(\phi(\zeta))^{m+1}}\right]\quad\text{in } W-V, \tag{7}$$

where W is a neighborhood of ∂D. (We also used our assumption that $f\in\mathcal{O}(\bar D)$.) Using (7), we can carry out the residue process:

$$\begin{aligned}\int\limits_{\zeta\in\partial D} f(\zeta)\Xi(\zeta)\wedge d\zeta_1\wedge d\zeta_2&=\lim_{\varepsilon\to 0}\int\limits_{\zeta\in(\partial D)\cap\{|\phi|>\varepsilon\}} f(\zeta)\Xi(\zeta)\wedge d\zeta_1\wedge d\zeta_2\\&=\lim_{\varepsilon\to 0}\int\limits_{\zeta\in(\partial D)\cap\{|\phi|>\varepsilon\}} d_\zeta\left[f(\zeta)\Theta(\zeta)\beta(\zeta)\wedge\frac{d\phi(\zeta)}{(\phi(\zeta))^{m+1}}\right]\\&=\lim_{\varepsilon\to 0}\int\limits_{\zeta\in(\partial D)\cap\{|\phi|=\varepsilon\}} f(\zeta)\Theta(\zeta)\beta(\zeta)\wedge\frac{d\phi(\zeta)}{(\phi(\zeta))^{m+1}}\\&=\lim_{\varepsilon\to 0}\int\limits_{|\tau|=\varepsilon}\left(\int\limits_{\zeta\in(\partial D)\cap\{|\phi|=\tau\}} f(\zeta)\Theta(\zeta)\beta(\zeta)\right)\frac{d\tau}{\tau^{m+1}}\\&=\frac{2\pi i}{m!}\frac{\partial^m}{\partial\tau^m}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{|\phi|=\tau\}} f(\zeta)\Theta(\zeta)\beta(\zeta)\right).\end{aligned}$$

In the above computation, we used Stokes's theorem, Fubini's theorem, and Lemma 1, applied to the function $\Psi(\tau)=\int_{(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Theta(\zeta)\beta(\zeta)$.

The proof of the general statement of the theorem is proved in the same exactly way. (Notice that this general statement does not follows directly from the first statement of the theorem, but it is proved similarly. The point here is that the differential forms $\bar\partial\Theta_j/\phi^{j+1}$ are not assumed to be defined on the whole ∂D—only their sum $\sum\bar\partial\Theta_j/\phi^{j+1}$ is.) $\square$

3. The formula for the first-order derivatives

The starting point is the identity

$$\begin{aligned}&\frac{1}{[\Gamma(\zeta,z)^2]}\det\begin{pmatrix}\gamma_1(\zeta,z) & \bar\partial_\zeta\gamma_1(\zeta,z)\\ \gamma_2(\zeta,z) & \bar\partial_\zeta\gamma_2(\zeta,z)\end{pmatrix}\\&\quad=-\frac{1}{\phi(\zeta_1,\zeta_2)-\phi(z_1,z_2)}\bar\partial_\zeta\left[\frac{1}{\Gamma(\zeta,z)}\det\begin{pmatrix}\gamma_1(\zeta,z) & \phi_1(\zeta,z)\\ \gamma_2(\zeta,z) & \phi_2(\zeta,z)\end{pmatrix}\right],\end{aligned}$$

which holds at points (ζ_1, ζ_2) and (z_1, z_2) wherever $\phi(\zeta_1, \zeta_2) - \phi(z_1, z_2) \neq 0$. The proof of this identity is a straightforward computation, which depends on (3) (see [1,2] for similar computations of more general type).

Thus

$$2\pi i K(\zeta, z) = \frac{1}{\phi(\zeta) - \phi(z)} \bar{\partial}_\zeta \big[\Omega(\zeta, z)\big]. \tag{8}$$

Differentiating (8) with respect to z_1, we obtain

$$2\pi i \frac{\partial K(\zeta, z)}{\partial z_1} = \frac{1}{\phi(\zeta) - \phi(z)} \bar{\partial}_\zeta \left[\frac{\partial \Omega(\zeta, z)}{\partial z_1}\right] + \frac{\partial\phi/\partial z_1}{(\phi(\zeta) - \phi(z))^2} \bar{\partial}_\zeta \big[\Omega(\zeta, z)\big]. \tag{9}$$

Therefore, if $\phi(\zeta) \neq 0$,

$$2\pi i \frac{\partial K(\zeta, z)}{\partial z_1} = \frac{1}{\phi(\zeta)} \bar{\partial}_\zeta \left[\frac{\partial \Omega(\zeta, z)}{\partial z_1}\right] + \frac{1}{(\phi(\zeta))^2} \bar{\partial}_\zeta \left[\frac{\partial\phi}{\partial z_1} \Omega(\zeta, z)\right] \quad \text{for } z \in M. \tag{10}$$

On the other hand, (5) with $s = (1, 0)$ gives

$$\frac{\partial f}{\partial z_1}(z) = \int_{\zeta \in \partial D} f(\zeta) \frac{\partial K(\zeta, z)}{\partial z_1} \wedge d\zeta_1 \wedge d\zeta_2.$$

Therefore, by (10) and Theorem 1, we obtain the following theorem.

Theorem 2. *For $f \in \mathcal{O}(\bar{D})$ and $z \in M$,*

$$\frac{\partial f}{\partial z_1}(z) = \int_{\zeta \in \partial M} f(\zeta) \frac{\partial \Omega(\zeta, z)}{\partial z_1} \beta(\zeta) + \frac{\partial\phi}{\partial z_1} \frac{\partial}{\partial \tau}\bigg|_{\tau=0} \left(\int_{\zeta \in (\partial D) \cap \{\phi = \tau\}} f(\zeta) \Omega(\zeta, z) \beta(\zeta) \right),$$

and a similar formula holds for the derivative $\partial f/\partial z_2$.

Remarks. (1) If we compute the quantity

$$\frac{\partial\phi}{\partial z_1}(z) \frac{\partial f}{\partial z_2}(z) - \frac{\partial\phi}{\partial z_2}(z) \frac{\partial f}{\partial z_1}(z),$$

by substituting the values of the derivatives $\partial f/\partial z_1$ and $\partial f/\partial z_2$, as these are given by Theorem 2, we obtain (4), since the quantities which contain the derivatives $(\partial/\partial\tau)_{\tau=0}$, cancel each other. In this sense, Theorem 2 is an extension of (4).

(2) Notice that if $\partial\phi/\partial z_1$ is zero at the point $z \in M$ then the term which contains the $(\partial/\partial\tau)_{\tau=0}$-derivative is not present in the formula. Thus at a point $z \in M$, where $(d\phi)(z) = 0$,

$$\frac{\partial f}{\partial z_j}(z) = \int_{\zeta \in \partial M} f(\zeta) \frac{\partial \Omega(\zeta, z)}{\partial z_j} \beta(\zeta) \quad \text{for } j = 1, 2.$$

(3) We arrived at the formula of Theorem 2, trying to compute the values of the derivatives $\partial f/\partial z_1$ and $\partial f/\partial z_2$. Of course, one way turn this around and view the formula of Theorem 2 as a method to compute the $(\partial/\partial\tau)_{\tau=0}$-derivative of the function

$\int_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Omega(\zeta,z)\beta(\zeta)$. Notice that this is possible if $(d\phi)(z)\neq 0$. For example, if at the point $z\in M$, $(\partial\phi/\partial z_2)(z)\neq 0$, then

$$\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Omega(\zeta,z)\beta(\zeta)\right)$$
$$=\frac{1}{(\partial\phi/\partial z_2)(z)}\left[\frac{\partial f}{\partial z_2}(z)-\int\limits_{\zeta\in\partial M} f(\zeta)\frac{\partial\Omega(\zeta,z)}{\partial z_2}\beta(\zeta)\right].$$

Here is a corollary of this formula: If U is a neighborhood of $\bar{D}$ and the sequence of functions $f_n\in\mathcal{O}(U)$ converges uniformly on U to a function f as $n\to\infty$, then

$$\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f_n(\zeta)\Omega(\zeta,z)\beta(\zeta)\right)$$
$$\to\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Omega(\zeta,z)\beta(\zeta)\right).$$

In particular, the map $\mathcal{O}(\mathbb{C}^2)\to\mathbb{C}$, defined by

$$f\to\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Omega(\zeta,z)\beta(\zeta)\right)\quad\text{for } f\in\mathcal{O}(\mathbb{C}^2),$$

is an analytic functional, carried by the compact set $\bar{M}$. (We proved this in the case $z\in M$ and $(d\phi)(z)\neq 0$, but something more general holds, as we will see in Section 5.)

(4) Applying formula (2) with the holomorphic function $\partial f/\partial\zeta_1$ in place of f, we obtain

$$\frac{\partial f}{\partial z_1}(z)=\int\limits_{\zeta\in\partial M}\frac{\partial f}{\partial\zeta_1}(\zeta)\Omega(\zeta,z)\beta(\zeta).$$

Therefore, the formula of Theorem 2 is written in the following way:

$$\int\limits_{\zeta\in\partial M}\frac{\partial f}{\partial\zeta_1}(\zeta)\Omega(\zeta,z)\beta(\zeta)$$
$$=\int\limits_{\zeta\in\partial M} f(\zeta)\frac{\partial\Omega(\zeta,z)}{\partial z_1}\beta(\zeta)+\frac{\partial\phi}{\partial z_1}\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Omega(\zeta,z)\beta(\zeta)\right),$$

which may be viewed as an "integration by parts" formula.

It also follows from this formula that the analytic functional, which we discussed in the previous remark, is carried by the curve ∂M.

4. The formula for the higher-order derivatives

For a fixed $s = (s_1, s_2)$, we apply the operator $\mathfrak{D}^s = \partial^{s_1+s_2}/\partial z_1^{s_1}\partial z_2^{s_2}$ to both sides of (8) and we obtain

$$2\pi i\,\mathfrak{D}^s K(\zeta,z) = \mathfrak{D}^s\left(\frac{1}{\phi(\zeta)-\phi(z)}\bar{\partial}_\zeta\big[\Omega(\zeta,z)\big]\right)$$
$$= \sum_{k\leqslant s}\binom{s}{k}\mathfrak{D}^k\left(\frac{1}{\phi(\zeta)-\phi(z)}\right)\bar{\partial}_\zeta\big[\mathfrak{D}^{s-k}\Omega(\zeta,z)\big],$$

where the sum is extended over the $k = (k_1, k_2) \leqslant s$, i.e., $0 \leqslant k_1 \leqslant s_1$ and $0 \leqslant k_2 \leqslant s_2$,

$$\binom{s}{k} = \binom{s_1}{k_1}\binom{s_2}{k_2} \quad \text{and} \quad s-k = (s_1-k_1, s_2-k_2).$$

Also

$$\mathfrak{D}^k\left(\frac{1}{\phi(\zeta)-\phi(z)}\right) = \sum_{j=0}^{|k|}\frac{A_j^{k,\phi}(z)}{(\phi(\zeta)-\phi(z))^{j+1}},$$

where each $A_j^{k,\phi}(z)$ is an easily computed and quite explicit combination of derivatives of the function ϕ, of order $\leqslant |k| = k_1 + k_2$, evaluated at z.

Thus

$$2\pi i\,\mathfrak{D}^s K(\zeta,z) = \sum_{k\leqslant s}\binom{s}{k}\sum_{j=0}^{|k|}\frac{A_j^{k,\phi}(z)}{(\phi(\zeta))^{j+1}}\bar{\partial}_\zeta\big[\mathfrak{D}^{s-k}\Omega(\zeta,z)\big] \quad \text{for } z \in M. \tag{11}$$

Combining (5), (11), and Theorem 1, we obtain the formula of the following theorem.

Theorem 3. *For $f \in \mathcal{O}(\bar{D})$ and $z \in M$,*

$$\mathfrak{D}^s f(z) = \sum_{k\leqslant s}\binom{s}{k}\sum_{j=0}^{|k|}\frac{A_j^{k,\phi}(z)}{j!}\frac{\partial^j}{\partial\tau^j}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\mathfrak{D}^{s-k}\Omega(\zeta,z)\beta(\zeta)\right).$$

Examples. (1) Keeping in mind that $K = K(\zeta, z)$ and $\Omega = \Omega(\zeta, z)$, we have

$$2\pi i\frac{\partial^2 K}{\partial z_1^2} = \frac{1}{\phi(\zeta)-\phi(z)}\bar{\partial}_\zeta\left[\frac{\partial^2\Omega}{\partial z_1^2}\right]$$
$$+\frac{1}{(\phi(\zeta)-\phi(z))^2}\bar{\partial}_\zeta\left[2\frac{\partial\phi}{\partial z_1}\frac{\partial\Omega}{\partial z_1}+\frac{\partial^2\phi}{\partial z_1^2}\Omega\right]+2\frac{(\partial\phi/\partial z_1)^2}{(\phi(\zeta)-\phi(z))^3}\bar{\partial}_\zeta[\Omega].$$

Therefore, for $f \in \mathcal{O}(\bar{D})$ and $z \in M$,

$$\frac{\partial^2 f}{\partial z_1^2}(z) = \int\limits_{\zeta\in\partial M} f(\zeta)\frac{\partial^2\Omega}{\partial z_1^2}\beta(\zeta)$$

$$+\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\left[2\frac{\partial\phi}{\partial z_1}\frac{\partial\Omega}{\partial z_1}+\frac{\partial^2\phi}{\partial z_1^2}\Omega\right]\beta(\zeta)\right)$$
$$+\frac{\partial^2}{\partial\tau^2}\bigg|_{\tau=0}\left(\left(\frac{\partial\phi}{\partial z_1}\right)^2\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Omega\beta(\zeta)\right).$$

In particular, if at a point $z\in M$, $\partial\phi/\partial z_1=0$ and $\partial^2\phi/\partial z_1^2=0$, then

$$\frac{\partial^2 f}{\partial z_1^2}(z)=\int\limits_{\zeta\in\partial M} f(\zeta)\frac{\partial^2\Omega}{\partial z_1^2}(\zeta,z)\beta(\zeta),$$

and also

$$\frac{\partial f}{\partial z_1}(z)=\int\limits_{\zeta\in\partial M} f(\zeta)\frac{\partial\Omega}{\partial z_1}(\zeta,z)\beta(\zeta).$$

For example, these formulas hold if $\phi(z_1,z_2)=z_1^3-z_2^5+z_2^3+z_2$ and $z=(0,0)$.

Here is another case. If $\phi(z_1,z_2)=z_1^2-z_2^5+z_2^3+z_2$ then

$$2\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Omega(\zeta,0)\beta(\zeta)\right)=\frac{\partial^2 f}{\partial z_1^2}(0)-\int\limits_{\zeta\in\partial M} f(\zeta)\frac{\partial^2\Omega}{\partial z_1^2}(\zeta,0)\beta(\zeta).$$

(2) Similarly,

$$\begin{aligned}\frac{\partial^3 f}{\partial z_1^2\partial z_2}(z)={}&\int\limits_{\zeta\in\partial M} f(\zeta)\frac{\partial^3\Omega}{\partial z_1^2\partial z_2}\beta(\zeta)\\
&+\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\left[\frac{\partial\phi}{\partial z_2}\frac{\partial^2\Omega}{\partial z_1^2}\right.\right.\\
&\qquad\left.\left.+\frac{\partial}{\partial z_2}\left(2\frac{\partial\phi}{\partial z_1}\frac{\partial\Omega}{\partial z_1}+\frac{\partial^2\phi}{\partial z_1^2}\Omega\right)\right]\beta(\zeta)\right)\\
&+\frac{\partial^2}{\partial\tau^2}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\left[\frac{\partial\phi}{\partial z_2}\left(2\frac{\partial\phi}{\partial z_1}\frac{\partial\Omega}{\partial z_1}+\frac{\partial^2\phi}{\partial z_1^2}\Omega\right)\right.\right.\\
&\qquad\left.\left.+\left(\frac{\partial\phi}{\partial z_1}\right)^2\frac{\partial\Omega}{\partial z_2}+2\frac{\partial\phi}{\partial z_1}\frac{\partial^2\phi}{\partial z_1\partial z_2}\Omega\right]\beta(\zeta)\right)\\
&+\frac{\partial^3}{\partial\tau^3}\bigg|_{\tau=0}\left(\left(\frac{\partial\phi}{\partial z_1}\right)^2\frac{\partial\phi}{\partial z_2}\int\limits_{\zeta\in(\partial D)\cap\{\phi=\tau\}} f(\zeta)\Omega\beta(\zeta)\right).\end{aligned}$$

(3) Let us examine the formulas of Theorem 3, when

$$D = \{(z_1, z_2) \in \mathbb{C}^2\colon |z_1|^2 + |z_2|^2 < R^2\},$$
$$\gamma(\zeta, z) = (\bar\zeta_1 - \bar z_1, \bar\zeta_2 - \bar z_2), \quad \text{and} \quad \phi(z_1, z_2) = z_2.$$

Then $\beta(\zeta_1, \zeta_2) = d\zeta_1$ and, choosing $\phi_1 = 0$ and $\phi_2 = 1$,

$$\Omega(\zeta_1, \zeta_2, z_1, z_2) = \frac{1}{2\pi i}\frac{\bar\zeta_1 - \bar z_1}{|\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2}.$$

Also (8) becomes

$$2\pi i K(\zeta, z) = \frac{1}{\zeta_2 - z_2}\bar\partial_\zeta\big[\Omega(\zeta, z)\big]$$

and gives the following equation:

$$2\pi i \frac{\partial^{s_1+s_2} K}{\partial z_1^{s_1}\partial z_2^{s_2}}\bigg|_{(z_1,z_2)=(0,0)} = \sum_{m=0}^{s_2}\binom{s_2}{m}\frac{m!}{\zeta_2^{m+1}}\bar\partial_\zeta\left[\frac{\partial^{s_1+s_2-m}\Omega}{\partial z_1^{s_1}\partial z_2^{s_2-m}}\bigg|_{(z_1,z_2)=(0,0)}\right].$$

This leads to the formula

$$\mathfrak{D}^s f(0) = \sum_{m=0}^{s_2}\binom{s_2}{m}\frac{\partial^m}{\partial\tau^m}\bigg|_{\tau=0}\left(\int\limits_{|\zeta_1|^2=R^2-|\tau|^2} f(\zeta_1, \tau)\mathfrak{D}^{(s_1,s_2-m)}\Omega(\zeta_1, \tau, 0, 0)\, d\zeta_1\right), \tag{12}$$

let us say, for $f \in \mathcal{O}(\mathbb{C}^2)$.

However, (12) can be simplified, as a computation shows. Indeed,

$$\frac{\partial^m}{\partial\tau^m}\bigg|_{\tau=0}\left(\int\limits_{|\zeta_1|^2=R^2-|\tau|^2} f(\zeta_1, \tau)\mathfrak{D}^{(s_1,s_2-m)}\Omega(\zeta_1, \tau, 0, 0)\, d\zeta_1\right) = 0, \tag{13}$$

unless $m = s_2$, and

$$\begin{aligned}&\frac{\partial^{s_2}}{\partial\tau^{s_2}}\bigg|_{\tau=0}\left(\int\limits_{|\zeta_1|^2=R^2-|\tau|^2} f(\zeta_1, \tau)\mathfrak{D}^{(s_1,0)}\Omega(\zeta_1, \tau, 0, 0)\, d\zeta_1\right)\\ &\quad= \int\limits_{|\zeta_1|=R}\frac{\partial^{s_2} f}{\partial\zeta_2^{s_2}}(\zeta_1, 0)\mathfrak{D}^{(s_1,0)}\Omega(\zeta_1, 0, 0, 0)\, d\zeta_1.\end{aligned} \tag{14}$$

These can be proved in the following way. First we compute

$$\mathfrak{D}^{(s_1,s_2-m)}\Omega(\zeta_1, \tau, 0, 0) = \frac{(s_1+s_2-m)!}{2\pi i}\frac{\bar\zeta_1^{s_1+1}\bar\tau^{s_2-m}}{(|\zeta_1|^2+|\tau|^2)^{s_1+s_2-m+1}}.$$

Therefore

$$\mathfrak{D}^{(s_1,s_2-m)}\Omega(\zeta_1, \tau, 0, 0) = \frac{(s_1+s_2-m)!}{2\pi i R^{2(s_1+s_2-m+1)}}\bar\zeta_1^{s_1+1}\bar\tau^{s_2-m} \quad \text{when } |\zeta_1|^2 = R^2 - |\tau|^2.$$

Substituting this in the line integrals and transforming these integrals by setting $\zeta_1 = (R^2 - |\tau|^2)^{1/2} e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, we can easily see why formulas (13) and (14) hold.

Hence (12) becomes

$$\begin{aligned} \frac{\partial^{s_1+s_2} f}{\partial z_1^{s_1} \partial z_2^{s_2}}(0) &= \int\limits_{|\zeta_1|=R} \frac{\partial^{s_2} f}{\partial \zeta_2^{s_2}}(\zeta_1, 0)\mathfrak{D}^{(s_1,0)}\Omega(\zeta_1, 0, 0, 0)\, d\zeta_1 \\ &= \frac{s_1!}{2\pi i} \int\limits_{|\zeta_1|=R} \frac{\partial^{s_2} f}{\partial \zeta_2^{s_2}}(\zeta_1, 0)\, \frac{d\zeta_1}{\zeta_1^{s_1+1}}, \end{aligned} \tag{15}$$

which of course can be checked independently (using only the one dimensional Cauchy formula). Thus the formula of Theorem 3 is a generalization of (15), from the case $\phi(z_1, z_2) = z_2$ to more general ϕ's.

5. Applications to analytic functionals

According to the following lemma, derivatives of integrals of the kind, which occur in the sum of the formula of Theorem 3, define analytic functionals. More precisely, in the setting of Theorem 3, we can prove

Lemma 2. *Let $X(\zeta)$ be any smooth function defined for $\zeta = (\zeta_1, \zeta_2)$ in a neighborhood of the curve $(\partial D) \cap \{\phi = 0\}$. Then the map $\mathcal{O}(\mathbb{C}^2) \to \mathbb{C}$, which assigns to each $f \in \mathcal{O}(\mathbb{C}^2)$ the quantity*

$$\frac{\partial^m}{\partial \tau^m}\bigg|_{\tau=0} \left(\int\limits_{\zeta \in (\partial D) \cap \{\phi = \tau\}} f(\zeta) X(\zeta) \beta(\zeta) \right), \tag{16}$$

is an analytic functional.

Proof. In the case

$$D = \{(z_1, z_2) \in \mathbb{C}^2\colon\ |z_1|^2 + |z_2|^2 < R^2\} \quad \text{and} \quad \phi(z_1, z_2) = z_2,$$

quantity (16) is equal to

$$\frac{\partial^m}{\partial \tau^m}\bigg|_{\tau=0} \left(\int\limits_{|\zeta_1|^2 = R^2 - |\tau|^2} f(\zeta_1, \tau) X(\zeta_1, \tau)\, d\zeta_1 \right).$$

Then, setting $\zeta_1 = (R^2 - |\tau|^2)^{1/2} e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, the line integral becomes

$$\begin{aligned} &\int\limits_{|\zeta_1|^2 = R^2 - |\tau|^2} f(\zeta_1, \tau) X(\zeta_1, \tau)\, d\zeta_1 \\ &\quad = \int\limits_{\theta=0}^{2\pi} f\big((R^2 - |\tau|^2)^{1/2} e^{i\theta}, \tau\big) X\big((R^2 - |\tau|^2)^{1/2} e^{i\theta}, \tau\big) \big(R^2 - |\tau|^2\big)^{1/2} e^{i\theta} i\, d\theta. \end{aligned}$$

Using this equation, it is easy to see that if the sequence $f_n \in \mathcal{O}(\mathbb{C}^2)$ converges to 0 uniformly on compact sets, then

$$\frac{\partial^m}{\partial \tau^m}\bigg|_{\tau=0}\left(\int\limits_{|\zeta_1|^2=R^2-|\tau|^2} f_n(\zeta_1,\tau)X(\zeta_1,\tau)\,d\zeta_1\right) \to 0 \quad \text{as } n\to\infty.$$

The proof in the general case is similar. All we have to do is to use a partition of unity in order to write the integral as a finite sum of integrals over small pieces of the curve $(\partial D)\cap\{\phi=\tau\}$ and to use appropriate local parametrizations of these pieces. $\square$

Now assuming that $\phi \in \mathcal{O}(\mathbb{C}^2)$ with $0 \in \{\phi = 0\}$ and using $\gamma(\zeta, z) = \bar{\zeta} - \bar{z}$, define $\mathcal{T}^R_{s_1,s_2} : \mathcal{O}(\mathbb{C}^2) \to \mathbb{C}$ by setting

$$\mathcal{T}^R_{s_1,s_2}(f) = \sum_{k\leqslant s}\binom{s}{k}\sum_{j=0}^{|k|}\frac{A_j^{k,\phi}(0)}{j!}\frac{\partial^j}{\partial\tau^j}\bigg|_{\tau=0}\left(\int\limits_{\zeta\in S_R\cap\{\phi=\tau\}} f(\zeta)\mathfrak{D}^{s-k}\Omega(\zeta,0)\beta(\zeta)\right)$$

for $f \in \mathcal{O}(\mathbb{C}^2)$, where s_1, s_2 are nonnegative integers and $R > 0$, so that $\{\phi = 0\}$ meets the sphere $S_R = \{|\zeta| = R\}$ transversally. Then each term in the sum, and therefore $\mathcal{T}^R_{s_1,s_2}$, is an analytic functional. Conversely, we will show that every analytic functional can be expanded in terms of these $\mathcal{T}^R_{s_1,s_2}$, provided that R is sufficiently large. More precisely, we will prove the following theorem.

Theorem 4. *Every analytic functional $\mathcal{T} : \mathcal{O}(\mathbb{C}^2) \to \mathbb{C}$ has an expansion of the form*

$$\mathcal{T} = \sum_{s_1,s_2\geqslant 0} c_{s_1,s_2}\mathcal{T}^R_{s_1,s_2}$$

for sufficiently large R. Furthermore, the coefficients are given by the formula

$$c_{s_1,s_2} = \frac{1}{s_1!s_2!}\frac{\partial^{s_1+s_2}F}{\partial w_1^{s_1}\partial w_2^{s_2}}\bigg|_{(w_1,w_2)=(0,0)},$$

where $F(w_1, w_2)$ is the Fourier–Laplace transform of $\mathcal{T}$.

Proof. It suffices to show that

$$\mathcal{T}(f) = \sum_{s_1,s_2\geqslant 0} c_{s_1,s_2}\mathcal{T}^R_{s_1,s_2}(f) \quad \text{for } f \in \mathcal{O}(\mathbb{C}^2). \tag{17}$$

Recall that

$$F(w_1, w_2) = \mathcal{T}_{\zeta_1,\zeta_2}\left[e^{\zeta_1 w_1+\zeta_2 w_2}\right],$$

the Fourier–Laplace transform of $\mathcal{T}$, is an entire function of exponential type, i.e.,

$$\left|F(w_1, w_2)\right| \leqslant Ae^{B(|w_1|+|w_2|)} \quad \text{for every } w_1, w_2, \tag{18}$$

for some positive constants A and B.

To deal with the convergence of the series in (17) (which is part of the conclusion), we will need an estimate for the coefficients c_{s_1,s_2}. It follows from (18) and Cauchy's inequalities (as in [3, p. 109]) that

$$|c_{s_1,s_2}| \leqslant A\frac{(\sqrt{2}eB)^{s_1+s_2}}{s_1^{s_1}s_2^{s_2}} \quad \text{for every } s_1, s_2,$$

and therefore

$$\sum_{s_1,s_2\geqslant 0}(s_1+s_2+1)!|c_{s_1,s_2}|r_1^{s_1}r_2^{s_2} < \infty, \quad \text{provided that } 0 < r_1, r_2 < \frac{1}{3\sqrt{2}eB}. \tag{19}$$

Applying (5) to the function $e^{\zeta_1 w_1+\zeta_2 w_2}$ of (ζ_1,ζ_2) with $\gamma(\zeta,z) = (\bar{\zeta}_1-\bar{z}_1, \bar{\zeta}_2-\bar{z}_2)$, we obtain

$$w_1^{s_1}w_2^{s_2} = \int\limits_{\zeta\in S_R} e^{\zeta_1 w_1+\zeta_2 w_2}\xi_{s_1,s_2}(\zeta_1,\zeta_2)\wedge d\zeta_1\wedge d\zeta_2 \quad \text{for } (w_1,w_2)\in\mathbb{C}^2, \tag{20}$$

where

$$\begin{aligned}
\xi_{s_1,s_2}(\zeta_1,\zeta_2) &= \frac{1}{4\pi^2}\frac{\partial^{s_1+s_2}}{\partial z_1^{s_1}\partial z_2^{s_2}}\left[\frac{(\bar{\zeta}_1-\bar{z}_1)\,d\bar{\zeta}_2-(\bar{\zeta}_2-\bar{z}_2)\,d\bar{\zeta}_1}{(|\zeta_1-z_1|^2+|\zeta_2-z_2|^2)^2}\right]\Bigg|_{(z_1,z_2)=(0,0)}\\
&= \frac{(s_1+s_2+1)!}{4\pi^2}\frac{\bar{\zeta}_1^{s_1}\bar{\zeta}_2^{s_2}}{(|\zeta_1|^2+|\zeta_2|^2)^{s_1+s_2+2}}(\bar{\zeta}_1\,d\bar{\zeta}_2-\bar{\zeta}_2\,d\bar{\zeta}_1)\\
&= \frac{(s_1+s_2+1)!}{4\pi^2}\frac{1}{(|\zeta_1|^2+|\zeta|^2)^2}\\
&\quad\times\left(\frac{\bar{\zeta}_1}{|\zeta_1|^2+|\zeta_2|^2}\right)^{s_1}\left(\frac{\bar{\zeta}_2}{|\zeta_1|^2+|\zeta_2|^2}\right)^{s_2}(\bar{\zeta}_1\,d\bar{\zeta}_2-\bar{\zeta}_2\,d\bar{\zeta}_1).
\end{aligned}\tag{21}$$

On the other hand,

$$F(w_1,w_2) = \sum_{s_1,s_2\geqslant 0} c_{s_1,s_2}w_1^{s_1}w_2^{s_2}. \tag{22}$$

Substituting (20) in (22) and interchanging the order of summation and integration, we obtain

$$\begin{aligned}
F(w_1,w_2) &= \sum_{s_1,s_2\geqslant 0} c_{s_1,s_2}\int\limits_{\zeta\in S_R} e^{\zeta_1 w_1+\zeta_2 w_2}\xi_{s_1,s_2}(\zeta_1,\zeta_2)\wedge d\zeta_1\wedge d\zeta_2\\
&= \int\limits_{\zeta\in S_R} e^{\zeta_1 w_1+\zeta_2 w_2}\left[\sum_{s_1,s_2\geqslant 0} c_{s_1,s_2}\xi_{s_1,s_2}(\zeta_1,\zeta_2)\right]\wedge d\zeta_1\wedge d\zeta_2,
\end{aligned}\tag{23}$$

provided that $R > 3\sqrt{2}eB$. At this point we are using the fact that the series

$$\sum_{s_1,s_2\geqslant 0} c_{s_1,s_2}\xi_{s_1,s_2}(\zeta_1,\zeta_2)$$

converges uniformly for $|\zeta_1|^2+|\zeta_2|^2 \geqslant R^2$, which follows from (19) and the expression of $\xi_{s_1,s_2}(\zeta_1,\zeta_2)$, as this is given by the last part of (21).

Since (23) holds for every w_1, w_2, it follows that, for $f \in \mathcal{O}(\mathbb{C}^2)$,

$$\mathcal{T}(f) = \int_{\zeta \in S_R} f(\zeta_1, \zeta_2)\Bigg[\sum_{s_1,s_2 \geqslant 0} c_{s_1,s_2}\xi_{s_1,s_2}(\zeta_1, \zeta_2)\Bigg] \wedge d\zeta_1 \wedge d\zeta_2$$
$$= \sum_{s_1,s_2 \geqslant 0} c_{s_1,s_2} \int_{\zeta \in S_R} f(\zeta_1, \zeta_2)\xi_{s_1,s_2}(\zeta_1, \zeta_2) \wedge d\zeta_1 \wedge d\zeta_2.$$

(We could not have concluded this, had we not interchanged the order of summation and integration in (23).)

Finally the computations, which led to the formula of Theorem 3, show that

$$\int_{\zeta \in S_R} f(\zeta_1, \zeta_2)\xi_{s_1,s_2}(\zeta_1, \zeta_2) \wedge d\zeta_1 \wedge d\zeta_2 = \mathcal{T}^R_{s_1,s_2}(f),$$

and the required expansion of the theorem follows. □

Comments. (1) As we pointed out in the above proof, part of the conclusion is the convergence of the series in (17). Notice also that it is not immediately clear that the sum $\sum c_{s_1,s_2}\mathcal{T}^R_{s_1,s_2}$ defines an analytic functional. This is justified only when (17) is proved.

(2) The above proof shows that any entire function of exponential type is the Fourier–Laplace transform of an analytic functional of the form $\sum c_{s_1,s_2}\mathcal{T}^R_{s_1,s_2}$. In particular, if μ is any measure with compact support in $\mathbb{C}^2$ then

$$\int_{\zeta \in \mathbb{C}^2} f(\zeta)\,d\mu(\zeta) = \sum_{s_1,s_2 \geqslant 0} c_{s_1,s_2}\mathcal{T}^R_{s_1,s_2}(f) \quad \text{for } f \in \mathcal{O}(\mathbb{C}^2).$$

In fact, c_{s_1,s_2} are the coefficients in the power series expansion of the entire function

$$\int_{(\zeta_1,\zeta_2) \in \mathbb{C}^2} e^{\zeta_1 w_1 + \zeta_2 w_2}\,d\mu(\zeta_1, \zeta_2), \quad \text{i.e.,} \quad c_{s_1,s_2} = \frac{1}{s_1!s_2!} \int_{(\zeta_1,\zeta_2) \in \mathbb{C}^2} \zeta_1^{s_1}\zeta_2^{s_2}\,d\mu(\zeta_1, \zeta_2).$$

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