Maps preserving the idempotency of products of operators

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Abstract

Let \( \mathcal{B}(X) \) be the algebra of all bounded linear operators on a complex Banach space \( X \). We give the concrete form of every unital surjective map \( \varphi \) on \( \mathcal{B}(X) \) such that \( AB \) is a non-zero idempotent if and only if \( \varphi(A)\varphi(B) \) is for all \( A, B \in \mathcal{B}(X) \) when the dimension of \( X \) is at least 3.

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1. Introduction

The problem of characterizing linear or additive maps on operator algebras preserving certain properties, subsets or relations has attracted the attention of many authors in the last decades (cf. [1,4,6,9,10]). Many results which have been obtained on this topic reveal both algebraic and geometric structures of the operator algebras from some new aspects. Very recently, some preserver problems concerning certain properties of products of operators have been considered...
Motivated by this point, we consider maps preserving idempotency of products of operators. Let $\mathcal{X}$ be a complex Banach space and let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on $\mathcal{X}$. $\mathcal{X}'$ denotes the dual space of $\mathcal{X}$ and $A'$ is the dual operator of $A \in \mathcal{B}(\mathcal{X})$. Let $\mathcal{I}(\mathcal{X}) = \{P \in \mathcal{B}(\mathcal{X}) : P^2 = P\}$ be the set of all idempotents and let $\mathcal{N}(\mathcal{X}) = \{N \in \mathcal{B}(\mathcal{X}) : N^k = 0 \text{ for some positive integer } k\}$ be the set of all nilpotent operators. We denote by $\mathcal{I}_1(\mathcal{X})$ and $\mathcal{N}_1(\mathcal{X})$ the set of all rank-1 idempotents and the set of all rank-1 nilpotent operators in $\mathcal{B}(\mathcal{X})$ respectively. If $\mathcal{X}$ has dimension $n$ with $3 \leq n < \infty$, then $\mathcal{B}(\mathcal{X})$ is identified with the algebra $\mathcal{M}_n$ of $n \times n$ complex matrices and $\mathcal{I}_n(\mathcal{X})$ refers the set of idempotent matrices in $\mathcal{M}_n$. In [7], authors characterized the structure of surjective maps $\varphi : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{X})$ having the property that for every pair $A, B \in \mathcal{B}(\mathcal{X})$,

$$AB \in \mathcal{N}(\mathcal{X}) \Leftrightarrow \varphi(A)\varphi(B) \in \mathcal{N}(\mathcal{X}).$$

We now are interested in determining the structure of unital surjective maps $\varphi : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{X})$ having the property that

$$AB \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \Leftrightarrow \varphi(A)\varphi(B) \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \quad (\forall A, B \in \mathcal{B}(\mathcal{X})). \quad (*)$$

The aim of this paper is to prove the following Theorems.

**Theorem 1.1.** Let $\mathcal{X}$ be an infinite dimensional Banach space. Then a unital surjective map $\varphi : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{X})$ satisfies that

$$AB \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \Leftrightarrow \varphi(A)\varphi(B) \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \quad (\forall A, B \in \mathcal{B}(\mathcal{X}))$$

if and only if there exists an invertible bounded linear or conjugate-linear operator $S : \mathcal{X} \rightarrow \mathcal{X}$ such that $\varphi(A) = SAS^{-1}$ for all $A \in \mathcal{B}(\mathcal{X})$.

**Theorem 1.2.** Let $n \geq 3$. Then a unital surjective map $\varphi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ satisfies that

$$AB \in \mathcal{I}_n(\mathcal{X}) \setminus \{0\} \Leftrightarrow \varphi(A)\varphi(B) \in \mathcal{I}_n(\mathcal{X}) \setminus \{0\} \quad (\forall A, B \in \mathcal{M}_n)$$

if and only if there exist a field automorphism $\xi : \mathbb{C} \rightarrow \mathbb{C}$ and an invertible matrix $S \in \mathcal{M}_n$ such that $\varphi(A) = SA_\xi S^{-1}$ for all $A \in \mathcal{M}_n$, where $A_\xi = [\xi(a_{ij})]$ if $A = [a_{ij}]$.

We recall some notations. Let $\mathcal{M}$ be a subspace of $\mathcal{X}$, we denote the dimension of $\mathcal{M}$ by $\dim \mathcal{M}$. For an operator $T \in \mathcal{B}(\mathcal{X})$, $\ker T$ denotes the kernel of $T$. Let $\mathcal{F}_1(\mathcal{X})$ denotes the set of rank-1 operators in $\mathcal{B}(\mathcal{X})$ and $I$ is the identity of $\mathcal{B}(\mathcal{X})$. For every non-zero $x \in \mathcal{X}$ and $f \in \mathcal{X}'$, the symbol $x \otimes f$ stands for the rank-1 bounded linear operator on $\mathcal{X}$ defined by $(x \otimes f)y = f(y)x$ for any $y \in \mathcal{X}$. Note that every rank-1 operator in $\mathcal{B}(\mathcal{X})$ can be written in this way. The rank-1 operator $x \otimes f$ is an idempotent if and only if $f(x) = 1$ and $x \otimes f$ is nilpotent if and only if $f(x) = 0$. Let $x \otimes f$ and $y \otimes g$ be two rank-one operators, we say that $x \otimes f \sim y \otimes g$ if $x$ and $y$ are linearly dependent or $f$ and $g$ are linearly dependent. Given $P, Q \in \mathcal{I}(\mathcal{X})$, we say $P \leq Q$ if $PQ =QP = P$ and we say $P < Q$ if $P \leq Q$ and $P \neq Q$. In addition, we say that $P$ and $Q$ are orthogonal if $PQ =QP = 0$. For any vectors $x$ and $y$ in a complex linear space, we denote by $\text{Gcv}\{x, y\} = \{\lambda x + (1 - \lambda)y : \lambda \in \mathbb{C}\}$ the generalized convex combination of $x$ and $y$, where $\mathbb{C}$ denotes the complex plane.
2. Preliminary results

In this section, we assume that $\mathcal{X}$ is a complex Banach space with dimension at least 3. We consider some elementary results which are useful in the proofs of main theorems.

**Lemma 2.1.** Let $A, B \in \mathcal{B}(\mathcal{X})$ be non-zero operators. Then the following are equivalent:

(i) $A = B$.

(ii) For every $T \in \mathcal{B}(\mathcal{X})$, $AT \in \mathcal{I}(\mathcal{X}) \setminus \{0\}$ if and only if $BT \in \mathcal{I}(\mathcal{X}) \setminus \{0\}$.

(iii) For every $T \in \mathcal{B}(\mathcal{X})$, $AT \in \mathcal{I}(\mathcal{X}) \setminus \{0\}$ whenever $BT \in \mathcal{I}(\mathcal{X}) \setminus \{0\}$.

**Proof.** In order to complete the proof, it is sufficient to prove that (iii) implies (i). For every pair $x \in \mathcal{X}$, $f \in \mathcal{X}'$ such that $f(Bx) = 1$ put $T = x \otimes f$. Then $BT \in \mathcal{I}(\mathcal{X}) \setminus \{0\}$. It follows from (iii) that $AT \in \mathcal{I}(\mathcal{X}) \setminus \{0\}$ which implies that $f(Ax) = 1$. In particular, for any $x \in \mathcal{X}$ such that $Bx \neq 0$, we have that $Ax$ and $Bx$ are linearly independent or that $Ax = Bx$. If there exists an $x \in \mathcal{X}$ such that $Ax$ and $Bx$ are linearly independent, then there exits an $f \in \mathcal{X}'$ such that $f(Bx) = 1$ but $f(Ax) = 0$. This is a contradiction. Hence for any $x \in \mathcal{X}$ such that $Bx \neq 0$, we have $Ax = Bx$. Assume $Bx = 0$. There is a vector $y \in \mathcal{X}$ such that $By \neq 0$ and $B(x + y) \neq 0$ since $B \neq 0$. Then by the proof above we have $Ay = By$ and $A(x + y) = B(x + y)$, which means that $Ax = 0$. Thus $A = B$. The proof is complete. $\square$

**Lemma 2.2.** Let $A, B \in \mathcal{B}(\mathcal{X})$ be non-scalar operators. Suppose that for every such $x \in \mathcal{X}$ that $x$ and $Ax$ are linearly independent or that $x = Ax$, $Bx \in \text{Gcv}(x, Ax)$. Then $B = \lambda I + (1 - \lambda)A$ for some $\lambda \in \mathbb{C} \setminus \{1\}$.

**Proof.** Assume first that the operators $A, B$ and $I$ are linearly dependent. Then $\alpha B + \beta A + \gamma I = 0$ for some complex scalar $\alpha, \beta$ and $\gamma$ with not all zero. Note that $\alpha \beta \neq 0$. Otherwise either $A$ or $B$ is a scalar operator, which is a contradiction. It follows that $B = -\frac{\beta}{\alpha}A - \frac{\gamma}{\alpha}I$. Take any $x \in \mathcal{X}$ such that $x$ and $Ax$ are linearly independent. Then, for such an $x$, we have that $Bx \in \text{Gcv}(x, Ax)$, which implies that $-\frac{\beta}{\alpha} + \frac{\gamma}{\alpha} = 1$.

In order to complete the proof we have to show that the assumption that $A, B$ and $I$ are linearly independent leads to a contradiction. Assume that $A, B$ and $I$ are linearly independent. Because $\dim \mathcal{X} \geq 3$, the identity has rank at least 3. With this observation and the assumptions on $A, B$ and $I$, we can apply Theorem 2.4 in [8] to conclude that there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha B + \beta A + \gamma I = z \otimes f$ for some rank-one operator $z \otimes f \in \mathcal{B}(\mathcal{X})$. There are two cases to be considered.

**Case 1.** $\alpha \neq 0$.

We first claim that $x$, $Ax$ and $z$ are linearly dependent for all $x \in \mathcal{X}$. Indeed, if $x$ and $Ax$ are linearly dependent, then the claim is proved. Otherwise, if $x$ and $Ax$ are linearly independent, then $Bx \in \text{Gcv}(x, Ax)$. Thus there is a $\lambda_x \in \mathbb{C}$ such that $f(x)z = (\alpha(1 - \lambda_x) + \beta)Ax + (\alpha \lambda_x + \gamma)x$. Assume that there exists an $x_0 \in \mathcal{X}$ such that $x_0$, $Ax_0$ and $z$ are linearly independent. Then, $f(x_0) = 0$. Taking $u \in \mathcal{X}$ such that $f(u) \neq 0$, there is a non-zero complex number $\mu$ such that $x_0 + \mu u, A(x_0 + \mu u)$ and $z$ are linearly independent by Lemma 2.1 in [2]. Thus, $f(x_0 + \mu u) = 0$. But $f(x_0 + \mu u) = f(x_0) + \mu f(u) = \mu f(u) \neq 0$, a contradiction.

Denote by $Q$ the canonical quotient map from $\mathcal{X}$ onto $\mathcal{X}/[z]$. Then $Qx$ and $QAx$ are linearly dependent for all $x \in \mathcal{X}$. By Theorem 2.3 in [2], $Q$ and $QA$ are linearly dependent. This yields that $Q(A - \delta I) = 0$ and $A = \delta I + z \otimes g$ for some complex $\delta$ and some functional $g \in \mathcal{X}'$. Since
A is a non-scalar operator, we know that $g \neq 0$ and $B = \frac{1}{\alpha} z \otimes f - \frac{\beta}{\alpha} A - \frac{\gamma}{\alpha} I = \lambda I + z \otimes h$ with $\lambda = -\frac{\beta}{\alpha} g - \frac{\gamma}{\alpha}$ and $h = \frac{1}{\alpha} f - \frac{\beta}{\alpha} g$. It is known that $h \neq 0$.

Next we show that $\ker(g) = \ker(h)$. Suppose that there exists a $z_1 \in \ker(h)$ such that $g(z_1) \neq 0$. If $z_1$ and $z$ are linearly independent, then $A z_1 = \delta z_1 + g(z_1) z$, which implies that $A z_1$ and $z_1$ are linearly independent. So, $B z_1 = \lambda z_1 + (1 - \lambda) A z_1 = (\lambda z_1 + (1 - \lambda) g)(z_1) z + B(z_1) = \lambda z_1 + h(z_1) z = \lambda z_1$. It follows that $(1 - \lambda z_1) g(z_1) = 0$ and $\lambda z_1 + (1 - \lambda) \delta = \lambda$. Since $g(z_1) \neq 0$, we have that $\lambda z_1 = \lambda = 1$ and $B = I + z \otimes h$. If $z_1$ and $z$ are linearly dependent, then there exists $z'_1 \in \ker(h)$ such that $z'_1$ and $z$ are linearly independent since $\dim \ker(h) \geq 2$. By Lemma 2.1 in [2], we can get that $\mu z_1 + z'_1$ and $z$ are linearly independent for all but finitely many values of $\mu \in \mathbb{C}$. Taking $\mu \in \mathbb{C} \setminus \{0\}$ such that $g(\mu z_1 + z'_1) \neq 0$, we have that $A(\mu z_1 + z'_1)$ and $\mu z_1 + z'_1$ are linearly independent. So, $B(\mu z_1 + z'_1) = \lambda \mu z_1 + (1 - \lambda \mu) A(\mu z_1 + z'_1) + (1 - \lambda \mu) g(\mu z_1 + z'_1) = (\lambda \mu + (1 - \lambda \mu) \delta)(\mu z_1 + z'_1) + (1 - \lambda \mu) g(\mu z_1 + z'_1) z$ for some $\lambda \mu \in \mathbb{C}$ and $B(\mu z_1 + z'_1) = \lambda \mu(z_1 + z'_1) z$. Thus, $(1 - \lambda \mu) g(\mu z_1 + z'_1) = 0$ and $\lambda = \lambda \mu + (1 - \lambda \mu) \delta$. Then, $\lambda \mu = \lambda = 1$ and $B = I + z \otimes h$. Moreover, we claim that $A = I + z \otimes g$ if there is also a $z_2 \in \ker(g)$ such that $h(z_2) \neq 0$. Once the mentioned vectors $z_1$ and $z_2$ exist, we know that at least one of $z_1$ and $z_2$ is linearly independent of $z$ saying $z_1$. By Lemma 2.1 in [2], we have that $t z_2 + z_1$ and $z$ are linearly independent for all but finitely many values of $t \in \mathbb{C}$. Since $A(t z_2 + z_1) = \delta(t z_2 + z_1) + g(t z_2 + z_1) z = \delta(t z_2 + z_1) + g(z_1) z$, we know that $t z_2 + z_1$ and $A(t z_2 + z_1)$ are linearly independent. It follows that there exists a $\lambda_t \in \mathbb{C}$ such that $B(t z_2 + z_1) = \lambda_t(t z_2 + z_1) + (1 - \lambda_t) A(t z_2 + z_1) = (\lambda_t + (1 - \lambda_t) \delta)(t z_2 + z_1) + (1 - \lambda_t) g(z_1) z$. Moreover, $B(t z_2 + z_1) = (t z_2 + z_1) + t h(z_2) z$. Hence, $\lambda_t + (1 - \lambda_t) \delta = 1$ and $(1 - \lambda_t) g(z_1) = t h(z_2) z$ for all but finitely many values of $t \in \mathbb{C}$. Therefore, $\delta = 1$ and $A = I + z \otimes g$. For every $x \in \ker(g)$ we have that $Ax = x$, which implies that $Bx = x + h(x) z \in Gcv\{x, Ax\}$ by the assumption. It follows that $h(x) = 0$ in this case. Thus $\ker(g) \subseteq \ker(h)$. However this contradicts to the fact that $z_2 \in \ker(g)$ with $h(z_2) \neq 0$. Then, we have that $\ker(g) \subseteq \ker(h)$ if there exists a $z_1 \in \ker(h)$ such that $g(z_1) \neq 0$. Hence, $\ker(h) = \mathcal{X}$ and $B = \lambda I$. This is a contradiction too. Hence, $z_1$ satisfying $z_1 \in \ker(h)$ but $g(z_1) \neq 0$ does not exist. We now get that $\ker(h) \subseteq \ker(g)$. Since $g \neq 0$, we know that $\ker(g) = \ker(h)$ and $h = \eta g$ for some non-zero complex number $\eta$. Then $A = \delta I + z \otimes g$ and $B = \lambda I + \eta z \otimes g$. This is a contradiction since $A$, $B$ and $I$ are linearly independent.

**Case 2.** $\alpha = 0$.

In this case, we have $\beta \neq 0$ and $A = \frac{1}{\alpha} z \otimes f - \frac{\gamma}{\alpha} I$. Then there exists a $z_0 \in \mathcal{X}$ such that $f(z_0) \neq 0$ and $z_0$ and $z$ are linearly independent. Thus $z_0$ and $A z_0$ are linearly independent, which implies that $B z_0 = \lambda z_0 + (1 - \lambda z_0) A z_0$ for some $\lambda z_0 \in \mathbb{C}$. For every fixed $x \in \ker(f)$, we know that $t z_0 + x$ and $z$ are linearly independent for all but finitely many values of $t \in \mathbb{C}$ by Lemma 2.1 in [2]. Thus, $B(t z_0 + x) = \lambda_t(t z_0 + x) + (1 - \lambda_t) A(t z_0 + x) = \lambda_t + (1 - \lambda_t)(-\frac{\gamma}{\beta}) (t z_0 + x) + (1 - \lambda_t) \frac{1}{\beta} t f(z_0) z$ for some $\lambda_t \in \mathbb{C}$. If there exists a subsequence $\{\lambda_{t_i}\}$ such that $\lim_{t \to 0} \lambda_{t_i} = \xi$ for some $\xi \in \mathbb{C}$, then we get that $B x = (\xi + (1 - \xi)(-\frac{\gamma}{\beta}) x = \xi x + (1 - \xi) A x$. If $\lim_{t \to 0} |\lambda_t| = \infty$, then $\frac{1}{\lambda_t} B(t z_0 + x) = (1 + (\frac{1}{\lambda_t} - 1)(-\frac{\gamma}{\beta}) (t z_0 + x) + \frac{1}{\lambda_t - 1}) t f(z_0) z$. By letting $t \to 0$, we can get that $(1 + \frac{1}{\lambda_t - 1}) x = 0$ and $x = -\frac{\gamma}{\beta} x = A x$. It follows that $B x = \lambda x + (1 - \lambda x) A x$ by assumption again. Thus $B x = \lambda x + (1 - \lambda x) A x$ for every $x \in \ker(f)$. If $x \notin \ker(f)$, then we also have that $B x = \lambda x + (1 - \lambda x) A x$ by similar way as above. Therefore, $B x = \lambda x x + (1 - \lambda x) A x$ for all $x \in \mathcal{X}$. Let $a = -\frac{\gamma}{\beta}$ and $g = \frac{1}{\beta} f$.

If $a = 1$, then $A = I + z \otimes g$ and $B = x + g(x) (1 - \lambda x) A x$ for all $x \in \mathcal{X}$. Take an $x_0$ such that $g(x_0) = 1$. Then $A x_0 = t x_0 + t g(x_0) z$ for all non-zero constants $t \in \mathbb{C}$. Then we have that
\[ Btx_0 = tx_0 + tg(x_0)(1 - \lambda tx_0)z = tBx_0 = tx_0 + tg(x_0)(1 - \lambda x_0)z. \] It follows that \( \lambda tx_0 = \lambda x_0 \) and \( Btx_0 = tx_0 + (1 - \lambda x_0)g(tx_0)z = (I + (1 - \lambda x_0)z \otimes g)(tx_0). \) On the other hand, we easily have \( Bx = Ax = x \) for any \( x \in \ker(g). \) Hence \( B(tx_0 + x) = (I + (1 - \lambda x_0)z \otimes g)(tx_0 + x) \) for all \( t \in \mathbb{C} \) and \( x \in \ker(g), \) which means that \( B = I + (1 - \lambda x_0)z \otimes g. \) This contradicts to the linear independence of \( A, B \) and \( I, \) since \( A = I + z \otimes g. \)

If \( a \neq 1, \) then
\[
(\lambda x + y - \lambda x)((1 - a)x - g(x)z) + (\lambda x + y - \lambda y)((1 - a)y - g(y)z) = 0
\]
for all \( x, y \in \mathcal{A}. \)

Let \( x, y \in \ker(g) \) be linearly independent. Then \( \lambda x + y - \lambda x = \lambda x + y - \lambda y = 0 \) and \( \lambda x + y = \lambda x = \lambda y. \) So, there is a \( \lambda \in \mathbb{C} \) such that \( \lambda x = \lambda \) for all \( x \in \ker(g). \) Let \( y \notin \ker(g), \) then there exists such \( x \in \ker(g) \) that \( x \) and \( (1 - a)y - g(y)z \) are linearly independent. Thus, we can get that \( \lambda x + y - \lambda x = \lambda x + y - \lambda y = 0 \) and \( \lambda y = \lambda \) for all \( y \notin \ker(g). \) Hence, \( \lambda x = \lambda \) for all \( x \in \mathcal{A}. \) This implies that \( B = \lambda I + (1 - \lambda)A. \) This contradicts to the linear independence of \( A, B \) and \( I, \) too.

Therefore, \( B = \lambda I + (1 - \lambda)A \) for some \( \lambda \in \mathbb{C}\setminus\{1\}. \) The proof is complete. \( \square \)

**Proposition 2.3.** Let \( A, B \in \mathcal{B}(\mathcal{A}) \) be non-scalar operators. If \( AP \in \mathcal{I}(\mathcal{A})\setminus\{0\} \) implies \( BP \in \mathcal{I}(\mathcal{A})\setminus\{0\} \) for every \( P \in \mathcal{I}_1(\mathcal{A}), \) then \( B = \lambda I + (1 - \lambda)A \) for some \( \lambda \in \mathbb{C}\setminus\{1\}. \)

**Proof.** We note that for any pair \( x \in \mathcal{A} \) and \( f \in \mathcal{A}' \) such that \( f(x) = f(Ax) = 1, \) \( P = x \otimes f \) and \( AP = Ax \otimes f \) are rank-1 idempotents, thus \( BP \in \mathcal{I}(\mathcal{A})\setminus\{0\} \) and \( f(Bx) = 1. \) We now prove that \( A \) and \( B \) satisfy the condition of Lemma 2.2.

**Case 1.** Let \( x \in \mathcal{A} \) such that \( x \) and \( Ax \) are linearly independent. If \( Bx \notin \text{span}\{x, Ax\}, \) then \( x, x - Ax \) and \( Bx \) are linearly independent. Thus, there exists an \( f \in \mathcal{A}' \) such that \( f(x) = 1 \) and \( f(Bx) = f(x - Ax) = 0. \) Hence, \( f(x) = f(Ax) = 1 \) but \( f(Bx) = 0 \neq 1. \) This is a contradiction by the preceding note. Then \( Bx = ax + \beta Ax \) for some \( \alpha, \beta \in \mathbb{C}. \) It is easily known that \( x \) and \( x - Ax \) are linearly independent, so there exists a \( g \in \mathcal{A}' \) such that \( g(x) = 1 \) and \( g(x - Ax) = 0, \) that is, \( g(x) = g(Ax) = 1. \) By preceding proof again, we have \( g(Bx) = \alpha + \beta = 1, \) which is what we desired.

**Case 2.** Let \( x \in \mathcal{A} \) such that \( x = Ax. \) If \( x \) and \( Bx \) are linearly independent, then there exists a \( g_1 \in \mathcal{A}' \) such that \( g_1(x) = 1 \) and \( g_1(Bx) = 0. \) Hence, \( g_1(x) = g_1(Ax) = 1 \) but \( g_1(Bx) = 0. \) This is a contradiction. Then \( Bx = ax \) for some \( \alpha \in \mathbb{C}. \) Take any \( g_2 \in \mathcal{A}' \) such that \( g_2(x) = 1. \) Then \( g_2(x) = g_2(Ax) = 1, \) which implies that \( g_2(Bx) = \alpha = 1. \)

Therefore we have \( B = \lambda I + (1 - \lambda)A \) for some \( \lambda \in \mathbb{C}\setminus\{1\} \) by Lemma 2.2. The proof is complete. \( \square \)

By Proposition 2.3, we easily have the following corollary.

**Corollary 2.4.** Let \( A, B \in \mathcal{B}(\mathcal{A}) \) be non-scalar operators. If \( AP \in \mathcal{I}(\mathcal{A})\setminus\{0\} \) implies \( BP \in \mathcal{I}(\mathcal{A})\setminus\{0\} \) for every \( P \in \mathcal{I}(\mathcal{A})\setminus\{0\}, \) then \( B = \lambda I + (1 - \lambda)A \) for some \( \lambda \in \mathbb{C}\setminus\{1\}. \)

**Lemma 2.5.** Let \( P, Q \in \mathcal{I}(\mathcal{A})\setminus\{0\} \) and \( \lambda \in \mathbb{C}\setminus\{0, 1\}. \) Then \( \lambda P + (1 - \lambda)Q \in \mathcal{I}(\mathcal{A})\setminus\{0\} \) if and only if \( P + Q = PQ + QP. \) Moreover, if there is a \( \lambda \in \mathbb{C}\setminus\{0, 1\} \) such that \( \lambda P + (1 - \lambda)Q \in \mathcal{I}(\mathcal{A})\setminus\{0\}, \) then \( \mu P + (1 - \mu)Q \in \mathcal{I}(\mathcal{X})\setminus\{0\} \) for any \( \mu \in \mathbb{C}. \)

**Proof.** We note that \( \lambda P + (1 - \lambda)Q \in \mathcal{I}(\mathcal{A})\setminus\{0\} \) is equivalent to \( \lambda P + (1 - \lambda)Q = \lambda^2 P + \lambda(1 - \lambda)(QP + PQ) + (1 - \lambda)^2 Q, \) that is, \( P + Q = PQ + QP. \)
Corollary 2.6. Let $\lambda \in \mathbb{C}\setminus \{0, 1\}$ and $P, Q \in \mathcal{I}(\mathcal{A})\setminus \{0\}$. If $PQ \in \mathcal{I}(\mathcal{A})\setminus \{0\}$ and $(\lambda I + (1 - \lambda)P)Q \in \mathcal{I}(\mathcal{A})\setminus \{0\}$, then $\text{rank } Q \leq \text{rank } P$.

**Proof.** Since $PQ \in \mathcal{I}(\mathcal{A})\setminus \{0\}$ and $(\lambda I + (1 - \lambda)P)Q \in \mathcal{I}(\mathcal{A})\setminus \{0\}$, we have that $Q = PQ$ by Lemma 2.5. It is clear that $\text{rank } Q \leq \text{rank } P$. □

We recall that $x \otimes f \sim y \otimes g$ if $x$ and $y$ are linearly dependent or $f$ and $g$ are linearly dependent.

**Proposition 2.7.** Let $A_1$ and $A_2$ be linearly independent rank-1 operators. Then the following are equivalent:

(i) $A_1 \sim A_2$.
(ii) There exists a $B \in \mathcal{F}_1(\mathcal{A})$ such that $B$ is linearly independent of $A_k$ ($k = 1, 2$) and for every $T \in \mathcal{B}(\mathcal{A})$ we have $A_kT \in \mathcal{I}(\mathcal{A})\setminus \{0\}$ ($k = 1, 2$) imply $BT \in \mathcal{I}(\mathcal{A})\setminus \{0\}$.

**Proof.** Let $A_1 = x \otimes f$ and $A_2 = y \otimes g$. If $A_1 \sim A_2$, then $y = \lambda x$ or $g = \lambda f$ for some non-zero complex number $\lambda$. For the case $y = \lambda x$, we take $B = \frac{1}{\lambda}(A_1 + A_2)$. Since $A_1$ and $A_2$ are linearly independent, $B$ is linearly independent of $A_k$ ($k = 1, 2$). If $A_kT \in \mathcal{I}(\mathcal{A})\setminus \{0\}$ ($k = 1, 2$), then $f(Tx) = g(\lambda Ty) = 1$. This yields that $f(Tx) + g(\lambda Ty) = 1$, which implies that $BT \in \mathcal{I}(\mathcal{A})\setminus \{0\}$. For another case $g = \lambda f$, we can use the similar discussion.

To prove the other direction, assume that $A_1 = x \otimes f$ and $A_2 = y \otimes g$ are rank-1 operators such that $x$ and $y$ as well as $f$ and $g$ are linearly independent. Suppose also that there exists a $B = u \otimes k$ satisfying the second condition. We will show that $k$ is a linear combination of $f$ and $g$. If $f$, $g$ and $k$ are linearly independent, then so are $f$, $k$ and $f - g$ and there exists a $z \in \mathcal{A}$ such that $f(z) = 1$ and $k(z) = (f - g)(z) = 0$. That is, $f(z) = g(z) = 1$ and $k(z) = 0$. If $x$, $y$ and $u$ are linearly independent, then we can find a $T \in \mathcal{B}(\mathcal{A})$ such that $Tx = Ty = Tu = z$. It is known that $f(Tx) = g(Ty) = 1$ and $k(Tu) = 0$. Thus, $A_kT \in \mathcal{I}(\mathcal{A})\setminus \{0\}$ ($k = 1, 2$) and $BT \notin \mathcal{I}(\mathcal{A})\setminus \{0\}$. This contradicts to the assumption. If $u = tx + sy$ for some $t, s \in \mathbb{C}$, then we can also find a $T \in \mathcal{B}(\mathcal{A})$ such that $Tx = Ty = z$. We also have $f(Tx) = g(Ty) = 1$ and $k(Tu) = k(T(x + sy)) = (t + s)k(z) = 0$, which implies that $A_kT \in \mathcal{I}(\mathcal{A})\setminus \{0\}$ ($k = 1, 2$) and $BT \notin \mathcal{I}(\mathcal{A})\setminus \{0\}$. This is a contradiction again. Hence, $k = \lambda f + \mu g$ for some complex numbers $\lambda, \mu \in \mathbb{C}$.

Next, we show that $u$ is a linear combination of $x$ and $y$. Assume that $u$, $x$ and $y$ are linearly independent. Since there exist $z_1, z_2 \in \mathcal{A}$ such that $f(z_1) = 1$ and $g(z_2) = 1$, we can find a $T \in \mathcal{B}(\mathcal{A})$ such that $Tx = z_1, Ty = z_2$ and $Tu = 0$. Then $f(Tx) = g(Ty) = 1$ and $k(Tu) = 0$. Thus, $A_kT \in \mathcal{I}(\mathcal{A})\setminus \{0\}$ ($k = 1, 2$) but $BT = u \otimes kT = u \otimes Tk \notin \mathcal{I}(\mathcal{A})\setminus \{0\}$. This contradiction implies that $u = \alpha x + \beta y$ for some complex numbers $\alpha, \beta \in \mathbb{C}$. Hence, $B = (\alpha x + \beta y) \otimes$
Let \( \eta \) and \( \nu \) be any complex numbers. As \( f \) and \( g \) are linearly independent, there exist \( w_1, w_2 \in X \) such that \( f(w_1) = 1, f(w_2) = \eta, g(w_1) = \nu \) and \( g(w_2) = 1 \). Since \( x \) and \( y \) are linearly independent, we can find a \( R \in B(X) \) satisfying \( Rx = w_1 \) and \( Ry = w_2 \). Then, \( f(Rx) = g(Ry) = 1 \), which means that \( A_k R(k, 2) \) are non-zero idempotents. Thus so is \( BR \), which implies that \( k(Ru) = (\lambda f + \mu g)(R(\alpha x + \beta y)) = \lambda \alpha + \beta \nu + \alpha \mu \eta + \beta \mu = 1 \). Note that \( \eta \) and \( \nu \) are arbitrary, we get that

\[
\begin{cases}
\lambda \beta = 0, \\
\alpha \mu = 0, \\
\alpha \lambda + \beta \mu = 1,
\end{cases}
\]

which implies that

\[
\begin{cases}
\lambda = 0, \\
\alpha = 0, \\
\beta = 0,
\end{cases}
\quad \text{or} \quad
\begin{cases}
\alpha = 0, \\
\mu = 0, \\
\beta = 0,
\end{cases}
\quad \text{or} \quad
\begin{cases}
\beta = 0, \\
\alpha = 0, \\
\mu = 0.
\end{cases}
\]

It follows that \( B \) is a multiple of either \( A_1 \) or \( A_2 \), a contradiction. Hence \( A_1 \sim A_2 \). The proof is complete. \( \Box \)

### 3. Proofs of main theorems

We continue to assume that \( X \) is a complex Banach space with \( \dim X \geq 3 \) and \( \varphi \) is a unital map on \( B(X) \) satisfying

\[
AB \in \mathcal{I}(X) \setminus \{0\} \iff \varphi(A) \varphi(B) \in \mathcal{I}(X) \setminus \{0\} \quad (\forall A, B \in B(X)).
\]

**Lemma 3.1.** Let \( \varphi \) be as above. Then

(i) \( \varphi(0) = 0 \).

(ii) \( \varphi \) is injective.

(iii) \( \varphi(\mathcal{I}(X)) = \mathcal{I}(X) \).

(iv) There is a bijective function \( \kappa(\lambda) \) on \( \mathbb{C} \) satisfying \( \kappa(0) = 0, \kappa(1) = 1 \) and \( \kappa(\lambda^{-1}) = \kappa(\lambda)^{-1} \) for all \( \lambda \neq 0 \) such that \( \varphi(\lambda I) = k(\lambda)I \) for all \( \lambda \in \mathbb{C} \).

**Proof.** (i) Let \( A \in B(X) \). Then \( A = 0 \) if and only if \( AT \notin \mathcal{I}(X) \setminus \{0\} \) for every \( T \in B(X) \). Note that \( \varphi \) is surjective, we easily have that \( \varphi(0) \varphi(T) \notin \mathcal{I}(X) \setminus \{0\} \) for every \( \varphi(T) \in B(X) \). Thus \( \varphi(0) = 0 \).

(ii) This follows from Lemma 2.1.

(iii) For every non-zero idempotent \( P = IP \in \mathcal{I}(X) \setminus \{0\} \), we have \( \varphi(I) \varphi(P) = \varphi(P) \in \mathcal{I}(X) \setminus \{0\} \) since \( \varphi(I) = I \). Thus, \( \varphi(\mathcal{I}(X)) \subseteq \mathcal{I}(X) \). On the other hand, we know that the inverse \( \varphi^{-1} \) is also a unital map on \( B(X) \) satisfying condition (*). Then we have \( \varphi(\mathcal{I}(X)) = \mathcal{I}(X) \).

(iv) Note that \( A \in \mathbb{C}^*I \) if and only if \( AP \notin \mathcal{I}(X) \setminus \{0\} \) for every \( P \in \mathcal{I}(X) \setminus \{0\} \), where \( \mathbb{C}^* = \mathbb{C} \setminus \{0, 1\} \). Then it easily follows from (i) and (ii) that \( \varphi(\mathbb{C}^*I) = \mathbb{C}^*I \). Thus there is a bijective function \( \kappa(\lambda) \) on \( \mathbb{C} \) satisfying \( \kappa(0) = 0 \) and \( \kappa(1) = 1 \) such that \( \varphi(\lambda I) = \kappa(\lambda)I \) for all \( \lambda \in \mathbb{C} \). It is trivial that \( \kappa(\lambda) \kappa(\lambda^{-1}) = 1 \). The proof is completed. \( \Box \)

We now know that both \( \varphi \) and \( \varphi^{-1} \) satisfy condition (*).

**Lemma 3.2.** Let \( A \in B(X) \). Then \( \varphi(\text{Gcv}(I, A)) = \text{Gcv}(I, \varphi(A)) \).

**Proof.** Let \( A_\lambda = \lambda I + (1 - \lambda)A \in \text{Gcv}(I, A) \) for some \( \lambda \in \mathbb{C} \). Then, there exists a \( \lambda_0 \in \mathbb{C} \setminus \{0, 1\} \) such that \( A_{\lambda_0} \) is invertible. For every \( P \in \mathcal{I}(X) \setminus \{0\} \), \( A_{\lambda_0} P \in \mathcal{I}(X) \setminus \{0\} \) implies that \( A_{\lambda_0} P A_{\lambda_0} P = A_{\lambda_0} P \) and then \( PA_{\lambda_0} P = P \). So, \( P = PA_{\lambda_0} P = \lambda_0 P + (1 - \lambda_0)PAP \). Thus, \( PAP = P \) and \( (AP)^2 = AP \). By Lemma 2.5, we know that \( A_\lambda P = \lambda P + (1 - \lambda)AP \in \mathcal{I}(X) \setminus \{0\} \) for any \( \lambda \in \mathbb{C} \).
On the other hand, for any \( Q \in \mathcal{I}(X) \setminus \{0\} \) there exists a \( P \in \mathcal{I}(X) \setminus \{0\} \) such that \( \varphi(P) = Q \). If \( \varphi(A_{\lambda_0}Q) = \varphi(A_{\lambda_0})P \in \mathcal{I}(X) \setminus \{0\} \), then \( A_{\lambda_0}P \in \mathcal{I}(X) \setminus \{0\} \). By the preceding note, \( A_{\lambda}P \in \mathcal{I}(X) \setminus \{0\} \) for any \( \lambda \in \mathbb{C} \). Thus we have that \( \varphi(A_{\lambda}) \varphi(P) = \varphi(A_{\lambda}Q) \in \mathcal{I}(X) \setminus \{0\} \) for all \( \lambda \in \mathbb{C} \). It follows that \( \varphi(A_{\lambda}) \) are non-zero idempotents for all \( \lambda \in \mathbb{C} \). Hence \( \varphi(P) = \mu \) is of rank-1. By considering \( \varphi(A_{\lambda}) \) we know that \( \lambda \) is of rank-1. By the assumption, \( \lambda \) is an at least rank-1 idempotent. Assume that rank \( \varphi(P) > n \). Then we have a rank-\( n \) idempotent \( \varphi(Q) < \varphi(P) \). Then \( \varphi^{-1}(Q) \) is also an at least rank-\( n \) idempotent by the assumption. Since \( (\lambda I+(1-\lambda)\varphi(P))Q = Q \) for all \( \lambda \in \mathbb{C} \), it follows that \( (\lambda I+(1-\lambda)P)\varphi^{-1}(Q) \) are non-zero idempotents for all \( \lambda \in \mathbb{C} \) by Lemma 3.2. Thus we have that \( \varphi^{-1}(Q) \) is of rank-\( n \) by Corollary 2.6. In particular, \( P\varphi^{-1}(Q) \in \mathcal{I}(X) \setminus \{0\} \) and \( \varphi^{-1}(Q) \varphi^{-1}(Q)P = \varphi^{-1}(Q)P\varphi^{-1}(Q) \), which implies that both \( P\varphi^{-1}(Q) \) and \( \varphi^{-1}(Q)P \) are rank-\( n \) idempotents. On the other hand, \( (P\varphi^{-1}(Q)P)^2 = P\varphi^{-1}(Q)P\varphi^{-1}(Q)P = P\varphi^{-1}(Q)P\varphi^{-1}(Q)P = P\varphi^{-1}(Q)P \) and \( P\varphi^{-1}(Q)P = P\varphi^{-1}(Q)P \). That is, \( P\varphi^{-1}(Q)P \) is an idempotent with \( P\varphi^{-1}(Q)P \leq P \). Noting that \( P\varphi^{-1}(Q)P = \varphi^{-1}(Q) \) is of rank-\( n \), so we get that \( P\varphi^{-1}(Q)P \) is a rank-\( n \) idempotent. Therefore \( P\varphi^{-1}(Q)P = P \) since \( P \) is of rank-\( n \). Thus, \( (\lambda I+(1-\lambda)\varphi^{-1}(Q))P \in \mathcal{I}(X) \setminus \{0\} \) by Lemma 2.5. It follows that \( (\lambda I+(1-\lambda)\varphi(P))P \in \mathcal{I}(X) \setminus \{0\} \) for any \( \lambda \in \mathbb{C} \) by Lemma 3.2. However, we have \( (\lambda I+(1-\lambda)Q)\varphi(P) = Q + \lambda(\varphi(P) - Q) \), which can not be idempotents for all \( \lambda \neq 1 \). This contradiction implies that \( \varphi(P) \) is of rank-\( n \). Thus \( \varphi \) preserves rank-\( n \) idempotents and so does \( \varphi^{-1} \). The proof is complete. \( \square \)

**Lemma 3.3.** \( \varphi \) preserves rank-\( n \) idempotents in both directions.

**Proof.** Note that \( \varphi \) preserves idempotents in both direction from Lemma 3.1. We use the induction to complete the proof. We first prove that \( \varphi \) preserves rank-1 idempotents in both direction.

Let \( P = x \otimes f \) for some \( x \in \mathcal{X}, f \in \mathcal{X}' \) with \( f(x) = 1 \). Then, \( \varphi(P) \) is a non-zero idempotent. If rank \( \varphi(P) > 2 \), then \( (\lambda I+(1-\lambda)\varphi(P))Q \) are non-zero idempotents for all \( \lambda \in \mathbb{C} \) and \( Q \in \mathcal{I}(X) \setminus \{0\} \). Thus we have that \( \varphi(A_{\lambda}) \varphi(P) = \varphi(A_{\lambda})Q \in \mathcal{I}(X) \setminus \{0\} \) for all \( \lambda \in \mathbb{C} \). Hence, \( \varphi(\text{Gcv}(I, \varphi(A_{\lambda}))) \subseteq \text{Gcv}(I, \varphi(A)) \). Moreover, \( \varphi^{-1} \) has the same property of \( \varphi \). Therefore, \( \varphi(\text{Gcv}(I, A)) = \text{Gcv}(I, \varphi(A)) \). The proof is complete. \( \square \)

**Lemma 3.4.** \( \varphi \) preserves rank-1 operators in both directions.

**Proof.** Let \( A = e \otimes f \) for some \( e \in \mathcal{X} \) and \( f \in \mathcal{X}' \) and \( B = \varphi(A) \). Since \( AP \in \mathcal{I}(X) \setminus \{0\} \) is equivalent to \( PA \in \mathcal{I}(X) \setminus \{0\} \) for every \( P \in \mathcal{I}(X) \setminus \{0\} \), \( B \varphi(P) \in \mathcal{I}(X) \setminus \{0\} \) is equivalent to \( \varphi(P)B \in \mathcal{I}(X) \setminus \{0\} \) for every \( P \in \mathcal{I}(X) \setminus \{0\} \). Moreover, \( \varphi(\mathcal{I}(X) \setminus \{0\}) = \mathcal{I}(X) \setminus \{0\} \) and \( \varphi \) is a bijective map. We know that \( BQ \in \mathcal{I}(X) \setminus \{0\} \) is equivalent to \( QB \in \mathcal{I}(X) \setminus \{0\} \) for every \( Q \in \mathcal{I}(X) \setminus \{0\} \). Since \( \dim \mathcal{X} \geq 3 \), there exist \( x, y \in \mathcal{X} \) such that \( e, x \) and \( y \) are linearly independent.
such that \( f(x) = 1 \). Under the direct sum decomposition \( \mathcal{X} = \text{span}\{e, x, y\} \oplus \mathcal{X}_1 \), we may define a bounded linear operator \( P_1 \) on \( \mathcal{X} \) in the way that \( P_1 e = x \), \( P_1 x = x \), \( P_1 y = y \) and \( P_1(\mathcal{X}_1) = 0 \). Then, \( P_1 \) is a rank-2 idempotent and \((e \otimes f)P_1 \in \mathcal{I}(\mathcal{X})\setminus\{0\} \). Then, \( \text{rank} \varphi(P_1) = 2 \) by Lemma 3.3 and \( B\varphi(P_1) \in \mathcal{I}(\mathcal{X})\setminus\{0\} \). In fact, \( \varphi(P_1)B\varphi(P_1) \neq \varphi(P_1) \). Assume that \( \varphi(P_1)B\varphi(P_1) = \varphi(P_1) \), then the fact that \( B\varphi(P_1) \in \mathcal{I}(\mathcal{X})\setminus\{0\} \) and Lemma 2.5 ensure that \((\lambda I + (1 - \lambda)B)\varphi(P_1) \in \mathcal{I}(\mathcal{X})\setminus\{0\} \) for all \( \lambda \in \mathbb{C} \). Thus, \((\lambda I + (1 - \lambda)A)P_1 \in \mathcal{I}(\mathcal{X})\setminus\{0\} \) for all \( \mu \in \mathbb{C} \) by Lemma 3.2. So, rank \( P_1 \leq \text{rank} P_1 A P_1 \leq \text{rank} A = 1 \) by Corollary 2.6. This is a contradiction. Then \( \varphi(P_1)B\varphi(P_1) \neq \varphi(P_1) \). On the other hand, it is known that \( \varphi(P_1)B\varphi(P_1) \) is a non-zero idempotent from the fact \( B\varphi(P_1) \in \mathcal{I}(\mathcal{X})\setminus\{0\} \) and \( \varphi(P_1)B\varphi(P_1) = \varphi(P_1)B\varphi(P_1) \). It now follows that \( \text{rank} \varphi(P_1)B\varphi(P_1) = \text{rank} \varphi(P_1) = 2 \) since \( \text{rank} \varphi(P_1) = 2 \).

Let \( \mathcal{M}_1 = (\varphi(P_1)B\varphi(P_1))(\mathcal{X}) \), \( \mathcal{M}_2 = (\varphi(P_1) - \varphi(P_1)B\varphi(P_1))(\mathcal{X}) \) and \( \mathcal{M}_3 = (I - \varphi(P_1))(\mathcal{X}) \). Then, \( \dim \mathcal{M}_1 = \dim \mathcal{M}_2 = 1 \). Under the direct sum decomposition \( \mathcal{X} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 \), we have

\[
B = \begin{pmatrix}
1 & 0 & B_{13} \\
0 & 0 & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{pmatrix}
\quad \text{and} \quad
\varphi(P_1) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Since \( B\varphi(P_1) \in \mathcal{I}(\mathcal{X})\setminus\{0\} \) and \( B\varphi(P_1) \in \mathcal{I}(\mathcal{X})\setminus\{0\} \) is equivalent to \( \varphi(P_1)B \in \mathcal{I}(\mathcal{X})\setminus\{0\} \), we get that \( B_{23} = B_{32} = 0 \). On the other hand,

\[
Q_F = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & F \\
0 & 0 & 0
\end{pmatrix}
\in \mathcal{I}(\mathcal{X})\setminus\{0\}
\]

for all \( F \in \mathcal{B}(\mathcal{M}_3, \mathcal{M}_2) \) and

\[
BQ_F = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
B_{31} & 0 & 0
\end{pmatrix}
\in \mathcal{I}(\mathcal{X})\setminus\{0\}.
\]

Then,

\[
Q_FB = \begin{pmatrix}
1 & 0 & B_{13} \\
FB_{13} & 0 & FB_{33} \\
0 & 0 & 0
\end{pmatrix}
\in \mathcal{I}(\mathcal{X})\setminus\{0\}.
\]

By an elementary calculation, we have that \( FB_{31}B_{13} = FB_{33} \) for all \( F \in \mathcal{B}(\mathcal{M}_3, \mathcal{M}_2) \) and thus \( B_{31}B_{13} = B_{33} \).

Taking

\[
T = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-B_{31} & 0 & 1
\end{pmatrix},
\]

we know that \( T \) is invertible and rank \( B = \text{rank} TB = \text{rank} \begin{pmatrix} 1 & 0 & B_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1 \). That is, \( \varphi \) preserves rank-1 operators. Moreover, \( \varphi^{-1} \) has the same property as \( \varphi \). Therefore, \( \varphi \) preserves rank-1 operators in both directions. The proof is complete. \( \square \)

**Lemma 3.5.** \( \varphi \) preserves rank-1 nilpotent operators in both directions.

**Proof.** Let \( A = x \otimes f \) be not a nilpotent operator. Then, \( A = f(x)(f(x))^{-1}x \otimes f \) and \( (f(x))^{-1}x \otimes f \in \mathcal{I}(\mathcal{X}) \). Since \( A(f(x))^{-1}I = (f(x))^{-1}x \otimes f \in \mathcal{I}(\mathcal{X}) \), we know that \( \varphi(A)\varphi((f(x))^{-1}I) \in \mathcal{I}(\mathcal{X})\setminus\{0\} \). By Lemma 3.1, we have that \( \varphi((f(x))^{-1}I) = \kappa((f(x))^{-1}I) \).
Thus, $\varphi(A) = f(x)Q$ with $Q \in \mathcal{I}_1(\mathcal{X})$. Hence, $\varphi(A)$ is not nilpotent. On the other hand, $\varphi^{-1}$ has the same property of $\varphi$. Therefore, $\varphi$ preserves the rank-1 nilpotent operators in both directions. \hfill \Box

**Lemma 3.6.** Let $\kappa(\lambda)$ be the function defined in Lemma 3.1. Then $\varphi(\lambda A) = \kappa(\lambda) \varphi(A)$ for any rank-1 operator $A$ and all $\lambda \in \mathbb{C}$.

**Proof.** In order to complete the proof, we divide the proof into two steps.

**Step 1.** $\varphi(\lambda P) = \kappa(\lambda) \varphi(P)$ for every rank-1 idempotent $P$ and all $\lambda \in \mathbb{C}$.

Since $P$ is a rank-1 idempotent and $\lambda P\lambda^{-1}I \in \mathcal{I}(\mathcal{X}) \setminus \{0\}$ for any $\lambda \in \mathbb{C} \setminus \{0\}$, we know that $\varphi(\lambda P)\varphi(\lambda^{-1}I) = \mathcal{I}$ for some $Q_{\lambda} \in \mathcal{I}(\mathcal{X})$ by Lemmas 3.1 and 3.4. Then $\varphi(\lambda P) = \kappa(\lambda) Q_{\lambda}$. We claim that $Q_{\lambda} = \varphi(P)$ for all $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Suppose there is a $\lambda \in \mathbb{C} \setminus \{0, 1\}$ such that $Q_{\lambda} \neq \varphi(P)$. Let $\varphi(P) = \xi \otimes g$ and $Q_{\lambda} = \eta \otimes h$. Then $g(\xi) = h(\eta) = 1$. There are two cases to be discussed.

**Case 1.** $\xi$ and $\eta$ are linearly dependent. We may assume that $\xi = \eta$ if we replace $h$ by a multiple of $h$. Then $g$ and $h$ are linear independent. Otherwise if $h = \beta g$ for some $\beta \in \mathbb{C}$, then $\eta \otimes h = \beta \xi \otimes g = \xi \otimes g$ since both $\xi \otimes g$ and $\eta \otimes g$ are idempotents. This contradicts to the assumption. Hence $g$ and $h$ are linearly independent. Thus there exists an $x \in \mathcal{X}$ such that $g(x) = 1$ and $h(x) = \kappa(\lambda)^{-1}$. Since $(\xi \otimes g)(\xi \otimes g) = x \otimes g$ and $(\xi \otimes g)(\kappa(\lambda)Q_{\lambda}) = (\xi \otimes g)(\kappa(\lambda)\eta \otimes h) = \kappa(\lambda)(x \otimes h)$ are non-zero idempotents, we get that both $\varphi^{-1}(x \otimes g)P$ and $\varphi^{-1}(x \otimes g)(\lambda P)$ are non-zero idempotents. This is a contradiction since $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

**Case 2.** $\xi$ and $\eta$ are linearly independent. If $g$ and $h$ are linearly dependent, then we may assume that $g = h$ and $\varphi(P) = \xi \otimes g$ and $Q_{\lambda} = \eta \otimes g$. Since there exists an $f \in \mathcal{X}'$ such that $f(\xi) = 1$ and $f(\eta) = \kappa(\lambda)^{-1}$, we know that $(\xi \otimes f)(\xi \otimes g)$ and $(\xi \otimes f)(\kappa(\lambda)\eta \otimes g)$ are non-zero idempotents. We get that $\varphi^{-1}(\xi \otimes f)P$ and $\varphi^{-1}(\xi \otimes f)(\lambda P)$ are non-zero idempotents. This is a contradiction again since $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Next we assume that $g$ and $h$ are linearly independent. If $h(\xi) \neq 0$, then there exists an $f \in \mathcal{X}'$ such that $f(\xi) = 1$ and $f(\eta) = (\kappa(\lambda)h(\xi))^{-1}$. Since both $(\xi \otimes f)(\xi \otimes g)$ and $(\xi \otimes f)(\kappa(\lambda)\eta \otimes h)$ are non-zero idempotents, we get that both $\varphi^{-1}(\xi \otimes f)P$ and $\varphi^{-1}(\xi \otimes f)(\lambda P)$ are non-zero idempotents. This is a contradiction since $\lambda \in \mathbb{C} \setminus \{0, 1\}$. We may similarly get a contradiction when $g(\eta) \neq 0$. If $h(\xi) = g(\eta) = 0$ and let $B = \xi \otimes g + (\kappa(\lambda))^{-1}\eta \otimes h$, then both $B(\xi \otimes g)$ and $B(\kappa(\lambda)Q_{\lambda})$ are non-zero idempotents. Thus so are $\varphi^{-1}(B)P$ and $\varphi^{-1}(B)(\lambda P)$. Hence, we get a contradiction again since $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

Therefore, $\varphi(\lambda P) = \kappa(\lambda) \varphi(P)$ for every rank-1 idempotent $P$.

**Step 2.** $\varphi(\lambda N) = \kappa(\lambda) \varphi(N)$ for every rank-1 nilpotent operator $N$ and any $\lambda \in \mathbb{C} \setminus \{0, 1\}$. We first prove that $\varphi(N)$ and $\varphi(\lambda N)$ are linearly dependent for every $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Suppose there is a $\lambda \in \mathbb{C} \setminus \{0, 1\}$ such that $\varphi(N)$ and $\varphi(\lambda N)$ are linearly independent and let $\varphi(N) = x \otimes f$ and $\varphi(\lambda N) = y \otimes g$ for some $x, y \in \mathcal{X}$ and $f, g \in \mathcal{X}'$ with $f(x) = g(y) = 0$.

**Case 1.** $x$ and $y$ are linearly dependent. We may assume that $x = y$. Then $f$ and $g$ are linearly independent. We know there is an $h \in \mathcal{X}'$ such that $h(x) = 1$. Clearly, $f, g$ and $h$ are linearly independent. Thus there is a $z \in \mathcal{X}$ such that $h(z) = 0$ and $f(z) = g(z) = 1$. Note that $(z \otimes h)(x \otimes f) = z \otimes f$ and $(z \otimes h)(x \otimes g) = z \otimes g$ are non-zero idempotents. So are $\varphi^{-1}(z \otimes h)N$ and $\varphi^{-1}(z \otimes h)(\lambda N)$. This is a contradiction.

**Case 2.** $x$ and $y$ are linearly independent. If $f$ and $g$ are linearly dependent, we similarly have a contradiction as Case 1. So we assume that $f$ and $g$ are linearly independent. If $f(y)g(x) \neq 0$ and let $A = (f(y)g(x))^{-1}(x + y) \otimes (f + g)$, then both $A(x \otimes f) = (f(y))^{-1}(x + y) \otimes f$ and $A(y \otimes g) = (g(x))^{-1}(x + y) \otimes g$ are idempotents, which will leads to a contradiction since $\varphi^{-1}(A)N$ and $\lambda \varphi^{-1}(A)N$ can not be idempotents at the same time. Thus we must have
Let $A_1, A_2 \in \mathcal{F}_1(X)$. Then $A_1 \sim A_2$ if and only if $\varphi(A_1) \sim \varphi(A_2)$.

**Proof.** Since $A_1 \sim A_2$. By Proposition 2.7, there exists a $B \in \mathcal{F}_1(X)$ such that $B$ is linearly independent of $A_k(k = 1, 2)$ and for every $T \in \mathcal{B}(X)$ we have $A_kT \in \mathcal{F}(X) \setminus \{0\}(k = 1, 2)$ imply that $BT \in \mathcal{F}(X) \setminus \{0\}$. According to Lemma 3.6, we get that $\varphi(B)$ is linearly independent of $\varphi(A_k)(k = 1, 2)$. Moreover, for every $T \in \mathcal{B}(X)$ we have $\varphi(A_k)T \in \mathcal{F}(X) \setminus \{0\}(k = 1, 2)$ imply that $\varphi(B)(\varphi(T)) \in \mathcal{F}(X) \setminus \{0\}$. Using Proposition 2.7 again and the surjection of $\varphi$, we have that $\varphi(A_1) \sim \varphi(A_2)$. By considering $\varphi^{-1}$, the converse is similar. The proof is complete. □

The following lemma was proved in [7].

**Lemma 3.8.** Let $P$ and $Q$ with $P \neq Q$ be rank-1 idempotents. Then the following are equivalent:

(i) $P$ and $Q$ are orthogonal.
(ii) There exist rank-1 nilpotent operators $M$ and $N$ such that $P \sim N$, $P \sim M$, $Q \sim N$, $Q \sim M$ and $M \sim N$.

**Proof of Theorem 1.1.** The sufficiency part is clear.

Suppose $\varphi : \mathcal{B}(X) \to \mathcal{B}(X)$ is a unital surjective map satisfying for every pair $A, B \in \mathcal{B}(X) AB \in \mathcal{F}(X) \setminus \{0\}$ if and only if $\varphi(A)\varphi(B) \in \mathcal{F}(X) \setminus \{0\}$. Then, $\varphi$ preserves rank-1 idempotents in both directions by Lemma 3.3 and $\varphi$ preserves rank-1 nilpotent operators in both directions by Lemma 3.5. Moreover, applying Lemma 3.7 and Lemma 3.8, we know that $\varphi$ preserves the orthogonality of rank-1 idempotents in both directions. By Theorem 2.4 in [10], either there exists a bounded invertible linear or conjugate-linear operator $S : X \to X$ such that $\varphi(P) = SPS^{-1}$ for every rank-1 idempotent $P$, or $X$ is reflexive and there exists a bounded invertible linear or conjugate-linear operator $S : \mathcal{X} \to \mathcal{X}$ such that $\varphi(P) = SP'S^{-1}$ for every rank-1 idempotent $P$. We will show that only the first possibility occur. The following proof needs dividing into several steps.
Step 1. \( \varphi(A) = SAS^{-1} \) for every rank-1 nilpotent operator \( A \).

Let \( A \in \mathcal{N}_1(\mathcal{X}) \). For any rank-1 idempotent \( P \), we have that

\[
AP \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \iff S(\lambda P)S^{-1} \in \mathcal{I}(X) \setminus \{0\}
\]
\[
\iff SAS^{-1}SPS^{-1} \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \iff SAS^{-1}\varphi(P) \in \mathcal{I}(\mathcal{X}) \setminus \{0\}
\]
\[
\iff \varphi(A)\varphi(P) \in \mathcal{I}(\mathcal{X}) \setminus \{0\}.
\]

Then, \( \varphi(A) = \lambda I + (1 - \lambda)SAS^{-1} \) for some \( \lambda \in \mathbb{C}\setminus\{1\} \) by Proposition 2.3 and \( \varphi(A) \in \mathcal{N}_1(\mathcal{X}) \) by Lemma 3.5. Since \( \sigma(\varphi(A)) = \lambda + (1 - \lambda)\sigma(SAS^{-1}) \) and \( \sigma(\varphi(A)) = \{0\} \), we get that \( \lambda = 0 \) and \( \varphi(A) = SAS^{-1} \).

Step 2. \( \varphi(A) = SAS^{-1} \) for every non-scalar operator \( A \in \mathcal{B}(\mathcal{X}) \).

Let \( A, B \in \mathcal{B}(\mathcal{X}) \) be any non-scalar operator \( A \in \mathcal{B}(\mathcal{X}) \). Then the fact that \( AR \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \iff BR \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \) for every \( R \in \mathcal{I}_1(\mathcal{X}) \cup \mathcal{N}_1(\mathcal{X}) \), implies \( A = B \). Indeed, from the fact that \( AR \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \iff BR \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \) for every \( R \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \), we know that \( B = \lambda I + (1 - \lambda)A \) for some \( \lambda \in \mathbb{C}\setminus\{1\} \) by Proposition 2.3. Choose an \( x \in \mathcal{X} \) such that \( x \) and \( Ax \) are linear independent. Then there is an \( f \in \mathcal{X} \) such that \( f(x) = 0 \) and \( f(Ax) = 1 \). Then \( R = x \otimes f \in \mathcal{N}_1(\mathcal{X}) \) and \( AR \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \). Thus we have that \( BR \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \), which implies that \( f(Bx) = \lambda f(x) + (1 - \lambda)f(Ax) = 1 - \lambda = 1 \). That is, \( \lambda = 0 \). Hence, \( A = B \). Since

\[
AR \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \iff S(AR)S^{-1} \in \mathcal{I}(\mathcal{X}) \setminus \{0\}
\]
\[
\iff SAS^{-1}\varphi(R) \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \iff \varphi(A)\varphi(R) \in \mathcal{I}(\mathcal{X}) \setminus \{0\}
\]

for every \( R \in \mathcal{I}_1(\mathcal{X}) \cup \mathcal{N}_1(\mathcal{X}) \), we have that \( \varphi(A) = SAS^{-1} \).

Step 3. \( \varphi(\lambda I) = S(\lambda I)S^{-1} \) for any \( \lambda \in \mathbb{C}\setminus\{0, 1\} \).

As we know that \( \varphi(\lambda P) = \kappa(\lambda)\varphi(P) \) for any rank-1 idempotent \( P \) and any \( \lambda \in \mathbb{C}\setminus\{0, 1\} \), it follows that \( S(\lambda P)S^{-1} = \kappa(\lambda)SAS^{-1} \) if \( S \) is linear or \( \kappa(\lambda)SAS^{-1} \) if \( S \) is conjugate-linear. Thus \( \kappa(\lambda) = \lambda \) or \( \kappa(\lambda) = \lambda \bar{\lambda} \) for any \( \lambda \in \mathbb{C}\setminus\{0, 1\} \). In both cases, we have \( \varphi(\lambda I) = S(\lambda I)S^{-1} \) for any \( \lambda \in \mathbb{C}\setminus\{0, 1\} \). Therefore, the desired conclusion follows.

Now we show that the second case can not occur. In fact, we know that for any \( A \in \mathcal{B}(\mathcal{X}) \) and any rank-1 operator \( R, AR \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \) if and only if \( RA \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \). Then by a similar way we may show that \( \varphi(A) = SAS^{-1} \) for all \( A \in \mathcal{B}(\mathcal{X}) \) if the second possibility occurs. However, since \( \dim \mathcal{X} \geq 3 \), \( \varphi \) with this form does not satisfy the condition \((\ast)\). Thus this case can not occur. The proof is complete. \( \square \)

Proof of Theorem 1.2. It is similar to that of Theorem 1 by applying Theorem 2.3 in [10]. \( \square \)

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References
