We consider the class $\mathcal{S}_n$ of all real positive semidefinite $n \times n$ matrices, and the subclass $\mathcal{S}_n^+$ of all $A \in \mathcal{S}_n$ with non-negative entries. For a positive, non-integer number $\alpha$ and some $A \in \mathcal{S}_n^+$, when will the fractional Hadamard power $A^{\alpha}$ again belong to $\mathcal{S}_n^+$? It is known that, for a specific $\alpha$, this holds for all $A \in \mathcal{S}_n^+$ if and only if $\alpha > n - 2$. Now let $A \in \mathcal{S}_n^+$ be of the form $A = T + V$, where $T \in \mathcal{S}_n^+$ has rank 1 and $V \in \mathcal{S}_n$ has rank $p \geq 1$. If the Hadamard quotient of $T$ and $V$ is Hadamard independent (‘in general position’) and $V$ has ‘sufficiently small’ entries, then a complete answer is given, depending on $n$, $p$, and $\alpha$. Special attention is given to the case that $p = 1$.

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1. Hadamard powers

Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of the same size, not necessarily square. Then their Hadamard product (also called Schur product) $A \odot B$ is defined by entrywise multiplication: $A \odot B = (a_{ij}b_{ij})$. The Hadamard unit matrix is the matrix $U$
all of whose entries are 1 (the size of $U$ being understood). A matrix $A$ is Hadamard invertible if all its entries are non-zero, and $A^{\Diamond(-1)} = (a^{-1}_{ij})$ is then called the Hadamard inverse of $A$. If $B$ is Hadamard invertible, then the Hadamard quotient $A^{\Diamond}/B$ of $A$ and $B$ is $(a_{ij}b_{ij}^{-1})$. The $k$-fold Hadamard product $A^{\Diamond k}$ of $A$ with itself $(k \geq 0)$ is called the $k$th Hadamard power of $A$; thus $(a_{ij})^{\Diamond k} = (a_{ij}^k)$. In particular, $A^{\Diamond 0} = U$ (we set $0^0 = 1$). If $A$ is Hadamard invertible, then $A^{\Diamond k}$ can be defined for negative integers as well, in an obvious manner. For more information on the Hadamard product, see [5] and [7, Chapter 5].

In this paper we restrict our attention to real matrices. If all entries of $A$ are non-negative, then we can also consider fractional Hadamard powers of $A$: if $A = (a_{ij})$ with $a_{ij} \geq 0$ for all $i, j$, and $\alpha \in \mathbb{R}, \alpha \geq 0$, then $A^{\Diamond \alpha} = (a_{ij}^\alpha)$. If $a_{ij} > 0$ for all $i, j$, then $A^{\Diamond \alpha}$ can be defined for all $\alpha \in \mathbb{R}$. Note that $(A^{\Diamond}B)^{\Diamond \alpha} = A^{\Diamond \alpha}B^{\Diamond \alpha}$ whenever all these fractional Hadamard powers are defined.

The transpose of a matrix $A$ is denoted by $A^*$. We identify matrices of size $n \times 1$ (column vectors) with elements of $\mathbb{R}^n$, and all the above notions apply to elements of $\mathbb{R}^n$. We denote the Hadamard unit vector by $u^*$; thus $u^* = (1, \ldots, 1)^*$. There is an interesting difference between matrices of rank one and matrices of higher rank: if a matrix $A$ with non-negative entries has rank one, then the same holds for all its (non-negative) fractional Hadamard powers. But if its rank is at least two, then $A^{\Diamond \alpha}$ has maximal rank for all but finitely many $\alpha > 0$, unless there are certain ‘obstructions’, such as a row or column of zeros, or two linearly dependent rows or columns.

2. Positive semidefinite matrices

A real $n \times n$ matrix $A$ is called positive semidefinite if $A$ is symmetric and $v^*Av \geq 0$ for all $v \in \mathbb{R}^n$. We write $\mathcal{S}_n$ for the set of all (real) positive semidefinite $n \times n$ matrices, and $\mathcal{S}_n^+$ for those matrices in $\mathcal{S}_n$ for which all entries are non-negative. The following proposition is a fundamental result of Schur [6, 7.5.3].

**Proposition 2.1.** If $A, B \in \mathcal{S}_n$, then also $A^{\Diamond}B \in \mathcal{S}_n$. In particular, if $A \in \mathcal{S}_n$, then $A^{\Diamond k} \in \mathcal{S}_n$ for all non-negative integers $k$.

Let $A = (a_{ij})$ be an $m \times n$ matrix. For $\emptyset \neq \lambda \subset \{1, \ldots, m\}$ and $\emptyset \neq \mu \subset \{1, \ldots, n\}$, the submatrix $A_{\lambda \mu}$ is defined as $A_{\lambda \mu} = (a_{ij})_{i \in \lambda, j \in \mu}$. If $m = n$ and $\emptyset \neq \lambda \subset \{1, \ldots, n\}$, then $A_{\lambda \lambda}$ is called a principal submatrix of $A$. The determinant of a square submatrix is called a minor, of a principal submatrix a principal minor of $A$ [6, Section 0.7.1]. For the next proposition we refer to [6, 7.1.2 and 7.2.5].

**Proposition 2.2.** A symmetric real $n \times n$ matrix $A$ is positive semidefinite if and only if all its principal minors are non-negative.
Now consider the following problem: for a matrix $A \in \mathcal{S}_n^+$ and a non-negative real number $\alpha$, when does $A^{\alpha}$ again belong to $\mathcal{S}_n^+$? For integer values of $\alpha$ this is always the case, by Proposition 2.1. It is further known that $A^{\alpha} \in \mathcal{S}_n^+$ holds for all $A \in \mathcal{S}_n^+$ if and only if $\alpha$ is an integer or $\alpha > n - 2$ [3, Theorem 2.2]. The sufficiency of this condition is proved by an inductive argument. Its necessity follows from an earlier more general result [4, Theorem 1.2 and Corollary 1.3], but a new proof is also given: an explicit example is constructed, for $\alpha < n - 2$ (and $\alpha$ not an integer) of a matrix $A \in \mathcal{S}_n^+$, of rank 2 and very close to the Hadamard unit matrix $U$, for which $A^{\alpha} \notin \mathcal{S}_n$ [3, p. 636]. This example inspired us to do the research of the present paper.

In fact, we study the following question: if a matrix $T \in \mathcal{S}_n^+$ of rank 1 and with strictly positive entries is approximated by matrices of the form $T + \varepsilon V$, with $V \in \mathcal{S}_n$, when will $(T + \varepsilon V)^{\alpha}$ have a negative determinant for sufficiently small $\varepsilon > 0$? In other words, we approach $T$ along straight lines in $T + \mathcal{S}_n$, the positive semidefinite cone at $T$, and study the behaviour of the determinant of the $\alpha$th power. Some restriction in the choice of the matrices $V$ is necessary, though, for the following reason. As was pointed out in Section 1, if $A \in \mathcal{S}_n^+$ has rank at least 2, then in general the rank of $A^{\alpha}$ will be maximal for most values of $\alpha > 0$. However, for matrices $A$ whose entries are “not in general position” it happens that no fractional power is of maximal rank. For instance, if $A$ has a row of zeros, or if two of its rows are linearly dependent, then the same is the case for $A^{\alpha}$. To avoid such exceptional matrices we introduce the notion of Hadamard independent matrices (Definition 5.11; see also Example 5.12). In this connection we introduce the notion of a cloud to represent a positive semidefinite matrix.

Theorems 7.2 and 7.5 provide an answer to the above question. Of course, the answer depends on the value of $\alpha$, but in fact, as we shall see, only on its integer part $[\alpha]$. It further turns out that the sign of the determinant (for $\varepsilon > 0$ small enough) depends only on the value $p$ of the rank of $V$ in relation to the size $n$ of the matrix. For instance, if the approximation is done with a matrix $V$ of rank $p$, then for $0 < \alpha < 1$ the determinant of the $\alpha$th power of $T + \varepsilon V$ will be positive for sufficiently small $\varepsilon > 0$ when $p = n$ or $p = n - 1$, but negative if $p = n - 2$.

**Remark 2.3.** We should mention here a result that is of interest in connection with the rank of fractional Hadamard powers of positive semidefinite matrices. In [1, Theorem 2.2] it is proved that if such a matrix $A$ is infinitely divisible (see [1] or [4] for a definition), then the matrices $A^{\alpha}$, $\alpha > 0$, all have the same rank.

### 3. Clouds

A matrix $A \in \mathcal{S}_n$ has rank one if and only if $A = vv^*$ for some non-zero $v \in \mathbb{R}^n$. More generally, $A$ has rank $p$, $1 \leq p \leq n$, if and only if there is an $n \times p$ matrix $V$ such that...
of rank $p$ such that $A = VV^*$. Denoting the column vectors of $V$ by $v_j$, $1 \leq j \leq p$, and its row vectors by $\tilde{v}_i$, $1 \leq i \leq n$, we get

$$A = v_1v_1^* + \cdots + v_pv_p^* = (\tilde{v}_i \cdot \tilde{v}_j)i,j, \quad (1)$$

where the dot denotes the usual inner product. The $v_j$ belong to $\mathbb{R}^n$, the $\tilde{v}_i$ to $\mathbb{R}^p$, and $A$ is called the Gram matrix of the $\tilde{v}_i$; see [6, p. 407].

If $V$ and $W$ are two $n \times p$ matrices of rank $p$, then $VV^* = WW^*$ if and only if there exists an orthogonal $p \times p$ matrix $S$ such that $W = VS$; this follows, as a special case, from [8, Theorem 3.1]. Consequently, two $n$-tuples $(\tilde{v}_1, \ldots, \tilde{v}_n)$ and $(\tilde{w}_1, \ldots, \tilde{w}_n)$ of vectors in $\mathbb{R}^p$ have the same Gram matrix if and only if there is an orthogonal $p \times p$ matrix $S$ (the adjoint of the previous $S$) such that $S\tilde{v}_i = \tilde{w}_i$, $1 \leq i \leq n$. This motivates the following definition.

**Definition 3.1.** Let $1 \leq p \leq n$ be fixed. Two $n$-tuples $(\tilde{v}_1, \ldots, \tilde{v}_n)$ and $(\tilde{w}_1, \ldots, \tilde{w}_n)$ of vectors in $\mathbb{R}^p$ are called equivalent if there is an orthogonal transformation $S$ of $\mathbb{R}^p$ such that $S\tilde{v}_i = \tilde{w}_i$ $(1 \leq i \leq n)$. The equivalence class to which $(\tilde{v}_1, \ldots, \tilde{v}_n)$ belongs is denoted by $[\tilde{v}_1, \ldots, \tilde{v}_n]$. A class $[\tilde{v}_1, \ldots, \tilde{v}_n]$ is called a cloud of size $n$ in $\mathbb{R}^p$ (or, simply, a cloud, when $n$ and $p$ are understood). Each representing element $(\tilde{v}_1, \ldots, \tilde{v}_n)$ of a cloud is called a positioning of the cloud. The dimension of the linear span of any positioning of a cloud is called the dimension of the cloud. The Gram matrix (1) of a positioning of a cloud is called the Gram matrix of the cloud.

It is evident that there is no ambiguity in the definition of the dimension of a cloud. We do not require that the dimension of a cloud in $\mathbb{R}^p$ is $p$. If a cloud in $\mathbb{R}^p$ has dimension $q < p$, then, clearly, its Gram matrix can also be represented by a cloud in $\mathbb{R}^q$.

To formulate our next theorem, let $K$ be any index set and let $\lambda$ be a subset of $K$ with $n$ elements. Let $v_k$, $k \in K$, be vectors in $\mathbb{R}^n$. The volume of the parallelepiped spanned by the vectors $v_k$ $(k \in \lambda)$ is denoted by $S_{\lambda}$. Explicitly: if $V_{\lambda}$ is a matrix whose columns are the $v_k$ with $k \in \lambda$, in any order, then $S_{\lambda} = |\det V_{\lambda}|$. Furthermore, for scalars $c_k$, $k \in K$, we set $c_{\lambda} = \prod_{k \in \lambda} c_k$.

**Theorem 3.2.** Let $K$ be any index set and let for each $k \in K$ a column vector $v_k \in \mathbb{R}^n$ and a scalar $c_k$ be given such that $\sum_{k \in K} c_k v_k v_k^*$ is entrywise absolutely convergent. Then:

$$\det \left( \sum_{k \in K} c_k v_k v_k^* \right) = \sum_{\lambda \subseteq K, |\lambda| = n} c_{\lambda} S_{\lambda}^2.$$

**Proof.** For finite $K$ this is Proposition 1 in [2]. The general case follows by continuity. □
We end this section with a notion that will be used in Sections 7 and 8. The multiplicative trace \( \text{mtr}(A) \) of a square matrix \( A = (a_{ij}) \) is defined by \( \text{mtr}(A) = \prod_i a_{ii} \). If \( A \in S_n \), then \( \text{mtr}(A) \geq 0 \), and \( \text{mtr}(A) = 0 \) occurs if and only if \( A \) has a complete row (and corresponding column) of zeros. Also, if \( A \in S_n^+ \) and \( \alpha \geq 0 \), then \( \text{mtr}(A^{\alpha}) = (\text{mtr}(A))^\alpha \).

**Lemma 3.3.** Suppose that \( A \in S_n \) has rank 1. Let \( B \) be any \( n \times n \) matrix. Then
\[
\det(A \odot B) = \text{mtr}(A) \det(B).
\]

**Proof.** Take \( a \in \mathbb{R}^n \) such that \( A = aa^* \). Let \( D \) be the diagonal matrix with the entries of \( a \) as entries on the diagonal. Then \( A \odot B = DBD \), where in the right-hand side we have the ordinary matrix product. The lemma follows. \( \square \)

### 4. Polynomial coefficients

Throughout, we adopt the convention that an empty sum is equal to zero, an empty product to one (e.g., \( 0^0 = 1 \)). We set \( \mathbb{N} = \{0, 1, 2, \ldots\} \). Let \( p \in \mathbb{N} \). For \( m = (m_1, \ldots, m_p) \in \mathbb{N}^p \) we define
\[
|m| = \sum_{i=1}^{p} m_i, \quad m! = \prod_{i=1}^{p} m_i!,
\]
and for \( \lambda \subset \mathbb{N}^p \) we set
\[
|\lambda| = \text{Card}(\lambda), \quad \|\lambda\| = \sum_{m \in \lambda} |m|.
\]

For convenience of notation, we denote the binomial coefficients \( \binom{m}{k} \) by \( (m \mid k) \), \( 0 \leq k \leq m \). It will further be useful to have a separate symbol, \( (m\|k) \), for the binomial coefficients \( (m + k \mid k) \). Explicitly, we define:
\[
(m\|k) = (m + k \mid k) = \frac{1}{k!} \prod_{i=1}^{k} (m + i) \quad (m, k \geq 0).
\]

Note that \( (m\|k) = (k\|m) \). The well-known formula \( (m + 1 \mid k) = (m \mid k) + (m \mid k - 1) \) yields
\[
(m + 1\|k) = (m\|k) + (m + 1\|k - 1).
\]

The following equalities are easily seen to hold for arbitrary integers \( p \geq 1 \) and \( k \geq 0 \).
\[
\text{Card}[m \in \mathbb{N}^p \mid |m| = k] = (p - 1\|k),
\]
\[
\text{Card}[m \in \mathbb{N}^p \mid |m| \leq k] = (p\|k).
\]
Definition 4.1. For \( p, n, l \in \mathbb{N} \) we define (there is no need to exclude the trivial case \( p = 0 \)):

(i) \( A(p, n) = \{ \lambda \mid \lambda \subset \mathbb{N}^p, |\lambda| = n \} \);
(ii) \( A(p, n, l) = \{ \lambda \mid \lambda \in A(p, n), \|\lambda\| = l \} \);
(iii) \( L(p, n) = \min \|\lambda\| \mid \lambda \in A(p, n) \} \).

Example 4.2. For \( n = 1 \) and arbitrary \( p \geq 1 \) we have \( A(p, 1) = \mathbb{N}^p \), if \( [m] \) and \( m \) are identified. Then \( A(p, 1, l) = \{ [m] \in \mathbb{N}^p \mid |[m]| = l \} \). In particular, \( A(p, 1, 0) = \{ [0, \ldots, 0] \} \) and \( L(p, 1) = 0 \). If \( 2 \leq n \leq p + 1 \), then \( L(p, n) = 1 \times 0 + (n - 1) \times 1 = n - 1 \). If \( n = p + 2 \), then \( L(p, n) = n \).

We shall mostly write \( L \) instead of \( L(p, n) \). For instance, the set \( A(p, n, L(p, n)) \) of all \( \lambda \in A(p, n) \) for which \( \|\lambda\| \) is minimal is denoted by \( A(p, n, L) \). Clearly:

\[
A(p, n) = \bigcup_{l=L}^{\infty} A(p, n, l). \tag{8}
\]

For \( i, j \in \mathbb{Z} \) we set \( [i, j] = \{ n \in \mathbb{Z} \mid i \leq n \leq j \} \), a (finite) interval in \( \mathbb{Z} \). Finite intervals in \( \mathbb{N} \) delimited by binomial coefficients \( (m\|k) \) play an important part later on. We therefore define:

\[
D_m(k) = [(m\|k), (m\|k + 1)] \quad (m \geq 1, k \geq 0). \tag{9}
\]

For each fixed \( m \geq 1 \) the \( D_m(k) \) \( (k \geq 0) \) are subsequent intervals, two adjacent intervals having one point in common. For instance, for \( m = 3 \) these intervals are: \([1, 4], [4, 10], [10, 20], \ldots \). Note that for \( m = 0 \) one would get \( D_0(k) = \{1\} \) for all \( k \).

The number of elements in \( D_m(k) \) (denoted as \( |D_m(k)| \)) is one more than the difference between its last and its first element. Applying (5) (with \( k \) and \( m \) interchanged) we therefore obtain:

\[
|D_m(k)| = 1 + (m - 1\|k + 1). \tag{10}
\]

Lemma 4.3. Let \( p \) and \( n \) be positive integers. Let \( k \geq 0 \) be such that \( n \in D_p(k) \).
Consider an element \( \lambda \in A(p, n) \). Then \( \lambda \in A(p, n, L) \) if and only if

\[
[m \in \mathbb{N}^p \mid |m| \leq k] \subset \lambda \subset [m \in \mathbb{N}^p \mid |m| \leq k + 1]. \tag{11}
\]

Proof. Set \([m \mid |m| \leq k] = M_k \). From (7), (i) of Definition 4.1, and the definition of \( k \) it follows that

\[
\text{Card } M_k \leq \text{Card } \lambda \leq \text{Card } M_{k+1}. \tag{12}
\]

If \( \lambda \in A(p, n) \) and \( M_k \not\subset \lambda \), then also \( \lambda \not\subset M_k \), by (12). Therefore, one can replace an element of \( \lambda \setminus M_k \) by an element of \( M_k \setminus \lambda \) and obtain an element \( \lambda' \in A(p, n) \) with \( \|\lambda'\| < \|\lambda\| \). It follows that \( \|\lambda\| \) is not minimal in \( A(p, n) \). A similar reasoning shows that \( \|\lambda\| \) is not minimal in \( A(p, n) \) if \( \lambda \not\subset M_{k+1} \). So, (11) is a necessary condition
for minimality of $\|\lambda\|$. But it is also sufficient, because for all $\lambda$ satisfying (11) $\|\lambda\|$ takes the same value. □

If $n = (p\|k + 1)$ for some $k \geq 0$, then $n \in D_p(k) \cap D_p(k + 1)$. In this case, however, $A(p, n, L)$ contains only one element: $\lambda = M_{k+1}$, so (11) holds for both $k$ and $k + 1$. In general, if $n \in D_p(k)$, then to obtain a $\lambda \subset \mathbb{N}_p$ with $|\lambda| = n$ and $\|\lambda\|$ minimal, one has to choose $n - (p\|k)$ elements from a set of $(p - 1\|k + 1)$ elements, by (7) and (6), respectively. Therefore,

$$\text{Card}(A(p, n, L)) = ((p - 1\|k + 1))|n - (p\|k)).$$

(13)

We further need some generalizations of the usual binomial coefficients. For $m \in \mathbb{N}_p$ ($p \geq 1$) we define the $p$-nomial coefficient

$$(|m||m|) = |m|! |m|! = \frac{(m_1 + \cdots + m_p)!}{m_1! \cdots m_p!},$$

(14)

which is the number of ways in which $|m|$ objects can be divided into $p$ numbered classes such that the $i$th class contains $m_i$ objects ($1 \leq i \leq p$). For $p = 2$ (14) is equal to $(m_1\|m_2)$. Definition (14) occurs in the well-known formula

$$(a_1 + \cdots + a_p)^k = \sum_{m \in \mathbb{N}_p : |m| = k} (k|m)a^m,$$

(15)

where for $a = (a_1, \ldots, a_p)$, $m = (m_1, \ldots, m_p)$ we use the notation:

$$a^m = \prod_{i=1}^p a_i^{m_i}.$$  

(16)

For $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$ one has the generalized binomial coefficients:

$$(\alpha|m) = \frac{\alpha(\alpha - 1) \cdots (\alpha - m + 1)}{m!},$$

(17)

which occur in the Taylor formula

$$(1 + x)^\alpha = \sum_{m=0}^{\infty} (\alpha|m)x^m \quad (\alpha \in \mathbb{R}, -1 < x < 1).$$

(18)

We now define for $m = (m_1, \ldots, m_p) \in \mathbb{N}_p$ ($p \geq 1$) the generalized $p$-nomial coefficients:

$$(\alpha|m) = \frac{\alpha(\alpha - 1) \cdots (\alpha - |m| + 1)}{|m|!}.$$ 

(19)

For $\alpha = |m|$ (19) coincides with (14). For $p = 1$ (19) coincides with (17), if $(\alpha|m)$ and $(\alpha|m)$ are identified. We note that for integers $k$ and $m$ with $k \geq m$ one has $(k|m) = (k[m, k - m])$. From (17), (14), and (19), respectively, one infers:

$$(\alpha||m)||(|m||m|) = (\alpha|m) \quad (\alpha \in \mathbb{R}, m \in \mathbb{N}_p (p \geq 1)).$$

(20)
Finally, for $\alpha \in \mathbb{R}$, $p \geq 1$, and $\lambda \subset \mathbb{N}^p$, $\lambda$ finite, we define:

$$ (\alpha | \lambda) = \prod_{m \in \lambda} (\alpha | m). \quad (21) $$

**Lemma 4.4.** Let $\alpha \in \mathbb{R}$ and let $p$ and $n$ be positive integers. Let the integer $k$ be such that $n \in D_p(k)$. If $\lambda \in A(p, n, L)$, then

$$ (\alpha | \lambda) = k \prod_{j=0}^{k} (\alpha - j)^{n - (j \parallel p)} / \prod_{m \in \lambda} m! . \quad (22) $$

**Proof.** Consider any $m \in \mathbb{N}^p$. The numerator of $(\alpha | m)$ as given in (19) contains a factor $\alpha - j$ if and only if $|m| > j$. For $\lambda \in A(p, n, L)$ there are $|\lambda| - \text{Card}\{m : |m| \leq j\} = n - (j \parallel p)$ such $m$ in $\lambda$. Therefore, for such a $\lambda$ the numerator of $(\alpha | \lambda)$ is equal to $\prod_{j=0}^{k} (\alpha - j)^{n - (j \parallel p)}$. Finally, the denominator of $(\alpha | \lambda)$, as given in (22), is obtained by combining (19) and (21). □

**Corollary 4.5.** Let $\alpha$, $p$, and $n$ be as in Lemma 4.4. Then for $\lambda \in A(p, n, L)$ the coefficients $(\alpha | \lambda)$ all have the same sign.

**Proof.** The numerator in (22) is the same for all $\lambda \in A(p, n, L)$. □

5. The Hadamard span

Let $E$ be any set. For $m = (m_v)_{v \in E} \in \mathbb{N}^E$ we define, analogous to (2):

$$ |m| = \sum_{v \in E} m_v. \quad (23) $$

If $E = \emptyset$, then $\mathbb{N}^E$ consists of one element $m$, satisfying $|m| = 0$. If $E$ is infinite, then $|m|$ can be $+\infty$. We denote by $\mathbb{N}_0^E$ the set of all $m \in \mathbb{N}^E$ for which $|m| < \infty$.

Now let $E$ be a subset of $\mathbb{R}^p$. Each $m \in \mathbb{N}_0^E$ defines a Hadamard product $v_m$ of elements of $E$, in the following way:

$$ v_m = \prod_{v \in E} \Diamond v \Diamond m_v \quad (m = (m_v) \in \mathbb{N}_0^E) \quad (24) $$

(the diamond attached to the product sign indicates that the product is taken in the Hadamard sense). If $E$ has $p$ elements, say $E = \{v_1, \ldots, v_p\}$, then $\mathbb{N}_0^E = \mathbb{N}^E \cong \mathbb{N}^p$ and (24) becomes

$$ v_m = \prod_{i=1}^{p} v_i \Diamond m_i = v_1 \Diamond m_1 \Diamond \cdots \Diamond v_p \Diamond m_p \quad (m \in \mathbb{N}^p). \quad (25) $$
The formulae (24) and (25) define mappings $m \mapsto v_m$ from $\mathbb{N}_0^E$ and $\mathbb{N}_0^p$, respectively, into $\mathbb{R}^n$.

**Definition 5.1.** Let $E$ be a subset of $\mathbb{R}^n$. The set $H(E) = \{v_m \mid m \in \mathbb{N}_0^E\}$ is called the **Hadamard span** of $E$. Note that $H(\emptyset) = \{u\}$.

The following terminology was introduced in [2].

**Definition 5.2.** A subset $E$ of a finite-dimensional vector space $X$ is called **quasi linearly independent**, or **qli** for shortness, if for all linear subspaces $Y$ of $X$ with $Y \neq X$ the set $Y \cap E$ contains at most $\dim(Y)$ elements. Otherwise, $E$ is called **quasi linearly dependent**, or **qld** for shortness.

If $Y = X$ is not excluded in Definition 5.2, then it reduces to ordinary linear (in)dependence. Some further easy observations are collected in the following proposition.

**Proposition 5.3.** Let $E$ be a subset of a finite-dimensional vector space $X$.

(i) $E$ is qli if and only if every subset of $E$ with at most $\dim X$ elements is linearly independent.

(ii) If $E$ has at least $\dim X$ elements, then $E$ is qli if and only if every subset of $E$ with $\dim X$ elements is a basis for $X$.

(iii) Write $E = \{v_i\}_i$. If $E$ is qli and $C = \{c_i\}_i$ is a set of non-zero real numbers, then also $EC = \{c_i v_i\}_i$ is qli.

**Example 5.4.** If $E \subset V$ is qli and $\dim V > 0$, then $0 \not\in E$. In $\mathbb{R}$ the set $\mathbb{R}\setminus\{0\}$ is qli. In $\mathbb{R}^2$ any curve not containing the origin and intersecting each line through the origin at most once, is qli.

The next definition is a combination of Definitions 5.1 and 5.2.

**Definition 5.5.** A subset $E$ of $\mathbb{R}^n$ is called **Hadamard quasi linearly independent** (**Hadamard qli** for shortness) if

(i) the mapping $m \mapsto v_m$ from $\mathbb{N}_0^E$ into $\mathbb{R}^n$ is injective;

(ii) the **Hadamard span** $H(E)$ of $E$ is quasi linearly independent.

If a set $E \subset \mathbb{R}^n$ is Hadamard qli, then so are all its subsets (including the empty set). It is therefore of interest to examine the case of a single vector $v \in \mathbb{R}^n$ more closely. When is a singleton $\{v\}$ Hadamard qli? Condition (i) of Definition 5.5 is satisfied unless all entries of $v$ are 0 or 1 or $-1$, in which case all even powers of $v$ are equal, likewise all odd powers, and even all powers if all non-zero entries of $v$
have the same sign. Condition (ii) is satisfied if the vectors \( u, v, v^\Diamond_2, v^\Diamond_3, \ldots \) form a qli set. For this latter condition we can give the following characterization.

**Proposition 5.6.** Let \( v \in \mathbb{R}^n \) be given. Then the vectors \( u, v, v^\Diamond_2, v^\Diamond_3, \ldots \) are qld if and only if there exists a non-zero polynomial \( P \) with at most \( n \) non-zero coefficients such that all entries of \( v \) are roots of \( P \).

**Proof.** By (i) of Proposition 5.3 the \( v^\Diamond_m, m \geq 0, \) are qld if and only if there are integers \( m_1, 1 \leq i \leq n, \) with \( 0 \leq m_1 < m_2 < \cdots < m_n \) such that the \( v^\Diamond_m, 1 \leq i \leq n, \) are linearly dependent. Now \( c_1 v^\Diamond_{m_1} + \cdots + c_n v^\Diamond_{m_n} = 0 \) holds if and only if \( c_1 x_{j_1} + \cdots + c_n x_{j_n} = 0 \) for all entries \( x_j, 1 \leq j \leq n, \) of \( v, \) hence if and only if all entries of \( v \) are roots of the polynomial \( P(x) = c_1 x_{m_1} + \cdots + c_n x_{m_n}. \) The \( v^\Diamond_m \) are linearly dependent if and only if there is a non-zero such \( P. \) \( \square \)

For the next proposition we need some notation. Let \( x_1, \ldots, x_n \) be the entries of some \( v \in \mathbb{R}^n. \) Denote the elementary symmetric sums of the \( x_j \) by \( e_i; \) thus

\[
e_i = \sum_{\lambda \subseteq \{1, \ldots, n\}, |\lambda| = i} \prod_{j \in \lambda} x_j \quad (1 \leq i \leq n).
\]

Let the polynomial \( Q_v \) be defined by \( Q_v(x) = \prod_{j=1}^n (x - x_j); \) then \( Q_v(x) = x^n + \sum_{i=1}^n (-1)^i e_i x^{n-i}. \)

**Proposition 5.7.** Let \( v \in \mathbb{R}^n \) be given, with entries \( x_1, \ldots, x_n \) and elementary symmetric sums \( e_1, \ldots, e_n. \) Then:

(i) The vectors \( u, v, v^\Diamond_2, \ldots, v^\Diamond(n-1) \) are linearly dependent if and only if not all entries of \( v \) are distinct.

(ii) Suppose that \( v \) has distinct entries. Then \( u, v, v^\Diamond_2, \ldots, v^\Diamond(n) \) are qld if and only if \( e_i = 0 \) for some \( i (1 \leq i \leq n). \) More specifically: if \( e_i = 0, \) then the \( v^\Diamond_m \) with \( 0 \leq m \leq n \) and \( m \neq n - i \) are linearly dependent.

(iii) If \( v \) has at least one entry equal to 0, then the vectors \( v, v^\Diamond_2, \ldots, v^\Diamond(n) \) are linearly dependent.

**Proof.** The determinant of the matrix with columns \( u, v, v^\Diamond_2, \ldots, v^\Diamond(n-1) \) is a Vandermonde determinant; its value is \( \prod_{1 \leq i < j \leq n} (x_i - x_j). \) This proves (i).

Concerning (ii), it follows as in the proof of Proposition 5.6 that the said vectors are qld if and only if there is a polynomial \( P \) of degree at most \( n \) and with at most \( n \) non-zero coefficients such that all entries of \( v \) are roots of \( P. \) Since the entries of \( v \) are distinct, \( P \) must be divisible by \( Q_v, \) and, as \( Q_v \) has degree \( n, \) \( P \) is a non-zero multiple of \( Q_v. \) Thus, quasi-linear dependence occurs if and only if \( Q_v \) has at most \( n \) non-zero coefficients, hence if and only if at least one of the \( e_i \) equals 0. This proves the first part of (ii). The specific case follows similarly.
Finally, if \( v \) has at least one entry equal to 0, then \( e_n = 0 \), and (iii) follows from (ii). Of course, (iii) is also evident without (ii): even all \( v^m \) \((m \geq 1)\) belong to the \((n - 1)\)-dimensional subspace of all vectors with that specific entry equal to 0. \( \square \)

It is instructive to verify the correctness of Proposition 5.7, and its proof, when \( n = 1 \).

**Theorem 5.8.** Let a vector \( v \in \mathbb{R}^n \) be given.

(i) If \( v \) has a zero entry, or if two of its entries are equal, then \( \{v\} \) is not Hadamard qli.

(ii) If the entries of \( v \) are distinct, non-zero, and of the same sign, then \( \{v\} \) is Hadamard qli.

**Proof.** In case (i) it follows from (iii) and (i) of Proposition 5.7, respectively, that the vectors \( u, v, v^2, \ldots, v^n \) are qld, and hence that \( \{v\} \) is not Hadamard qli.

Now suppose that \( v \) satisfies the conditions given in (ii). Then either all its entries are strictly positive, or they are strictly negative. Assume the former. Consider any polynomial \( P \) with at most \( n \) non-zero coefficients. By Descartes’ rule of signs— the number of positive roots of a real polynomial is at most equal to the number of changes of sign in its sequence of non-zero coefficients; if it is less, then by an even number (see [9, Part V, Chapter 1])—such a polynomial can have at most \( n - 1 \) positive roots. The desired result now follows from Proposition 5.6. If all entries of \( v \) are negative, the result follows by considering \(-v\) (iii) of Proposition 5.3) or by applying Descartes’ rule to \( P(-x) \). \( \square \)

**Remark 5.9.** In [2] we prove the stronger result that for a vector \( v \) as in (ii) of Theorem 5.8 even the set \( \{v^\alpha \mid \alpha \in \mathbb{R}\} \) is qli; see [2, Lemma 1 and the remark that follows it]. The preceding proof of the more restricted result is simpler and more direct.

**Remark 5.10.** For a singleton \( \{v\} \) to be Hadamard qli it is not enough to require only that its entries are non-zero and distinct in absolute value. For instance, if \( v^* = (1, 3, -4) \), then, by (ii) of Proposition 5.7 (or by direct verification), the vectors \( u, v, v^2, v^3 \) are linearly dependent (because \( e_1 = 0 \)); and if \( v^* = (1, 3, -3/4) \), then \( u, v^2, v^3 \) are linearly dependent (because \( e_2 = 0 \)). As another example, consider the polynomial \( x^3 - 2x + 1 \). It has three real roots, say \( x_1, x_2, x_3 \), with \( x_1 < -1 \) and \( 1/2 < x_2 < x_3 = 1 \). Let \( v \) be the vector with entries \( x_1, x_2, x_3 \). One verifies that the corresponding numbers \( e_1, e_2, e_3 \) are non-zero: \( e_1 \) is positive, while \( e_2 \) and \( e_3 \) are negative. Hence \( u, v, v^2, v^3 \) are qli, by (ii) of Proposition 5.7. But the vectors \( u, v, v^3 \) are linearly dependent (explicitly: \( v^3 - 2v + u = 0 \)); thus \( \{v\} \) is not Hadamard qli. This example shows that the last condition in (ii) of Theorem 5.8 cannot be dropped.
Now consider the case of an arbitrary finite subset \( E \) of \( \mathbb{R}^p \) with \( p \) elements, say \( E = \{v_1, \ldots, v_p\} \). When is \( E \) Hadamard qli? As we have seen, a necessary condition is that each singleton \( \{v_k\} \) (\( 1 \leq k \leq p \)) is Hadamard qli. The elements of \( H(E) \) are the vectors \( v_m \) with \( m \in \mathbb{N}^p \). Let us write \( v_m^p = (x_{1k}, \ldots, x_{nk}) \) (\( 1 \leq k \leq p \)). If \( m = (m_1, \ldots, m_p) \), then the entries of \( v_m \) are \( \prod_{k=1}^{p} x_{ik} \) (\( 1 \leq i \leq n \)). Now \( H(E) \) is Hadamard qli unless there are \( n \) distinct elements \( m_j \in \mathbb{N}^p \) (\( 1 \leq j \leq n \)) such that the vectors \( v_{m_j} \) are linearly dependent. Writing \( m_j = (m_{j1}, \ldots, m_{jp}) \) (\( 1 \leq j \leq n \)), this is the case if and only if

\[
\det \left( \prod_{k=1}^{p} x_{ik}^{m_{jk}} \right)_{ij} = 0. \tag{26}
\]

The left hand side of (26) is a polynomial in the \( np \) variables \( x_{ik} \) (\( 1 \leq i \leq n, 1 \leq k \leq p \)). Moreover, this polynomial is homogeneous of degree \( \sum_{k=1}^{p} \sum_{j=1}^{n} m_{jk} = \sum_{j=1}^{n} |m_j| \). Eq. (26) determines a conic manifold in \( \mathbb{R}^{np} \), call it \( W_\mu \), where \( \mu = (m_1, \ldots, m_n) = (m_{jk}) \in (\mathbb{N}^p)^n \cong \mathbb{N}^{np} \). We observe that \( W_\mu \) has Lebesgue measure zero, and that hence so has the set \( W = \bigcup_{\mu \in \mathbb{N}^{np}} W_\mu \). Representing, in a natural way, the set \( E \) as an element of \( \mathbb{R}^{np} \), we conclude that \( E \) is Hadamard qli unless \( E \in W \), a negligible subset of \( \mathbb{R}^{np} \).

Any \( A \in \mathcal{S}_n \) of rank \( p \) corresponds to a unique cloud of size \( n \) in \( \mathbb{R}^p \). Each positioning \( (\tilde{v}_1, \ldots, \tilde{v}_p) \) of this cloud in \( \mathbb{R}^p \) determines, via (1), a \( p \)-tuple \( \{v_1, \ldots, v_p\} \) of vectors in \( \mathbb{R}^n \) such that \( A = v_1 v_1^* + \cdots + v_p v_p^* \). It may well happen that some of these \( p \)-tuples are Hadamard qld, whereas most others are not. One can try to find a positioning of the cloud of \( A \) in such a way that the corresponding \( p \)-tuple is Hadamard qli. These considerations motivate the following definition.

**Definition 5.11.** A matrix \( A \in \mathcal{S}_n \) is called **Hadamard independent** if \( A = v_1 v_1^* + \cdots + v_p v_p^* \) for some Hadamard qli set \( \{v_1, \ldots, v_p\} \).

For a matrix \( A \in \mathcal{S}_n \) to be Hadamard independent it is certainly necessary that the \( n \) vectors in (a positioning of) its cloud are distinct and all different from zero. Indeed, if \( \tilde{v}_i = \tilde{v}_j \), then the \( i \)th and \( j \)th row of \( A \) are equal (the columns as well), and the equality of the two vectors persists after an orthogonal transformation; likewise if one of the vectors in the cloud is zero. Now consider a matrix \( A \in \mathcal{S}_n \) of rank \( p \) whose cloud consists of distinct non-zero vectors. If \( p = 1 \), then it can still happen that \( A \) is not Hadamard independent, due to the fact that a cloud in \( \mathbb{R} \) has only two positionings. For instance, if \( A = v v^* \) with \( v^* = (1, 3, -4) \), as in Remark 5.10, then the positionings of its cloud \([1, 3, -4] \) are \( \pm (1, 3, -4) \). However, if \( p \geq 2 \), it seems plausible that a positioning of the cloud can be found for which the corresponding \( p \)-tuple of vectors is Hadamard qli. The following example illustrates this.
Example 5.12. Take $n = 2$ and consider the cloud $[(1, 1), (a, b)]$ with $a$ and $b$ positive and different from 1. The Gram matrix $A$ of this cloud is given by $A = v_1 v_1^* + v_2 v_2^*$ with $v_1^* = (1, a)$ and $v_2^* = (1, b)$; thus

$$A = \begin{pmatrix} 2 & a + b \\ a + b & a^2 + b^2 \end{pmatrix}.$$ 

Set $E = \{v_1, v_2\} \subset \mathbb{R}^2$. The Hadamard span $H(E)$ consists of all vectors $v_1^{\otimes m} v_2^{\otimes k} = (1, a^m b^k)^*$ with $m, k \in \mathbb{N}$. In this particular case condition (ii) in Definition 5.5 is trivially satisfied because $H(E)$ is contained in the line $\{(1, y) \mid y \in \mathbb{R}\}$ (Example 5.4). One finds that condition (i) is satisfied unless $b$ is a rational power of $a$. More generally, if $v_1$ and $v_2$ each have positive and distinct entries, say $v_1 = (x_1, x_2)^*$, $v_2 = (y_1, y_2)^*$, then $E$ is Hadamard qli unless $x_1 / x_2$ and $y_1 / y_2$ are rational powers of one another.

As a specific example, let us take $a = 2$ and $b = 4$. Then

$$A = \begin{pmatrix} 2 & 6 \\ 6 & 20 \end{pmatrix},$$

its rank is 2, but its generating set $E = \{v_1, v_2\}$ is not Hadamard qli because $b = a^2$. Now apply an orthogonal transformation, say a rotation over $\theta$. Then a generating set $F = \{w_1, w_2\}$ is obtained with $w_1 = \cos \theta v_1 - \sin \theta v_2$, $w_2 = \sin \theta v_1 + \cos \theta v_2$, thus $w_1^* = (\cos \theta - \sin \theta, 2 \cos \theta - 4 \sin \theta)$, $w_2^* = (\cos \theta + \sin \theta, 4 \cos \theta + 2 \sin \theta)$ (note that indeed $w_1 w_1^* + w_2 w_2^* = A$). It then follows that $F$ is Hadamard qli, for instance, for values of $\theta$ close to 0 for which the quotient

$$\frac{\log(\cos \theta - \sin \theta) - \log(2 \cos \theta - 4 \sin \theta)}{\log(\cos \theta + \sin \theta) - \log(4 \cos \theta + 2 \sin \theta)}$$

is irrational.

6. A Taylor expansion

In this section we obtain an expansion of Taylor type for the determinant of fractional powers of matrices near the unit matrix $U$.

Lemma 6.1. Let $V$ be an $n \times n$ matrix all of whose entries are less than 1 in absolute value. Let $\alpha$ be a real number. Then the $\alpha$th fractional Hadamard power of $U + V$ is given by:

$$(U + V)^{\hat{\alpha}} = \sum_{k=0}^{\infty} (\alpha(k)) V^{\hat{\alpha} k}.$$

Proof. Apply (18) entrywise.
Lemma 6.2. Let \( v_1, \ldots, v_p \) be vectors in \( \mathbb{R}^n \) and let the matrix \( V \) be given by
\[
V = v_1 v_1^* + \cdots + v_p v_p^*.
\]
Then the integer Hadamard powers \( V^{\otimes k} (k \geq 0) \) are given by the formula:
\[
V^{\otimes k} = \sum_{m \in \mathbb{N}^p, |m| = k} (k|m) v_m v_m^*.
\]

Proof. Set \( v_l = (x_{il})_l (1 \leq l \leq p) \); then
\[
V = (\sum_{l=1}^p x_{il} x_{lj})_{i,j}.
\]
Now apply (15) and (16) first, and then (25).

Lemma 6.3. Let \( v_1, \ldots, v_p \) and \( V \) be as in Lemma 6.2. Let \( \alpha \) be an arbitrary real number. Then for \( \epsilon \) sufficiently close to zero the \( \alpha \)th fractional Hadamard power of the matrix \( U + \epsilon V \) is given by
\[
(U + \epsilon V)^{\hat{\alpha}} = \sum_{m \in \mathbb{N}^p} (\alpha|m) \epsilon |m| v_m v_m^*.
\] (27)

Proof. Take \( \epsilon \) so close to zero that all entries of \( \epsilon V \) are less than 1 in absolute value. Now combine Lemmas 6.1 and 6.2 and apply (20).

The following key result can now be proved.

Theorem 6.4. Let \( v_1, \ldots, v_p \) be vectors in \( \mathbb{R}^n \) and let \( V = v_1 v_1^* + \cdots + v_p v_p^* \). Then for \( \epsilon \) sufficiently close to zero and any \( \alpha \in \mathbb{R} \) the determinant of the matrix \( (U + \epsilon V)^{\hat{\alpha}} \) is given by the formula
\[
\det((U + \epsilon V)^{\hat{\alpha}}) = \sum_{l=L}^\infty C_l \epsilon^l,
\] (28)
where \( L = L(p, n) \) and the \( C_l \) (\( l \geq L \)) are given by
\[
C_l = \sum_{\lambda \in \mathcal{A}(p, n, l)} (\alpha|\lambda|) S^2_{\lambda}.
\] (29)

Proof. Applying Theorem 3.2 to (27) we obtain
\[
\det((U + \epsilon V)^{\hat{\alpha}}) = \sum_{\lambda \subset \mathbb{N}^p, |\lambda| = n} \left( \prod_{m \in \lambda} (\alpha|m) \epsilon |m| \right) S^2_{\lambda},
\]
which by (21) and (3) entails
\[
\det((U + \epsilon V)^{\hat{\alpha}}) = \sum_{\lambda \in \mathcal{A}(p, n)} (\alpha|\lambda|) \epsilon^{|\lambda|} S^2_{\lambda}.
\] (30)
Rearranging this sum according to (8), (28) follows.
Remark 6.5. From Theorem 6.4 it is clear why integer values for the exponent $\alpha$ are exceptional in the sense that integer powers of positive semidefinite matrices are again positive semidefinite. Indeed, in this case the factors $(\alpha | m)$ do not take negative values: they are positive for $|m| \leq \alpha$ and zero for $|m| > \alpha + 1$; consequently, the coefficients $(\alpha | \lambda)$ are 0 for $\lambda$ sufficiently large, the series (28) is finite, and the sum is positive.

The reader may find it instructive to compute, for the case $n = 2$, $\det(U + \varepsilon V)^\alpha$ directly: first with $p = 1$, say $V = vv^* = (x^2 x y y^2)$, and then for $p = 2$.

7. The main theorem

We first define a certain pattern of plus and minus signs; see (4) and (9) for the notation.

Definition 7.1. For integers $p \geq 1$ and $a \geq 0$ the function $T_{p,a} : [p, +\infty) \to \{1, -1\}$ is defined by the following rules:

(i) $T_{p,a}(n) = 1$ if $p \leq n \leq (p|a + 1)$, i.e., if $n \in \{p\} \cup D_p(1) \cup \cdots \cup D_p(a)$;
(ii) for $t \geq 0$, $T_{p,a}$ is constant on $D_p(a + t)$ when $t$ is even, and alternating when $t$ is odd.

Subsequent sets $D_p(a + t)$ and $D_p(a + t + 1)$ have precisely one element in common; therefore, $n \mapsto T_{p,a}(n)$ is defined successively on the sets $D_p(a + t)(t \geq 0)$. The smallest $n$ for which $T_{p,a}(n)$ is negative is the second element of $D_p(a + 1)$. Denoting this element by $N_{p,a}$, we get:

$$N_{p,a} = 1 + (a + 1 \| p) = 1 + \frac{(a + 2)(a + 3) \cdots (a + p + 1)}{p!}. \quad (31)$$

For instance, $N_{1,a} = a + 3$, and $N_{2,a} = (a^2 + 5a + 8)/2$. Likewise, $N_{p,0} = p + 2$ and $N_{p,1} = (p^2 + 3p + 4)/2$. The symmetry of $(\|)$ yields

$$N_{p,a} = N_{a+1,p-1}. \quad (32)$$

We can now state our main result; here $\sgn(x) = x/|x|$ ($x \neq 0$), $\sgn(0) = 0$.

Theorem 7.2. Let $p$ and $n$ be integers with $1 \leq p \leq n$. Let $V$ be a Hadamard independent positive semidefinite $n \times n$ matrix of rank $p$. Let $\alpha$ be a positive non-integer real number. Then

$$\lim_{\varepsilon \downarrow 0} (\sgn(\det((U + \varepsilon V)^\alpha))) = T_{p,[\alpha]}(n). \quad (33)$$
Proof. By Definition 5.11, there is a Hadamard qli set of vectors $v_1, \ldots, v_p$ in $\mathbb{R}^n$ such that $V = v_1v_1^* + \cdots + v_pv_p^*$. Write $\det((U + \varepsilon V)^{\alpha})$ as in (28) and (29).

The Hadamard span $\{v_m \mid m \in \mathbb{N}^p\}$ is qli. In view of Definition 5.5. Hence it follows from (ii) of Proposition 5.3 that $S_j \neq 0$ for all $\lambda \in A(p, n)$.

By Definition 5.11, there is a Hadamard qli set of vectors $P$. Fischer, J.D. Stegeman / Linear Algebra and its Applications 371 (2003) 53–74

It remains to show that this sign is equal to $T_p$, $\alpha > n > p$.

$(33)$ is the sign of $\lambda_j$ in (32). From (ii) of Proposition 5.3 that $S_j \neq 0$ for all $\lambda \in A(p, n)$. Moreover, for $\lambda \in A(p, n, L)$ all coefficients $(\alpha|\lambda)$ have the same sign (Corollary 4.5). This implies that $C_L \neq 0$. When $\varepsilon \downarrow 0$, the term $C_L \epsilon^k$ in (28) becomes dominant, hence the limit in (33) is the sign of $C_L$. By (29) this is equal to the sign of the $(\alpha|\lambda)$ for $\lambda \in A(p, n, L)$.

It remains to show that this sign is equal to $T_p|\alpha|(r)$ for all $n \geq p$. We do this by induction.

Let us denote the smallest elements of $\mathbb{N}^p$ as follows: $m_0 = (0, \ldots, 0)$, $m_j = (\delta_1, \ldots, \delta_n)$, where $\delta_i$ is 1 if $i = j$ and 0 if $i \neq j$. Then $m_0 = (0)$ and $m_0 = 1$ (1 $\leq j \leq p$). Set $\lambda_0 = \{m_0, m_1, \ldots, m_p\}$ and $\lambda_j = \lambda_0 \setminus \{m_j\}$ ($1 \leq j \leq p$). For instance, if $p = 1$ then $\lambda_0 = \{(0), (1)\}$ and $\lambda_1 = \{(0)\}$, and if $p = 2$, then $\lambda_0 = \{(0, 0), (1, 0), (0, 1)\}$, $\lambda_1 = \{(0, 0), (0, 1)\}$ and $\lambda_2 = \{(0, 0), (1, 0)\}$. For $1 \leq j \leq p$ we have $|\lambda_j| = p$, hence $\lambda_j \in A(p, p)$, and $\|\lambda_j\| = p - 1$. It is also clear that these $\lambda_j$ are the only $\lambda \in A(p, p)$ for which $\langle\lambda\rangle$ is minimal; thus $L(p, p) = p - 1$ and $A(p, p, p - 1) = \{\lambda_1, \ldots, \lambda_p\}$. Similarly, $|\lambda_0| = p + 1$ and $\|\lambda_0\| = p$, and $\lambda_0$ is the only $\lambda \in A(p, p + 1)$ for which $\langle\lambda\rangle$ is minimal; thus $L(p, p + 1) = p$ and $A(p, p + 1, p) = \{\lambda_0\}$.

In the terminology of the proof of Lemma 4.3: $M_0 = \{m_0\}$, $M_1 = \{m_0, m_1, \ldots, m_p\}$, and $M_0 \subset \lambda_j \subset \lambda_0 = M_1 (0 \leq j \leq p)$.

For any $\alpha \in \mathbb{R}$, we have:

$$ (\alpha|\lambda_0) = \prod_{k=0}^{p} (\alpha|m_k) = \alpha^p, \quad (\alpha|\lambda_j) = (\alpha|\lambda_0)/(\alpha|m_j) = \alpha^{p-1}(1 \leq j \leq p). $$

For $\alpha > 0$ this implies that $C_L = C_{p-1} > 0$ when $n = p$ and $C_L = C_p > 0$ when $n = p + 1$; thus (33) is proved for $n = p$ (and for $n = p + 1$). Now suppose that $n > p$ (or, if one prefers, $n > p + 1$), and that (33) has been proved already for $n - 1$ instead of $n$. Let $t \geq 0$ be the integer such that

$$ ([\alpha] + t || p) \leq n - 1 < n \leq ([\alpha] + t + 1 || p). $$

Then both $n - 1$ and $n$ belong to $D_p([\alpha] + t)$. By Lemma 4.4, with $k = [\alpha] + t$, the factor $\alpha - j$ is positive if and only if $j \leq [\alpha]$. Therefore, the number of negative factors in the product in the numerator of (22) is equal to

$$ \sum_{j=[\alpha]+1}^{k} (n - (j || p)) = (k - [\alpha])n - \sum_{j=[\alpha]+1}^{k} (j || p), $$

(recall that $\alpha$ is not an integer). This number is $k - [\alpha]$ more than the corresponding number for $n - 1$. This shows that the limiting sign in (28) obeys rule (ii) in Definition 7.1. □

Note that for a given size $n$ of the matrices and a fixed value of $\alpha$, the left-hand side of (33) depends only on the rank of $V$, not on its particular shape otherwise.
Example 7.3. Equality (32) implies the following fact. The smallest matrix size for which the limiting sign of \( \det((U + \varepsilon V)^{\alpha}) \) is negative if \( \text{rank}(V) = p \) and \( a < \alpha < a + 1 \) is equal to the smallest size for which the limiting sign is negative if \( \text{rank}(V) = a + 1 \) and \( p - 1 < \alpha < p \). For instance, that smallest size when \( V \) has rank 10 and \( 5 < \alpha < 6 \) is the same as when \( V \) has rank 6 and \( 9 < \alpha < 10 \). This size is \( 1 + (10\|6) = 8009 \), the second element of \( D_{10}(6) \), which is equal to the second element of \( D_6(10) \). However, the final elements of \( D_{10}(6) \) and \( D_6(10) \) are not the same: for the former it is \( (10\|7) = 19,448 \), whereas for the latter it is \( (6\|11) = 12,376 \). This implies, by (ii) of Definition 7.1, that in the former case the limiting sign is negative for all odd sizes from 8009 up to and including 19,447, whereas in the latter case the limiting sign is negative for all odd sizes from 8009 up to and including 12,375 only. After these final values there is again a long stretch of size values \( n \) for which the limiting sign is positive: in the former case from 19,448 up to \( (10\|8) = 43,758 \), in the latter case from 12,376 up to \( (6\|12) = 18,564 \). We say a little more about such ‘sign patterns’ in Section 9.

Remark 7.4. In Theorem 7.2 the requirement that \( V \) be Hadamard independent was made to guarantee that for all \( \lambda \subset \mathbb{N}^p \) with \( |\lambda| = n \) the vectors \( u_m \) (\( m \in \lambda \)) would be linearly independent, so that all \( S_\lambda \) would be non-zero and \( C_L \neq 0 \) would hold. But in each specific case a much weaker condition already suffices. In fact, (33) holds as soon as \( S_\lambda \neq 0 \) holds for at least one \( \lambda \in \mathcal{A}(p, n, L) \). In particular, it does not matter if Hadamard dependency occurs for products of higher powers of the generating vectors \( v_j \). To illustrate this, let us examine the cases \( n = p \) and \( n = p + 1 \) considered in the proof of Theorem 7.2. We saw there that \( L(p, p) = p - 1 \) and \( L(p, p + 1) = p \). If \( n = p \), then

\[
C_L = \sum_{\lambda \in \mathcal{A}(p, p, p-1)} (\alpha|\lambda|)S_\lambda^2 = \alpha^{p-1} \sum_{j=1}^{p} S_{\lambda_j}^2,
\]

where \( S_{\lambda_j} \) is the volume of the parallelepiped spanned by the vectors \( u, v_1, \ldots, v_p \) except \( v_j \). If \( n = p + 1 \), then

\[
C_L = \sum_{\lambda \in \mathcal{A}(p, p+1, p)} (\alpha|\lambda|)S_\lambda^2 = \alpha^p S_{\lambda_0}^2,
\]

where \( S_{\lambda_0} \) is the volume of the parallelepiped spanned by the vectors \( u, v_1, \ldots, v_p \). To conclude that \( C_L \neq 0 \) we do not need Hadamard quasi linear independence of the vectors \( v_1, \ldots, v_p \). Indeed, from the way they are chosen it follows that these vectors are linearly independent. If \( p = n \), then they form a basis of \( \mathbb{R}^p \), and hence there is at least one index \( j, 1 \leq j \leq p \), such that when \( v_j \) is replaced by \( u \) we have again a basis, and hence \( S_{\lambda_j} \neq 0 \). If \( p = n - 1 \), then the only requirement is that \( u \) does not belong to the space spanned by the vectors \( v_1, \ldots, v_p \). Observe that this is implied
by Hadamard quasi linear independence. Loosely speaking, the larger \( n - p \) is, the more the Hadamard quasi linear independence of the \( v_j \) is needed.

Using Lemma 3.3 we can obtain a more general result, where \( U \) is replaced by an arbitrary matrix \( T \in \mathcal{S}_n^+ \) of rank 1 and with strictly positive entries. Let \( T \) be such a matrix, and let \( V \in \mathcal{S}_n \) be a matrix of rank \( p \) for some \( p \leq n \). Consider \( A = T + V \), an element of the positive semidefinite cone at \( T \). Let us say that \( A \) is sufficiently close to \( T \) with respect to a real number \( \alpha \) if \( \det(T + \varepsilon V)^{\alpha} \) is of constant sign for \( 0 < \varepsilon \leq 1 \).

**Theorem 7.5.** Suppose that \( T \in \mathcal{S}_n^+ \) has rank 1 and strictly positive entries. Let \( V \in \mathcal{S}_n \) have rank \( p \) for some \( p \leq n \). Let \( \alpha \) be a non-integer positive real number. If the Hadamard quotient \( V \odot T \) is Hadamard independent, and if \( A = T + V \) is sufficiently close to \( T \) with respect to \( \alpha \), then the sign of \( \det A^{\odot \alpha} \) is equal to \( T_{p,[\alpha]}(n) \).

**Proof.** Take vectors \( v_0, v_1, \ldots, v_p \) such that \( T = v_0 v_0^\ast \) and \( V = v_1 v_1^\ast + \cdots + v_p v_p^\ast \). All entries of \( v_0 \) are different from 0; therefore we can consider the vectors \( w_j = v_j \odot v_0 \) (\( 1 \leq j \leq p \)). Set \( W = w_1 w_1^\ast + \cdots + w_p w_p^\ast \). Then \( V = T \odot W \), so \( A = T \odot (U + W) \). We have \( A^{\odot \alpha} = T^{\odot \alpha} \odot (U + W)^{\odot \alpha} \), and hence, by Lemma 3.3:

\[
\det A^{\odot \alpha} = (\text{mtr}(T))^{\odot \alpha} \cdot \det(U + W)^{\odot \alpha}.
\]

Now \( (\text{mtr}(T))^{\odot \alpha} > 0 \); therefore, \( \det(A^{\odot \alpha}) \) has the same sign as \( \det(U + W)^{\odot \alpha} \), and the latter determinant is equal to \( T_{p,[\alpha]}(n) \), by Theorem 7.2. \( \square \)

### 8. The case of lowest rank

In this section we take \( p = 1 \), i.e., we examine the case that the approximation is done with a matrix \( V \) of rank 1, say \( V = vv^\ast \) with \( v \in \mathbb{R}^n \), \( v \neq 0 \). We identify \( \mathbb{N}^p \) with \( \mathbb{N} \) by identifying \( m = (m) \) with \( m \). The set \( A(1, n) \) consists of all subsets \( \lambda \) of \( \mathbb{N} \) with \( n \) elements, and \( A(1, n, 1) \) consists of those \( \lambda \in A(1, n) \) for which \( ||\lambda|| \) is equal to \( l \). Clearly, \( [0, 1, 2, \ldots, n-1] \) is the element \( \lambda \) of \( A(1, n) \) for which \( ||\lambda|| \) is minimal. Let us denote this element by \( \lambda_0 \) (for \( n = 2 \) this is the same \( \lambda_0 \) as in the proof of Theorem 7.2). Then \( L(1, n) = ||\lambda_0|| = n(n-1)/2 \), and \( A(1, n, L) = \{ \lambda_0 \} \). For \( p = 1 \) we now obtain from Theorem 7.2 the following result.

**Theorem 8.1.** Let \( n \geq 2 \) be given and let \( 0 < \alpha \in \mathbb{R} \setminus \mathbb{Z} \). Let \( v \in \mathbb{R}^n \) have pairwise distinct entries. Set \( V = vv^\ast \). Then \( \lim_{\varepsilon \downarrow 0} (\text{sgn}(\det((U + \varepsilon V)^{\odot \alpha}))) \) is positive if \( \alpha > n - 2 \) or \( n - 2 < \alpha + 4k < n \) for some integer \( k \geq 1 \), and negative if \( n - 4 < \alpha + 4k < n - 2 \) for some \( k \geq 0 \).

**Proof.** As just observed, we have \( A(1, n, L) = \{ \lambda_0 \} \). As was pointed out in Remark 7.4, the conclusion of Theorem 7.2 (equality (33)) is valid if \( S_k \neq 0 \) for at least one element of \( A(1, n, L) \); hence if \( S_{\lambda_0} \neq 0 \). Now \( S_{\lambda_0} = |\det(u, v, \ldots, v^{\odot(n-1)})| \).
and it follows from (i) of Proposition 5.7 that $S_{\lambda_0} \neq 0$ if and only if the entries of $v$ are distinct. Furthermore, $D_1(k) = \{k + 1, k + 2\}$ ($k \geq 0$). Therefore, $T_{1,\alpha}(n)$ is positive if $1 \leq n \leq \alpha + 2$, while for $n = \alpha + 3$ it is negative (thus $N_{1,\alpha} = \alpha + 3$, cf. (31)). It is again negative for $n = \alpha + 4$, for the next two values of $n$ it is positive, then negative for the next two values, and so on. □

**Remark 8.2.** The result in Theorem 8.1 can be verified in a more direct way: if $\lambda = \{m_1, \ldots, m_n\} \in \mathbb{A}(1, n)$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, then

\[
(\alpha | \lambda) = \prod_{m \in \lambda} (\alpha | m) = \prod_{i=1}^{n} \frac{\alpha(\alpha - 1) \cdots (\alpha - m_i + 1)}{m_i!}.
\]

In particular:

\[
(\alpha | \lambda_0) = \frac{\alpha^{n-1}(\alpha - 1)^{n-2} \cdots (\alpha - n + 2)(\alpha - n + 2)}{1! \cdot 2! \cdots (n-1)!}.
\]

It follows that $(\alpha | \lambda_0)$ is positive if $\alpha - n + 2 > 0$, i.e., for $n \leq [\alpha] + 2$. When $n = [\alpha] + 3$ we get a minus sign, and we get two more minus signs for the next $n$. Therefore, $(\alpha | \lambda_0)$ is negative for $n = [\alpha] + 3$ and also for $n = [\alpha] + 4$. For the next two values of $n$ it is positive again because we get three and four extra minus signs, respectively. Continuing this argument, the same pattern as in the theorem is obtained.

In the next corollary the meaning of 'sufficiently small' is: so small that the sign of the determinant considered does not change any more when the $x_i$ are replaced by $\varepsilon x_i$ with $0 < \varepsilon < 1$.

**Corollary 8.3.** Let $n \geq 2$ be given, and let $\alpha$ be a positive non-integer real number. Let $x_1, \ldots, x_n$ be distinct real numbers such that either all $x_i$ are non-negative or all $x_i$ are at most one in absolute value. Let $A$ be the $n \times n$ matrix whose general element is $(1 + x_i x_j) \sqrt{1 + x_i^2}(1 + x_j^2)$. Then if the $x_i$ are sufficiently small, $\det(A^{\alpha})$ is positive if $\alpha > n - 2$ or $n - 2 < \alpha + 4k < n$ for some integer $k \geq 1$, and negative if $n - 4 < \alpha + 4k < n - 2$ for some $k \geq 0$.

**Proof.** Set $v^* = (x_1, \ldots, x_n)$ and $V = vv^*$. We compute:

\[
U + V = (1 + x_i x_j)
\]

\[
= \left(\sqrt{1 + x_i^2}(1 + x_j^2)\right) \circ \left(1 + x_i x_j \sqrt{1 + x_i^2}(1 + x_j^2)\right) = T \circ A,
\]

and hence $(U + V)^{\alpha} = T^{\alpha} \circ A^{\alpha}$. Now $T$ has rank 1, hence so has $T^{\alpha}$. We have mtr$(T^{\alpha}) = \prod_i (1 + x_i^2)^{\alpha}$, and therefore Lemma 3.3 implies that $\det A^{\alpha}$ has the same sign as $\det(U + V)^{\alpha}$. The result now follows from Theorem 8.1. □
Example 8.4. We return to the example in [3], mentioned in Section 2. Let \( n \geq 3 \) be given and let \( \alpha \) be a positive non-integer real number satisfying \( \alpha < n - 2 \). We can take \( m \in \mathbb{Z} \) with \( m < n \) such that \( m - 3 < \alpha < m - 2 \). Take \( A_\varepsilon = U + \varepsilon V \) as in Theorem 8.1, with \( \varepsilon > 0 \) sufficiently small. It then follows from Theorem 8.1 that all principal minors of \( A_\alpha \) of size \( m \) are negative, and hence, by Proposition 2.2, \( A_\alpha \notin S_n \), as desired. The principal minors with sizes \( m + 1, m + 4, m + 5, m + 8, \ldots \) are also negative, whereas those with size less than \( m \) or with size \( m + 2, m + 3, m + 6, m + 7, \ldots \) are positive. It is worthwhile to compare this reasoning with the specific example in [3, p. 636]; there \( V = vv^* \) with \( v^* = (1, 2, \ldots, n) \) (all entries distinct!).

It is essential that the matrices \( A_\varepsilon \) that produce negative principal minors (when taken to a fractional power) have rank 2, i.e., are of as low a rank as possible. And the larger \( \alpha \) is, the more essential this low rank is. Indeed, if instead of \( p = 1 \) we take \( p = 2 \) (so that the \( A_\varepsilon \) have rank 3), then, as \( N_{2,a} = (a^2 + 5a + 8)/2 \), no negative principal minor is obtained when \( n \leq (a + 2)(a + 3)/2 \). For instance, if \( 10 < \alpha < 11 \), the size \( n \) has to be at least 79 to obtain (by our technique) negative principal minors of \( A_\alpha \) (and when \( p = 3 \) this minimal size is already 365). One might say: ‘the larger the rank of a positive semidefinite matrix, the more stable is its positive semidefiniteness under taking fractional powers’.

9. Sign patterns

Let us consider the patterns of plus and minus signs determined by the functions \( T_{p,a} \). From Definition 7.1 we know that for fixed \( p \) and \( a \) the function \( n \mapsto T_{p,a}(n) \) is constant (and positive) for \( p \leq n \leq (p\|a + 1) \), after which it is alternating for a while (on \( D_p(a + 1) \), then constant again (on \( D_p(a + 2) \)), and so on. Let us call an interval \( D_p(a + t) \) an ‘interval of constancy’ (relative to \( p \) and \( a \)) when \( t \) is even, and an ‘interval of alternation’ when \( t \) is odd.

On two subsequent intervals of constancy \( T_{p,a} \) has the same value if and only if the interval of alternation between these two intervals has an odd number of elements. Indeed, then \( T_{p,a} \) has the same value at both endpoints of that interval, and hence on the two adjacent intervals of constancy (recall that adjacent intervals have one point in common).

Binomial coefficients are more often even than odd. For instance, among the \( 4^N \) coefficients \( (m\|k) \) with \( 0 \leq m, k < 2^N \) there are only \( 3^N \) odd ones, and hence, from \( N = 3 \) on, there are more even than odd ones. As a consequence, by (10), for matrices of not too small rank it will be more common to have subsequent intervals of constancy of equal sign than of opposite sign.

Can \( T_{p,a} \) have the same (necessarily positive) sign on all intervals of constancy? It turns out that this happens surprisingly often; in fact, in one quarter of all cases. To
prove this, we need a lemma on the parity of binomial coefficients; its (elementary) proof is left to the reader.

**Lemma 9.1.** Suppose that \( k \) and \( i \) are integers satisfying \( 0 \leq i \leq k \). Then:

(i) if \( k \) is even and \( i \) is odd, then \( (k|i) \) is even;
(ii) for even \( i \) there are arbitrarily large even \( k \) such that \( (k|i) \) is odd;
(iii) for arbitrary \( i \) there are arbitrarily large odd \( k \) such that \( (k|i) \) is odd.

Let us call an integer \( n \) an isolated element of a set of integers \( E \) if \( n \in E \) but \( n - 1 \not\in E \) and \( n + 1 \not\in E \). The following remarkable result holds.

**Theorem 9.2.** Let \( p \geq 1 \) and \( a \geq 0 \) be given. The set of integers \( n \geq p \) for which \( T_{p,a}(n) = -1 \) has only isolated elements if and only if \( p \) is even and \( a \) is odd.

**Proof.** Let \( p \geq 1 \) and \( a \geq 0 \) be fixed. Let \( N_+ \) and \( N_- \) be the sets of integers \( n \geq p \) for which \( T_{p,a}(n) \) is positive, respectively negative. We have then the following chain of equivalent statements: \( N_- \) has only isolated elements; all intervals of constancy belong to \( N_+ \); all intervals of alternation have an odd number of elements; \(|D_{p}(a + t)|\) is odd for all odd \( t \geq 1 \); \((a + t + 1)p - 1\) is even for all odd \( t \geq 1 \) (by (10)); \((a + t + p)|p - 1\) is even for all odd \( t \geq 1 \) (by (4)); \( a + t + p \) is even and \( p - 1 \) is odd for all odd \( t \geq 1 \) (by (i)–(iii) of Lemma 9.1); \( p \) is even and \( a \) is odd. \( \square \)

**Example 9.3.** If \( p = 2^N \) for some positive \( N \), then \( (p - 1)! \) contains relatively few factors 2, and hence \((a + t + 1)|p - 1\) has a good chance to be even. Actually, it is easily verified that the only \( m \geq 0 \) for which \((m|2^N - 1)\) is odd are of the form \( m = k2^N \) with \( k \geq 0 \). For instance, the first interval of alternation for \( T_{2N,0} \) with an even number of elements is \( D_{2N}(2^N - 1) \), and hence \( D_{2N}(2^N) \) is the first interval of constancy where \( T_{2N,0} \) takes the value \(-1\). Note that the existence of such intervals follows from Theorem 9.2 because \( a = 0 \).

As an example, consider the case that \( N = 4 \), thus \( p = 16 \). For \( 16 \leq n \leq 601,080,389 \) the value of \( T_{16,0}(n) \) is negative for \( n \) equal to 18, 20, \ldots, 152, 970, 972, \ldots, 4844, 20,350, \ldots, i.e., for all even \( n \) such that \((2s - 1|16) < n < (2s|16)\) for an \( s \) with \( 1 \leq s \leq 8 \), altogether for \( \frac{1}{2} \sum_{s=1}^{8} (2s|15) = 199,650,082 \) values of \( n \) (all even and all isolated), and positive for the other 401,430,292 values. But for the next \( 565,722,721 \) values of \( n \) the \( T_{p,a}(n) \) are all negative. The number of negative values hence increases to \( 765,372,803 \), an increase from 33.2% to 65.6%. To state a very concrete case: imagine a (Hadamard independent) matrix of size \( 10^9 \), of rank 17, in the positive semidefinite cone of the unit matrix \( U \) and sufficiently close to \( U \): a vast field of \( 10^{18} \) numbers, all very close to 1. Then the determinant of the Hadamard square root of that matrix is negative. The same is the case for sizes \( n \) satisfying \( 601,080,390 \leq n \leq 1,166,803,110 \); but not so for the previous, nor for the next \( n \).
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References