Spectral zeta function of the sub-Laplacian on two step nilmanifolds

W. Bauer a,*,1, K. Furutani b,2, C. Iwasaki c,3

a Mathematisches Institut, Georg-August-Universität, Bunsen-str. 3-5, 37073 Göttingen, Germany
b Department of Mathematics, Tokyo University of Science, 2641 Yamazaki, Noda, Chiba 278-8510, Japan
c Department of Mathematics, University of Hyogo, 2167 Shosha Himeji, Hyogo 671-2201, Japan

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Abstract

We study the heat kernel trace and the spectral zeta function of an intrinsic sub-Laplace operator $\Delta_{L \setminus G}^{\text{sub}}$ on a two step compact nilmanifold $L \setminus G$. Here $G$ is an arbitrary nilpotent Lie group of step 2 and we assume the existence of a lattice $L \subset G$. We essentially use the well-known heat kernel expressions of the sub-Laplace operator on $G$ due to Beals, Gaveau and Greiner. In contrast to the spectral zeta function of the Laplacian on $L \setminus G$ which can have infinitely many simple poles it turns out that in case of the sub-Laplacian only one simple pole occurs. Its residue divided by the volume of $L \setminus G$ is independent of $L$ and can be expressed by the Lie group structure of $G$. By standard arguments this result is equivalent to a specific asymptotic behaviour of the heat kernel trace of $\Delta_{L \setminus G}^{\text{sub}}$ as time tends to zero. As an example we explicitly calculate the spectrum of the sub-Laplacian $\Delta_{L \setminus G}^{\text{sub}}$ in case of the six-dimensional free nilpotent Lie group $G$ and a standard lattice $L \subset G$ by using a decomposition of $\Delta_{L \setminus G}^{\text{sub}}$ into a family of elliptic operators.

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Résumé

On étudie la trace du noyau de la chaleur et de la fonction zéta spectrale d’un opérateur sous-laplacien intrinsèque $\Delta_{L \setminus G}^{\text{sub}}$ sur une nilvariété compacte $L \setminus G$ : $G$ est un groupe de Lie nilpotent de classe 2 arbitraire, on suppose qu’il existe un réseau $L \subset G$. Pour le noyau de la chaleur du sous-laplacien sur $G$, on utilise l’expression donnée par Beals, Gaveau et Greiner. Par contraste avec la fonction zéta spectrale du laplacien sur $L \setminus G$, qui a un nombre infini de pôles simples, on montre que, dans ce cas, le sous-laplacien possède un unique pôle simple dont le résidu divisé par le volume de $L \setminus G$ est indépendant de $L$ et est déterminé par la structure de groupe de Lie de $G$. En utilisant des arguments classiques, ce résultat est équivalent à un comportement asymptotique spécifique de la trace du noyau de la chaleur de $\Delta_{L \setminus G}^{\text{sub}}$ au temps zéro. Pour illustrer ce résultat par un exemple, on calcule explicitement le spectre du sous-laplacien $\Delta_{L \setminus G}^{\text{sub}}$ dans le cas d’un groupe de Lie libre 2-nilpotent de dimension 6 d’un réseau standard $L \subset G$, en décomposant $\Delta_{L \setminus G}^{\text{sub}}$ en une famille d’opérateurs elliptiques.

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1. Introduction

Differential operators on manifolds which are associated to a geometric structure and their spectral functions have found significant applications in various fields of mathematics and physics such as geometric analysis, differential geometry or quantum theory. The Laplace–Beltrami operator in Riemannian geometry or the Dirac operator associated to a spin geometry form important examples of such objects and deep relations between their spectral invariants and the geometric structure of the underlying manifolds have been discovered. For various subclasses and corresponding geometric operators methods are available that allow to calculate their spectrum exactly. In these cases an explicit analysis of the corresponding spectral functions, such as the heat kernel trace and the spectral zeta function, is possible and leads to interesting examples of the general theory. In particular, the functional determinants of such operators which cannot easily be obtained from the asymptotic behaviour of the spectrum can be calculated. The spectral zeta function of a complex sub-Laplacian on odd-dimensional spheres has been studied in [6]. In low-dimensional cases functional equations could be derived that are linked to the classical functional equation of the Riemann zeta function. Furthermore, the analysis of spectral inverse problems in sub-Riemannian geometries is a natural task for a future investigation.

We study the spectral zeta function \( \zeta_{L,G}^{\text{sub}}(s) \) of an intrinsic sub-Laplace operator \( \Delta_{L,G}^{\text{sub}} \) associated to a trivializable sub-Riemannian structure (cf. [1,2,5,15]) on a compact nilmanifold \( L \setminus G \). Here \( L \) is a uniform discrete subgroup of a general two step nilpotent Lie group \( G \) (we assume that \( L \) exists). The sub-Laplace operator is non-elliptic in general and defined via a sum of squares of global left invariant vector fields on \( G \). In [1,2] we have treated the special case of Heisenberg manifolds where \( G \) is the (2N+1)-dimensional Heisenberg group and \( L \subset H_{2N+1} \) is a standard lattice. The spectral zeta function could be obtained explicitly in the form:

\[
\zeta_{L,H_{2N+1}}(s) = \frac{2^{N+1}}{(2\pi)^s} \zeta(s-N) \sum_{j_1,\ldots,j_N=0}^{\infty} \frac{1}{(2j_1 + \cdots + 2j_N + N)^s} + \frac{1}{(2\pi^2)^s} \zeta_{Ep}^{(2N)}(s),
\]

where we have written \( \zeta(s) \) for the Riemann and \( \zeta_{Ep}^{(2N)}(s) \) for the Epstein zeta function. It was shown in [1] that (1) admits a meromorphic extension to the complex plane which is holomorphic in a zero neighbourhood and has only one simple pole in \( s = N + 1 \). The sub-Riemannian structure on \( L \setminus G \) extends to a Riemannian structure with corresponding left invariant Laplacian \( \Delta_{L,G} \) and spectral zeta function \( \zeta_{L,G}(s) \). In contrast to the above result it previously was observed by the second author and S. de Gosson in [12] that \( \zeta_{L,H_{2N+1}}(s) \) in general has infinitely many simple poles located on the real axis, cf. [13].

The aim of the present paper is to generalize this type of results to arbitrary compact two step nilmanifolds and to provide explicit examples in the case of low-dimensional manifolds that are not of Heisenberg type. More precisely, let \( \mathfrak{g} \) denote the Lie algebra of \( G \) and let \( [\mathfrak{g}, \mathfrak{g}] \) be its first derived ideal. Then, we can prove:

**Theorem A.** The spectral zeta function of the sub-Laplace \( \Delta_{L,G}^{\text{sub}} \) on the compact nilmanifold \( L \setminus G \) has exactly one simple pole located in \( s_p = \frac{1}{2} \dim G + \frac{1}{2} \dim [\mathfrak{g}, \mathfrak{g}] \). The residue in \( s_p \) can be calculated as

\[
\text{Res}_{s=s_p} \zeta_{L,G}^{\text{sub}}(s) = \frac{\text{vol}(L \setminus G)}{(2\pi)^{2d+\frac{1}{2}}} \int_{\mathbb{R}^{2d}} W(\tau) d\tau.
\]

Here the integrand \( W(\tau) \) is the volume form in the heat kernel expression of the sub-Laplacian on \( G \). It can be expressed explicitly and encodes the structure constants of the Lie group.

We point out that the above theorem can be used to calculate the values of \( \zeta_{L,G}^{\text{sub}}(s) \) at non-positive integers and to show that they are independent of \( L \) and \( G \). As is well-known the spectral zeta function of the left invariant Laplace operator \( \Delta_{L,G} \) on \( L \setminus G \) has only simple poles which are located at most at points \( s_j \in \{ \dim G/2 - j \mid j \in \mathbb{N}_0 \} \). In
general, finitely many (more than one) or even infinitely many poles appear and the values \( \zeta_{L \setminus G}(-k) \) where \( k \in \mathbb{N}_0 \) may depend on \( G \) in case of an even-dimensional group \( G \). In Section 4 we calculate the residues of \( \zeta_{L \setminus G}(s) \) in \( s = s_j \) for \( j \in \mathbb{N}_0 \), cf. Theorem 4.4.

As a crucial ingredient to the proof of Theorem A we derive an inequality (Proposition 2.5) on the heat kernel \( K(t, g, \tilde{g}) \) of the sub-Laplacian on two step nilpotent Lie groups \( G \) (see also [11]). Recently precise upper and lower pointwise estimates on the sub-elliptic heat kernel in the special situation where \( G \) is a Heisenberg group or an \( H \)-type group have been obtained in [10] (see also [8]). However, in order to obtain the result in Theorem A we need to deal with general two step nilpotent groups and we essentially use an integral expression of \( K(t, g, \tilde{g}) \) due to Beals, Gaveau and Greiner in [3,4]. Especially the estimate given in Proposition 3.1 based on this inequality (Proposition 2.5) is essential for the proof of Theorem A. Then an asymptotic expansion of the heat kernel trace of the sub-Laplacian on the compact manifold \( L \setminus G \) is proven which through the Mellin transform is equivalent to the statement in Theorem A. Since we have started with a general two-step nilpotent Lie group \( G \) we have to assume the existence of a uniform discrete subgroup \( L \subset G \). On the one hand it is known that there are nilpotent Lie groups without uniform discrete subgroups even among the groups of step two, cf. [9,16]. On the other hand there are interesting classes of Lie groups including the \( H \)-type Lie groups (cf. [7,14]) for which the existence of lattices is known.

Compact nilmanifolds of 2-step nilpotent Lie groups can be interpreted as the total space of a principal bundle whose fiber and base spaces both are tori. In a natural way the sub-Riemannian structure studied in this paper defines a connection for this principal bundle. Then not only the sub-Laplacian can be descended to the base space and induces the Laplacian on the flat torus, but also there is a family of elliptic operators acting on line bundles associated to characters of the structure group (which again is a torus). From this point of view the sub-Laplacian can be seen as the \( L_2 \)-infinite sum of elliptic operators on the base space of a principal bundle. To provide an example that is not covered by previous results and allows an explicit spectral decomposition of the sub-Laplacian we choose \( G \) to be a free two-step nilpotent group of dimension \( n(n+1)/2 \) where \( n \in \mathbb{N} \) and we fix a standard lattice \( L \subset G \). In case of \( \dim G = 6 \) and simplifying our method in [2] we calculate the spectrum of the sub-Laplacian \( \Delta^{\text{sub}}_{L \setminus G} \). Even in such a low-dimensional case the corresponding spectral zeta function is far more complicated than (1). It decomposes into an infinite sum of spectral zeta functions \( \zeta_n(s) \), where \( n \in \mathbb{Z} \), of elliptic operators and each function \( \zeta_n(s) \) is meromorphic with in general infinitely many simple poles. However, it follows from Theorem A that after the infinite summation all the poles disappear, and one new pole at \( s = 9/2 \) appears. Finally, we mention that the regularized determinants of the operators in Section 6 can be calculated, but we do not provide the details here (see [1] for an example of an explicit determination of the zeta regularized determinant of a sub-Laplacian).

The structure of the paper is as follows. In Section 2 we recall an expression for the heat kernel \( K \) of the sub-Laplacian acting on \( C^\infty(G) \) where \( G \) is a general two step nilpotent Lie group. Then in Proposition 2.5 we derive some estimates on \( K \) from above.

Section 3 contains the proof of Theorem A, which is derived from the asymptotic expansion of the heat trace of the sub-Laplacian on a general compact two step nilmanifold \( L \setminus G \) (we only assume the existence of a lattice \( L \subset G \)).

In Section 4 we deal with the left invariant Laplacian on \( L \setminus G \) and we calculate the residues of the corresponding spectral zeta function in all its poles.

In the short Section 5 as an application to Theorem A we calculate the values of the spectral zeta function for the sub-Laplacian on \( L \setminus G \) in all non-positive integers and we compare the result with the elliptic case of the Laplacian. In particular, Theorem 5.1 gives a generalization of Theorem 8.4 in [1], where we only dealt with Heisenberg manifolds.

Finally, in Section 6 we deal with explicit examples and choose \( G \) to be a free nilpotent two step Lie group with standard lattice \( L \subset G \). In case of \( \dim G = 6 \) we calculate the spectral zeta function of the sub-Laplacian. The expression we obtain in this last example has already been given in [2], however, we present an essentially simplified method of the calculation here.

Appendix A complements the text, where we have collected some identities and lemmas which are useful throughout the paper.

2. Heat kernel estimates for a sub-Laplacian on 2-step nilpotent Lie groups

We start by recalling an integral expression for the heat kernel of the sub-Laplacian on a general 2-step nilpotent Lie group \( G \) in [2,3,11]. Special cases of these spaces and operators have been treated in [1,2], see also [12] for results on the Laplacian.
Let $G$ be an $(n+d)$-dimensional connected and simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$ where $d$ is the dimension of the first derived ideal $[\mathfrak{g}, \mathfrak{g}]$. We identify the group $G$ and its Lie algebra through the exponential map
\[
\exp : \mathfrak{g} \rightarrow G.
\]
Let $\{Z_k\}_{k=1}^d$ and $\{X_i\}_{i=1}^n$ be suitable bases of $[\mathfrak{g}, \mathfrak{g}]$ and a complement of it, respectively. Put
\[
[X_i, X_j] = 2 \sum_{k=1}^d a_{ij}^k Z_k,
\]
where $a_{ij}^k = -a_{ji}^k$ for $i, j = 1, \ldots, n$. With respect to the above bases and the identification $\mathfrak{g} \cong G$ we express an element $g \in G$ in the form $g = (x, z) \in \mathbb{R}^{n+d}$. If we use the notation $a_{ij} := (a_{ij}^1, \ldots, a_{ij}^d)^t \in \mathbb{R}^d$, then the product $\ast$ on $\mathfrak{g} \cong G$ is given by
\[
(x, z) \ast (u, v) = \left( x + u, z + v + \sum_{i,j=1}^n x_i u_j a_{ij} \right).
\]
(3)

With $\tau = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d \cong [\mathfrak{g}, \mathfrak{g}]$ and the Euclidean inner-product $\langle y, \tau \rangle = y_1 \tau_1 + \cdots + y_d \tau_d$ on $\mathbb{R}^d$ we write
\[
\Omega(\tau) = \left( \begin{array}{c} \langle a_{11}, \tau \rangle \\ \vdots \\ \langle a_{n1}, \tau \rangle \end{array} \right) = \tau_1 \Omega_1 + \cdots + \tau_d \Omega_d \in \mathbb{R}^{n \times n}
\]
(4)
with the matrices $\Omega_\ell := (a_{ij}^\ell) \in \mathbb{R}^{n \times n}$. Consider the sub-Laplacian on $G$ defined by
\[
\Delta_{\text{sub}}^G = -\frac{1}{2} \sum_{i=1}^n \tilde{X}_i^2,
\]
where $\tilde{X}_i$ denotes the left invariant vector field associated to $X_i$. Since the vector fields $\{\tilde{X}_i\}_{i=1}^n$ fulfill the Hörmander condition (= bracket generating condition) it is known that the sub-Laplacian defines a hypo-elliptic but in general non-elliptic operator. The following theorem is proven in [3] (see also [2,11]):

**Theorem 2.1.** The heat kernel $K \in C^\infty(\mathbb{R}^+ \times G \times G)$ of the sub-Laplacian $\Delta_{\text{sub}}^G$ has the form
\[
K(t; (x, z), (\tilde{x}, \tilde{z})) = k_t((\tilde{x}, \tilde{z})^{-1} \ast (x, z)),
\]
where $k_t \in C^\infty(\mathbb{R}^+ \times G)$ is given by
\[
k_t(x, z) = \frac{1}{(2\pi t)^{\frac{n+d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{\langle x, \tau \rangle}{2t}} \sqrt{\det \Omega(\sqrt{-1} \tau) \sinh \Omega(\sqrt{-1} \tau)} \, d\tau.
\]
(5)
and the action function $f(x, z, \tau)$ in the integrand takes the form
\[
f(x, z, \tau) = \sqrt{-1}(\tau, z) + \frac{1}{2} \Omega(\sqrt{-1} \tau) \coth \Omega(\sqrt{-1} \tau) \cdot x, x).
\]

As in [11] we write
\[
W(\tau) := \sqrt{\det \Omega(\sqrt{-1} \tau) \sinh \Omega(\sqrt{-1} \tau)}
\]
(6)
for the so-called volume element in the above integral expression of the heat kernel. As is known, $W(\tau)$ satisfies the following first order transport equation (cf. [3,11])
\[
\sum_{i} \tau_i \frac{\partial W(\tau)}{\partial \tau_i} - \left( \Delta_{\text{sub}}^G(f) + \frac{d}{2} \right) \cdot W(\tau) = 0.
\]
However, in order to get a precise estimate on the behaviour of $W(\tau)$ as $\tau \to \infty$, we need to prove a differential equation for $\sigma(\tau) := W(\tau)^2$. With $k \in \mathbb{N}$ let $S(\mathbb{R}^k)$ denote the usual Schwartz-space over $\mathbb{R}^k$. Moreover, we write $C^\infty_b(\mathbb{R}^k)$ for the space of all smooth functions on $\mathbb{R}^k$ which have bounded derivatives of all orders.

**Lemma 2.2.** For all $k \in \{1, \ldots, d\}$ there is $\alpha_k(\tau) \in C^\infty_b(\mathbb{R}^d)$ such that

$$\frac{\partial \sigma}{\partial \tau_k}(\tau) = \alpha_k(\tau)\sigma(\tau).$$

**Proof.** We write $g(s) := \frac{s}{\sinh s}$ which defines an even, non-negative and real analytic function in $S(\mathbb{R})$. Then according to the expression

$$g(\Omega(\sqrt{-1}\tau)) = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma} g(\lambda) [\lambda - \Omega(\sqrt{-1}\tau)]^{-1} d\lambda,$$

it is apparent that the matrix function $A(\tau) := g(\Omega(\sqrt{-1}\tau))$ smoothly depends on $\tau$. The contour $\Gamma$ is taken suitably surrounding the spectrum of $\Omega(\sqrt{-1}\tau)$. The spectrum of $A(\tau)$ is contained in $(0, \infty)$ and one can write

$$\sigma(\tau) := W(\tau)^2 = \det A(\tau) = \det e^{\log A(\tau)} = e^{\text{trace} \log A(\tau)}.$$

By taking the partial derivative of $\sigma(\tau)$ with respect to $\tau_k$ it follows

$$\frac{\partial \sigma}{\partial \tau_k}(\tau) = \text{trace} \left\{ A^{-1}(\tau) \frac{\partial}{\partial \tau_k} A(\tau) \right\} \sigma(\tau).$$

With our notation $\Omega(\tau) = \tau_1 \Omega_1 + \cdots + \tau_d \Omega_d$ and by making use of the resolvent equation we calculate

$$\text{trace} \left\{ A^{-1}(\tau) \frac{\partial}{\partial \tau_k} A(\tau) \right\} = \text{trace} \left\{ \sqrt{-1} \Omega_k \cdot \left( \frac{g'}{g} \right) (\Omega(\sqrt{-1}\tau)) \right\},$$

where for $s \neq 0$ the function $h := g'g^{-1}$ is given by

$$h(s) = \frac{g'(s)}{g(s)} = \frac{1}{s} - \frac{e^s + e^{-s}}{e^s - e^{-s}} = \frac{1}{s} - \frac{\cosh s}{\sinh s}.$$  

Of course, this expression is odd and has an analytic extension to $s = 0$. Also we can see that the function $h$ is bounded and all its derivatives $h^{(j)}(s)$, $j \in \mathbb{N}$, are vanishing as $|s| \to \infty$. In particular, it follows that the functions

$$\mathbb{R}^d \ni \tau \mapsto \text{trace} \left\{ h^{(j)}(\Omega(\sqrt{-1}\tau)) \right\}$$

are bounded for all $j \in \mathbb{N}_0$. Together with (8), this implies the assertion. □

As a corollary to Lemma 2.2 we obtain:

**Corollary 2.3.** It holds $W(\tau) \in S(\mathbb{R}^d)$.

**Proof.** Let $r(\tau) = \|\Omega(\sqrt{-1}\tau)\|$ denote the spectral radius of $\Omega(\sqrt{-1}\tau)$. Since norms on $\mathbb{R}^{n \times n}$ are equivalent, there is $c > 0$ such that

$$r(\tau)^2 \geq c \sum_{i,j=1}^{n} |(a_{ij}, \tau)|^2, \quad \tau \in \mathbb{R}^d.$$

As a function of $\tau$ the right-hand side of this inequality is non-vanishing and continuous on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ and therefore we can find $\tilde{c} > 0$ independently of $\tau \in S^{d-1}$ such that

$$\max\{ |\lambda(\tau)| : \lambda(\tau) \text{ is eigenvalue of } \Omega(\sqrt{-1}\tau) \} \geq \tilde{c}.$$

Therefore, we have for all $\tau \in \mathbb{R}^d$

$$\max\{ |\lambda(\tau)| : \lambda(\tau) \text{ is eigenvalue of } \Omega(\sqrt{-1}\tau) \} \geq \tilde{c} |\tau|.$$
Let \( \{ \lambda_j(\tau) \} \) denote the eigenvalues of \( \Omega(\sqrt{-1}\tau) \) with multiplicities \( m_j(\tau) \in \mathbb{N} \). Since \( g(s) = \frac{s}{\sinh s} \) is bounded, even and monotony decreasing for \( s > 0 \) sufficiently large, we conclude that there is a constant \( \beta > 0 \) with
\[
\sigma(\tau) = \det g(\Omega(\sqrt{-1}\tau)) = \prod[g \circ \lambda_j(\tau)]^{m_j(\tau)} \leq \beta \cdot g(\tilde{c}|\tau|).
\]
The right-hand side of this inequality decays to infinite order as \( \tau \to \infty \) and therefore \( \sigma(\tau) = O(|\tau|^{-j}) \) for \( j > 0 \) arbitrary. Finally, by using (7) in Lemma 2.2 it follows that
\[
\frac{\partial W}{\partial \tau_k}(\tau) = \frac{\partial \sqrt{\sigma}}{\partial \tau_k}(\tau) = \frac{1}{2\sqrt{\sigma(\tau)}} \frac{\partial \sigma}{\partial \tau_k}(\tau) = \frac{\alpha_k(\tau)}{2} W(\tau),
\]
where \( \alpha_k(\tau) \in C_0^\infty(\mathbb{R}^d) \). Since \( W(\tau) = \sqrt{\sigma(\tau)} = O(|\tau|^{-j}) \) for \( j > 0 \) arbitrary and by applying the above relation between \( W(\tau) \) and its derivatives one inductively concludes the assertion. \( \square \)

Next, we analyze the action function \( f(x, z, \tau) \) in Theorem 2.1. We use our notation before and write \( m(s) := s \coth s \).

**Lemma 2.4.** There is a constant \( c > 0 \) independent of \( x \) such that
\[
\inf\{m(\Omega(\sqrt{-1}\tau)): x \cdot x \in \mathbb{R}^d \} \geq c|x|^2. \tag{9}
\]
Moreover, all the maps \( \mathbb{R}^d \ni \tau \mapsto \beta_j(\tau) := \|m^{(j)}(\Omega(\sqrt{-1}\tau))\| \) with \( j \in \mathbb{N} \) are bounded. Here \( m^{(j)} \) denotes the \( j \)-th derivative of \( m \).

**Proof.** We write \( B(\tau) := m(\Omega(\sqrt{-1}\tau)) \in \mathbb{R}^{n \times n} \) which is a self-adjoint matrix for all \( \tau \in \mathbb{R}^d \). Let us denote by \( \{\lambda_j(\tau)\} \subset \mathbb{R} \) the eigenvalues of \( \Omega(\sqrt{-1}\tau) \). Then the eigenvalues of \( B(\tau) \) are clearly contained in \( m(\mathbb{R}) \subset (0, \infty) \). We write
\[
c := \min\{m(s) \mid s \in \mathbb{R}\} > 0.
\]
For each \( \tau \in \mathbb{R}^d \) let us denote by \( \{e_j(\tau) \mid j = 1, \ldots, n\} \) an orthonormal basis of \( \mathbb{R}^n \) consisting of eigenvectors of \( B(\tau) \). Then we have
\[
\langle B(\tau) \cdot x, x \rangle = \sum_{j=1}^n \|x, e_j(\tau)\|^2 m \circ \lambda_j(\tau) \geq c \sum_{j=1}^n \|x, e_j(\tau)\|^2 = c|x|^2,
\]
which proves the first assertion. Now, we prove the boundedness of \( \beta_j(\tau) \). Since \( m^{(j)}(\Omega(\sqrt{-1}\tau)) \) is self-adjoint it follows for all \( \tau \in \mathbb{R}^d \) that
\[
\beta_j(\tau) = \max\{m^{(j)}(\lambda(\tau)) \mid \lambda(\tau) \text{ is eigenvalue of } \Omega(\sqrt{-1}\tau)\},
\]
and it is sufficient to show that all derivatives \( m^{(j)}(s) \) are bounded on \( \mathbb{R} \). By the same argument as in Lemma 2.2 this follows from the fact that all derivatives of \( s \mapsto \coth s \) vanish exponentially as \( |s| \to \infty \). \( \square \)

With our notation in (5) we now can show:

**Proposition 2.5.** Let \( g = (x, z) \in G \) and \( \delta > 0 \).

(a) There are constants \( C, c > 0 \) such that
\[
\left| f_t(x, z) \right| \leq Ct^{-\frac{n}{2} - d} e^{-\frac{ct^2}{t}}.
\]
(b) Let \( g = (x, z) \) with \( |z| \geq \delta \). Then for any \( N \in \mathbb{N} \) we can choose \( C_N > 0 \) and \( c > 0 \) such that
\[
\left| f_t(x, z) \right| \leq C_N \frac{(1 + |x|)^{2N}}{|z|^{2N}} t^{2N - \frac{d}{2} - d} e^{-\frac{ct^2}{t} |x|^2}.
\]
Proof. We distinguish two cases:

(a) It directly follows from Theorem 2.1, Lemma 2.4 and $W(\tau) \in S(\mathbb{R}^d)$ that there are $C, c > 0$ independent of $t$ and $x$ such that

$$|k_t(x, z)| \leq (2\pi t)^{-\frac{n}{2} - d} e^{-\frac{c|x|^2}{t}} \int_{\mathbb{R}^d} W(\tau) d\tau = Ct^{-\frac{n}{2} - d} e^{-\frac{c|x|^2}{t}}.$$

(b) Let $g = (x, z)$ with $|z| > \delta$ and by $\Delta_{\tau} = \sum_{j=1}^d \frac{\partial^2}{\partial \tau_j^2}$ denote the Laplace operator with respect to the variable $\tau$. Then, with any integer $N \in \mathbb{N}$ we have

$$e^{-\sqrt{-1} \langle \tau, z \rangle t} = (-1)^N \left| \frac{2\pi}{|z|} \right|^2 \Delta_{\tau}^N e^{-\sqrt{-1} \langle \tau, z \rangle t}.$$

Hence $k_t(x, z)$ can be written as

$$k_t(x, z) = \frac{(-1)^N t^{2N - \frac{n}{2} - d}}{(2\pi)^{\frac{n}{2} + d} |z|^{2N}} \int_{\mathbb{R}^d} \Delta_{\tau}^N e^{-\sqrt{-1} \langle \tau, z \rangle t} \cdot \exp \left\{ -\frac{1}{2t} \left\{ \Omega(\sqrt{-1} \tau) \coth(\sqrt{-1} \tau) \cdot x, x \right\} W(\tau) d\tau. $$

According to Lemma 2.4 all the functions $\beta_j(\tau) := \|m_{(j)}(\Omega(\sqrt{-1} \tau))\|$ with $m(s) = s \coth s$ are bounded on $\mathbb{R}^d$ and therefore, after partial integrations in the above integral and using (9), we obtain the estimate

$$|k_t(x, z)| \leq C_N \left| \frac{1}{|z|^{2N}} \right|^2 \frac{1}{2N} \frac{2N - \frac{n}{2} - d}{2N} e^{-\frac{c}{2} |x|^2},$$

where $C_N, c > 0$ are suitable constants independent of $x, z$ and $t$.

3. Spectral zeta function of the sub-Laplacian on compact nilmanifold

In Section 3 we calculate the small time asymptotic of the heat kernel trace for the sub-Laplacian on a compact nilmanifold where the underlying Lie group $G$ is nilpotent of step two. As a corollary we obtain the pole distribution of the corresponding spectral zeta function through the Mellin transform and we provide an expression for the residue. We start from a quite general framework by only assuming the existence of a lattice $L (= \text{uniform discrete subgroup})$ in $G$. Then for the heat kernel trace of the sub-Laplacian on $L \setminus G$ we have

$$K_{L\setminus G}(t) = \sum_{\gamma \in L} \int k_t\left((x, z)^{-1} * \gamma * (x, z)\right) dx \, dz = \text{vol}(\mathcal{F}_L)k_t(0) + R(t),$$

where we write

$$R(t) := \sum_{\gamma \in L \setminus \{0\} \mathcal{F}_L} k_t\left((x, z)^{-1} * \gamma * (x, z)\right) dx \, dz.$$

Here, the integration is taken over a fundamental domain $\mathcal{F}_L$ of $G$ with respect to the lattice $L$. The following proposition is the essential step in proving Theorem A of the introduction.

Proposition 3.1. The remainder $R(t)$ in (10) vanishes to infinite order as $t \downarrow 0$.

Proof. Let $\gamma = (p, q) \in L \setminus \{0\}$. Then we see from the multiplication rule in (3) that

$$(x, z)^{-1} * \gamma * (x, z) = (x, z)^{-1} * (p, q) * (x, z) = (p, \tilde{q}(p, q, x)),$$

where by a straightforward calculation and using the notation in (3) one has

$$\tilde{q}(p, q, x) = q + \sum_{i,j=1}^n (p_i x_j - x_i p_j)a_{ij} \in \mathbb{R}^d.$$

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We distinguish the following cases:

**Case I:** Set \( L_1 := \{(0, q) \mid q \in \mathbb{R}^d\} \cap L \). Since \( \tilde{q}(0, q, x) = q \) is independent of \( x \) it follows from Proposition 2.5(b) and with suitable constants \( C_N, c > 0 \) that

\[
\sum_{y \in L_1 \cap [0,t]} \int |k_t(0, \tilde{q}(y, x))| \, dx \, dz \leq C_N t^{2N-\frac{d}{2}} \sum_{(0,q) \in L_1 \cap [0,t]} |q|^{-2N}.
\]

The subgroup \( L_1 \) is discrete by assumption and therefore the sum on the right converges for sufficiently large \( N \). Since \( N \) can be chosen arbitrarily the expression on the left vanishes of infinite order as \( t \downarrow 0 \).

**Case II:** Let \( B_\delta(z) \subset \mathbb{R}^d \) denote the \( d \)-dimensional open ball of radius \( \delta > 0 \), and centered in \( z \). Let us write \( L_2 \subset L \) for all \( (p,q) \in L \) with \( p \neq 0 \) such that

\[
\{ \tilde{q}(p,q,x) \mid (x,z) \in F_L \} \cap B_1(0) \neq \emptyset.
\]

Put \( S := \{ 0 \neq p \in \mathbb{R}^n \mid \exists (p,q) \in L_2 \} \) and for fixed \( p \in S \) set \( R_p := \{ q \mid (p,q) \in L_2 \} \). Recall that

\[
\tilde{q}(p,q,x) = q + \sum_{i,j=1}^n (p_i x_j - x_i p_j) a_{ij},
\]

where \( B_p(x) \) is linear in \( x \) and there is \( D_1 > 0 \) with \( \|B_p\| \leq D_1 |p| \). We estimate the number of elements in \( R_p \) for fixed \( p \in S \). Since \( F_L \subset \mathbb{R}^{n+d} \) is bounded we can choose a suitable constant \( D_2 > 0 \) such that

\[
\{ \tilde{q}(p,q,x) \mid (x,z) \in F_L \} \subset q + B_{D_2|p|}(0).
\]

Therefore, the condition \( (p,q) \in L_2 \) requires that \( \{ q + B_{D_2|p|}(0) \} \cap B_1(0) \neq \emptyset \) which in return implies that \( |q| \leq 1 + D_2 |p| \). Since by assumption \( L \) and hence \( R_p \) both are discrete there is \( D_3 > 0 \) with

\[
\alpha_p := \# R_p \leq D_3 (1 + |p|)^d.
\]

According to Proposition 2.5(a) we can choose constants \( C, c > 0 \) such that

\[
\sum_{(p,q) \in L_2 \cap F_L} \int |k_t(p, \tilde{q}(p,q,x))| \, dx \, dz \leq \sum_{p \in S} \sum_{q \in R_p} \int |k_t(p, \tilde{q}(p,q,x))| \, dx \, dz
\]

\[
\leq C t^{\frac{n}{2} - d} \sum_{p \in S} \sum_{q \in R_p} e^{-c|p|^2} \leq C D_3 t^{\frac{n}{2} - d} \sum_{p \in S} (1 + |p|)^d e^{-c|p|^2}.
\]

In the last inequality we have used (13). Since the set \( S \) is discrete, the right-hand side vanishes to infinite order as \( t \downarrow 0 \).

**Case III:** By \( L_3 \subset L \) we denote all lattice points \( (p,q) \in L \), \( p \neq 0 \) such that

\[
\{ \tilde{q}(p,q,x) \mid (x,z) \in F_L \} \cap B_1(0) = \emptyset.
\]

By applying Proposition 2.5(b) with \( \delta = 1 \) we see that for any \( N \in \mathbb{N} \) we can choose \( C_N > 0 \) and \( c > 0 \) such that with \( (p,q) \in L_3 \)

\[
|k_t(p, \tilde{q}(p,q,x))| \leq C_N \frac{(1 + |p|)^{2N}}{|\tilde{q}(p,q,x)|^{2N-\frac{2}{2} - d} e^{-c|p|^2}}.
\]

Put \( \tilde{S} := \{ 0 \neq p \in \mathbb{R}^n \mid \exists (p,q) \in L_3 \} \) and for fixed \( p \in \tilde{S} \) set \( \tilde{R}_p = \{ q \mid (p,q) \in L_3 \} \). It follows that
\[
\sum_{(p,q) \in L_3} \int \left| k_t(p, \tilde{q}(p, q, x)) \right| \, dx \, dz \\
\leq C_N t^{2N-\frac{d}{2}} \sum_{p \in \tilde{S}} (1 + |p|)^{2N} e^{-\frac{\gamma}{2} |p|^2} \int_{F_L} \frac{1}{|q + B_p(x)|^{2N}} \, dx \, dz.
\]

We estimate the sum over \(\tilde{R}_p\). Note that from (14) we have for all \((x, z) \in F_L\)
\[
\delta := \min \{ |q + B_p(x)| : q \in \tilde{R}_p \} \geq 1.
\]
According to Lemma A.1 we can choose \(N\) sufficiently large and a constant \(D_1 > 0\) independent of \(p\) and \(x\) with
\[
\sum_{q \in \tilde{R}_p} (1 + |p|)^{2N} e^{-\gamma t |p|^2} \leq D_1.
\]
Since \(F_L\) is bounded there is \(D_2 > 0\) such that
\[
\sum_{(p,q) \in L_3} \int \left| k_t(p, \tilde{q}(p, q, x)) \right| \, dx \, dz 
\leq D_2 t^{2N-\frac{d}{2}} \sum_{p \in \tilde{S}} (1 + |p|)^{2N} e^{-\gamma t |p|^2}.
\]
The right-hand side vanishes to infinite order as \(t \downarrow 0\) and we have also completed the third case. Finally, the assertion follows by combining the cases I–III.

**Remark 3.2.** In general the set \(\exp^{-1} L = \tilde{L}\) does not form a lattice in \(g\) (considered as a vector space). However, we can regard \(\tilde{L}\) as a subset of a lattice generated by \(\tilde{L}\) according to Theorem 2.12 in [16].

Let \(\Delta_{L \setminus G}^{\text{sub}}\) be the sub-Laplacian on the compact nilmanifold \(L \setminus G\). As before we write
\[
K_{L \setminus G}(t) := \text{trace} e^{-t \Delta_{L \setminus G}^{\text{sub}}}
\]
for its heat kernel trace. Then it follows from (10) and Proposition 3.1:

**Theorem 3.3.** The heat kernel trace of \(\Delta_{L \setminus G}^{\text{sub}}\) has the following asymptotic expansion as \(t \downarrow 0\):
\[
K_{L \setminus G}(t) = (2\pi t)^{-\frac{d}{2}} \text{vol}(F_L) \int_{R^d} W(\tau) \, d\tau + O(t^\infty).
\]

**Proof.** This follows from (10) and \(k_t(0) = (2\pi t)^{-\frac{d}{2}} \int_{R^d} W(\tau) \, d\tau\). \(\square\)

Let \(\Lambda\) denote the spectrum of the sub-Laplacian \(\Delta_{L \setminus G}^{\text{sub}}\). Then the corresponding spectral zeta function is defined by
\[
\zeta_{L \setminus G}^{\text{sub}}(s) = \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^s}.
\]
Let \(0 < \gamma < \min\{\lambda \in \Lambda \mid \lambda \neq 0\}\) and apply \(\dim(\text{kernel} \, \Delta_{L \setminus G}^{\text{sub}}) = 1\). Then there is \(C > 0\) such that for all \(t \geq 1\),
\[
|K_{L \setminus G}(t) - 1| = \sum_{\lambda \in \Lambda \setminus \{0\}} e^{-\lambda t} \sum_{\lambda \in \Lambda \setminus \{0\}} e^{-\gamma t} \leq C e^{-\gamma t},
\]
and it follows that \(K_{L \setminus G}(t) - 1\) vanishes to infinite order as \(t \to \infty\). Via the Mellin transform we have the relation
\[
\zeta_{L \setminus G}^{\text{sub}}(s) = \frac{1}{\Gamma(s)} \int_0^1 \{K_{L \setminus G}(t) - 1\} t^{s-1} \, dt + H(s),
\]
where $H(s)$ denotes an entire function. We set

$$\alpha := (2\pi)^{-\frac{n}{2} - d} \text{vol}(\mathcal{F}_L) \int_{\mathbb{R}^d} W(\tau) \, d\tau,$$

then the small time asymptotic expansion of the heat kernel trace in Theorem 3.3 shows that there is an entire function $H_1(s)$ with

$$\zeta_{s,L,G}(s) = \frac{1}{\Gamma(s)} \frac{\alpha}{s - \frac{n}{2} - d} + H_1(s) + H(s).$$

**Theorem 3.4.** The spectral zeta function of the sub-Laplacian $\Delta_{s,L,G}$ on the compact nilmanifold $L \setminus G$ has one simple pole located in $s_p = \frac{n}{2} + d$. The residue in $s_p$ is given by

$$\text{Res}_{s=s_p} \zeta_{s,L,G}(s) = \frac{\text{vol}(\mathcal{F}_L)}{(2\pi)^{\frac{n}{2} + d} \Gamma(\frac{n}{2} + d)} \int_{\mathbb{R}^d} W(\tau) \, d\tau. \quad (15)$$

In particular, the pole distribution of $\zeta_{s,L,G}(s)$ is independent of the lattice $L$, and only depends on the dimensions $n$ of $[g, g]^{\perp}$ and $d$ of $[g, g]$.

### 4. Heat trace expansion of the Laplacian on $L \setminus G$

We use the notation of the previous section and assume that $G$ is equipped with the left invariant metric such that $[X_i, Z_k \mid i = 1, \ldots, n; k = 1, \ldots, d]$ forms an orthonormal basis at the identity element. The Laplacian $\Delta_G$ on the Lie group $G$ with respect to this metric is given by

$$\Delta_G = \Delta_G^\text{sub} - \frac{1}{2} \sum_{k=1}^{d} \frac{\partial^2}{\partial z_k^2}. \quad (16)$$

The following has been shown in [3]:

**Theorem 4.1.** Let $W(\tau)$ be the volume element in (6). Then the heat kernel $K_{\Delta} \in C^{\infty}(\mathbb{R}^+ \times G \times G)$ of the Laplacian is given by

$$K_{\Delta}(t; (x, z), (\tilde{x}, \tilde{z})) = k_t^{\Delta}((\tilde{x}, \tilde{z})^{-1} * (x, z)),$$

where $k_t^{\Delta} \in C^{\infty}(\mathbb{R}^+ \times G)$ has the form:

$$k_t^{\Delta}(x, z) = \frac{1}{(2\pi t)^{\frac{n}{2} + d}} \int_{\mathbb{R}^d} e^{-\frac{f^{\Delta}(x, z, \tau)}{t}} W(\tau) \, d\tau, \quad (17)$$

and

$$f^{\Delta}(x, z, \tau) = \sqrt{-1}(\tau, z) + \frac{1}{2} \Omega(\sqrt{-1}\tau) \cdot x, x \} + \frac{1}{2} |\tau|^2 = f(x, z, \tau) + \frac{1}{2} |\tau|^2.$$

Again we assume that there is a lattice $L$ in $G$. Then the heat kernel trace $K_{L \setminus G}^{\Delta}(t)$ of the Laplacian $\Delta_{L \setminus G}$ on $L \setminus G$ can be decomposed as follows

$$K_{L \setminus G}^{\Delta}(t) = \sum_{\gamma \in L, \mathcal{F}_L} \int k_t^{\Delta}((x, z)^{-1} * \gamma * (x, z)) \, dx \, dz = \text{vol}(\mathcal{F}_L)k_t^{\Delta}(0) + R(t),$$

where $\mathcal{F}_L$ is a fundamental domain of the lattice $L$. By similar arguments as before one can show that the remainder

$$R(t) = \sum_{\gamma \in L \setminus [0], \mathcal{F}_L} \int k_t^{\Delta}((x, z)^{-1} * \gamma * (x, z)) \, dx \, dz$$
vanishes to infinite order as $t \downarrow 0$. Hence we have as $t \downarrow 0$

$$
K_{L,G}(t) = \frac{\text{vol}(F_L)}{(2\pi t)^{\frac{n}{2} + d}} \int_{\mathbb{R}^d} e^{-\frac{|\tau|^2}{2t}} W(\tau) \, d\tau + O(t^\infty)
$$

with the definition

$$
\alpha(t) := \int_{\mathbb{R}^d} e^{-\frac{|\tau|^2}{2t}} W(\sqrt{t}\cdot\tau) \, d\tau,
$$

which is bounded on $[0, \infty)$. By defining $\alpha(-t) := \alpha(t)$ we extend $\alpha(t)$ to a function on the real line.

**Lemma 4.2.** It holds $\alpha(t) \in C^\infty(\mathbb{R})$ and for all $\ell \in \mathbb{N}_0$ there is $k = k(\ell) \in \mathbb{N}$ and $C > 0$ such that

$$
\sup_{t > 0} \left| \frac{\partial^\ell W}{\partial t^\ell} (\sqrt{t}\tau) \right| \leq C |\tau|^k.
$$

**Proof.** First, we prove smoothness of $\alpha(t)$ in $t = 0$. Let $\{\lambda_j(\tau)\}$ denote the eigenvalues of $\Omega(\sqrt{-1}\tau)$ with multiplicities $m_j(\tau) \in \mathbb{N}$. Applying the same calculation as in the proof of Corollary 2.3 it follows

$$
W(\sqrt{t}\tau) = \prod_j \left[ g(\sqrt{t}\lambda_j(\tau)) \right]^{m_j(\tau)} = \prod_j \left[ g(\sqrt{t}\cdot\lambda_j(\tau)) \right]^{m_j(\tau)},
$$

(21)

where $g(s) = \frac{s}{\sinh s}$. Since the matrix $\Omega(\tau) \in \mathbb{R}^{n \times n}$ in (4) is skew-symmetric we conclude that its eigenvalues are purely imaginary and if $\sqrt{-1}\lambda \in \sqrt{-1}\mathbb{R}$ is an eigenvalue then also $-\sqrt{-1}\lambda \in \sqrt{-1}\mathbb{R}$ is an eigenvalue. The function $g(s)$ is even and therefore, we can write

$$
W(\sqrt{t}\tau) = \prod_{\lambda_j(\tau) > 0} \left[ g(\sqrt{t}\lambda_j(\tau)) \right]^{m_j(\tau)}.
$$

(22)

Note that the power series expansion of $g(s)$ in $s = 0$ has only even powers and therefore the assignment $t \mapsto g(\sqrt{t}\lambda_j(\tau))$ is smooth in $t = 0$. As a consequence, we see that $t \mapsto W(\sqrt{t}\tau)$ is smooth in $t = 0$ for any fixed $\tau \in \mathbb{R}^d$. Moreover, by applying Proposition A.3 and (22) we conclude that for all $\ell \in \mathbb{N}$ there are $k_j = k_j(\ell)$, $k = k(\ell) \in \mathbb{N}$ and $C_1, C > 0$ independent of $t > 0$ such that

$$
\left| \frac{\partial^\ell W}{\partial t^\ell} (\sqrt{t}\tau) \right| \leq C_1 \prod_{\lambda_j(\tau) > 0} \lambda_j(\tau)^{k_j} \leq C |\tau|^k.
$$

By using standard results on the differentiability of parameter integrals it follows that $\alpha(t)$ defines a smooth function in $t = 0$. □

Consider the Taylor expansion of the smooth map $t \mapsto W(\sqrt{t}\tau)$ in $t = 0$,

$$
W(\sqrt{t}\tau) = \sum_{j=0}^N b_j(\tau) t^j + \tilde{r}_N(t, \tau).
$$

According to Lemma 4.2 we have:

**Lemma 4.3.** For $N \in \mathbb{N}$ the function $\alpha(t)$ has the following asymptotic expansion

$$
\alpha(t) = \sum_{j=0}^N \tilde{b}_j t^j + O(t^{N+1}) \quad \text{as } t \downarrow 0,
$$

(23)

where $\tilde{b}_j := \int_{\mathbb{R}^d} b_j(\tau) e^{-\frac{|\tau|^2}{2t}} \, d\tau$. 

We write $\zeta_{L\setminus G}(s)$ for the spectral zeta function of the Laplacian on $L \setminus G$. Since $K^\Delta_{L\setminus G}(t) - 1$ vanishes to infinite order as $t \uparrow \infty$ and by applying (18) it follows that

$$
\zeta_{L\setminus G}(s) = \frac{1}{\Gamma(s)} \int_0^1 \{K^\Delta_{L\setminus G}(t) - 1\} t^{s-1} dt + H_0(s)
$$

$$
= \frac{\text{vol}(\mathcal{F}_L)}{(2\pi)^{\frac{n+d}{2}} \Gamma(s)} \int_0^1 \alpha(t) \cdot t^{s-\frac{n+d}{2}-1} dt + H_1(s),
$$

where $H_0(s)$ and $H_1(s)$ are entire functions on the complex plane. We insert the expansion (23) of $\alpha(t)$ for small values $t > 0$

$$
\zeta_{L\setminus G}(s) = \frac{\text{vol}(\mathcal{F}_L)}{(2\pi)^{\frac{n+d}{2}} \Gamma(s)} \sum_{j=0}^N \frac{\tilde{b}_j}{s+j-\frac{n+d}{2}} + H_2(s).
$$

The function $H_2(s)$ is holomorphic for $\text{Re}(s) > \frac{n+d}{2} - N - 1$. As is well known all poles of $\zeta_{L\setminus G}(s)$ are simple and located at most at $\{\dim G/2 - j | j \in \mathbb{N}_0\}$. We calculate the residues in these poles:

**Theorem 4.4.** Put $s_p := \frac{1}{2}(n+d)$ then the residues of $\zeta_{L\setminus G}(s)$ at points $\{s_p - j | j \in \mathbb{N}\}$ are given by

$$
\text{Res}_{s=s_p} \zeta_{L\setminus G}(s) = \frac{\text{vol}(\mathcal{F}_L)}{(2\pi)^{\frac{n+d}{2}} \Gamma(\frac{n+d}{2})} \tilde{b}_j.
$$

Here the values $\Gamma(-k)^{-1}$ with $k \in \mathbb{N}_0$ should be interpreted as zero. In particular, the residue in $s_p$ is

$$
\text{Res}_{s=s_p} \zeta_{L\setminus G}(s) = \frac{\text{vol}(\mathcal{F}_L)}{(2\pi)^{\frac{n+d}{2}} \Gamma(\frac{n+d}{2})}.
$$

**Proof.** We only show (26). Note that $\tilde{b}_0 = \alpha(0) = \int_{\mathbb{R}^d} e^{-|\tau|^2} \, d\tau = (2\pi)^{\frac{d}{2}}$. From (24) the assertion follows. \qed

5. On certain values of the spectral zeta function for the sub-Laplacian

Let $\Lambda_M := \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots \}$ denote the spectrum of the sub-Laplacian $\Delta^\text{sub}_M$ on a two step compact nilmanifold $M = L \setminus G$ of dimension $\dim M = n + d$. By $\mathbb{S}^1$ we denote the unit circle equipped with the Laplacian $-\frac{d^2}{dt^2}$. Consider the sub-Laplace operator on the product manifold $M \times \mathbb{S}^1$,

$$
\Delta^\text{sub}_{M \times \mathbb{S}^1} = \Delta^\text{sub}_M \otimes \text{Id} - \text{Id} \otimes \frac{d^2}{dt^2}.
$$

Then $\Delta^\text{sub}_{M \times \mathbb{S}^1}$ clearly has the spectrum

$$
\Lambda_{M \times \mathbb{S}^1} = \{\lambda_n + k^2 | n \in \mathbb{N}_0, k \in \mathbb{Z}\}.
$$

In the following we write $\zeta^\text{sub}_{M \times \mathbb{S}^1}(s)$, $\zeta^\text{sub}_M(s)$ and $\zeta_{\mathbb{S}^1}(s) = 2\zeta(2s)$ for the spectral zeta functions of $\Delta^\text{sub}_{M \times \mathbb{S}^1}$, $\Delta^\text{sub}_M$ and $\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}$, respectively. We have

$$
\zeta^\text{sub}_{M \times \mathbb{S}^1}(s) = \sum_{0 \neq (n,k) \in \mathbb{N}_0 \times \mathbb{Z}} \frac{1}{(\lambda_n + k^2)^s} = \zeta^\text{sub}_M(s) + 2\zeta(2s) + \sum_{n,k=1}^{\infty} \frac{2}{(\lambda_n + k^2)^s}.
$$

By writing $Z(s) := \sum_{n,k=1}^{\infty} \frac{2}{(\lambda_n + k^2)^s}$ and applying the Mellin transform, one obtains
\[
Z(s) = \frac{2}{\Gamma(s)} \sum_{n,k=1}^{\infty} \int_{0}^{\infty} e^{-(\lambda_n + k^2)t} t^{s-1} dt
= \frac{1}{\Gamma(s)} \int_{0}^{\infty} \left( k_M(t) - 1 \right) \left( k_{\mathbb{Z}}(t) - 1 \right) t^{s-1} dt,
\]

where \( k_M(t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \) and \( k_{\mathbb{Z}}(t) = \sum_{k \in \mathbb{Z}} e^{-k^2 t} \) denote the heat kernel traces of \( \Delta_M^{\text{sub}} \) and \( \Delta_{\mathbb{Z}}^{\text{sub}} \), respectively. Next, we use the Poisson summation formula:

\[
k_{\mathbb{Z}}(t) - 1 = \sum_{k \in \mathbb{Z}} e^{-k^2 t} - 1 = \sqrt{\frac{\pi}{t}} \sum_{k \in \mathbb{Z}} e^{-\frac{\pi^2}{t} k^2} - 1.
\]

Together with (27) we obtain:

\[
Z(s) + \zeta_M^{\text{sub}}(s) = \frac{\sqrt{\pi}}{\Gamma(s)} \int_{0}^{\infty} \left( k_M(t) - 1 \right) \sum_{k \in \mathbb{Z}} e^{-\frac{\pi^2}{t} k^2} t^{s-\frac{1}{2}} dt
= \frac{2\sqrt{\pi}}{\Gamma(s)} \int_{0}^{\infty} \left( k_M(t) - 1 \right) \sum_{k=1}^{\infty} e^{-\frac{\pi^2}{t} k^2} t^{s-\frac{1}{2}} dt + \sqrt{\frac{\pi}{\Gamma(s)}} \Gamma(s - \frac{1}{2}) \zeta_M^{\text{sub}} \left( s - \frac{1}{2} \right).
\]

It is easy to check that \( H(s) \) defines an entire function on the complex plane and we have shown that

\[
\zeta_M^{\text{sub}}_{L \setminus G}(s) = H(s) + 2\zeta(2s) + \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta_M^{\text{sub}} \left( s - \frac{1}{2} \right).
\]

**Theorem 5.1.** With \( M = L \setminus G \) and the previous notation it follows:

(a) The spectral zeta function \( \zeta_M^{\text{sub}}_{L \setminus G}(s) \) vanishes at negative integers \( \{-1, -2, \ldots\} \).

(b) The value of \( \zeta_M^{\text{sub}}_{L \setminus G}(s) \) at \( s = 0 \) is given by \( \zeta_M^{\text{sub}}_{L \setminus G}(0) = -1 \).

In particular, these values are independent of \( L \) and \( G \).

**Proof.** (a) From Theorem 3.4 we know that the left-hand side of (28) has no poles in \( s - \frac{1}{2} \in \{-1, -2, -3, \ldots\} \). Hence the product \( \Gamma(s - \frac{1}{2}) \zeta_M^{\text{sub}}(s - \frac{1}{2}) \) is finite for such \( s \), which shows that \( \zeta_M^{\text{sub}}(-k) = 0 \) for \( k \in \mathbb{N} \).

(b) From Theorem 3.4 we know that the left-hand side of (28) possesses only one simple pole in \( s = \frac{n+1}{2} + d > \frac{1}{2} \). Therefore, \( \zeta_M^{\text{sub}}_{L \setminus G}(0) = \text{Res}_{s = \frac{n+1}{2}} \zeta_M^{\text{sub}}_{L \setminus G}(s) = -1 \). \( \square \)

**Remark 5.2.** Note that Theorem 5.1 is a generalization of Theorem 8.4 in [1] where we only dealt with Heisenberg manifolds \( M_H = L \setminus H_{2n+1} \) and standard lattices \( L \subset H_{2n+1} \). Heisenberg manifold only exists in odd dimensions. In case of the Laplace operator \( \Delta_N \) on a closed odd-dimensional Riemannian manifold \( N \) with spectral zeta function \( \zeta_N(s) \) one can show by a similar argument that \( \zeta_N(-k) = 0 \) for \( k \in \mathbb{N} \) and \( \zeta_N(0) = -1 \). However, different from the case of the Laplace operator the statement in Theorem 5.1 remains also true for an even-dimensional group \( G \) (cf. the example of the following section).

6. Nilmanifold over free two step nilpotent Lie groups

In Section 6 we provide some examples different from the earlier ones in [1] and we discuss a class of nilmanifolds \( M := L \setminus G \) that are not of Heisenberg type. In particular, \( M \) can be even-dimensional. We explicitly calculate the spectral zeta function \( \zeta_M^{\text{sub}}(s) \) of the sub-Laplace operator on a nilmanifold over the free nilpotent Lie group of
dimension six. Even in such a low-dimensional case $\zeta_{M}^{\text{sub}}(s)$ takes a complicated form. However, some of its analytic properties can be obtained from the general results in Theorems 3.4 and 5.1. Fix $n \in \mathbb{N}$ and let

$$F(n+\ell) \cong \mathbb{R}^n \oplus \mathbb{R}^\ell,$$

be a connected and simply connected free 2-step nilpotent Lie group with Lie algebra $\mathfrak{f}_{n+\ell}$. We fix a basis $\{X_i, Z_{i,j} \mid 1 \leq i, j \leq n, \ i < j\}$ of $\mathfrak{f}_{n+\ell}$ and assume the bracket relations

$$[X_i, X_j] = 2Z_{ij},$$

where $1 \leq i < j \leq n$. Identifying the Lie group and Lie algebra via the exponential map the group multiplication is given by the formula (3).

**Remark 6.1.** Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra with basis $\{X_i\}_{i=1}^n$ of a complement of the derived ideal $[\mathfrak{g}, \mathfrak{g}]$ and $\{Z_k\}_{k=1}^d$ a basis of $[\mathfrak{g}, \mathfrak{g}]$. Put

$$[X_i, X_j] = 2 \sum_{k=1}^d a_{ij}^k Z_k,$$

where $a_{ij}^k = -a_{ji}^k$ for $i, j = 1, \ldots, n$. Then we can define a Lie algebra homomorphism $\rho: \mathfrak{f}_{n+\ell} \to \mathfrak{g}$ by

$$\rho \left( \sum_{i=1}^n x_i X_i + \sum_{i,j} z_{ij} Z_{ij} \right) := \sum_{i=1}^n x_i X_i + \sum_{i,j} \sum_{k=1}^d z_{ij} a_{ij}^k Z_k.$$

In this sense any 2-step nilpotent Lie group (Lie algebra) is covered by a free 2-step nilpotent Lie group (Lie algebra).

Let $\tilde{X}_i$ be the left invariant vector field on $F(n+\ell)$ which corresponds to $X_i$ ($i = 1, \ldots, n$). More precisely,

$$[\tilde{X}_i, f](g) = \frac{d}{dt} f(g \ast e^{tX_i}) \bigg|_{t=0} = \frac{\partial f}{\partial x_i}(g) + \sum_{j<i} x_j \frac{\partial f}{\partial z_{ij}}(g) - \sum_{j>i} x_j \frac{\partial f}{\partial z_{ij}}(g),$$

where $g = (x, z) \in \mathbb{R}^n \oplus \mathbb{R}^\ell$ and $f \in C^\infty(F(n+\ell))$. With our previous notation the skew-symmetric matrix $\Omega(\tau) \in \mathbb{R}^{n \times n}$ is given by

$$\Omega(\tau) = \begin{pmatrix}
0 & \tau_{12} & \cdots & \tau_{1n} \\
-\tau_{12} & 0 & \tau_{23} & \cdots & \tau_{2n} \\
\vdots & & & & \\
-\tau_{1n} & -\tau_{2n} & \cdots & 0 \\
\end{pmatrix} \in \mathbb{R}^{n \times n},$$

with $\tau = (\tau_{ij} \mid i < j) \in \mathbb{R}^\ell$. The heat kernel of the sub-Laplacian

$$\Delta_{F(n+\ell)}^{\text{sub}} = -\frac{1}{2} \sum_{i=1}^n \tilde{X}_i^2$$

on $F(n+\ell)$ is expressed in Theorem 2.1. We fix the standard lattice

$$L = \left\{(m_1, \ldots, m_n, k_1, \ldots, k_\ell) \mid m_i, k_i \in \mathbb{Z} \right\} \subset F(n+\ell).$$

In order to give more explicit formulas we specialize to the case of the 6-dimensional free nilpotent Lie group $F_{(3+3)}$. In this case, the matrix $\Omega(\tau)$ with $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ is

$$\Omega(\tau) = \begin{pmatrix}
0 & \tau_1 & \tau_2 \\
-\tau_1 & 0 & \tau_3 \\
-\tau_2 & -\tau_3 & 0 \\
\end{pmatrix} \in \mathbb{R}^{3 \times 3},$$
which has the eigenvalues 0 and $\pm \sqrt{|\tau|}$. So the volume element $W(\tau)$ can be calculated explicitly, cf. [2]:

$$W(\tau) = \sqrt{\det \frac{\Omega(\sqrt{-1}\tau)}{\sinh \Omega(\sqrt{-1}\tau)}} = \frac{|\tau|}{\sinh |\tau|}. \tag{29}$$

The asymptotic behaviour of the heat kernel trace of $\Delta_{F(3+3)}^{\text{sub}}$ was given in Theorem 3.3. To calculate the coefficients in this expansion precisely we need to evaluate the integral over the volume element using the identity

$$\int_{\mathbb{R}^d} \frac{|\tau|}{\sinh |\tau|} d\tau = 4\pi^\frac{d}{2} \frac{\Gamma(d+1)}{\Gamma(\frac{d}{2})} (1 - 2^{-(d+1)}) \zeta(d+1) \quad (d \in \mathbb{N})$$

(cf. Lemma A.2). Let $K_{L \setminus F(3+3)}(t)$ denote the heat kernel trace of $\Delta_{L \setminus F(3+3)}^{\text{sub}}$. A fundamental domain $F_L$ of $L$ is

$$F_L = \{(x_1, x_2, x_3, z_1, z_2, z_3) \mid 0 \leq x_j < 1 \text{ and } 0 \leq z_j < 1\}$$

and it holds $\text{vol}(F_L) = 1$. We conclude from Theorem 3.3.

**Proposition 6.2.** The heat kernel trace of $\Delta_{L \setminus F(3+3)}^{\text{sub}}$ has an asymptotic expansions as $t \downarrow 0$ given by

$$K_{L \setminus F(3+3)}(t) = \frac{\sqrt{\pi}}{32\sqrt{2}} t^{-\frac{9}{2}} + O(t^\infty).$$

In particular, the spectral zeta function $\zeta_{L \setminus F(3+3)}^{\text{sub}}(s)$ is meromorphic on the complex plane with only one simple pole in $s_p = \frac{9}{2}$, and residue

$$\text{Res}_{s=s_p} \zeta_{L \setminus F(3+3)}^{\text{sub}}(s) = \frac{\sqrt{\pi}}{32\sqrt{2} \Gamma(\frac{9}{2})} = \frac{1}{210\sqrt{2}}.$$

**Proof.** According to Theorem 3.3 we have $K_{L \setminus F(3+3)}(t) = \alpha t^{-\frac{9}{2}} + O(t^\infty)$ with

$$\alpha = \frac{1}{(2\pi)^{\frac{9}{2}}} \int_{\mathbb{R}^3} \frac{|\tau|}{\sinh |\tau|} d\tau = \frac{\sqrt{\pi}}{32\sqrt{2}},$$

where we have used Lemma A.2(iii). The second assertion directly follows from Corollary 3.4. $\Box$

In the example below we calculate the residues (25) of the spectral zeta function $\zeta_{L \setminus F(3+3)}(s)$ for the Laplacian on $L \setminus F(3+3)$. Since $n + d = 6$ is even, only three poles appear in $s = 1, 2, 3$.

**Example 6.3.** From (29) we see that the function $\alpha(t)$ in (19) is more explicitly given by

$$\alpha(t) = \int_{\mathbb{R}^3} e^{-\frac{|\tau|^2}{2}} \frac{|\sqrt{t}\tau|}{\sinh |\sqrt{t}\tau|} d\tau.$$

By $B_j(x)$ we denote the $j$-th Bernoulli polynomial and we use the well-known Taylor expansion

$$\frac{x}{\sinh x} = \sum_{j=0}^\infty B_{2j} \left(\frac{1}{2}\right) \frac{(2x)^{2j}}{(2j)!},$$

which implies that for $N \in \mathbb{N}$

$$W(\sqrt{t}\tau) = \sum_{j=0}^N B_{2j} \left(\frac{1}{2}\right) \frac{4^j |\tau|^{2j}}{(2j)!} t^j + \tilde{r}_N(t, \tau).$$
Then the numbers $\tilde{b}_j$ for $j \in \mathbb{N}_0$ in (24) which determine the residues of the spectral zeta function of the Laplacian on $L \setminus F_{(3+)}$ in (25) are given by
\[
\tilde{b}_j = \frac{2^j}{(2j)!} B_{2j} \left( \frac{1}{2} \right) \frac{\Gamma \left( j + \frac{3}{2} \right)}{\pi}.
\]
According to (25) we obtain the following non-zero residues for $j = 0, 1, 2$:
\[
\text{Res}_{s=3-j} \xi_{L \setminus F_{(3+)}}(s) = \frac{2^{3j-2}}{(2j)! \pi^2} B_{2j} \left( \frac{1}{2} \right) \frac{\Gamma \left( j + \frac{3}{2} \right)}{\Gamma(3-j)}.
\]

Finally, we derive an explicit expression of the spectral zeta function for the sub-Laplacian $\Delta_{L/F_{(3+)}}$ on $L \setminus F_{(3+)}$ by decomposing the operator into an infinite series of elliptic operators acting on sections into line bundles. Recall that two step nilmanifolds can be interpreted as the total space of a principal bundle where the fiber and the base space both are tori. The center $Z(F_{(3+)})$ of the group $F_{(3+)}$ is
\[
Z(F_{(3+)}) = \left\{ (0, 0, 0, s) \mid s = (s_1, s_2, s_3) \in \mathbb{R}^3 \right\}.
\]
If we define $\rho : F_{(3+)} \cong \mathbb{R}^3 \to F_{(3+)} / Z(F_{(3+)}) \cong \mathbb{R}^3$ by
\[
\rho(x_1, x_2, x_3, z_1, z_2, z_3) := (x_1, x_2, x_3) \in \mathbb{R}^3,
\]
then $\rho$ induces the projection of the principal bundle
\[
L \setminus F_{(3+)} \to \rho(L) \setminus F_{(3+)}/Z(F_{(3+)}) \cong \mathbb{R}^3/\mathbb{Z}^3 = T^3
\]
with structure group $Z(F_{(3+)}) / \rho(L) \cong U(1)^3$. With
\[
\lambda = (e^{2\pi \sqrt{-1}s_1}, e^{2\pi \sqrt{-1}s_2}, e^{2\pi \sqrt{-1}s_3}) \in T^3
\]
and $n \in \mathbb{Z}^3$ let $\chi_n : U(1)^3 \to U(1)$ be the character $\chi_n(\lambda) = \lambda_1^{n_1} \lambda_2^{n_2} \lambda_3^{n_3} = \lambda^n$. Let $\mathcal{F}(n)$ with $n \in \mathbb{Z}^3$ be the following subspace of $C^\infty(L \setminus F_{(3+)})$
\[
\mathcal{F}(n) = \left\{ f \in C^\infty(L \setminus F_{(3+)}) \mid f((x, z) * (0, 0, 0, s)) = \chi_n(\lambda)^{-1} f(x, z) \right\},
\]
where we write $x = (x_1, x_2, x_3), z = (z_1, z_2, z_3)$ and $s = (s_1, s_2, s_3)$. Then we have a decomposition of $C^\infty(L \setminus F_{(3+)})$
\[
C^\infty(L \setminus F_{(3+)}) = \sum_{n \in \mathbb{Z}^3} \mathcal{F}(n).
\]
Let $E(n)$ be the complex line bundle on the base space
\[
\rho(L) \setminus F_{(3+)}/Z(F_{(3+)}) \cong T^3
\]
associated to the character $\chi_n$. Then $E(n)$ is identified with the quotient space of $F_{3+} \times \mathbb{C}$ by the equivalence relation
\[
(x_1, x_2, x_3, z_1, z_2, z_3; w)
\sim (x_1 + k_1, x_2 + k_2, x_3 + k_3, z_1 + m_1 + x_1 k_2 - x_2 k_1 - s_1, z_2 + m_2 + x_1 k_3 - x_3 k_1 - s_2, z_3 + m_3 + x_2 k_3 - x_3 k_2 - s_3; \chi_n(\lambda) \cdot w),
\]
where $(k_1, k_2, k_3, m_1, m_2, m_3) \in L$ and $s = (s_1, s_2, s_3) \in \mathbb{R}^3$. The subspace $\mathcal{F}(n)$ is identified with the space of smooth sections $\Gamma(T^3, E(n))$ and the sub-Laplacian $\Delta_{L/F_{(3+)}}$ leaves these spaces invariant and can be seen as a differential operator on the line bundle. We denote the restriction of the sub-Laplacian $\Delta_{L/F_{(3+)}}$ to $\mathcal{F}(n)$ by $\mathcal{D}(n)$.

We shall express its principal symbol $\sigma(\mathcal{D}(n))$. A fiber $E_{(x_1, x_2, x_3)}(n)$ of the line bundle $E(n)$ in a point $[x_1, x_2, x_3] \in T^3$ is identified with $\{(x, z; w) \mid w \in \mathbb{C}\}$ through the diagram
\[
\begin{array}{ccc}
F_{(3+)} \times \mathbb{C} & \to & L \setminus F_{(3+)} \times \mathbb{C} \\
\downarrow & & \downarrow \\
F_{(3+)} & \to & L \setminus F_{(3+)} \\
\end{array}
\]
\[
F_{(3+)} \to L \setminus F_{(3+)} \to T^3,
\]

(30)
The principal symbol $\sigma(D_{(n)})_{([x], \theta)} : E_{[x]}^{(n)} \to E_{[x]}^{(n)}$ at a point

$$\{[x], \theta\} = \left([x], \sum_{i=1}^{3} \xi_i dx_i\right) \in T^*([x]_3)$$

is the map $\sigma(D_{(n)})_{([x], \theta)}(x, z; w) = (x, z, |\xi|^2 \cdot w)$ which shows the ellipticity of $D_{(n)}$. The heat kernel of the operator $D_{(n)}$ is expressed in an integral form

$$\text{Kernel function of } e^{-tD_{(n)}} = \int_{[0,1]^3} K_{L\setminus F_{(3+3)}}^{\text{sub}}(t, g \ast \lambda, \mathring{g}) \chi_{(n)}^{-1}(\lambda) ds_1 ds_2 ds_3,$$

where $g = (x, z), \mathring{g} = (\mathring{x}, \mathring{z}) \in F_{(3+3)}$. Its trace is given by

$$\frac{1}{(2\pi t)^{3/2}} \int_{[0,1]^3} \int_{[0,1]^3} \int_{[0,1]^3} \sum_{(k,m) \in \mathbb{Z}^6} \exp\left\{-\frac{\sqrt{t}}{t} \left(\tau, m - 2[x,k] + s\right)\right\} \left\{\left(\text{Id} + \frac{|\tau| \coth |\tau| - 1}{|\tau|^2} \Omega \left(\sqrt{-1} \tau^2\right)\right) \cdot k, k\right\}$$

$$\times \frac{|\tau|}{\sinh |\tau|} d\tau dx dz \chi_{(n)}(\lambda)^{-1} ds,$$

where we put $\lambda = (e^{2\pi \sqrt{-1}s_1}, e^{2\pi \sqrt{-1}s_2}, e^{2\pi \sqrt{-1}s_3})$, and we write

$$[x, k] = (x_1 k_2 - x_2 k_1, x_1 k_3 - x_3 k_1, x_2 k_3 - x_3 k_2).$$

This is calculated as

$$\text{trace } e^{-tD_{(n)}} = \frac{1}{(2\pi t)^{3/2}} \sum_{k \in \mathbb{Z}^3} \left| e^{\frac{1}{2t} \left\{(\text{Id} + \frac{2\pi i |m| \coth 2\pi |m| - 1}{(2\pi i |m|)^2} \Omega (2\pi i \sqrt{-1} n)\right)^2 k, k\right\} \right.$$}

$$\times \frac{2\pi t |n|}{\sinh 2\pi t |n|} \exp\left\{4\pi \sqrt{-1} (x, \Omega (n) \cdot k)\right\} dx.$$}

Since $\langle [x, k], n \rangle = 0$ for $x \in [0, 1]^3$ is equivalent to $\Omega (n) \cdot k = 0$, the integral reduces to

$$\text{trace } e^{-tD_{(n)}} = \frac{1}{\sqrt{2\pi t}} \sum_{k \in \mathbb{Z}^3} \frac{|n|}{\Omega (n) \cdot k = 0} e^{\frac{|n|}{2\pi t \sinh 2\pi t |n|},}$$

according to the choice of $\mathbb{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$. The spectral zeta function $\zeta_{D_{(n)}}(s)$ of the elliptic operator $D_{(n)}$ acting on the line bundle $E^{(0)}$ is meromorphic with only simple poles, the largest one is located at $s = 3/2$. The spectral zeta function of the sub-Laplacian $\Delta_{L\setminus F_{(3+3)}}^{\text{sub}}$ is given by

$$\zeta_{L\setminus F_{(3+3)}}^{\text{sub}}(s) = \zeta_{D_{(0,0,0)}}(s) + \sum_{\mathbb{n} \in \mathbb{Z}^3 \setminus \{0\}} \zeta_{D_{(n)}}(s). \quad (31)$$

Let $\langle |n_1|, |n_2| \rangle$ and $\langle |n_1|, |n_2|, |n_3| \rangle$ denote the least common divisor of its entries where $\mathbb{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$. In order to calculate the operator traces above we distinguish the case $\mathbb{n} = (0, 0, 0) = 0$ and $\mathbb{n} \neq 0$ with the convention that $\langle |n_1|, |n_2|, 0 \rangle = (|n_1|, |n_2|)$, if $n_3 = 0$ and $\langle |n_1|, 0, 0 \rangle = |n_1|$ if $n_2 = n_3 = 0$, and so on.

1. Let $\mathbb{n} = 0$, then

$$\text{trace } e^{-tD_{(n)}} = \frac{1}{(2\pi t)^{3/2}} \sum_{k \in \mathbb{Z}^3} e^{-\frac{k_1^2 + k_2^2 + k_3^2}{2\pi t}} = \sum_{k \in \mathbb{Z}^3} e^{-2\pi^2 t (k_1^2 + k_2^2 + k_3^2)}.$$}

2. Let $\mathbb{n} \neq 0$, then

$$\text{trace } e^{-tD_{(n)}} = \sum_{\ell \in \mathbb{Z}} \langle |n_1|, |n_2|, |n_3| \rangle e^{-2\pi^2 t (n_1 |n_2| |n_3|)^2} \frac{1}{\sinh 2\pi t \sqrt{n_1^2 + n_2^2 + n_3^2}}.$$
Now, we can show:

**Proposition 6.4.** Let \( n \neq 0 \), then the trace of \( e^{-tD(n)} \) is given by

\[
\text{trace } e^{-tD(n)} = \sum_{\ell \in \mathbb{Z}} \sum_{j=0}^{\infty} 2(|n_1|, |n_2|, |n_3|) e^{-2\pi t \frac{(|n_1|, |n_2|, |n_3|)^2}{|n|^2} \ell^2 \pi + |n|(2j+1)}.
\]

Hence, the eigenvalues of \( D(n) \) are

\[
\lambda_{\ell,j}^{(n)} = 2\pi \left( \frac{(|n_1|, |n_2|, |n_3|)^2}{|n|^2} \ell^2 \pi + |n|(2j+1) \right),
\]

where \((\ell, j) \in \mathbb{Z} \times \mathbb{N}_0\). Its multiplicities are given by \( m(\lambda_{\ell,j}^{(n)}) = 2(|n_1|, |n_2|, |n_3|) \).

As an application of Proposition 6.4 and using (31) we can write the spectral zeta function \( \zeta_{\text{sub} L,F(3+3)}(s) \) in form of a Dirichlet series.

**Theorem 6.5.** The following Dirichlet series \( Z(s) \) is convergent for \( \text{Re}(s) > \frac{9}{2} \):

\[
Z(s) = \frac{2}{(2\pi)^s} \sum_{m \in \mathbb{Z}^3 \setminus \{0\}} \sum_{\ell \in \mathbb{Z}} \sum_{j=0}^{\infty} \frac{(|m_1|, |m_2|, |m_3|)}{\ell^2 |m_1| + (2j+1)|m|^s}.
\]

Moreover, \( Z(s) \) has a meromorphic extension to the complex plane such that

\[
\zeta_{\text{sub} L,F(3+3)}^{\text{sub}}(s) = \frac{1}{(2\pi)^s} \zeta_{\mathbb{R}^3/\mathbb{Z}^3}(s) + Z(s).
\]

Here the Epstein zeta function \( \zeta_{\mathbb{R}^3/\mathbb{Z}^3}(s) \) (which is the spectral zeta function of the Laplacian in the 3-torus \( \mathbb{R}^3/\mathbb{Z}^3 \)) is defined by:

\[
\zeta_{\mathbb{R}^3/\mathbb{Z}^3}(s) := \sum_{m \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{(m_1^2 + m_2^2 + m_3^2)^s}.
\]

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**Appendix A**

We collect some formulas and results which are useful throughout the text.

**Lemma A.1.** Let \( L \subset \mathbb{R}^d \) be discrete where \( d \in \mathbb{N} \). Assume that \( N \in \mathbb{N} \) is sufficiently large such that the sum below converges and consider the function

\[
F(x) := \sum_{p \in L} \frac{1}{|p - x|^N}, \quad x \in \mathbb{R}^d \setminus L.
\]

If we put \( \delta := \min\{|p - x|: p \in L\} \), then there is \( C > 0 \) such that \( |F(x)| \leq C\delta^{-N} \).

**Proof.** Let \( 0 \neq p \in L \), then it follows from

\[
\frac{1}{|1 - \frac{|x|}{|p|}|} \geq \frac{|p|}{|p - x|} \geq \frac{1}{1 + \frac{|x|}{|p|}}
\]
that one can choose $C_1 > 2$ with $|p||p-x|^{-1} \leq 2$ for all $p \in L$ such that $|p| > C_1$. Moreover, in case of $|p| < C_1$ we have

$$\frac{|p|}{|p-x|} \leq |p|\delta^{-1} \leq C_1\delta^{-1}.$$ 

Without loss of generality we assume that $\delta < 1$ which implies $C_1^\delta^{-1} > 2$ and it follows that

$$F(x) = \sum_{p \in L} \frac{|p|^N}{|p-x|^N} \leq C_1^N \sum_{p \in L} |p|^{-N}.$$ 

Finally, put $C := C_1^N \sum_{p \in L} |p|^{-N}$.

We need to evaluate the following integral:

**Lemma A.2.** With $d \in \mathbb{N}$ we have

$$\int_{\mathbb{R}^d} \frac{|	au|}{\sinh |	au|} d\tau = 4\pi^\frac{d}{2} \frac{\Gamma(d+1)}{\Gamma\left(\frac{d}{2}\right)} (1 - 2^{-(d+1)}) \zeta(d+1).$$

In particular, it follows

(i) $\int_{\mathbb{R}} \frac{|	au|}{\sinh |	au|} d\tau = \frac{\pi^2}{2},$

(ii) $\int_{\mathbb{R}^2} \frac{|	au|}{\sinh |	au|} d\tau = 7\pi \zeta(3),$

(iii) $\int_{\mathbb{R}^3} \frac{|	au|}{\sinh |	au|} d\tau = \frac{\pi^5}{2}.$

**Proof.** Since the integrand is a radial symmetric function, we have

$$\int_{\mathbb{R}^d} \frac{|	au|}{\sinh |	au|} d\tau = V_d \int_0^\infty \frac{r^d}{\sinh r} dr = 2V_d \int_0^\infty \frac{r^d e^{-r}}{1 - e^{-2r}} dr,$$

where $V_d = 2\pi^\frac{d}{2} / \Gamma\left(\frac{d}{2}\right)$ is the surface volume of the unit sphere of dimension $d - 1$ and

$$\int_0^\infty \frac{r^d e^{-r}}{1 - e^{-2r}} dr = \sum_{j=0}^\infty \int_0^\infty r^d e^{-(1+2j)r} dr = \sum_{j=0}^\infty \frac{1}{(1+2j)^{d+1}} \int_0^\infty r^d e^{-r} dr$$

$$= 2^{-(d+1)} \Gamma(d+1) \left(\zeta(d+1) - \frac{1}{2}\right)$$

$$= \Gamma(d+1) \left(1 - 2^{-(d+1)}\right) \zeta(d+1).$$

The formulas (i)--(iii) now are obtained by using the identities $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$. 

The following estimate is needed in the proof of Lemma 4.2. With $t, s \geq 0$ consider the function

$$F(t,s) = H(ts^2), \quad \text{where } H(x) := \frac{\sqrt{x}}{\sinh \sqrt{x}}.$$
Proposition A.3. For all $\ell \in \mathbb{N}$ there is $C_{\ell} > 0$ such that for $s > 0$

$$\sup_{t \in \mathbb{R}} \left| \frac{\partial \ell F}{\partial t}(t,s) \right| \leq C_{\ell} s^{2\ell}. \tag{A.1}$$

Proof. For $t, s \in \mathbb{R}$ we have $\frac{\partial \ell F}{\partial t}(t,s) = s^{2\ell} H^{(\ell)}(ts^2)$ which shows that (A.1) follows from the estimate

$$\sup_{x \in \mathbb{R}} \left| H^{(\ell)}(x) \right| \leq C_{\ell} \tag{A.2}$$

for $\ell \in \mathbb{N}$ and with a suitable constant $C_{\ell} > 0$. Hence we only show (A.2) and we distinguish two cases:

(i) The boundedness of all derivatives $H^{(\ell)}(x)$ in $|x| \leq 1$ directly follows from the smoothness of

$$H(x) = \sum_{j=0}^{\infty} B_{2j} \left( \frac{1}{2} \right)^j \frac{4^j x^j}{(2j)!}.$$ 

(ii) In case of $|x| \geq 1$ we prove the boundedness of $H^{(\ell)}(x)$ by induction. The relation $H(x) \sinh \sqrt{x} = \sqrt{x}$ shows

$$H^{(\ell)}(x) \sinh \sqrt{x} + \sum_{k=0}^{\ell-1} \binom{\ell}{k} H^{(k)}(x) (\sinh \sqrt{x})^{(\ell-k)} = (\sqrt{x})^{(\ell)}.$$ 

Hence we find that

$$H^{(\ell)}(x) = -\sum_{k=0}^{\ell-1} \binom{\ell}{k} H^{(k)}(x) \frac{(\sinh \sqrt{x})^{(\ell-k)}}{\sinh \sqrt{x}} + \frac{\sqrt{x}^{(\ell)}}{\sinh \sqrt{x}}.$$ 

By induction and using the boundedness of the functions $H^{(k)}(x)$, $(\sinh \sqrt{x})^{(\ell-k)}$ and $(\sqrt{x})^{(n)}$ for $|x| \geq 1$ we have the conclusion. \( \square \)

References


