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On Some Fundamental Partial Integral Inequalities

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This paper presents several new partial integral inequalities in two independent variables which can be used in the analysis of various problems in the theory of partial differential and integral equations as ready and powerful tools. An elementary technique of reducing the integral inequality to second order partial differential inequality and then integrating it by Riemann's method is used to establish our results.

1. INTRODUCTION

Integral inequalities originally due to Peano and Gronwall, and their various generalizations (see, [2, 3, 11]) have been extensively used in obtaining a priori bounds for solutions of differential and integral equations. An interesting and useful but apparently neglected generalization of Gronwall's inequality in two independent variables is due to Wendroff given in [1, p. 154]. Wendroff's inequality which has its origin in the field of partial differential equations has recently evoked lively interest as may be seen from the recent papers of Snow [22, 23], Young [25], Ghoshal and Masood [7], Headley [9], Chandra and Davis [5], Bondge and Pachpatte [4] and Pachpatte [19, 20], see also the monograph [24, pp. 130–147] of W. Walter, which are motivated by certain applications in the theory of hyperbolic partial differential and integrodifferential equations. Our objective here is to present a number of partial integral inequalities involving two independent variables which claim the following integral inequality as their origin.

LEMMA 1 (Pachpatte [12]). *Let $u(t)$, $f(t)$ and $g(t)$ be real-valued nonnegative continuous functions defined on $I = [0, \infty)$ for which the inequality*

$$u(t) \leqslant u_0 + \int_0^t f(s)\, u(s)\, ds + \int_0^t f(s) \left(\int_0^s g(\tau)\, u(\tau)\, d\tau\right) ds, \qquad t \in I,$$

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holds, where u_0 is a nonnegative constant. Then

$$u(t) \leqslant u_0 \left[1 + \int_0^t f(s) \exp\left(\int_0^s [f(\tau) + g(\tau)]\, d\tau\right) ds\right], \qquad t \in I.$$

Various applications of Lemma 1 and its variants may be found in [16—18] and in many other recent papers of the present author. Recently, J. C. Helton [10] has obtained some useful generalizations of Lemma 1 by using product integration. In this paper an elementary method used by Snow [22] will be used to establish several fundamental integral inequalities in two independent variables. The resultant new class of inequalities will bring a great number of inequalities under one proof, so to speak, and may in general, be applied to study the boundedness, uniqueness, continuous dependence and other problems in the theory of partial integral and integrodifferential equations of the more general type.

2. Main Results

In this section we state and prove our main results on partial integral inequalities related to the integral inequality established in Lemma 1. The proofs of the main results are along the lines for the one-variable case and involves second order partial differential inequalities which are integrated by using Riemann's method [21, p. 120].

In our subsequent discussion we assume the following:

(H_1) $u(x, y)$, $a(x, y)$, $b(x, y)$, $c(x, y)$, $p(x, y)$ and $q(x, y)$ are real-valued nonnegative continuous functions defined on a domain D.

(H_2) $P_0(x_0, y_0)$ and $P(x, y)$ be two points in D such that $(x - x_0)(y - y_0) > 0$ and R the rectangular region whose opposite corners are the points P_0 and P.

A useful two independent variable generalization of Lemma 1 is embodied in the following theorem.

THEOREM 1. *Suppose* (H_1) *and* (H_2) *are true. Let $v(s, t; x, y)$ and $w(s, t: x, y)$ be the solutions of the characteristic initial value problem*

$$\begin{aligned} L[v] = v_{st} - [p(s, t) + b(s, t)(c(s, t) + q(s, t))]\, v = 0,& \\ v(s, y) = v(x, t) = 1,& \end{aligned} \tag{1}$$

and

$$\begin{aligned} M[w] = w_{st} - [b(s, t)\, c(s, t) - p(s, t)]\, w = 0,& \\ w(s, y) = w(x, t) = 1,& \end{aligned} \tag{2}$$

respectively and let D^+ be a connected subdomain of D which contains P and on which $v > 0$ and $w > 0$. Then, if $R \subset D^+$ and $u(x, y)$ satisfies

$$u(x,y) \leqslant a(x,y) + b(x,y)\left[\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\,u(s,t)\,ds\,dt \right.$$
$$\left. + \int_{x_0}^{x}\int_{y_0}^{y} p(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} q(\xi,\eta)\,u(\xi,\eta)\,d\xi\,d\eta\right) ds\,dt\right], \tag{3}$$

then $u(x, y)$ also satisfies

$$u(x,y) \leqslant a(x,y) + b(x,y)\left[\int_{x_0}^{x}\int_{y_0}^{y} w(s,t;x,y)\left\{a(s,t)\,c(s,t) + p(s,t)\int_{x_0}^{s}\int_{y_0}^{t} a(\xi,\eta)\right.\right.$$
$$\left.\left. \times [c(\xi,\eta) + q(\xi,\eta)]\,v(\xi,\eta;s,t)\,d\xi\,d\eta\right\} ds\,dt\right]. \tag{4}$$

The proof of this theorem is obtained by reducing the integral inequality (3) to a partial differential inequality and then integrating it by Riemann's method for hyperbolic partial differential equations [21, p. 120]. The functions $v(s, t; x, y)$ and $w(s, t; x, y)$ involved in theorem are Riemann functions relative to the point $P(x, y)$ for the self adjoint operators L and M respectively. There are

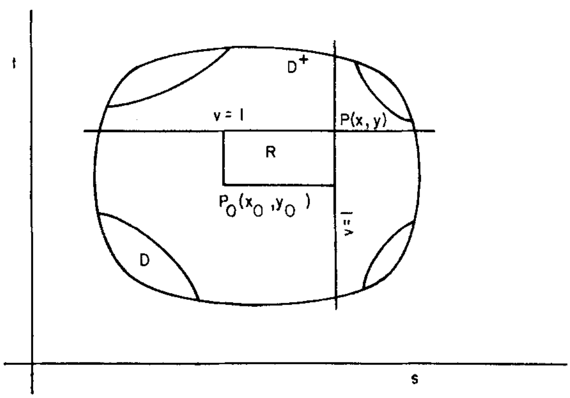

FIGURE 1

such functions and a domain D^+ on which $v > 0$ since $v = 1$ and $w > 0$ since $w = 1$ on the vertical and horizontal lines through P and since v and w are continuous. The existence and continuity of the Riemann function is well known and may be demonstrated by the method of successive approximation (See [6]).

Proof. Define a function $\phi(x, y)$ such that

$$\phi(x, y) = \int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\, u(s, t)\, ds\, dt + \int_{x_0}^{x}\int_{y_0}^{y} p(s, t) \left(\int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta)\, u(\xi, \eta)\, d\xi\, d\eta\right) ds\, dt,$$

$$\phi(x_0, y) = \phi(x, y_0) = 0,$$

then we have

$$\phi_{xy}(x, y) = c(x, y)\, u(x, y) + p(x, y) \left(\int_{x_0}^{x}\int_{y_0}^{y} q(\xi, \eta)\, u(\xi, \eta)\, d\xi\, d\eta\right),$$

which in view of (3) implies

$$\begin{aligned}\phi_{xy}(x, y) \leqslant\ & c(x, y)\, [a(x, y) + b(x, y)\, \phi(x, y)] \\ & + p(x, y) \left(\int_{x_0}^{x}\int_{y_0}^{y} q(\xi, \eta)\, [a(\xi, \eta) + b(\xi, \eta)\, \phi(\xi, \eta)]\, d\xi\, d\eta\right).\end{aligned}$$

Adding $p(x, y)\, \phi(x, y)$ to both sides of the above inequality we have

$$\begin{aligned}&\phi_{xy}(x, y) + p(x, y)\, \phi(x, y) \\ &\quad \leqslant c(x, y)\, [a(x, y) + b(x, y)\, \phi(x, y)] \\ &\qquad + p(x, y) \left[\phi(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} q(\xi, \eta)\, [a(\xi, \eta) + b(\xi, \eta)\, \phi(\xi, \eta)]\, d\xi\, d\eta\right].\end{aligned} \tag{5}$$

If we put

$$\begin{aligned}\psi(x, y) &= \phi(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} q(\xi, \eta)\, [a(\xi, \eta) + b(\xi, \eta)\, \phi(\xi, \eta)]\, d\xi\, d\eta, \\ \psi(x_0, y) &= \psi(x, y_0) = 0,\end{aligned} \tag{6}$$

then we obtain

$$\psi_{xy}(x, y) = \phi_{xy}(x, y) + q(x, y)\, [a(x, y) + b(x, y)\, \phi(x, y)]. \tag{7}$$

Using $\phi_{xy}(x, y) \leqslant c(x, y)\, [a(x, y) + b(x, y)\, \phi(x, y)] + p(x, y)\, \psi(x, y)$ from (5) and $\phi(x, y) \leqslant \psi(x, y)$ from (6) in (7) we have

$$\begin{aligned}\psi_{xy}(x, y) \leqslant\ & a(x, y)\, [c(x, y) + q(x, y)] \\ & + [p(x, y) + b(x, y)\, (c(x, y) + q(x, y))]\, \psi(x, y)\end{aligned}$$

i.e.

$$\begin{aligned}L[\psi] &= \psi_{xy}(x, y) - [p(x, y) + b(x, y)\, (c(x, y) + q(x, y))]\, \psi(x, y) \\ &\leqslant a(x, y)\, [c(x, y) + q(x, y)].\end{aligned} \tag{8}$$

The operator L is self-adjoint and hyperbolic. For any twice continuously differentiable ψ and v the operator L satisfies the identity

$$vL[\psi] - \psi L[v] = -(\psi v_y)_x + (v\psi_x)_y \, . \tag{9}$$

Let P_0 and P be any points as in theorem and label the directed sides and corners of the rectangle R as shown in Fig. 2.

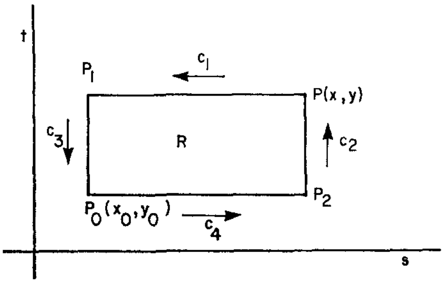

FIGURE 2

Using s and t as the independent variables, we integrate the identity (9) over R and use Green's theorem to obtain

$$\iint_R (vL[\psi] - \psi L[v]) \, ds \, dt = -\int_{c_1+c_2+c_3+c_4} (v\psi_s \, ds + \psi v_t \, dt)$$
$$= -\int_{c_1+c_4} v\psi_s \, ds - \int_{c_2+c_3} \psi v_t \, dt.$$

This holds for any functions in C^2.

For the particular function ψ defined earlier we have $\psi = 0$ on c_3 and $\psi = \psi_s = 0$ on c_4, so the right hand side in the above identity reduces to

$$-\int_{c_1} v\psi_s \, ds - \int_{c_2} \psi v_t \, dt. \tag{10}$$

Now suppose v satisfies

$$L[v] = v_{st} - [p(s, t) + b(s, t)\,(c(s, t) + q(s, t)]\, v = 0, \tag{11}$$

$$v = 1 \qquad \text{on} \qquad c_1 \, , \tag{12}$$

$$v_t = 0 \qquad \text{on} \qquad c_2 \, . \tag{13}$$

Then (12) and (13) imply that

$$v = 1 \qquad \text{on} \qquad c_2 \, . \tag{14}$$

Since $v \geqslant 0$ on R and $\psi(P_1) = 0$, by using (8) identity (10) becomes

$$\psi(P) \leqslant \iint_R v[a(s, t)\,[c(s, t) + q(s, t)]]\, ds\, dt,$$

i.e.,

$$\psi(x, y) \leqslant \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)\,[c(s, t) + q(s, t)]\, v(s, t; x, y)\, ds\, dt.$$

Substituting this bound on $\psi(x, y)$ in (5) we obtain

$$\begin{aligned} M[\phi] &= \phi_{xy}(x, y) - [b(x, y)\, c(x, y) - p(x, y)]\, \phi(x, y) \\ &\leqslant \left[a(x, y)\, c(x, y) + p(x, y) \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)\ [c(s, t) + q(s, t)]\, v(s, t; x, y)\, ds\, dt\right]. \end{aligned}$$

Again by following the same argument as above we obtain the estimate for $\phi(x, y)$ such that

$$\begin{aligned} \phi(x, y) \leqslant \int_{x_0}^{x}\int_{y_0}^{y} w(s, t; x, y)\, \Big[a(s, t)\, c(s, t) \\ + p(s, t) \int_{x_0}^{s}\int_{y_0}^{t} a(\xi, \eta)\, [c(\xi, \eta) + q(\xi, \eta)]\, v(\xi, \eta; s, t)\, d\xi\, d\eta\Big]\, ds\, dt. \end{aligned}$$

Now substituting this bound on $\phi(x, y)$ in (3) we obtain the desired bound in (4).

Another interesting and useful generalization of Lemma 1 is embodied in the following theorem.

THEOREM 2. *Suppose* (H_1) *and* (H_2) *are true. Let* $v(s, t; x, y)$ *and* $w(s, t; x, y)$ *be the solutions of the characteristic initial value problem*

$$\begin{aligned} L[v] = v_{st} - b(s, t)\, [c(s, t) + p(s, t) + q(s, t)]\ v &= 0, \\ v(s, y) = v(x, t) &= 1 \end{aligned} \tag{15}$$

and

$$M[w] = w_{st} - b(s, t)\, c(s, t)\ w = 0, \qquad w(s, y) = w(x, t) = 1, \tag{16}$$

respectively and let D^+ *be a connected subdomain of* D *which contains* P *and on which* $v > 0$ *and* $w > 0$. *Then, if* $R \subset D^+$ *and* $u(x, y)$ *satisfies*

$$\begin{aligned} u(x, y) \leqslant a(x, y) + b(x, y) \Big[\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\, u(s, t)\, ds\, dt \\ + \int_{x_0}^{x}\int_{y_0}^{y} p(s, t) \Big(u(s, t) + b(s, t) \int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta)\, u(\xi, \eta)\, d\xi\, d\eta\Big)\, ds\, dt\Big], \end{aligned} \tag{17}$$

then $u(x, y)$ also satisfies

$$u(x, y) \leqslant a(x, y) + b(x, y) \Big[\int_{x_0}^{x}\int_{y_0}^{y} w(s, t; x, y) \Big\{a(s, t) (c(s, t) + p(s, t))$$

$$+ b(s, t) p(s, t) \int_{x_0}^{s}\int_{y_0}^{t} a(\xi, \eta) [c(\xi, \eta) + p(\xi, \eta) + q(\xi, \eta)] v(\xi, \eta; s, t)$$

$$\times d\xi \, d\eta\Big\} \, ds \, dt\Big] . \tag{18}$$

The proof of this theorem follows by an argument similar to that in the proof of Theorem 1 with suitable modifications. We omit the details.

We now apply Theorem 1 to establish the following interesting and useful integral inequalities which in turn are the further generalizations of the integral inequalities which in turn are the further generalizations of the integral inequalities recently established by Gollwitzer [8, Theorem 1] and Pachpatte [13, Theorem 2].

THEOREM 3. *Suppose* (H_1) *and* (H_2) *are true. Let $G(r)$ be continuous, strictly increasing, convex and submultiplicative function for $r \geqslant 0$, $G(0) = 0$, $\lim_{r\to\infty} G(r) = \infty$ for all (x, y) in D, $\alpha(x, y)$, $\beta(x, y)$ be positive continuous functions defined on a domain D, and $\alpha(x, y) + \beta(x, y) = 1$. Let $v(s, t; x, y)$ and $w(s, t; x, y)$ be the solutions of the characteristic initial value problem*

$$\begin{gathered} L[v] = v_{st} - [p(s, t) + \beta(s, t) \, G(b(s, t) \, \beta^{-1}(s, t)) \, (c(s, t) + q(s, t))] \, v = 0, \\ v(s, y) = v(x, t) = 1, \end{gathered} \tag{19}$$

and

$$\begin{gathered} M[w] = m_{st} - [\beta(s, t) \, G(b(s, t) \, \beta^{-1}(s, t)) \, c(s, t) - p(s, t)] \, w = 0, \\ w(s, y) = w(x, t) = 1, \end{gathered} \tag{20}$$

respectively and let D^+ be a connected subdomain of D which contains P and on which $v > 0$ and $w > 0$. Then, if $R \subset D^+$ and $u(x, y)$ satisfies

$$\begin{aligned} u(x, y) \leqslant a(x, y) + b(x, y) \, G^{-1} \Big[\int_{x_0}^{x}\int_{y_0}^{y} c(s, t) \, G(u(s, t)) \, ds \, dt \\ + \int_{x_0}^{x}\int_{y_0}^{y} p(s, t) \Big(\int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta) \, G(u(\xi, \eta)) \, d\xi \, d\eta\Big) \, ds \, dt\Big] , \end{aligned} \tag{21}$$

then $u(x, y)$ also satisfies

$$u(x, y) \leqslant a(x, y) + b(x, y)\, G^{-1}\Big[\int_{x_0}^{x}\int_{y_0}^{y} w(s, t; x, y) \Big\{\alpha(s, t)\, G(a(s, t)\, \alpha^{-1}(s, t))\, c(s, t)$$

$$+ p(s, t) \int_{x_0}^{s}\int_{y_0}^{t} \alpha(\xi, \eta)\, G(a(\xi, \eta)\, \alpha^{-1}(\xi, \eta))\, [c(\xi, \eta) + q(\xi, \eta)]$$

$$\times v(\xi, \eta; s, t)\, d\xi\, d\eta\Big\}\, ds\, dt\Big]. \tag{22}$$

Proof. Rewrite (21) as

$$u(x, y) \leqslant \alpha(x, y)\, a(x, y)\, \alpha^{-1}(x, y) + \beta(x, y)\, b(x, y)\, \beta^{-1}(x, y)$$

$$\times G^{-1}\Big[\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\, G(u(s, t))\, ds\, dt$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y} p(s, t) \Big(\int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta)\, G(u(\xi, \eta))\, d\xi\, d\eta\Big)\, ds\, dt\Big].$$

Since G is convex, submultiplicative and monotonic we have

$$G(u(x, y)) \leqslant \alpha(x, y)\, G(a(x, y)\, \alpha^{-1}(x, y)) + \beta(x, y)\, G(b(x, y)\, \beta^{-1}(x, y))$$

$$\times \Big[\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\, G(u(s, t))\, ds\, dt \tag{23}$$

$$+ \int_{x_0}^{x}\int_{y_0}^{y} p(s, t) \Big(\int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta)\, G(u(\xi, \eta))\, d\xi\, d\eta\Big)\, ds\, dt\Big].$$

The estimate given in (22) follows by first applying Theorem 1 with $a(x, y) = \alpha(x, y)\, G(a(x, y)\, \alpha^{-1}(x, y))$, $b(x, y) = \beta(x, y)\, G(b(x, y)\, \beta^{-1}(x, y))$ and $u(x, y) = G(u(x, y))$ and then applying G^{-1} to both sides of the resulting inequality.

THEOREM 4. *Suppose* (H_1) *and* (H_2) *are true. Let $G(r)$ be a positive, continuous, strictly increasing, subadditive and submultiplicative function for $r > 0$, $G(0) = 0$ for all $(x, y) \in D$, and G^{-1} is the inverse function of G. Let $v(s, t; x, y)$ and $w(s, t; x, y)$ be the solutions of the characteristic initial value problem*

$$\begin{gathered} L[v] = v_{st} - [p(s, t) + G(b(s, t))\, (c(s, t) + q(s, t))]\, v = 0, \\ v(s, y) = v(x, t) = 1. \end{gathered} \tag{24}$$

and

$$\begin{gathered} M[w] = w_{st} - [G(b(s, t))\, c(s, t) - p(s, t)]\, w = 0, \\ w(s, y) = w(x, t) = 1, \end{gathered} \tag{25}$$

respectively and let D^+ be a connected subdomain of D which contains P and on which $v > 0$ and $w > o$. Then, if $R \subset D^+$ and $u(x, y)$ satisfies

$$u(x, y) \leqslant a(x, y) + b(x, y)\, G^{-1} \left[\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\, G(u(s, t))\, ds\, dt \right.$$
$$\left. + \int_{x_0}^{x}\int_{y_0}^{y} p(s, t) \left(\int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta)\, G(u(\xi, \eta))\, d\xi\, d\eta\right) ds\, dt\right], \tag{26}$$

then $u(x, y)$ also satisfies

$$u(x, y) \leqslant G^{-1} \left[G(a(x, y)) + G(b(x, y)) \left[\int_{x_0}^{x}\int_{y_0}^{y} w(s, t; x, y) \left\{G(a(s, t))\, c(s, t) \right.\right.\right.$$
$$\left.\left.\left. + p(s, t) \int_{x_0}^{s}\int_{y_0}^{t} G(a(\xi, \eta))\, [c(\xi, \eta) + q(\xi, \eta)]\, v(\xi, \eta; s, t)\, d\xi\, d\eta\right\} ds\, dt\right]\right]. \tag{27}$$

Proof. Since G is subadditive, submultiplicative and monotonic, we have from (26)

$$G(u(x, y)) \leqslant G(a(x, y)) + G(b(x, y)) \left[\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\, G(u(s, t))\, ds\, dt \right.$$
$$\left. + \int_{x_0}^{x}\int_{y_0}^{y} p(s, t) \left(\int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta)\, G(u(\xi, \eta))\, d\xi\, d\eta\right) ds\, dt\right]. \tag{28}$$

The desired bound in (27) follows by first applying Theorem 1 to (28) with $a(x, y) = G(a(x, y))$, $b(x, y) = G(b(x, y))$ and $u(x, y) = G(u(x, y))$ and then applying G^{-1} to both sides of the resulting inequality.

Before closing this section, we apply Theorem 2 to establish the following integral inequalities similar to that proved in Theorems 3 and 4 which can be used in some applications. The proofs of the Theorems 3 and 4 can be adapted readily into this context.

THEOREM 5. *Suppose* (H_1) *and* (H_2) *are true. Let $G(r)$, $\alpha(x, y)$, $\beta(x, y)$ be the same functions as defined in Theorem 3. Let $v(s, t; x, y)$ and $w(s, t; x, y)$ be the solutions of the characteristic initial value problem*

$$L[v] = v_{st} - \beta(s, t)\, G(b(s, t)\, \beta^{-1}(s, t))\, [c(s, t) + p(s, t) + q(s, t)]\, v = 0, \quad v(s, y) = v(x, t) = 1, \tag{29}$$

and

$$M[w] = w_{st} - \beta(s, t)\, G(b(s, t)\, \beta^{-1}(s, t))\, c(s, t)\, w = 0, \quad w(s, y) = w(x, t) = 1, \tag{30}$$

respectively and let D^+ be a connected subdomain of D which contains P and on which $v > 0$ and $w > 0$. Then, if $R \subset D^+$ and $u(x, y)$ satisfies

$$\begin{aligned} u(x, y) \leqslant a(x, y) + b(x, y)\, G^{-1} \Big[&\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\, G(u(s, t))\, ds\, dt \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} p(s, t)\Big(G(u(s, t)) + \beta(s, t)\, G(b(s, t)\, \beta^{-1}(s, t)) \\ &\times \int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta)\, G(u(\xi, \eta))\, d\xi\, d\eta\Big)\, ds\, dt\Big], \end{aligned} \tag{31}$$

then $u(x, y)$ also satisfies

$$\begin{aligned} u(x, y) \leqslant a(x, y) + b(x, y)\, G^{-1} \Big[&\int_{x_0}^{x}\int_{y_0}^{y} w(s, t; x, y) \Big\{\alpha(s, t)\, G(a(s, t)\, \alpha^{-1}s, t)) \\ &\times [c(s, t) + p(s, t)] + \beta(s, t)\, G(b(s, t)\, \beta^{-1}(s, t))\, p(s, t) \\ &\times \int_{x_0}^{x}\int_{y_0}^{y} \alpha(\xi, \eta)\, G(a(\xi, \eta)\, \alpha^{-1}(\xi, \eta)) \\ &\times [c(\xi, \eta) + p(\xi, \eta) + q(\xi, \eta)]\, v(\xi, \eta; s, t)\, d\xi\, d\eta\Big\}\, ds\, dt\Big]. \end{aligned} \tag{32}$$

THEOREM 6. *Suppose* (H_1) *and* (H_2) *are true. Let G and G^{-1} be the same functions as defined in Theorem 4. Let $v(s, t; x, y)$ and $w(s, t; x, y)$ be the solutions of the characteristic initial value problem*

$$\begin{aligned} L[v] = v_{st} - G(b(s, t))\, [c(s, t) + p(s, t) + q(s, t)]\, v = 0, \\ v(s, y) = v(x, t) = 1, \end{aligned} \tag{33}$$

and

$$\begin{aligned} M[w] = w_{st} - G(b(s, t))\, c(s, t)\, w = 0, \\ w(s, y) = w(x, t) = 1, \end{aligned} \tag{34}$$

respectively and let D^+ be a connected subdomain of D which contains P and on which $v > 0$ and $w > 0$. Then, if $R \subset D^+$ and $u(x, y)$ satisfies

$$\begin{aligned} u(x, y) \leqslant a(x, y) + b(x, y)\, G^{-1} \Big[&\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\, G(u(s, t))\, ds\, dt \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} p(s, t)\, \Big(G(u(s, t)) + G(b(s, t)) \\ &\times \int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta)\, G(u(\xi, \eta))\, d\xi\, d\eta\Big)\, ds\, dt\Big], \end{aligned} \tag{35}$$

then $u(x, y)$ *also satisfies*

$$
\begin{aligned}
&u(x, y)\\
&\quad \leqslant G^{-1}\Big[G(a(x, y)) + G(b(x, y))\Big[\int_{x_0}^{x}\int_{y_0}^{y} w(s, t; x, y)\Big\{G(a(s, t))\,[c(s, t) + p(s, t)]\\
&\qquad + G(b(s, t))\,p(s, t)\int_{x_0}^{s}\int_{y_0}^{t} G(a(\xi, \eta))\,[c(\xi, \eta) + p(\xi, \eta) + q(\xi, \eta)]\\
&\qquad \times v(\xi, \eta; s, t)\,d\xi\,d\eta\Big\}\,ds\,dt\Big]\Big]. \qquad (36)
\end{aligned}
$$

We note that in the special case when $p(x, y) = q(x, y) = 0$, Theorems 1–6 reduces to the further generalizations of the integral inequality recently established by Snow [22]. In the special case when $c(x, y) = 0$, our results in Theorems 1–6 are new to the literature.

3. Some Applications

In this section we present some applications of our results to study the boundedness and uniqueness of the solutions of some nonlinear hyperbolic partial integrodifferential equations. These applications are not stated as theorems so as to obscure the main ideas with technical details. It appears that these inequalities will have as many applications for partial integral and integrodifferential equations as the classical integral inequality given in Lemma 1 and its various generalizations have had for ordinary integrodifferential and integral equations.

Example 1. As a first application, we obtain the bound on the solution of a nonlinear hyperbolic partial integrodifferential equation

$$
u_{xy}(x, y) = f(x, y, u(x, y)) + h\left[x, y, u(x, y), \int_{x_0}^{x}\int_{y_0}^{y} k(x, y, s, t, u(s, t))\,ds\,dt\right], \qquad (37)
$$

with the given boundary conditions

$$
u(x, y_0) = a_1(x), \qquad u(x_0, y) = a_2(y), \qquad a_1(x_0) = a_2(y_0) = 0,
$$

where all the functions are real-valued, continuous and defined on a domain D and such that

$$
|f(x, y, u)| \leqslant c(x, y)\,|u|, \qquad (38)
$$

$$
|k(x, y, s, t, u)| \leqslant q(s, t)\,|u|, \qquad (39)
$$

$$
|h[x, y, u, v]| \leqslant p(x, y)\,[|u| + |v|], \qquad (40)
$$

where $c(x, y)$, $p(x, y)$ and $q(x, y)$ are as in (H_1). The equation (37) is equivalent to the Volterra integral equation

$$u(x, y) = a_1(x) + a_2(y) + \int_{x_0}^{x}\int_{y_0}^{y} f(s, t, u(s, t))\, ds\, dt$$
$$+ \int_{x_0}^{x}\int_{y_0}^{y} h\left[s, t, u(s, t), \int_{x_0}^{s}\int_{y_0}^{t} k(s, t, \xi, \eta, u(\xi, \eta))\, d\xi\, d\eta\right] ds\, dt, \tag{41}$$

where $u(x, y)$ is any solution of (37). Using (38)–(40) in (41) and assuming that $|a_1(x)| + |a_2(y)| \leqslant a(x, y)$, where $a(x, y)$ is as defined in (H_1), we have

$$|u(x, y)| \leqslant a(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\, |u(s, t)|\, ds\, dt + \int_{x_0}^{x}\int_{y_0}^{y} p(s, t) \Big(|u(s, t)|$$
$$+ \int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta)\, |u(\xi, \eta)|\, d\xi\, d\eta\Big)\, ds\, dt.$$

Now an application of Theorem 2 with $b(x, y) = 1$ yields

$$|u(x, y)| \leqslant a(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} w(s, t; x, y) \Big\{a(s, t)\,[c(s, t) + p(s, t)]$$
$$+ p(s, t) \int_{x_0}^{s}\int_{y_0}^{t} a(\xi, \eta)\,[c(\xi, \eta) + p(\xi, n) + q(\xi, \eta)] \tag{42}$$
$$\times v(\xi, \eta; s, t)\, d\xi\, d\eta\Big\}\, ds\, dt,$$

where $v(s, t; x, y)$ and $w(s, t; x, y)$ are the solutions of the characteristic initial value problems (15) and (16) respectively with $b(x, y) = 1$. Thus the right hand side in (42) gives us the bound on the solution $u(x, y)$ of (37) in terms of the known functions.

If $|a_1(x)| + |a_2(y)| \leqslant \epsilon$, where $\epsilon > 0$ is arbitrary, then the bound obtained in (42) reduces to

$$|u(x, y)| \leqslant \epsilon \Big\{1 + \int_{x_0}^{x}\int_{y_0}^{y} w(s, t; x, y) \Big[[c(s, t) + p(s, t)]$$
$$+ p(s, t) \int_{x_0}^{s}\int_{y_0}^{t} [c(\xi, \eta) + p(\xi, \eta) + q(\xi, \eta)]\, v(\xi, \eta; s, t)\, d\xi\, d\eta\Big]\, ds\, dt. \tag{43}$$

In this case we note that, Example 1 implies not only the boundedness but the stability of the solution $u(x, y)$ of (37), if the bound obtained on the right side in (43) is small enough.

EXAMPLE 2. As a second application, we discuss the uniqueness of the solution of the nonlinear hyperbolic partial integrodifferential equation (37). We assume that the functions f, k and h in (37) satisfy

$$|f(x, y, u) - f(x, y, \bar{u})| \leqslant c(x, y) \, | u - \bar{u} | , \tag{44}$$

$$| k(x, y, s, t, u) - k(x, y, s, t, \bar{u})| \leqslant q(s, t) \, | u - \bar{u} | , \tag{45}$$

$$| h[x, y, u, r] - h[x, y, \bar{u}, \bar{r}]| \leqslant p(x, y) \, [| u - \bar{u} | + | r - \bar{r} |] \tag{46}$$

where $c(x, y)$, $p(x, y)$ and $q(x, y)$ are as in (H_1). The problem (37) is equivalent to the Volterra integral equation (41). Now if $u(x, y)$ and $\bar{u}(x, y)$ be two solutions of the given boundary value problem (37) with the same boundary conditions then we have

$$\begin{aligned} u - \bar{u} = {} & \int_{x_0}^{x}\int_{y_0}^{y} \{f(s, t, u) - f(s, t, \bar{u})\} \, ds \, dt \\ & + \int_{x_0}^{x}\int_{y_0}^{y} \Big\{ h\left[s, t, u, \int_{x_0}^{s}\int_{y_0}^{t} k(s, t, \xi, \eta, u) \, d\xi \, d\eta\right] \\ & - h\left[s, t, \bar{u}, \int_{x_0}^{s}\int_{y_0}^{t} k(s, t, \xi, \eta, \bar{u}) \, d\xi \, d\eta\right]\Big\} \, ds \, dt. \end{aligned} \tag{47}$$

Using (44)–(46) in (47) we have

$$\begin{aligned} | u - \bar{u} | \leqslant {} & \int_{x_0}^{x}\int_{y_0}^{y} c(s, t) \, | u - \bar{u} | \, ds \, dt \\ & + \int_{x_0}^{x}\int_{y_0}^{y} p(s, t) \left(| u - \bar{u} | + \int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta) \, | u - \bar{u} | \, d\xi \, d\eta\right) ds \, dt. \end{aligned}$$

Now a suitable application of Theorem 2 yields, $| u - \bar{u} | \leqslant 0$. Therefore $u = \bar{u}$; i.e., there is atmost one solution of the problem.

In concluding this paper we note that the inequalities and their applications presented here can be extended very easily to the corresponding vector problems as in [7] and [23]. We also note that there is no essential difficulty in obtaining n independent variable generalizations of the inequalities established in Theorems 1–6 by using the technique used by Young in [25]. Since this translation is quite straight forward in view of the results of this paper and we omit the details.

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