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On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators ☆

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In this paper we are interested in the existence of a principal eigenfunction of a nonlocal operator which appears in the description of various phenomena ranging from population dynamics to micro-magnetism. More precisely, we study the following eigenvalue problem:

$$\int_{\Omega} J\left(\frac{x-y}{g(y)}\right)\frac{\phi(y)}{g^n(y)}\,dy + a(x)\phi = \rho\phi,$$

where $\Omega \subset \mathbb{R}^n$ is an open connected set, J a non-negative kernel and g a positive function. First, we establish a criterion for the existence of a principal eigenpair (λ_p, ϕ_p). We also explore the relation between the sign of the largest element of the spectrum with a strong maximum property satisfied by the operator. As an application of these results we construct and characterise the solutions of some nonlinear nonlocal reaction diffusion equations.

1. Introduction and main results

In the past few years much attention has been drawn to the study of nonlocal reaction diffusion equations, where the usual elliptic diffusion operator is replaced by a nonlocal operator of the form

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$$\mathcal{M}[u] := \int_{\Omega} k(x,y)u(y)\,dy - b(x)u, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $k \geqslant 0$ satisfies $\int_{\mathbb{R}^n} k(y,x)\,dy < \infty$ for all $x \in \mathbb{R}^n$ and $b(x) \in C(\Omega)$; see among other references [1–3,10–12,14–16,19,22–24,33,34,39]. Such type of diffusion process has been widely used to describe the dispersal of a population through its environment in the following sense. As stated in [29,30,32] if $u(y,t)$ is thought of as a density at a location y at a time t and $k(x,y)$ as the probability distribution of jumping from a location y to a location x, then the rate at which the individuals from all other places are arriving to the location x is

$$\int_{\Omega} k(x,y)u(y,t)\,dy.$$

On the other hand, the rate at which the individuals are leaving the location x is $-b(x)u(x,t)$. This formulation of the dispersal of individuals finds its justification in many ecological problems of seed dispersion; see for example [9,13,25,33,34,39].

In this paper, we study the properties of the principal eigenvalue of the operator $\mathcal{M}$, when the kernel $k(x,y)$ takes the form

$$k(x,y) = J\left(\frac{x-y}{g(y)}\right)\frac{1}{g^n(y)}, \tag{1.2}$$

where J is a continuous probability density and the function g is bounded and positive. That is to say, we investigate the following eigenvalue problem:

$$\int_{\Omega} J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g^n(y)}\,dy - b(x)u = -\lambda u \quad \text{in } \Omega. \tag{1.3}$$

Such type of diffusion kernel was recently introduced by Cortázar et al. [14] in order to model a non-homogeneous dispersal process. Along this paper, with no further specifications, we will always make the following assumptions on Ω, J, g and b:

$$\Omega \subset \mathbb{R}^n \text{ is an open connected set}, \tag{$H1$}$$

$$J \in C_c(\mathbb{R}^n),\ J \geqslant 0,\ J(0) > 0, \tag{$H2$}$$

$$g \in L^\infty(\Omega),\ 0 < \alpha \leqslant g \leqslant \beta, \tag{$H3$}$$

$$b \in C(\overline{\Omega}) \cap L^\infty(\Omega), \tag{$H4$}$$

where $C_c(\mathbb{R}^n)$ denotes the set of continuous functions with compact support.

The existence and a variational characterisation of the principal eigenvalue λ_p of $\mathcal{M}$ is known from a long time, see for example Donsker and Varadhan [26]. However, as Donsker and Varadhan [26] have already noticed, λ_p is in general not an eigenvalue, that is to say, there exists no positive function ϕ_p such that (λ_p, ϕ_p) is a solution of (1.3). In this paper, we are interested in finding some conditions on $\mathcal{M}$ ensuring the existence of a principal eigenpair (λ_p, ϕ_p) of (1.3) such that $\phi_p \in C(\Omega)$ and $\phi_p > 0$. Such type of solution is commonly used to analyse the long-time behaviour of some nonlocal evolution problems [10,14] and had proven to be a very efficient tool in the analysis of nonlinear integrodifferential problems; see for example [21,31].

To our knowledge, besides some particular situations the existence of an eigenpair (λ_p, ϕ_p) for Eq. (1.3) is still an open question and many of the known results concern these two cases:

(1) $b(x) \equiv Constant$.
(2) The operator $\mathcal{M}$ satisfies a mass preserving property, i.e. $\forall u \in C(\Omega)$,

$$\int_\Omega \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g^n(y)}\,dy\,dx - \int_\Omega b(x)u(x)\,dx = 0.$$

In both cases, the principal eigenvalue problem (1.3) is either reduced to the analysis of the spectrum of the positive operator $\mathcal{L}_\Omega$ defined below:

$$\mathcal{L}_\Omega[u] := \int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g^n(y)}\,dy$$

or the principal eigenvalue is explicitly known, i.e. $\lambda_p = 0$ and the principal eigenfunction ϕ_p is also the positive solution of the following eigenvalue problem

$$\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{\psi(y)}{g^n(y)}\,dy = \rho b(x)\psi.$$

Note that even in these two simplified cases, showing the existence of an eigenfunction is still a difficult task when the domain Ω is unbounded.

As observed in [19], Eq. (1.1) shares many properties with the usual elliptic operators

$$\mathcal{E} := \sigma_{ij}(x)\partial_{ij} + \beta_i(x)\partial_i + c(x).$$

In particular, acting on smooth functions, we can rewrite $\mathcal{M}$

$$\mathcal{M}[u] = \mathcal{E}[u] + \mathcal{R}[u]$$

with $\mathcal{R}$ an operator involving derivatives of higher order that in $\mathcal{E}$.

Indeed, we have

$$\mathcal{M}[u] = \int_\Omega k(x,y)\big[u(y) - u(x)\big]\,dy - c(x)u,$$

with $c(x) := b(x) - \int_\Omega k(x,y)\,dy$. Using the change of variables $z = x - y$ and performing a formal Taylor expansion of u in the integral, we can rewrite the nonlocal operator as follows

$$\int_{x-\Omega} k(x, x-z)\big[u(x-z) - u(x)\big]\,dy = \sigma_{ij}(x)\partial_{ij}u + \beta_i(x)\partial_i u + \mathcal{R}[u]$$

where we use the Einstein summation convention and $\sigma_{ij}(x)$, $\beta_i(x)$, and $\mathcal{R}$ are defined by the following expressions

$$\sigma_{ij}(x) = \frac{1}{2}\int_{x-\Omega} k(x,x-z)z_i z_j\,dz,$$

$$\beta_i(x) = \int_{x-\Omega} k(x,x-z)z_i\,dz,$$

$$\mathcal{R}[u] := \int_0^1\int_0^1\int_0^1\int_{x-\Omega} k(x,x-z)z_i z_j t^2 s\partial_{ijk}u(x+ts\tau z)\,dt\,ds\,d\tau\,dz.$$

For the second order elliptic operator $\mathcal{E}$, the existence of a principal eigenpair (λ_p,ϕ_p) is well known and various variational formulas characterising the principal eigenvalue exist, see for example [5,26,28,36–38]. In particular, Berestycki, Nirenberg and Varadhan [5] give a very simple and general definition of the principal eigenvalue of $\mathcal{E}$ that we recall below. Namely, they define the principal eigenvalue of the elliptic operator $\mathcal{E}$ by the following quantity:

$$\lambda_1 := \sup\bigl\{\lambda\in\mathbb{R}\bigm|\exists\phi\in C(\Omega),\ \phi>0,\ \text{such that}\ \mathcal{E}[\phi]+\lambda\phi\leqslant 0\bigr\}. \tag{1.4}$$

In this paper, we adopt the definition of Berestycki, Nirenberg and Varadhan for the definition of the principal eigenvalue of the operator $\mathcal{M}$. The principal eigenvalue of the operator $\mathcal{M}$ is then given by the following quantity:

$$\lambda_p(\mathcal{M}) := \sup\bigl\{\lambda\in\mathbb{R}\bigm|\exists\phi\in C(\Omega),\ \phi>0,\ \text{such that}\ \mathcal{M}[\phi]+\lambda\phi\leqslant 0\bigr\}.$$

To make more explicit the dependence of the different parameters and to simplify the presentation of the results, we shall adopt the following notations:

- Let A and B be two sets, we denote $A\Subset B$ the compact inclusion $A\subset\subset B$.
- $a(x) := -b(x)$.
- $\sigma := \sup_\Omega a(x)$.
- $d\mu$ is the measure defined by $d\mu := \frac{dx}{g^n(x)}$.
- $\mathcal{L}_\Omega[u] := \int_\Omega J(\frac{x-y}{g(y)})\frac{u(y)}{g^n(y)}\,dy = \int_\Omega J(\frac{x-y}{g(y)})u(y)\,d\mu$.
- $\mathcal{M} := \mathcal{M}_\Omega := \mathcal{L}_\Omega + a(x)Id$.

With this new notation the principal eigenvalue of $\mathcal{M}_\Omega$ can be rewritten as follows

$$\lambda_p(\mathcal{M}_\Omega) := \sup\bigl\{\lambda\in\mathbb{R}\bigm|\exists\phi\in C(\Omega),\ \phi>0,\ \text{such that}\ \mathcal{L}_\Omega[\phi]+\bigl(a(x)+\lambda\bigr)\phi\leqslant 0\bigr\}. \tag{1.5}$$

Under the assumptions $(H1)$–$(H4)$, the principal eigenvalue $\lambda_p(\mathcal{M}_\Omega)$ is well defined, see Appendix A for the details.

Obviously, λ_p is monotone with respect to the domain, the zero order term $a(x)$ and J. Moreover, λ_p is a concave function of its argument and is Lipschitz continuous with respect to $a(x)$. More precisely, we have

Proposition 1.1.

(i) *Assume $\Omega_1\subset\Omega_2$, then*

$$\lambda_p\bigl(\mathcal{L}_{\Omega_1}+a(x)\bigr)\geqslant\lambda_p\bigl(\mathcal{L}_{\Omega_2}+a(x)\bigr).$$

(ii) *Fix Ω and assume that $a_1(x) \geqslant a_2(x)$, then*

$$\lambda_p\big(\mathcal{L}_\Omega + a_2(x)\big) \geqslant \lambda_p\big(\mathcal{L}_\Omega + a_1(x)\big).$$

Moreover, if $a_1(x) \geqslant a_2(x) + \delta$ for some $\delta > 0$ then

$$\lambda_p\big(\mathcal{L}_\Omega + a_2(x)\big) > \lambda_p\big(\mathcal{L}_\Omega + a_1(x)\big).$$

(iii) *$\lambda_p(\mathcal{L}_\Omega + a(x))$ is Lipschitz continuous in $a(x)$. More precisely,*

$$\big|\lambda_p\big(\mathcal{L}_\Omega + a(x)\big) - \lambda_p\big(\mathcal{L}_\Omega + b(x)\big)\big| \leqslant \big\|a(x) - b(x)\big\|_\infty.$$

(iv) *Let $J_1 \leqslant J_2$ be two positive continuous integrable functions and let us denote respectively by $\mathcal{L}_{1,\Omega}$ and $\mathcal{L}_{2,\Omega}$ the corresponding operators. Then we have*

$$\lambda_p\big(\mathcal{L}_{1,\Omega} + a(x)\big) > \lambda_p\big(\mathcal{L}_{2,\Omega} + a(x)\big).$$

Let us state our first result concerning a sufficient condition for the existence of a principal eigenpair (λ_p, ϕ_p) for the operator $\mathcal{M}$.

Theorem 1.1 *(Sufficient condition). Assume that Ω, J, g and a satisfy $(H1)$–$(H4)$. Let us denote $\sigma := \sup_{\overline{\Omega}} a(x)$ and assume further that the function $a(x)$ satisfies $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\Omega_0)$ for some bounded domain $\Omega_0 \subset \overline{\Omega}$. Then there exists a principal eigenpair (λ_p, ϕ_p) solution of (1.3). Moreover, $\phi_p \in C(\Omega)$, $\phi_p > 0$ and we have the following estimate*

$$-\sigma' < \lambda_p < -\sigma,$$

where $\sigma' := \sup_{x\in\Omega}[a(x) + \int_\Omega J(\frac{y-x}{g(x)})\frac{dy}{g^n(x)}]$.

Note that the theorem holds true whenever Ω is bounded or not.

The condition $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\Omega_0)$ is sharp in the sense that if $\frac{1}{\sigma - a(x)} \in L^1_{d\mu,loc}(\Omega)$ then we can construct an operator $\mathcal{M}_\Omega$ such that Eq. (1.3) does not have a principal eigenpair. This is discussed in Section 5, where such an operator is constructed. We want also to stress that the boundedness of the open set Ω does not ensure the existence of an eigenfunction, see the counterexample in Section 5.

In contrast with the elliptic case, the sufficient condition has nothing to do with the regularity of the functions $a(x)$, J or g. This means that in general improving the regularity of the coefficients does not ensure at all the existence of an eigenpair. However, in low dimension of space $n = 1, 2$ the condition $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\Omega_0)$ can be related to a regularity condition on the coefficient $a(x)$. Indeed, in one dimension if a is Lipschitz continuous and achieves a maximum in Ω then the condition $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\Omega_0)$ is automatically satisfied. Similarly, when $n = 2$ the non-integrability condition is always satisfied when $a(x) \in C^{1,1}(\Omega)$ and achieves a maximum in Ω. More precisely, we have the following:

Theorem 1.2. *Assume that Ω, J, g and a satisfy $(H1)$–$(H4)$, that a achieves a global maximum at some point $x_0 \in \Omega$. Then there exists a principal eigenpair (λ_p, ϕ_p) solution of (1.3) in the following situations*

- $n = 1$, $a(x) \in C^{0,1}(\Omega)$,
- $n = 2$, $a(x) \in C^{1,1}(\Omega)$,
- $n \geqslant 3$, $a(x) \in C^{n-1,1}(\Omega)$, $\forall k < n$, $\partial^k a(x_0) = 0$.

One of the most interesting properties of the principal eigenvalue for an elliptic operator $\mathcal{E}$ is its relation with the existence of a maximum principle for $\mathcal{E}$. Indeed, Berestycki et al. [5] have shown that there exists a strong relation between the sign of this principal eigenvalue and the existence of a maximum principle for the elliptic operator $\mathcal{E}$. Namely, they have proved

Theorem 1.3 *(BNV). Let Ω be a bounded open set, then $\mathcal{E}$ satisfies a refined maximum principle if and only if $\lambda_1 > 0$.*

It turns out that when the principal eigenpair exists for $\mathcal{M}$, we can also obtain a similar relation between the sign of the principal eigenvalue of $\mathcal{M}$ and some maximum principle property. More precisely, let us first define the maximum principle property satisfied by $\mathcal{M}$:

Definition 1.4 *(Maximum principle).* When Ω is bounded, we say that the maximum principle is satisfied by an operator $\mathcal{M}_\Omega$ if for all function $u \in C(\overline{\Omega})$ satisfying

$$\mathcal{M}_\Omega[u] \leqslant 0 \quad \text{in } \Omega,$$
$$u \geqslant 0 \quad \text{in } \partial\Omega,$$

then $u \geqslant 0$ in Ω.

With this definition of maximum principle, we show

Theorem 1.5. *Assume that Ω is a bounded set and let J, g and a be as in Theorem* 1.1. *Then the maximum principle is satisfied by $\mathcal{M}_\Omega$ if and only if $\lambda_p(\mathcal{M}_\Omega) \geqslant 0$.*

Note that there is a slight difference between the criteria for elliptic operators and for nonlocal ones. To have a maximum principle for nonlocal operator it is sufficient to have a non-negative principal eigenvalue, which is untrue for an elliptic operator where a strict sign of λ_p is required.

Our last result is an application of the sufficient condition for the existence of a principal eigenpair to obtain a simple criterion for the existence/non-existence of a positive solution of the following semilinear problem:

$$\mathcal{M}_\Omega[u] + f(x,u) = 0 \quad \text{in } \Omega, \tag{1.6}$$

where f is a KPP type nonlinearity. Such type of equation naturally appears in some ecological problems when in addition to the dispersion of the individuals in the environment, the birth and death of these individuals are also modelled, see [31–34].

On f we assume that:

$$\begin{cases} f \in C\big(\mathbb{R} \times [0,\infty)\big) \text{ and is differentiable with respect to } u, \\ f_u(\cdot,0) \text{ is Lipschitz}, \\ f(\cdot,0) \equiv 0 \text{ and } f(x,u)/u \text{ is decreasing with respect to } u, \\ \text{there exists } M > 0 \text{ such that } f(x,u) \leqslant 0 \text{ for all } u \geqslant M \text{ and all } x. \end{cases} \tag{1.7}$$

The simplest example of such a nonlinearity is

$$f(x,u) = u\big(\mu(x) - u\big),$$

where $\mu(x)$ is a Lipschitz function.

Such type of problem has received recently a lot of attention, see for example [4,32–34] and reference therein. In particular, for Ω bounded and for a symmetric kernel J Hutson et al. [32] have shown that there exists a unique non-trivial stationary solution (1.6) provided that some principal eigenvalue of the linearised operator around the solution 0 is positive. This result can be extended to more general kernel J using the definition of principal eigenvalue (1.5). More precisely, we show that:

Theorem 1.6. *Assume Ω, J, g and a satisfy (H1)–(H4), Ω is bounded, $a(x) \leqslant 0$ and f satisfies (1.7). Then there exists a unique non-trivial solution of (1.6) when*

$$\lambda_p\big(\mathcal{M}_\Omega + f_u(x,0)\big) < 0,$$

where λ_p is the principal eigenvalue of the linear operator $\mathcal{M}_\Omega + f_u(x,0)$. Moreover, if $\lambda_p \geqslant 0$ then any non-negative uniformly bounded solution of (1.6) is identically zero.

As a consequence, we can derive the asymptotic behaviour of the solution of the evolution problem associated to (1.6):

$$\frac{\partial u}{\partial t} = \mathcal{M}_\Omega[u] + f(x,u) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.8}$$

$$u(0,x) = u_0(x) \quad \text{in } \Omega. \tag{1.9}$$

Namely, the asymptotic behaviour of $u(t,x)$ as $t \to +\infty$ is described in the following theorem:

Theorem 1.7. *Let Ω, J, g, b and f be as in Theorem 1.6. Let u_0 be an arbitrary bounded and continuous function in Ω such that $u_0 \geqslant 0$, $u_0 \not\equiv 0$. Let $u(t,x)$ be the solution of (1.8) with initial datum $u(0,x) = u_0(x)$. Then, we have:*

(1) *If 0 is an unstable solution of (1.6) (that is $\lambda_p < 0$), then $u(t,x) \to p(x)$ pointwise as $t \to \infty$, where p is the unique positive solution of (1.6) given by Theorem 1.6.*
(2) *If 0 is a stable solution of (1.6) (that is $\lambda_p \geqslant 0$), then $u(t,x) \to 0$ pointwise in Ω as $t \to +\infty$.*

Note that this criterion involves only the sign of λ_p and does not require any conditions on the function $f_u(x,0)$ ensuring the existence of a principal eigenfunction. Therefore, even in a situation where no principal eigenfunction exists for the operator $\mathcal{M}_\Omega + f_u(x,0)$ we still have information on the survival or the extinction of the considered species. Observe also that the condition obtained on the principal eigenvalue of the linearised operator is sufficient and necessary for the existence of a non-trivial solution.

Before going into the proofs of these results, let us make some comments. We first point out that the proofs we have given apply to a more general situation. More precisely, the above results can be easily extended to the case of a dispersal kernel $k(x,y)$ which satisfies the following conditions:

$$k(x,y) \in C_c(\Omega \times \Omega),\ k \geqslant 0,\ \int_\Omega k(x,y)\,dy < +\infty\ \forall x \in \Omega, \tag{$\tilde{H}1$}$$

$$\exists c_0 > 0,\ \epsilon_0 > 0 \text{ such that } \min_{x\in\Omega}\Big(\min_{y\in B(x,\epsilon_0)} k(x,y)\Big) > c_0. \tag{$\tilde{H}2$}$$

An example of such kernel is given by

$$k(x,y) = J\left(\frac{x_1-y_1}{g_1(y)}; \frac{x_2-y_2}{g_2(y)}; \dots; \frac{x_n-y_n}{g_n(y)}\right)\frac{1}{\prod_{i=1}^n g_i(y)},$$

with $0 < \alpha_i \leqslant g_i \leqslant \beta_i$.

We want also to emphasize that the condition that J or k has a compact support is only needed to construct an eigenpair when Ω is unbounded. For a bounded domain, all the results will also hold true if J is not assume compactly supported in Ω.

Note that the assumption $J(0) > 0$ implies that the operator $\mathcal{L}_\Omega$ is not trivial on any open subset $\omega \subset \Omega$, i.e. $\forall \omega \subset \Omega$, $\forall u \in C(\Omega)$, $\mathcal{L}_\Omega[u] \not\equiv 0$ for $x \in \omega$. This condition makes sure that the principal eigenfunction ϕ_p is positive in Ω, which is a necessary condition for the existence of such principal eigenfunction. Indeed, when there exists an open subset $\omega \subset \Omega$ such that $\mathcal{L}_\Omega$ is trivial, there is no guarantee that a principal eigenpair exists. For example, this is the case for the operator $\mathcal{M}_\Omega$ where $\Omega := (-1, 1)$, J is such that $\mathrm{supp}(J) \subset (\frac{1}{2}, 1)$ and $3 \leqslant g \leqslant 4$. In this situation, we easily see that for any $x \in (-\frac{1}{4}, \frac{1}{4})$ and for any function $u \in C(\Omega)$, we have $\mathcal{L}_\Omega[u](x) = 0$. Therefore, the existence of an eigenfunction will strongly depend on the behaviour of the function $a(x)$ on this subset, i.e. $(\lambda_p + a(x))\phi \equiv 0$ for $x \in (-\frac{1}{4}, \frac{1}{4})$. If $(\lambda_p + a(x)) \neq 0$ then $\phi \equiv 0$ in $(-\frac{1}{4}, \frac{1}{4})$. In this situation there is clearly no existence of a positive principal eigenfunction. However, the condition $J(0) > 0$ can still be relaxed and the above theorems hold also true if we only assume that the kernel J is such that there exists a positive integer $p \in \mathbb{N}_0$ such that the following kernel $J_p(x, y)$ satisfies $(\tilde{H}2)$ where $J_p(x, y)$ is defined by the recursion

$$J_1(x, y) := J\left(\frac{x-y}{g(y)}\right)\frac{1}{g^n(y)},$$

$$J_{p+1}(x, y) := \int_\Omega J_1(x, z) J_p(z, y)\, dz \quad \text{for } p \geqslant 1.$$

The above condition is slightly more general that $J(0) > 0$ and we see that $J(0) > 0$ implies that J_1 satisfies $(\tilde{H}2)$. In particular, as showed for example in [17], for a convolution operator $K(x, y) := J(x - y)$, this new condition is optimal and can be related to a geometric condition on the convex hull of $\{y \in \mathbb{R}^n \mid J(y) > 0\}$:

There exists $p \in \mathbb{N}^*$, such that J_p satisfies $(\tilde{H}2)$ if and only if the convex hull of $\{y \in \mathbb{R}^n \mid J(y) > 0\}$ contains 0.

We also want to stress that we can easily extend the results of Theorems 1.6 and 1.7 to a periodic setting using the above generalisation on general non-negative kernel. Namely, if we consider the following problem

$$\frac{\partial u}{\partial t} = \mathcal{M}_{\mathbb{R}^n}[u] + f(x, u) \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{1.10}$$

where g and $f(., u)$ are assumed to be periodic functions then the existence of a unique non-trivial periodic solution of (1.10) is uniquely conditioned by the sign of the periodic principal eigenvalue $\lambda_{p,per}(\mathcal{M}_{\mathbb{R}^n} + f_u(x, 0))$, where $\lambda_{p,per}$ is defined as follows:

$$\lambda_{p,per}(\mathcal{M}) := \sup\bigl\{\lambda \in \mathbb{R} \bigm| \exists \psi > 0,\ \psi \in C_{per}\bigl(\mathbb{R}^n\bigr) \text{ such that } \mathcal{M}_{\mathbb{R}^n}[\psi] + \lambda\psi \leqslant 0\bigr\}.$$

It is worth noticing that in this context, using the periodicity, we have

$$\lambda_{p,per}\bigl(\mathcal{M}_{\mathbb{R}^n} + f_u(x, 0)\bigr) = \lambda_p\bigl(\mathcal{L}_Q + f_u(x, 0), Q\bigr),$$

where Q is the unit periodic cell and $\mathcal{L}_Q[\psi] := \int_Q k(x, y)u(y)\, dy$ with k a positive kernel satisfying $(\tilde{H}1)$ and $(\tilde{H}2)$. Hence the analysis of the existence/non-existence of stationary solutions of (1.10) will be handled through the analysis of the existence/non-existence of stationary solutions of a semilinear KPP problem defined on a bounded domain.

Finally, along our analysis, provided a more restrictive assumption on the coefficient $a(x)$ is made, we also observe that Theorem 1.1 holds as well when we relax the assumption on the function g and allow g to touch 0. More precisely, assuming that g satisfies

$$g \in L^\infty(\Omega),\ 0 \leqslant g \leqslant \beta,\ \frac{1}{g^n} \in L^p_{loc}(\overline{\Omega}) \text{ with } p > 1 \qquad (\tilde{H}3)$$

then for a bounded domain Ω, we have the following result:

Theorem 1.8. *Assume that Ω, J and a satisfy $(H1)$, $(H2)$, $(\tilde{H}3)$, $(H4)$, Ω is bounded and g satisfies $(\tilde{H}3)$. Let us denote $\sigma := \sup_{\overline{\Omega}} a(x)$ and let Γ be the following set*

$$\Gamma := \{x \in \overline{\Omega} \mid a(x) = \sigma\}.$$

Assume further that $\mathring{\Gamma} \neq \emptyset$. Then there exists a principal eigenpair (λ_p, ϕ_p) solution of (1.3). Moreover, $\phi_p \in C(\Omega)$, $\phi_p > 0$ and we have the following estimate

$$-\sigma' < \lambda_p < -\sigma,$$

where $\sigma' := \sup_{x\in\Omega}[a(x) + \int_\Omega J(\frac{y-x}{g(x)})\frac{dy}{g^n(x)}]$.

As a consequence the criterion on the survival/extinction of a species obtained in Theorems 1.6 and 1.7 can be extended to such type of dispersal kernel. More precisely, we have

Theorem 1.9. *Assume Ω, J and g satisfy $(H1)$, $(\tilde{H}2)$, $(\tilde{H}3)$, Ω is bounded and f satisfies (1.7). Then there exists a unique non-trivial solution of (1.6) if*

$$\lambda_p(\mathcal{M}_\Omega + f_u(x,0)) < 0,$$

where λ_p is the principal eigenvalue of the linear operator $\mathcal{M}_\Omega + f_u(x,0)$. Moreover, if $\lambda_p \geqslant 0$ then any non-negative uniformly bounded solution is identically zero.

And

Theorem 1.10. *Let Ω, J, g, b and f be as in Theorem 1.9. Let u_0 be an arbitrary bounded and continuous function in Ω such that $u_0 \geqslant 0$, $u_0 \not\equiv 0$. Let $u(t,x)$ be the solution of (1.8) with initial datum $u(0,x) = u_0(x)$. Then, we have:*

(1) *If 0 is an unstable solution of (1.6) (that is $\lambda_p < 0$), then $u(t,x) \to p(x)$ pointwise as $t \to \infty$, where p is the unique positive solution of (1.6) given by Theorem 1.9.*
(2) *If 0 is a stable solution of (1.6) (that is $\lambda_p \geqslant 0$), then $u(t,x) \to 0$ pointwise in Ω as $t \to +\infty$.*

In this context, the existence of a simple sufficient condition for the existence of a principal eigenpair when Ω is an unbounded domain is more involved and we have to make a technical assumption on the set $\Sigma := \{x \in \overline{\Omega} \mid g(x) = 0\}$. More precisely, we show

Theorem 1.11. *Assume that Ω, J and a satisfy $(H1)$, $(\tilde{H}2)$, $(H4)$ and g satisfies $(\tilde{H}3)$. Let us denote $\sigma := \sup_{\overline{\Omega}} a(x)$ and let Γ, Σ be the following sets*

$$\Gamma := \{x \in \overline{\Omega} \mid a(x) = \sigma\},$$
$$\Sigma := \{x \in \overline{\Omega} \mid g(x) = 0\}.$$

Assume further that $\Omega \cap \Sigma \Subset \Omega$ *and* $\mathring{\Gamma} \neq \emptyset$. *Then there exists a principal eigenpair* (λ_p, ϕ_p) *solution of* (1.3). *Moreover,* $\phi_p > 0$ *and we have the following estimate*

$$-\sigma' < \lambda_p < -\sigma,$$

where $\sigma' := \sup_{x \in \Omega}[a(x) + \int_\Omega J(\frac{y-x}{g(x)}) \frac{dy}{g^n(x)}]$.

The paper is organised as follows. In Section 2 we review some spectral theory of positive operators and we recall some Harnack's inequalities satisfied by a positive solution of integral equation. Then, we prove Theorems 1.1 and 1.8 in Section 3. The relation between the maximum principle and the sign of the principal eigenvalue (Theorem 1.5) and a counterexample to the existence of a principal eigenpair are obtained respectively in Section 4 and in Section 5. The last two sections is devoted to the derivation of the survival/extinction criteria (Theorems 1.6, 1.7, 1.9).

2. Preliminaries

In this section we first recall some results on the spectral theory of positive operators and some Harnack's inequalities satisfied by a positive solution of

$$\mathcal{L}_\Omega[u] - b(x)u = 0, \tag{2.1}$$

where $\mathcal{L}_\Omega$ is defined as above and $b(x)$ is a positive continuous function in Ω. Let us start with the spectral theory.

2.1. *Spectral theory of positive operators*

Let us recall some basic spectral results for positive operators due to Edmunds, Potter and Stuart [27] which are extensions of the Krein–Rutman theorem for positive non-compact operators.

A cone in a real Banach space X is a non-empty closed set K such that for all $x, y \in K$ and all $\alpha \geqslant 0$ one has $x + \alpha y \in K$, and if $x \in K$, $-x \in K$ then $x = 0$. A cone K is called reproducing if $X = K - K$. A cone K induces a partial ordering in X by the relation $x \leqslant y$ if and only if $x - y \in K$. A linear map or operator $T : X \to X$ is called positive if $T(K) \subseteq K$. The dual cone K^* is the set of functional $x^* \in X^*$ which are positive, that is, such that $x^*(K) \subset [0, \infty)$.

If $T : X \to X$ is a bounded linear map on a complex Banach space X, its essential spectrum (according to Browder [8]) consists of those λ in the spectrum of T such that at least one of the following conditions holds: (1) the range of $\lambda I - T$ is not closed, (2) λ is a limit point of the spectrum of A, (3) $\bigcup_{n=1}^\infty \ker((\lambda I - T)^n)$ is infinite dimensional. The radius of the essential spectrum of T, denoted by $r_e(T)$, is the largest value of $|\lambda|$ with λ in the essential spectrum of T. For more properties of $r_e(T)$ see [35].

Theorem 2.1 *(Edmunds, Potter, Stuart). Let* K *be a reproducing cone in a real Banach space* X, *and let* $T \in \mathcal{L}(X)$ *be a positive operator such that* $T^p(u) \geqslant cu$ *for some* $u \in K$ *with* $\|u\| = 1$, *some positive integer* p *and some positive number* c. *Then if* $c^{\frac{1}{p}} > r_e(T_c)$, T *has an eigenvector* $v \in K$ *with associated eigenvalue* $\rho \geqslant c^{\frac{1}{p}}$ *and* T^* *has eigenvector* $v^* \in K^*$ *corresponding to the eigenvalue* ρ. *Moreover,* ρ *is unique.*

A proof of this theorem can be found in [27].

2.2. Harnack's inequality

Let us now present some Harnack's inequality satisfied by any positive continuous solution of the nonlocal equation (2.1).

Theorem 2.2 *(Harnack inequality). Assume that Ω, J, g and $b > 0$ satisfy* $(H1)$, $(\tilde{H}2)$, $(H3)$, $(H4)$. *Let $\omega \Subset \Omega$ be a compact set. Then there exists $C(J, \omega, b, g)$ such that for all positive continuous bounded solutions u of* (2.1) *we have*

$$u(x) \leqslant C u(y) \quad \text{for all } x, y \in \omega.$$

When the assumption on g is relaxed the above Harnack's estimate does not hold any more but a uniform estimate still holds. Namely,

Theorem 2.3 *(Local uniform estimate). Assume that Ω, J, g and $b > 0$ satisfy* $(H1)$, $(\tilde{H}2)$, $(\tilde{H}3)$, $(H4)$. *Assume that $\Omega \cap \Sigma \Subset \Omega$ and let $\omega \subset \overline{\Omega}$ be a compact set. Let $\Omega(\omega)$ denote the following set*

$$\Omega(\omega) := \bigcup_{x \in \omega} B(x, \beta).$$

Then there exists a positive constant η^* such that, for any $0 < \eta \leqslant \eta^*$, there exist a compact set $\omega' \Subset \Omega(\omega) \cap \Omega$ and a constant $C(J, \omega, \Omega, \omega', b, g, \eta)$ such that the following assertions are verified:

(i) $\{x \in \Omega(\omega) \cap W_\eta \mid d(x, \partial(\Omega(\omega) \cap W_\eta)) > \eta\} \subset \omega'$, *where* $W_\eta := \{x \in \Omega \mid g(x) > \eta\}$,
(ii) *for all positive continuous solution u of* (2.1), *the following inequality holds*:

$$u(x) \leqslant C u(y) \quad \text{for all } x \in \omega,\ y \in \omega' \cap \omega.$$

Next, we present a contraction lemma which guarantees that when Ω is bounded then any continuous positive solution u of Eq. (2.1) is bounded in $\overline{\Omega}$.

Lemma 2.4 *(Contraction lemma). Let $\Omega \subset \mathbb{R}^n$ and $u \in C(\Omega)$ be respectively an open set and a positive solution of* (2.1). *Then there exists $\epsilon^* > 0$ such that for all $\epsilon \leqslant \epsilon^*$, there exists Ω_ϵ and $C(\alpha, \beta, J, \epsilon, b)$ such that*

$$\int_{\Omega_\epsilon} u(y)\,dy \geqslant C \int_{\Omega} u(y)\,dy.$$

Moreover, Ω_ϵ satisfies the following chain of inclusion

$$\left\{x \in \Omega \,\middle|\, d(x, \partial\Omega) > \alpha\epsilon\right\} \subset \Omega_\epsilon \subset \left\{x \in \Omega \,\middle|\, d(x, \partial\Omega) > \frac{\alpha\epsilon}{2}\right\}.$$

A proof of these results can be found in [19].

3. Construction of a principal eigenpair

In this section we prove the criterion of existence of a principal eigenpair (Theorems 1.1, 1.8 and 1.11). That is, we prove the existence of a solution (λ_p, ϕ_p) of the equation

$$\mathcal{L}_\Omega[\phi_p] + a(x)\phi_p = -\lambda_p \phi_p \quad \text{in } \Omega \tag{3.1}$$

with $\phi_p > 0$, $\phi_p \in C(\Omega)$ and λ_p is the principal eigenvalue of $\mathcal{L}_\Omega + a(x)$ defined by (1.5). In this task, we first restrict our analysis to the case of a bounded domain Ω and then prove the criterion for unbounded domains. We split this section into two subsections, each of them dedicated to one situation.

3.1. Existence of a principal eigenpair when Ω is a bounded domain

To simplify the presentation, we will first concentrate our attention on the construction of a principal eigenpair when J, g, b satisfy the assumptions $(H2)$–$(H4)$ (Theorem 1.1). Then we provide an argumentation for the construction of a principal eigenpair when the assumptions on g are relaxed (Theorem 1.8).

In a first step, let us show that the eigenvalue problem (3.1) admits a positive solution, i.e. there exists $(\mu_{1,0}, \phi_1)$ with $\phi_1 > 0$, $\phi_1 \in L^\infty(\Omega) \cap C(\overline{\Omega})$ solution of (3.1). More precisely, we prove

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and assume that J, g, and $a(x)$ satisfy $(H1)$–$(H4)$. Let us denote $\sigma := \sup_{\overline{\Omega}} a(x)$ and $\Omega_\theta := \{x \in \Omega \mid d(x, \partial\Omega) > \theta\}$. Assume further that the function $a(x)$ satisfies $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\overline{\Omega})$. Then there exists $\theta_0 > 0$ such that for all $\theta \leqslant \theta_0$ the operator $\mathcal{L}_{\Omega_\theta} + a(x)$ has a unique eigenvalue $\mu_{1,\theta}$ in $C(\Omega_\theta)$, that is to say, there is a unique $\mu_{1,\theta} \in \mathbb{R}$ such that*

$$\mathcal{L}_{\Omega_\theta}[\phi_1] + a(x)\phi_1 = -\mu_{1,\theta}\phi_1 \quad \text{in } \Omega_\theta \tag{3.2}$$

admits a positive solution $\phi_1 \in C(\overline{\Omega}_\theta)$. Moreover, $\mu_{1,\theta}$ is simple (i.e. the space of $C(\overline{\Omega}_\theta)$ solutions to (3.1) is one-dimensional) and satisfies

$$\mu_{1,\theta} < -\max_{\overline{\Omega}_\theta} a(x).$$

Suppose for the moment that the above theorem holds true. To conclude the proof of Theorem 1.1 which establishes the criterion of existence of an eigenpair, we are left to show that the principal eigenvalue defined by (1.5) is the same as the one obtained in Theorem 3.1 for $\theta = 0$. Namely, we are reduced to prove of the following results.

Lemma 3.2. *Let $a(x)$ be as in Theorem 3.1 then we have $\lambda_p = \mu_{1,0}$ where λ_p and $\mu_{1,0}$ are respectively the principal eigenvalue of $\mathcal{L}_\Omega + a(x)$ defined by (1.5) and the eigenvalue of $\mathcal{L}_\Omega + a(x)$ obtained in Theorem 3.1.*

Before proving Theorem 3.1, let us prove the above lemma.

Proof of Lemma 3.2. First, let us define the following quantity

$$\lambda_p' := \sup\bigl\{\lambda \in \mathbb{R} \bigm| \exists \phi > 0,\ \phi \in C(\overline{\Omega}) \text{ so that } \mathcal{L}_\Omega[\phi] + a(x)\phi + \lambda\phi \leqslant 0 \text{ in } \overline{\Omega}\bigr\}.$$

Obviously λ_p' is well defined and is sharing the same properties than λ_p. Moreover, we have $\lambda_p' \leqslant \lambda_p$. Let us now show that $\lambda_p' = \mu_{1,0}$. First by definition of λ_p' we easily have $\lambda_p' \geqslant \mu_{1,0}$. Now to obtain the equality $\lambda_p' = \mu_{1,0}$ we argue by contradiction. Assume that $\lambda_p' > \mu_{1,0}$. By definition of λ_p' there exists $\psi > 0$, $\psi \in C(\overline{\Omega})$ such that

$$\mathcal{L}_\Omega[\psi] + \bigl(a(x) + \lambda\bigr)\psi \leqslant 0 \quad \text{in } \overline{\Omega}. \tag{3.3}$$

Observe that we can rewrite $\mathcal{L}_\Omega[\phi_1] + a(x)\phi_1$ as follows

$$\mathcal{L}_\Omega[\phi_1] + a(x)\phi_1 = \int_\Omega J\left[\frac{x-y}{g(y)}\right]\frac{\phi_1(y)}{g(y)}\,dy + a(x)\phi_1$$
$$= \int_\Omega J\left[\frac{x-y}{g(y)}\right]\frac{\psi(y)\phi_1(y)}{\psi(y)g(y)}\,dy + a(x)\frac{\phi_1(x)}{\psi(x)}\psi(x).$$

From (3.3), we find that

$$a(x)\psi \leqslant -\mathcal{L}_\Omega[\psi] - \lambda\psi$$

and it follows that

$$\mathcal{L}_\Omega[\phi_1] + a(x)\phi_1 \leqslant \int_\Omega J\left[\frac{x-y}{g(y)}\right]\frac{\psi(y)}{g(y)}\left[\frac{\phi_1(y)}{\psi(y)} - \frac{\phi_1(x)}{\psi(x)}\right]dy - \lambda\frac{\phi_1(x)}{\psi(x)}\psi(x).$$

By using the definition of $\mu_{1,0}$, we end up with the following inequality

$$\int_\Omega J\left[\frac{x-y}{g(y)}\right]\frac{\psi(y)}{g(y)}\left[\frac{\phi_1(y)}{\psi(y)} - \frac{\phi_1(x)}{\psi(x)}\right]dy \geqslant (\lambda - \mu_{1,0})\phi_1 > 0. \tag{3.4}$$

Let us denote $w := \frac{\phi_1}{\psi}$. Observe that by (3.3) $w \in L^\infty \cap C(\overline{\Omega})$, therefore w achieves a global maximum somewhere in $\overline{\Omega}$, say at $\bar{x}$. By using the inequality (3.4) at the point $\bar{x}$, we find the following contradiction

$$0 < \int_\Omega J\left[\frac{\bar{x}-y}{g(y)}\right]\frac{\psi(y)}{g(y)}\big[w(y) - w(\bar{x})\big]\,dy \leqslant 0.$$

Thus $\mu_{1,0} = \lambda_p'$.

Observe now that if there exists a positive eigenfunction $\psi \in C(\Omega) \cap L^\infty(\Omega)$ associated to the principal eigenvalue λ_p, i.e. $\mathcal{L}_\Omega[\psi] + (a(x) + \lambda_p)\psi = 0$, then we have $\psi \in C(\overline{\Omega})$. Therefore, using the definition of λ_p' it follows that $\lambda_p \leqslant \lambda_p' = \mu_{1,0} \leqslant \lambda_p$. To conclude the proof, we are left to show that such bounded function ψ exists.

So let $(\theta_n)_{n\in\mathbb{N}}$ be a positive sequence which converges to 0 and consider the sequence of set $(\Omega_{\theta_n})_{n\in\mathbb{N}}$ defined in Theorem 3.1. By construction, using the monotonicity property of the principal eigenvalue with respect to the domain ((i) of Proposition 1.1) we deduce that $(\lambda_p'(\mathcal{L}_{\Omega_{\theta_n}} + a(x)))_{n\in\mathbb{N}}$ is a non-increasing bounded sequence. Namely, we have for all $n \in \mathbb{N}$

$$\lambda_p\big(\mathcal{L}_\Omega + a(x)\big) \leqslant \lambda_p'\big(\mathcal{L}_{\Omega_{\theta_{n+1}}} + a(x)\big) \leqslant \lambda_p'\big(\mathcal{L}_{\Omega_{\theta_n}} + a(x)\big).$$

Thus, as n goes to infinity $\lambda_p'(\mathcal{L}_{\Omega_{\theta_n}} + a(x))$ converges to some $\bar{\lambda} \geqslant \lambda_p$.

On another hand since θ_n tends to 0, by Theorem 3.1, there exists n_0 so that for all $n \geqslant n_0$, a principal eigenpair $(\mu_{1,\theta_n}, \phi_n)$ exists for the operator $\mathcal{L}_{\Omega_{\theta_n}} + a(x)$. Arguing as above, we conclude that $\mu_{1,\theta_n} = \lambda_p'(\mathcal{L}_{\Omega_{\theta_n}} + a(x))$.

We claim that:

Claim 3.1. *There exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$ we have $\mu_{1,\theta_n} < -\sigma = -\sup_\Omega a(x)$.*

Assume for the moment that the claim holds. Then the final argumentation goes as follows. Next, let us normalised ϕ_n so that $\sup_{\Omega_{\theta_n}} \phi_n = 1$. With this normalisation $(\phi_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence of continuous functions. So by a standard diagonal extraction argument, there exists a subsequence still denoted $(\phi_n)_{n\in\mathbb{N}}$ such that $(\phi_n)_{n\in\mathbb{N}}$ converges locally uniformly to a non-negative bounded continuous function ψ. Furthermore, ψ satisfies

$$\mathcal{L}_\Omega[\psi] + \big(a(x) + \bar{\lambda}\big)\psi = 0.$$

Now recall that $(\mu_{1,\theta_n}, \phi_n)$ satisfies

$$\mathcal{L}_{\Omega_{\theta_n}}[\phi_n] + a(x)\phi_n + \mu_{1,\theta_n}\phi_n = 0.$$

Using the above claim, we have $\mu_{1,\theta_n} < -\sigma = -\sup_\Omega a(x) \leqslant -\sup_{\Omega_{\theta_n}} a(x)$ for n big enough, so $\sup_{\Omega_{\theta_n}}(a(x) + \mu_{1,\theta_n}) < 0$ and the uniform estimates i.e. Theorem 2.3 applies to ϕ_n. Thus we have for $\eta > 0$ small fixed independently of n

$$1 \leqslant C(\eta)\phi_n(x) \quad \text{for all } x \in \big\{x \in \Omega_{\theta_n} \,\big|\, d(x, \partial\Omega_{\theta_n}) > \eta\big\}.$$

Therefore ψ is non-trivial and $(\bar{\lambda}, \psi)$ solves the eigenvalue problem (3.1). Using once again the equation satisfied by ψ and the definition of λ_p, we easily obtain that $\bar{\lambda} \leqslant \lambda_p \leqslant \bar{\lambda}$ which proves that ψ is our desired eigenfunction associated to λ_p. □

Let us turn our attention to the proof of Claim 3.1. But before proving the claim let us establish the following useful estimate.

Lemma 3.3. *There exist positive constants r and c_0 so that*

$$\forall x \in \overline{\Omega}, \quad \int\limits_{B_r(x)\cap\overline{\Omega}} J\left(\frac{x-y}{g(y)}\right) u(y)\, d\mu(y) \geqslant c_0 \int\limits_{B_r(x)\cap\overline{\Omega}} u(y)\, d\mu(y).$$

Proof. Since J is continuous and $J(0) > 0$, there exist $\delta > 0$ and $c_0 > 0$ so that for all $z \in B(0,\delta)$ we have $J(z) \geqslant c_0$.

Observe that for all $(x, y) \in \overline{\Omega} \times B_r(x)$ with $r < \frac{\delta\alpha}{2}$, using that $g \geqslant \alpha > 0$, we have

$$\left\| \frac{x-y}{g(y)} \right\| \leqslant \frac{2r}{\alpha} \leqslant \delta.$$

Thus, for $r < \frac{\delta\alpha}{2}$ and $y \in B_r(x)$ we have $J(\frac{x-y}{g(y)}) > c_0$, and the estimate follows. □

We are now in position to prove Claim 3.1.

Proof of Claim 3.1. Let us denote by σ the maximum of $a(x)$ in $\overline{\Omega}$. By assumption, we have $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu,loc}(\overline{\Omega})$. So there exists $x_0 \in \overline{\Omega}$ such that $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(B_r(x_0) \cap \overline{\Omega})$ and for ϵ small enough, say $\epsilon \leqslant \epsilon_0$, we have

$$c_0 \int\limits_{\overline{\Omega}\cap B(x_0,r)} \frac{d\mu}{-(a(x) - \sigma + \epsilon)} \geqslant 4.$$

Choose n_1 big enough, so that for all $n \geqslant n_1$, $B_r(x_0) \cap \overline{\Omega}_{\theta_n} \neq \emptyset$. For $\epsilon \leqslant \epsilon_0$, since $\Omega_{\theta_n} \to \Omega$, we can increase n_1 if necessary to achieve for all $n \geqslant n_1$

$$c_0 \int_{\overline{\Omega}_{\theta_n} \cap B(x_0,r)} \frac{d\mu}{-(a(x) - \sigma - \epsilon)} \geqslant 2. \tag{3.5}$$

Recall now that for n big enough, say $n \geqslant n_2$, there exists $(\mu_{1,\theta_n}, \phi_n)$ that satisfies the equation

$$\mathcal{L}_{\Omega_{\theta_n}}[\phi_n] + a(x)\phi_n + \mu_{1,\theta_n}\phi_n = 0.$$

Since ϕ_n is positive we have

$$\mathcal{L}_{\overline{\Omega}_{\theta_n} \cap B(x_0,r)}[\phi_n] \leqslant -\big(a(x) + \mu_{1,\theta_n}\big)\phi_n.$$

Using Lemma 3.3, we see that

$$\frac{c_0}{-(a(x) + \mu_{1,\theta_n})} \int_{\overline{\Omega}_{\theta_n} \cap B(x_0,r)} \phi_n(y)\, d\mu \leqslant \phi_n(x).$$

Integrating the above inequality on $\overline{\Omega}_{\theta_n} \cap B(x_0,r)$ it follows that

$$\int_{\overline{\Omega}_{\theta_n} \cap B(x_0,r)} \left(\frac{c_0}{-(a(x) + \mu_{1,\theta_n})} \int_{\overline{\Omega}_{\theta_n} \cap B(x_0,r)} \phi_n(y)\, d\mu \right) d\mu \leqslant \int_{\overline{\Omega}_{\theta_n} \cap B(x_0,r)} \phi_n(x)\, d\mu,$$

$$\int_{\overline{\Omega}_{\theta_n} \cap B(x_0,r)} \left(\frac{c_0}{-(a(x) + \mu_{1,\theta_n})} \right) d\mu \int_{\overline{\Omega}_{\theta_n} \cap B(x_0,r)} \phi_n(y)\, d\mu \leqslant \int_{\overline{\Omega}_{\theta_n} \cap B(x_0,r)} \phi_n(x)\, d\mu.$$

Thus,

$$\int_{\overline{\Omega}_{\theta_n} \cap B(x_0,r)} \left(\frac{c_0}{-(a(x) + \mu_{1,\theta_n})} \right) d\mu \leqslant 1.$$

From (3.5), it follows that for all $n \geqslant \sup(n_1, n_2)$ we have

$$\mu_{1,\theta_n} \leqslant -\sigma - \epsilon. \qquad \square$$

Remark 3.4. Observe that if $\sup_{\Omega} a(x)$ is achieved in Ω then the estimation $\mu(1,\theta)$ follows immediately from the monotonicity properties of the principal eigenvalue. Indeed, for θ small enough, say $\theta \leqslant \theta_0$ we have $\sup_{\Omega_\theta} a(x) = \sup_{\Omega} a(x)$. Hence,

$$\lambda_p'\big(\mathcal{L}_{\Omega_\theta} + a(x)\big) \leqslant \lambda_p'\big(\mathcal{L}_{\Omega_{\theta_0}} + a(x)\big) < -\sup_{\Omega_{\theta_0}} a(x) = -\sigma.$$

Let us now turn our attention to the proof of Theorem 3.1.

For convenience, in this proof we write the eigenvalue problem

$$\mathcal{L}_{\Omega_\theta}[u] + a(x)u = -\mu u$$

in the form

$$\mathcal{L}_{\Omega_\theta}[u] + \bar{a}(x)u = \rho u \tag{3.6}$$

where

$$\bar{a}(x) = a(x) + k, \qquad \rho = -\mu + k$$

and $k > 0$ is a constant such that $\inf_{\Omega_\theta} \bar{a} > 0$.

Let us now prove the following useful result:

Lemma 3.5. *Let Ω, J, g and a be as in Theorem 3.1. Then there exists $\theta_0 > 0$ so that for all $\theta \leqslant \theta_0$ there exist $\delta > 0$ and $u \in C(\overline{\Omega}_\theta)$, $u \geqslant 0$, $u \not\equiv 0$, such that in $\overline{\Omega}_\theta$*

$$\mathcal{L}_{\Omega_\theta}[u] + \bar{a}(x)u \geqslant (\overline{\sigma} + \delta)u,$$

where $\overline{\sigma}(\theta) := \max_{\overline{\Omega}_\theta} \bar{a}(x)$.

Observe that the proof of Theorem 3.1 easily follows from the above lemma. Indeed, if the lemma holds true, since under the assumptions $(H1)$–$(H4)$ the operator $\mathcal{L}_\Omega : C(\overline{\Omega}_\theta) \to C(\overline{\Omega}_\theta)$ is compact, we have $r_e(\mathcal{L}_{\Omega_\theta} + \bar{a}(x)) = r_e(\bar{a}(x)) = \overline{\sigma}(\theta)$. Thus $(\overline{\sigma}(\theta) + \delta) > r_e(\mathcal{L}_{\Omega_\theta} + \bar{a}(x))$ and the existence theorem of Edmunds et al. (Theorem 2.1) applies.

Finally we observe that the principal eigenvalue is simple since for a bounded domain Ω the cone of positive continuous functions has a non-empty interior and, for a sufficiently large p, the operator $(\mathcal{L}_{\Omega_\theta} + \bar{a})^p$ is strongly positive, that is, it maps $u \geqslant 0$, $u \not\equiv 0$ to a strictly positive function, see [40].

Remark 3.6. Note that the simplicity of the eigenvalue μ_θ requires that Ω_θ is a connected set. Indeed, when open set Ω is not connected, it may happen that the operator $(\mathcal{L}_{\Omega_\theta} + \bar{a})^p$ is never strongly positive in $C(\overline{\Omega})$ and several non-positive eigenfunction exists with no positive eigenfunction.

Let us now turn our attention to the proof of Lemma 3.5:

Proof of Lemma 3.5. Let us denote by Γ the closed set where the continuous function $\bar{a}$ takes its maximum $\overline{\sigma}$ in $\overline{\Omega}$:

$$\Gamma := \{ z \in \overline{\Omega} \mid \bar{a}(z) = \overline{\sigma} \}.$$

Since $\bar{a}$ is a continuous function and Ω is bounded, Γ is a compact set. Therefore Γ can be covered by a finite number of balls of radius r, i.e. $\Gamma \subset \bigcup_{i=1}^N B_r(x_i)$ with $x_i \in \Gamma$. By construction, we have $\frac{1}{\overline{\sigma} - \bar{a}(x)} = \frac{1}{\sigma - a(x)} \notin L^1_{d\mu,loc}(\overline{\Omega})$. Therefore $\frac{1}{\overline{\sigma} - \bar{a}(x)} \notin L^1_{d\mu}(\bigcup_{i=1}^N B_r(x_i) \cap \overline{\Omega})$ and there exists $-\lambda_0 > \overline{\sigma}$ so that for some x_i we have

$$\int\limits_{B_r(x_i) \cap \overline{\Omega}} \frac{c_0}{-\lambda_0 - \bar{a}(x)}\, d\mu \geqslant 4. \tag{3.7}$$

Since $\Omega_\theta \to \Omega$ as θ tends to 0 there exists θ_0 so that for all $\theta \leqslant \theta_0$ we have

$$\int_{B_r(x_i)\cap\overline{\Omega}_\theta} \frac{c_0}{-\lambda_0 - \bar{a}(x)}\, d\mu \geqslant 2. \tag{3.8}$$

Let us fix x_i such that (3.8) holds true and let us denote $\omega_\theta := B_r(x_i) \cap \overline{\Omega}_\theta$. We consider now the following eigenvalue problem

$$c_0 \int_{\omega_\theta} u(y)\, d\mu(y) + \bar{a}(x)u(x) + \lambda u(x) = 0, \tag{3.9}$$

where c_0 is the constant obtained in Lemma 3.3.

We claim that:

Claim 3.2. *There exists* (λ_1, ϕ_1) *solution of* (3.9) *so that* $\phi_1 \in L^\infty(\omega_\theta) \cap C(\omega_\theta)$ *and* $\phi_1 > 0$.

Observe that by proving this claim we end the proof of the lemma. Indeed, fix $\theta < \theta_0$ and assume for the moment that this claim holds true. Then there exists (λ_1, ϕ_1) such that

$$c_0 \int_{\omega_\theta} \phi_1(y)\, d\mu(y) + \bar{a}(x)\phi_1(x) + \lambda_1\phi_1(x) = 0. \tag{3.10}$$

Obviously, for any positive constant ρ, $(\lambda_1, \rho\phi_1)$ is also a solution of Eq. (3.10). Therefore without any loss of generality we can assume that ϕ_1 is such that $\phi_1 \leqslant 1$. Set $\tilde{c}_0 := c_0 \int_{\omega_\theta} \phi_1(y)\, d\mu(y)$. From Eq. (3.10), since $0 < \phi_1 \leqslant 1$ we see easily that

$$-\big(\lambda_1 + \bar{a}(x)\big) > \tilde{c}_0.$$

Therefore there exists a positive constant d_0 such that

$$\phi_1 \geqslant d_0 \quad \text{in } \omega \tag{3.11}$$

and

$$-\big(\lambda_1 + \overline{\sigma}(\theta)\big) \geqslant \tilde{c}_0 > 0. \tag{3.12}$$

Let us now consider a set $\omega_\epsilon \Subset \omega_\theta$ which verifies

$$\int_{\omega_\theta \setminus \omega_\epsilon} d\mu \leqslant \frac{d_0|\lambda_1 + \overline{\sigma}(\theta)|}{2c_0}. \tag{3.13}$$

Since by construction $\overline{\Omega}_\theta \setminus \omega_\theta$ and $\overline{\omega}_\epsilon$ are two disjoint closed subsets of $\overline{\Omega}_\theta$, the Urysohn's lemma applies and there exists a positive continuous function η such that $0 \leqslant \eta \leqslant 1$, $\eta(x) = 1$ in ω_ϵ, $\eta(x) = 0$ in $\overline{\Omega}_\theta \setminus \omega_\theta$.

Next, we define $w := \phi_1\eta$ and we compute $\mathcal{L}_{\Omega_\theta}[w] + b(x)w$.

Since $w \equiv 0$ in $\overline{\Omega}_\theta \setminus \omega_\theta$, we have

$$\mathcal{L}_{\Omega_\theta}[w] + \bar{a}(x)w = \int\limits_{\omega_\theta} J\left(\frac{x-y}{g(y)}\right) w(y)\,d\mu(y) \geqslant \left(\overline{\sigma}(\theta) + \delta\right) w = 0$$

for any $\delta > 0$.

On another hand, in ω_θ, by using Lemma 3.3 we see that

$$\mathcal{L}_{\Omega_\theta}[w] + \bar{a}(x)w = \int\limits_{\omega_\theta} J\left(\frac{x-y}{g(y)}\right) w(y)\,d\mu(y) + \bar{a}(x)w \tag{3.14}$$

$$\geqslant c_0 \int\limits_{\omega_\theta} w(y)\,d\mu(y) + \bar{a}(x)w \tag{3.15}$$

$$\geqslant c_0 \int\limits_{\omega_\epsilon} \phi_1(y)\,d\mu(y) + \bar{a}(x)w. \tag{3.16}$$

Since ϕ_1 satisfies Eq. (3.10), using the estimates (3.11), (3.12) and (3.13) we deduce from the inequality (3.16) that

$$\mathcal{L}_{\Omega_\theta}[w] + \bar{a}(x)w$$

$$\geqslant -\left(\lambda_1 + \bar{a}(x)\right)\phi_1 + \bar{a}(x)w - c_0 \int\limits_{\omega_\theta \setminus \omega_\epsilon} \phi_1(y)\,d\mu(y) \tag{3.17}$$

$$\geqslant \frac{|\lambda_1 + \overline{\sigma}(\theta)|}{2}\phi_1 + \left(\overline{\sigma}(\theta) - \bar{a}(x)\right)\phi_1 + \bar{a}(x)w + \frac{d_0|\lambda_1 + \overline{\sigma}(\theta)|}{2} - c_0 \int\limits_{\omega_\theta \setminus \omega_\epsilon} \phi_1(y)\,d\mu(y) \tag{3.18}$$

$$\geqslant \left(\frac{|\lambda_1 + \overline{\sigma}(\theta)|}{2}\right)\phi_1 + \left(\overline{\sigma}(\theta) - \bar{a}(x)\right)\phi_1 + \bar{a}(x)w, \tag{3.19}$$

where we use in the last inequality, that $\phi_1 \leqslant 1$ and the estimate (3.13).

Since $(\overline{\sigma}(\theta) - \bar{a}(x))$ and $\frac{|\lambda_1 + \overline{\sigma}(\theta)|}{2}$ are two positive quantities and $\phi_1 \geqslant w$, we conclude that

$$\mathcal{L}_{\Omega_\theta}[w] + \bar{a}(x)w \geqslant \left(\frac{|\lambda_1 + \overline{\sigma}(\theta)|}{2} + \overline{\sigma}(\theta)\right) w. \tag{3.20}$$

Hence, in $\overline{\Omega}_\theta$, w satisfies

$$\mathcal{L}_{\Omega_\theta}[w] + \bar{a}(x)w \geqslant \left(\overline{\sigma}(\theta) + \delta\right) w,$$

with $\delta = \frac{|\lambda_1 + \overline{\sigma}(\theta)|}{2}$, which proves the lemma. □

Let us now prove Claim 3.2.

Proof of Claim 3.2. Fix $\theta \leqslant \theta_0$. For $\lambda < -\overline{\sigma}(\theta)$, consider the positive function $\phi_\lambda := \frac{c_0}{-\lambda - \bar{a}(x)}$. Let us substitute ϕ_λ into Eq. (3.9), then we have

$$c_0 \int_{\omega_\theta} \phi_\lambda \, d\mu - c_0 = 0.$$

Therefore, we end the proof of Claim 3.2 by finding λ such that $\int_{\omega_\theta} \phi_\lambda \, d\mu = 1$. Observe that the functional $F(\lambda) := \int_{\omega_\theta} \phi_\lambda \, d\mu$ is continuous and monotone increasing with respect to λ in $(-\infty, -\overline{\sigma})$. Moreover, by construction, we have:

$$\lim_{\lambda \to -\infty} F(\lambda) = 0 \quad \text{and} \quad F(\lambda_0) \geqslant 2.$$

Hence by continuity there exists a λ_1 such that $F(\lambda_1) = 1$. □

Now we expose the argumentation for the construction of a principal eigenpair when the assumptions on g are relaxed and prove Theorem 1.8. To show Theorem 1.8 we follow the scheme of the argument developed above.

Proof of Theorem 1.8. As above, we can rewrite the eigenvalue problem (3.1) as follows

$$\mathcal{L}_{\Omega_\theta}[u] + \bar{a}(x)u = \rho u \tag{3.21}$$

with

$$\bar{a}(x) = a(x) + k, \qquad \rho = -\mu + k$$

and $k > 0$ is a constant such that $\inf_{\Omega_\theta} \bar{a} > 0$.

Observe that under the assumptions $(H1)$, $(H2)$, $(\tilde{H}3)$, $(H4)$ the following family

$$\mathcal{L}_{\Omega_\theta}(B_1) := \big\{ \mathcal{L}_{\Omega_\theta}[f] / f : \Omega \to \mathbb{R},\ \|f\|_\infty \leqslant 1 \big\}$$

is equicontinuous. Indeed, let $\epsilon > 0$ be fixed. Since $\frac{1}{g^n} \in L^p_{loc}(\overline{\Omega}_\theta)$, there exists $\eta > 0$ such that

$$\int_{\Omega_\theta \cap \{g < \eta\}} \frac{dy}{g^n(y)} < \frac{\epsilon}{4\|J\|_\infty}. \tag{3.22}$$

From the uniform continuity of J in the unit ball $B(0,1)$, we deduce that there exists $\gamma > 0$ such that for $|w - \overline{w}| < \gamma/\eta$,

$$\big| J(w) - J(\overline{w}) \big| < \epsilon \eta^n / 2|\Omega_\theta|. \tag{3.23}$$

A short computation using (3.22) and (3.23) shows that for $|x - z| < \gamma$

$$\left|\mathcal{L}_{\Omega_\theta}[f](x)-\mathcal{L}_{\Omega_\theta}[f](z)\right| \leqslant \int_{\Omega_\theta}\left|J\left[\frac{x-y}{g(y)}\right]-J\left[\frac{z-y}{g(y)}\right]\right|\left|\frac{f(y)}{g^n(y)}\right|dy$$
$$\leqslant 2\|J\|_\infty \int_{\Omega_\theta\cap\{g<\eta\}} \frac{1}{g^n(y)}\,dy + \frac{1}{\delta^n}\int_{\Omega_\theta\cap\{g\geqslant\eta\}}\left|J\left[\frac{x-y}{g(y)}\right]-J\left[\frac{z-y}{g(y)}\right]\right|dy$$
$$\leqslant \epsilon.$$

Hence, $\mathcal{L}_{\Omega_\theta}(B_1)$ is equicontinuous and $\mathcal{L}_{\Omega_\theta}: C(\overline{\Omega}_\theta)\to C(\overline{\Omega}_\theta)$ is a compact operator.

Next, we show the following

Lemma 3.7. *Let Ω, J, g and a be as in Theorem* 1.8. *Then there exists θ_0 so that for all $\theta \leqslant \theta_0$ there exists $\delta > 0$ and $u \in C(\overline{\Omega}_\theta)$, $u \geqslant 0$, $u \not\equiv 0$, such that in $\overline{\Omega}_\theta$*

$$\mathcal{L}_{\Omega_\theta}[u] + \bar{a}(x)u \geqslant (\overline{\sigma}+\delta)u.$$

As above the existence of a positive eigenpair (ρ,ϕ) easily follows from Lemma 3.7. Arguing as above, we see that $\mu_{1,0} = \lambda_p(\mathcal{L}_\Omega + a(x))$, which concludes the proof of Theorem 1.8. □

Let us turn our attention to the proof of Lemma 3.7.

Proof of Lemma 3.7. First let us recall that by assumption $\mathring{\Gamma} \neq \emptyset$ where $\Gamma := \{x \in \overline{\Omega} \mid a(x) = \sigma\}$ and let us define the following set $\Sigma_\eta := \{x \in \Omega \mid g(x) \geqslant \eta\}$.

By construction, we easily see that $\mathring{\Gamma}' \neq \emptyset$ where $\Gamma' := \{x \in \overline{\Omega} \mid \bar{a}(x) = \overline{\sigma}\}$. Therefore, there exist $x_0 \in \Omega$ and $\epsilon > 0$ such that $B_\epsilon(x_0) \subset (\mathring{\Gamma}' \cap \Omega)$. Moreover, for θ small, say $\theta \leqslant \theta_0$ we have $B_\epsilon(x_0) \subset (\mathring{\Gamma}' \cap \Omega_\theta)$.

Let us define $\omega_\eta := B_\epsilon(x_0) \cap \Sigma_\eta$. By assumption we have $\frac{1}{g^n} \in L^p(\Omega)$, so for η small enough ω_η is a non-void open subset of Ω_θ for $\theta \leqslant \theta_0$.

Let us now consider the eigenvalue problem (3.21) with $\Omega = \omega_\eta$, i.e.

$$\mathcal{L}_{\omega_\eta}[u] + \bar{a}(x)u = \rho u \quad \text{in } \omega_\eta.$$

By construction, in $B_\epsilon(x_0)$ we have $\bar{a}(x) \equiv \overline{\sigma}$. So the above equation reduces to:

$$\mathcal{L}_{\omega_\eta}[u] = \overline{\rho} u \quad \text{in } \omega_\eta, \tag{3.24}$$

where $\overline{\rho} = (\rho - \overline{\sigma})$.

Since $\mathcal{L}_{\omega_\eta}$ is a compact strictly positive operator in $C(\overline{\omega}_\eta)$, using Krein–Rutman theorem there exists a positive eigenvalue $\overline{\rho}_1 > 0$ and a positive eigenfunction $\phi_1 \in C(\overline{\omega}_\eta)$ such that $(\overline{\rho}_1, \phi_1)$ satisfies (3.24), i.e.

$$\mathcal{L}_{\omega_\eta}[\phi_1] = \overline{\rho}\phi_1.$$

Arguing as in Lemma 3.5, for all $\theta \leqslant \theta_0$ we can construct a non-negative test function u such that in $\overline{\Omega}_\theta$

$$\mathcal{L}_{\Omega_\theta}[u] + \bar{a}(x)u \geqslant (\delta + \overline{\sigma})u,$$

for a $\delta > 0$ small enough. □

Remark 3.8. Observe that all the previous constructions can be easily adapted to an operator $\mathcal{T}+a(x)$ where $\mathcal{T}$ is an integral operator with a continuous non-negative kernel $k(x,y)$ that satisfies $(\tilde{H}2)$, i.e.

$$\exists c_0>0,\ \epsilon_0>0 \quad \text{such that} \quad \min_{x\in\Omega}\Big(\min_{y\in B(x,\epsilon_0)} k(x,y)\Big)>c_0.$$

In particular, we can extend the criterion of existence of a principal eigenpair for an operator $\mathcal{T}+a(x)$ where $\mathcal{T}$ is an integral operator with a kernel $k(x,y)$ that only satisfies that there exists a positive integer N, so that the kernel $k_N(x,y)$ satisfies $(\tilde{H}2)$ where k_N is defined by the recursion:

$$k_1(x,y):=k(x,y),$$
$$k_{N+1}(x,y):=\int_{\Omega} k_1(x,z)k_N(z,y)\,dz \quad \text{for } N\geqslant 1.$$

Indeed, in this situation the construction of a test function u (Lemma 3.5 or Lemma 3.7) holds also for the operator $\mathcal{T}^N+\bar{a}^N(x)$. Using that $\bar{a}\geqslant 0$, we deduce

$$\big(\mathcal{T}+\bar{a}(x)\big)^N[u]\geqslant \mathcal{T}^N u+\bar{a}^N(x)u\geqslant\big(\overline{\sigma}^N+\delta\big)u.$$

Since in this situation $\mathcal{T}$ is a compact operator, we also have $r_e((\mathcal{T}+\bar{a}(x))^N)=r_e(\bar{a}(x)^N)$. Thus $(\overline{\sigma}^N+\delta)>r_e((\mathcal{T}+\bar{a}(x))^N)$ and Theorem 2.1 applies. Hence, there exists a unique principal eigenpair (λ_p,ϕ_p) of the following problem

$$\big(\mathcal{T}+\bar{a}(x)\big)^N\phi_p=-\lambda_p\phi_p.$$

To obtain a principal eigenpair for $\mathcal{T}+a$ we argue as follows. Applying $\mathcal{T}+a(x)$ to the above equation it follows that

$$\big(\mathcal{T}+\bar{a}(x)\big)^{N+1}\phi_p=-\lambda_p\big(\mathcal{T}+a(x)\big)\phi_p,$$
$$\big(\mathcal{T}+\bar{a}(x)\big)^N\psi=-\lambda_p\psi$$

with $\psi:=(\mathcal{T}+a(x))\phi_p$. Since $(\mathcal{T}+\bar{a})^N$ is positive operator in $C(\overline{\Omega})$, λ_p is simple, we have $\psi=\rho\phi_p$. Hence, $((-\lambda_p)^{\frac{1}{N}},\phi_p)$ is the principal eigenpair of $\mathcal{T}+\bar{a}(x)$.

3.2. Construction of a principal eigenpair when Ω is an unbounded domain

For simplicity in the presentation of the arguments and since the proof of the existence of a principal eigenpair under the relaxed assumptions does not significantly differ, we will only present the case where Ω, J, g and a satisfy the assumptions $(H1)$–$(H4)$.

To construct an eigenpair (λ_p,ϕ_p) in this situation, we proceed using a standard approximation scheme.

First let us recall that, by assumption, there exists $\Omega_0\subset\overline{\Omega}$ a bounded subset such that $\frac{1}{\sigma-a(x)}\notin L^1_{d\mu}(\overline{\Omega}_0)$. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of bounded increasing connected set which covers Ω, i.e.

$$\omega_n\subset\omega_{n+1},\quad \bigcup_{n\in\mathbb{N}}\omega_n=\Omega.$$

Without loss of generality, we can also assume that $\Omega_0 \subset \omega_0$ and therefore $\frac{1}{\sigma - a(x)} \notin L^1_{d\mu}(\overline{\omega}_n)$ for all $n \in \mathbb{N}$. Observe that for each ω_n Theorem 3.1 and Lemma 3.2 apply. Therefore for each n there exists a principal eigenpair $(\lambda_{p,n}, \phi_{p,n})$ to the eigenvalue problem (3.1) with ω_n instead of Ω.

By construction, using the monotonicity of the sequence of $(\omega_n)_{n\in\mathbb{N}}$ and the assertion (i) of Proposition 1.1 we deduce that $(\lambda_{p,n})_{n\in\mathbb{N}}$ is a monotone non-increasing sequence which is bounded from below. Thus $\lambda_{p,n}$ converges to some $\bar{\lambda} \geqslant \lambda_p(\mathcal{L}_\Omega + a(x))$. Moreover, we also have that for all $n \in \mathbb{N}$

$$\lambda_p\big(\mathcal{L}_\Omega + a(x)\big) \leqslant \bar{\lambda} \leqslant \lambda_{p,n} < \lambda_{p,0} < -\sup_{\overline{\Omega}} a(x) = \sigma.$$

Let us now fix $x_1 \in \omega_0 \cap \Omega$. Observe that since for each integer n the eigenvalue $\lambda_{p,n}$ is simple we can normalise $\phi_{p,n}$ by $\phi_{p,n}(x_1) = 1$.

Let us now define $b_n(x) := -\lambda_{p,n} - a(x)$. Then $\phi_{p,n}$ satisfies

$$\mathcal{L}_{\omega_n}[\phi_{p,n}] = b_n(x)\phi_{p,n} \quad \text{in } \omega_n. \tag{3.25}$$

By construction for all $n \in \mathbb{N}$ we have $b_n(x) \geqslant -\lambda_{p,0} - \sigma > 0$, therefore the Harnack inequality (Theorem 2.2) applies to $\phi_{p,n}$. Thus for n fixed and for all compact set $\omega' \Subset \omega_n$ there exists a constant $C_n(\omega')$ such that

$$\phi_{p,n}(x) \leqslant C_n\big(\omega'\big)\phi_{p,n}(y) \quad \forall x, y \in \omega'.$$

Moreover, the constant $C_n(\omega')$ only depends on $\bigcup_{x\in\omega} B(x,\beta)$ and is monotone decreasing with respect to $\inf_{x\in\omega_n} b_n(x)$. For all n, the function $b_n(x)$ being uniformly bounded from below by a constant independent of n, the constant C_n is bounded from above independently of n by a constant $C(\omega')$. Thus we have

$$\phi_{p,n}(x) \leqslant C\big(\omega'\big)\phi_{p,n}(y) \quad \forall x, y \in \omega'.$$

From a standard argumentation, using the normalisation $\phi_{p,n}(x_1) = 1$, we deduce that the sequence $(\phi_{p,n})_{n\in\mathbb{N}}$ is bounded in $C_{loc}(\Omega)$ topology. Moreover, from a standard diagonal extraction argument, there exists a subsequence still denoted $(\phi_{p,n})_{n\in\mathbb{N}}$ such that $(\phi_{p,n})_{n\in\mathbb{N}}$ converges locally uniformly to a continuous function ϕ. Furthermore, ϕ is a non-negative non-trivial function and $\phi(x_1) = 1$.

Since J has a compact support we can pass to the limit in Eq. (3.25) using the Lebesgue monotone convergence theorem and get

$$\int_\Omega J\left(\frac{x-y}{g(y)}\right)\phi(y)\,d\mu(y) + \big(\bar{\lambda} + a(x)\big)\phi(x) = 0 \quad \text{in } \Omega.$$

As above using the equation, we deduce that $\phi > 0$ in Ω. Lastly, from the definition of λ_p using $(\bar{\lambda}, \phi)$ as a test function, we see that $\bar{\lambda} \leqslant \lambda_p \leqslant \bar{\lambda}$. Hence, $(\bar{\lambda}, \phi)$ is our desired eigenpair.

Remark 3.9. Note that our proof of the existence of a principal eigenpair in this situation relies only on the Harnack estimate which for some form holds true when the assumption on J and g are relaxed.

Remark 3.10. From the above proofs, using the properties of the principal eigenvalue, we can derive a practical dichotomy for λ_p. Indeed, either $\lambda_p = -\sigma$ or $\lambda_p < -\sigma$ and there exists a principal positive eigenfunction ϕ_p associated to λ_p.

4. Existence of a maximum principle

In this section, we explore the relation between a maximum principle property satisfied by an operator $\mathcal{M}$ and the sign of its principal eigenvalue. Namely, we prove Theorem 1.5 that we recall below.

Theorem 4.1. *Assume that Ω is a bounded set and let J, g and a be as in Theorem* 1.1. *Then the maximum principle is satisfied by $\mathcal{M}_\Omega$ if and only if $\lambda_p(\mathcal{M}_\Omega) \geqslant 0$.*

Proof. Assume first that the operator satisfies the maximum principle. From Theorem 1.1, there exists (λ_p, ϕ_p) such that $\phi_p \in C(\overline{\Omega})$, $\phi_p > 0$ and

$$\mathcal{L}_\Omega[\phi_p] + a(x)\phi_p + \lambda_p\phi_p = 0.$$

As in the previous section, we can normalise ϕ_p so that we have $1 \geqslant \phi_p \geqslant c_0$. Furthermore, there exists $\delta > 0$ so that $-\lambda_p - \sigma \geqslant \delta > 0$ where σ denotes the maximum of a in $\overline{\Omega}$.

Assuming by contradiction that $\lambda_p < 0$ we have

$$\mathcal{L}_\Omega[\phi_p] + a(x)\phi_p = -\lambda_p\phi_p > 0.$$

Let us choose $\omega \Subset \Omega$ such that

$$\int_{\Omega\setminus\omega} d\mu(y) \leqslant \frac{c_0 \inf\{\delta, |\lambda_p|\}}{2\|J\|_\infty}.$$

As in the previous section, we can construct a continuous function η such that $0 \leqslant \eta \leqslant 1$, $\eta(x) = 1$ in ω, $\eta(x) = 0$ in $\partial\Omega$. Consider now $\phi_p\eta$ and let us compute $\mathcal{L}_\Omega[\phi_p\eta] + a(x)\phi_p\eta$. Then we have

$$\begin{aligned}
\mathcal{L}_\Omega[\phi_p\eta] + a(x)\phi_p\eta &\geqslant -\lambda_p\phi_p - \|J\| \int_{\Omega\setminus\omega} d\mu(y) - a(x)\phi_p(1-\eta) \\
&\geqslant -\lambda_p\phi_p - \frac{c_0 \inf\{\delta, |\lambda_p|\}}{2} - a(x)\phi_p(1-\eta) \\
&\geqslant -\lambda_p\phi_p - \frac{c_0 \inf\{\delta, |\lambda_p|\}}{2} - \max\{\sigma, 0\}\phi_p \\
&\geqslant -\big(\lambda_p + \max\{\sigma, 0\}\big)\phi_p - \frac{c_0 \inf\{\delta, |\lambda_p|\}}{2}.
\end{aligned}$$

Since by assumption $-\lambda_p > 0$ and $-\lambda_p - \sigma \geqslant 0$ it follows from the above inequality that

$$\begin{aligned}
\mathcal{L}_\Omega[\phi_p\eta] + a(x)\phi_p\eta &\geqslant -\big(\lambda_p + \max\{\sigma, 0\}\big)c_0 - \frac{c_0 \inf\{\delta, |\lambda_p|\}}{2} \\
&\geqslant \frac{c_0 \inf\{\delta, |\lambda_p|\}}{2} \geqslant 0.
\end{aligned}$$

By construction we have $\phi_p\eta \in C(\Omega)$ that satisfies

$$\mathcal{L}_\Omega[\phi_p\eta] + a(x)\phi_p\eta \geqslant 0 \quad \text{in } \Omega,$$

$$\phi_p\eta = 0 \quad \text{on } \partial\Omega.$$

Therefore, by the maximum principle 1.4, $\phi_p \eta \leqslant 0$ in Ω which is a contradiction. Hence, $\lambda_p \geqslant 0$.

Let us now show the converse implication. Assume that $\lambda_p(\mathcal{L}_\Omega + a(x)) \geqslant 0$, then we will show that the operator satisfies the maximum principle. Let $u \not\equiv 0$, $u \in C(\overline{\Omega})$ such that $u \geqslant 0$ on $\partial\Omega$ and

$$\mathcal{L}_\Omega[u] + a(x)u \leqslant 0.$$

Let us show that $u > 0$ in Ω.

By Theorem 1.1, there exists $\phi_p > 0$ such that

$$\mathcal{L}_\Omega[\phi_p] + a(x)\phi_p = -\lambda_p \phi_p \leqslant 0.$$

Let us rewrite $\mathcal{L}_\Omega[u] + a(x)u$ as follows

$$\begin{aligned}
\mathcal{L}_\Omega[u] + a(x)u &= \int_\Omega J\left[\frac{x-y}{g(y)}\right]\frac{\phi_p(y)}{g(y)}\frac{u(y)}{\phi_p(y)}\,dy + a(x)\phi_p(x)\frac{u(x)}{\phi_p(x)} \\
&= \int_\Omega J\left[\frac{x-y}{g(y)}\right]\frac{\phi_p(y)}{g^n(y)}\left(\frac{u(y)}{\phi_p(y)} - \frac{u(x)}{\phi_p(x)}\right)dy - \lambda_p\phi_p\frac{u(x)}{\phi_p(x)}.
\end{aligned}$$

Let us set $w := \frac{u}{\phi_p}$, then we have the following inequality in Ω

$$\int_\Omega J\left[\frac{x-y}{g(y)}\right]\frac{\phi_p(y)}{g^n(y)}\big(w(y) - w(x)\big)\,dy - \lambda_p\phi_p w(x) \leqslant 0.$$

From the above inequality we deduce that w cannot achieve a non-positive minimum in Ω without being constant. Therefore it follows that either $w > 0$ in Ω or $w \equiv 0$. Since $u \not\equiv 0$, we have $w > 0$. Hence, $\frac{u}{\phi_p} > 0$ which implies that $u > 0$. □

Remark 4.2. From the proof, we can observe that to show the implication

$$\text{“}\lambda_p\big(\mathcal{L}_\Omega + a(x)\big) \geqslant 0 \quad\Longrightarrow\quad \mathcal{L}_\Omega + a(x) \text{ satisfies the maximum principle”}$$

we do not need the existence of a principal eigenfunction ϕ_p when $\lambda_p(\mathcal{L}_\Omega + a(x)) > 0$. Indeed, in this situation we can replace in our argumentation the principal eigenfunction ϕ_p by a well-chosen positive function ψ, i.e. $\psi > 0$ such that there exists $0 < \lambda \leqslant \lambda_p$ satisfying $\mathcal{L}_\Omega[\psi] + (a(x) + \lambda)\psi \leqslant 0$ which is always possible since $\lambda_p(\mathcal{L}_\Omega + a(x)) > 0$.

5. A counterexample

In this section, we provide an example of nonlocal equation where no positive bounded eigenfunction exists. Let Ω be a bounded domain and let us consider the following principal eigenvalue problem:

$$\rho\int_\Omega u\,dx + a(x)u = \lambda u, \tag{5.1}$$

where $\sigma = a(x_0) = \max_{\overline{\Omega}} a(x)$, ρ is a positive constant and $a(x) \in C^0(\overline{\Omega})$ satisfies the condition $\frac{1}{\sigma - a(x)} \in L^1_{loc}(\Omega)$. For this eigenvalue problem, we show the following result:

Theorem 5.1. *If ρ is so that $\rho\int_\Omega \frac{dx}{\sigma-a(x)}<1$, then there exists no bounded continuous positive principal eigenfunction ϕ to* (5.1)*.*

Proof. We argue by contradiction. Let us assume that there exists a bounded positive continuous eigenfunction ϕ associated with λ_p that we normalise by $\int_\Omega \phi(x)\,dx=1$. By substituting ϕ into Eq. (5.1) it follows that

$$\rho=\big(\lambda_p-a(x)\big)\phi.$$

Since $\rho>0$, from the above equation we conclude that $\lambda_p-\sigma\geqslant\tau>0$. Therefore

$$\phi=\frac{\rho}{\lambda_p-a(x)}.$$

Next, using the normalisation we obtain

$$1=\rho\int\limits_\Omega \frac{dx}{\lambda_p-a(x)}.$$

By construction $\lambda_p\geqslant\sigma$, therefore we have

$$1=\rho\int\limits_\Omega \frac{dx}{\lambda_p-a(x)}\leqslant\rho\int\limits_\Omega \frac{dx}{\sigma-a(x)}.$$

Since $\rho\int_\Omega \frac{dx}{\sigma-a(x)}<1$ we end up with the following contradiction

$$1=\rho\int\limits_\Omega \frac{dx}{\lambda_p-a(x)}\leqslant\rho\int\limits_\Omega \frac{dx}{\sigma-a(x)}<1.$$

Hence there exists no positive bounded eigenfunction ϕ associated to λ_p. $\square$

6. Existence/non-existence of solution of (1.6)

In this section we prove Theorem 1.6. That is to say, we investigate the existence/non-existence of solution of the following problem:

$$\mathcal{M}_\Omega[u]+f(x,u)=0\quad\text{in }\Omega \tag{6.1}$$

where f is of KPP type. We show that the existence of a non-trivial solution of (1.6) is governed by the sign of the principal eigenvalue of the following operator $\mathcal{M}_\Omega+f_u(x,0)$. Moreover, when a non-trivial solution exists, then it is unique.

To show the existence/non-existence of solutions of (1.6) and their properties, we follow and adapt the arguments developed in [6,7,20].

6.1. Existence of a non-trivial solution

Let us assume that

$$\lambda_p\big(\mathcal{M}_\Omega + f_u(x,0)\big) < 0.$$

Then we will show that there exists a non-trivial solution to (1.6).

Before going to the construction of a non-trivial solution, let us first define some quantities. First let us denote $a(x) := f_u(x,0) - b(x)$ and $\sigma := \sup_\Omega a(x)$. Observe that with this notation, we have $\lambda_p(\mathcal{M}_\Omega + f_u(x,0)) = \lambda_p(\mathcal{L}_\Omega + a(x))$.

From the definition of σ there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \Omega$ and $|\sigma - a(x_n)| \leqslant \frac{1}{n}$.

Then by continuity of $a(x)$, for each n there exists η_n such that for all $x \in B_{\eta_n}(x_n)$ we have $|\sigma - a(x)| \leqslant \frac{2}{n}$.

Now let us consider a sequence of real numbers $(\epsilon_n)_{n\in\mathbb{N}}$ which converges to zero such that $\epsilon_n \leqslant \frac{\eta_n}{2}$.

Next, let $(\chi_n)_{n\in\mathbb{N}}$ be the following sequence of cut-off functions: $\chi_n(x) := \chi(\frac{\|x-x_n\|}{\epsilon_n})$ where χ is a smooth function such that $0 \leqslant \chi \leqslant 1$, $\chi(x) = 0$ for $|x| \geqslant 2$ and $\chi(x) = 1$ for $|x| \leqslant 1$.

Finally, let us consider the following sequence of continuous functions $(a_n)_{n\in\mathbb{N}}$, defined by $a_n(x) := \sup\{a(x), \sigma\chi_n\}$. Observe that by construction the sequence $(a_n)_{n\in\mathbb{N}}$ is such that $\|a(x) - a_n(x)\|_\infty \to 0$.

Let us now proceed to the construction of a non-trivial solution.

By construction, for each n, the function a_n satisfies $\sup_\Omega a_n = \sigma$ and $a_n \equiv \sigma$ in $B_{\frac{\epsilon_n}{2}}(x_n)$. Therefore, the sequence a_n satisfies $\frac{1}{\sigma - a_n} \notin L^1_{loc}(\Omega)$ and by Theorem 1.1 there exists a principal eigenpair (λ_p^n, ϕ_n) solution of the eigenvalue problem:

$$\mathcal{L}_\Omega[\phi] + a_n(x)\phi + \lambda\phi = 0,$$

such that $\phi_n \in L^\infty(\Omega) \cap C(\Omega)$.

Next, using that $\|a_n(x) - a(x)\|_\infty \to 0$ as $n \to \infty$, from (iii) of Proposition 1.1 it follows that for n big enough, say $n \geqslant n_0$, we have

$$\lambda_p^n < \frac{\lambda_p(\mathcal{L}_\Omega + a(x))}{2} < 0.$$

Moreover, by choosing n_0 bigger if necessary, we achieve for $n \geqslant n_0$

$$\lambda_p^n + \big\|a_n(x) - a(x)\big\|_\infty \leqslant \frac{\lambda_p(\mathcal{L}_\Omega + a(x))}{4}.$$

Let us now compute $\mathcal{M}_\Omega[\epsilon\phi_n] + f(x,\epsilon\phi_n)$. For $n \geqslant n_0$, we have

$$\begin{aligned}
\mathcal{M}_\Omega[\epsilon\phi_n] + f(x,\epsilon\phi_n) &= f(x,\epsilon\phi_n) - \big(b(x) + a_n(x)\big)\epsilon\phi_n - \epsilon\lambda_p^n\phi_n \\
&= \big(f_u(x,0) - \big(a_n(x) + b(x)\big)\big)\epsilon\phi_n - \epsilon\lambda_p^n\phi_n + o(\epsilon\phi_n) \\
&\geqslant \big(-\big\|a(x) - a_n(x)\big\|_\infty - \lambda_p^n\big)\epsilon\phi_n + o(\epsilon\phi_n) \\
&\geqslant -\frac{\lambda_p(\mathcal{M}_\Omega + f_u(x,0))}{4}\epsilon\phi_n + o(\epsilon\phi_n) > 0.
\end{aligned}$$

Therefore, for $\epsilon > 0$ sufficiently small and n big enough, $\epsilon\phi_n$ is a sub-solution of (1.6). By definition of f, any large enough constant M is a super-solution of (1.6). By choosing M so large that $\epsilon\phi_n \leqslant M$ and using a basic iterative scheme we obtain the existence of a positive non-trivial solution u of (1.6).

6.2. Non-existence of positive bounded solutions

Let now turn our attention to the non-existence result. Let us prove that when $\lambda_p(\mathcal{M}_\Omega + f_u(x,0)) \geqslant 0$ then there exists no non-trivial solution to (1.6).

Assume by contradiction that $\lambda_p(\mathcal{M}_\Omega + f_u(x,0)) \geqslant 0$ and there exists a positive bounded solution u to Eq. (1.6).

Obviously, since u is non-negative and bounded, using (1.6) we have for all $x \in \overline{\Omega}$

$$0 \leqslant \mathcal{L}_\Omega[u] = \left(b(x) - \frac{f(x,u)}{u}\right)u. \tag{6.2}$$

Let us denote $h(x) := \mathcal{L}_\Omega[u]$. By construction, h is a non-negative continuous function in $\overline{\Omega}$. Therefore, since $\overline{\Omega}$ is compact, h achieves at some point $x_0 \in \overline{\Omega}$ a non-negative minimum. A short argument shows that $h(x_0) > 0$. Indeed, otherwise we have

$$\int_\Omega J\left(\frac{x_0 - y}{g(y)}\right)\frac{u(y)}{g^n(y)}\,dy = 0.$$

Thus, since J, g and u are non-negative quantities, from the above equality we deduce that $u(y) = 0$ for almost every $y \in \{z \in \overline{\Omega} \mid \frac{x_0 - z}{g(z)} \in \operatorname{supp}(J)\}$. By iterating this argument and using the assumption $J(0) > 0$, we can show that $u(y) = 0$ for almost every $y \in \overline{\Omega}$, which implies that $u \equiv 0$ since u is continuous.

As a consequence $\inf_{x \in \Omega}(b(x) - \frac{f(x,u)}{u}) \geqslant \delta$ for some $\delta > 0$ and there exists a positive constant c_0 so that $u > c_0$ in $\overline{\Omega}$. From the monotone properties of $f(x,.)$, we deduce that $\frac{f(x,u)}{u} \leqslant \frac{f(x,c_0)}{c_0} < f_u(x,0)$. Let us now denote $\gamma(x) = \frac{f(x,c_0)}{c_0} - b(x)$. By construction, we have $\gamma(x) < a(x)$ and therefore by (ii) of Proposition 1.1,

$$\lambda_p\big(\mathcal{L}_\Omega + \gamma(x)\big) > \lambda_p\big(\mathcal{L}_\Omega + a(x)\big) \geqslant 0.$$

Moreover, since u is a solution of (1.6), we have

$$\mathcal{L}_\Omega[u] + \gamma(x)u \geqslant \mathcal{M}_\Omega[u] + f(x,u) = 0.$$

By definition of $\lambda_p(\mathcal{L}_\Omega + \gamma(x))$, for all positive $\lambda_p(\mathcal{L}_\Omega + a(x)) < \lambda < \lambda_p(\mathcal{L}_\Omega + \gamma(x))$ there exists a positive continuous function ϕ_λ such that

$$\mathcal{L}_\Omega[\phi_\lambda] + \gamma(x)\phi_\lambda \leqslant -\lambda\phi_\lambda \leqslant 0.$$

Arguing as above, we can see that $\phi_\lambda \geqslant \delta$ for some positive δ. Let us define the following quantity

$$\tau^* := \inf\{\tau > 0 \mid u \leqslant \tau\phi_\lambda\}.$$

Obviously, we end the proof of the theorem by proving that $\tau^* = 0$. Assume that $\tau^* > 0$. Then by definition of τ^*, there exists $x_0 \in \overline{\Omega}$ such that $\tau^*\phi_p(x_0) = u(x_0) > 0$. At this point x_0, we have

$$0 \leqslant \mathcal{L}_\Omega[w](x_0) = \mathcal{L}_\Omega\big[\big(\tau^*\phi_\lambda - u\big)\big](x_0) \leqslant 0.$$

Therefore, since $w \geqslant 0$, using a similar argumentation as above, we have $w(y) = 0$ for almost every $y \in \overline{\Omega}$. Thus, we end up with $\tau^*\phi_1 \equiv u$ and we get the following contradiction,

$$0 \leqslant \mathcal{L}_\Omega[u] + \gamma(x)u = \mathcal{L}_\Omega\big[\tau^*\phi_\lambda\big] + \gamma(x)\tau^*\phi_\lambda < 0.$$

Hence $\tau^* = 0$.

6.3. Uniqueness of the solution

Lastly, we show that when a solution of (1.6) exists then it is unique. The proof of the uniqueness of the solution is obtained as follows.

Let u and v be two non-negative bounded solutions of (1.6). Arguing as in the above subsection, we see that there exist two positive constants c_0 and c_1 such that

$$u \geqslant c_0 \quad \text{in } \overline{\Omega},$$
$$v \geqslant c_1 \quad \text{in } \overline{\Omega}.$$

Since u and v are bounded and strictly positive, the following quantity is well defined

$$\gamma^* := \inf\{\gamma > 0 \mid \gamma u \geqslant v\}.$$

We claim that $\gamma^* \leqslant 1$. Indeed, assume by contradiction that $\gamma^* > 1$. From (1.6) we see that

$$\mathcal{M}_\Omega\big[\gamma^*u\big] + f\big(x, \gamma^*u\big) = f\big(x, \gamma^*u\big) - \gamma^* f(x,u) \tag{6.3}$$
$$= \gamma^*u\left(\frac{f(x,\gamma^*u)}{\gamma^*u} - \frac{f(x,u)}{u}\right) \leqslant 0. \tag{6.4}$$

Now, by definition of γ^*, there exists $x_0 \in \overline{\Omega}$ so that $\gamma u(x_0) = v(x_0)$ and from (1.6) we can easily see that

$$\mathcal{M}_\Omega\big[\gamma^*u\big](x_0) + f\big(x, \gamma^*u(x_0)\big) = \mathcal{L}_\Omega\big[\gamma^*u - v\big] \geqslant 0. \tag{6.5}$$

From (6.4) and (6.5) we deduce that

$$\mathcal{L}_\Omega\big[\gamma^*u - v\big](x_0) = 0.$$

Therefore, arguing as in the above subsection it follows that $\gamma^*u = v$. Using now (6.4), we deduce that

$$0 = \mathcal{M}_\Omega[v] + f(x,v) = \mathcal{M}_\Omega\big[\gamma^*u\big] + f\big(x,\gamma^*u\big) = \gamma^*u\left(\frac{f(x,\gamma^*u)}{\gamma^*u} - \frac{f(x,u)}{u}\right) \leqslant 0,$$

which implies that for all $x \in \Omega$ $f(x,\gamma^*u) \equiv f(x,u)$. This later is impossible since $\gamma^* > 1$. Hence, $\gamma^* \leqslant 1$ and as a consequence $u \geqslant v$.

Observe that the role of u and v can be interchanged in the above argumentation. So we also have $v \geqslant u$, which shows the uniqueness of the solution.

7. Asymptotic behaviour of the solution of (1.8)

Lastly, in this section, we prove Theorem 1.7 which establishes the asymptotic behaviour of the solution of

$$\frac{\partial u}{\partial t} = \mathcal{M}_{\Omega}[u] + f(x,u) \quad \text{in } \mathbb{R}^{+} \times \Omega,$$
$$u(0,x) = u_0(x) \quad \text{in } \Omega.$$

Proof of Theorem 1.7. The existence of a solution defined for all time t follows from a standard argument and will not be exposed. Moreover, since $u_0 \geqslant 0$ and $u_0 \not\equiv 0$, using the parabolic maximum principle, there exists a positive constant δ such that $u(1,x) > \delta$ in $\overline{\Omega}$. Let us first assume that $\lambda_p < 0$. By following the argument developed in above section, we can construct a bounded continuous function ψ so that $\epsilon\psi$ is a sub-solution of (1.8) for ϵ small enough. Since, $u(1,x) \geqslant \delta$ and ψ is bounded, by choosing ϵ smaller if necessary we achieve also that $\epsilon\psi \leqslant u(1,x)$. Now, let us denote by $\underline{\Psi}(x,t)$ the solution of evolution problem (1.8) with initial datum $\epsilon\psi$. By construction, using a standard argument, $\underline{\Psi}(t,x)$ is a non-decreasing function of the time and $\Psi(t,x) \leqslant u(t+1,x)$. On the other hand, since for M big enough M is a super-solution of (1.8) and u_0 is bounded, we have also $u(t,x) \leqslant \overline{\Psi}(t,x)$, where $\overline{\Psi}(x,t)$ denotes the solution of evolution problem (1.8) with initial datum $\overline{\Psi}(0,x) = M \geqslant u_0$. A standard argument using the parabolic comparison principle shows that $\overline{\Psi}$ is a non-increasing function of t. Thus we have for all time t

$$\epsilon\psi \leqslant \underline{\Psi}(t,x) \leqslant u(t+1,x) \leqslant \overline{\Psi}(t+1,x).$$

Since $\underline{\Psi}(t,x)$ (resp. $\overline{\Psi}(t,x)$) is a uniformly bounded monotonic function of t, $\underline{\Psi}$ (resp. $\overline{\Psi}$) converges pointwise to $\underline{p}$ (resp. $\overline{p}$) which is a solution of (1.6). From $\underline{\Psi}(t,x) \not\equiv 0$, using the uniqueness of a non-trivial solution (Theorem 1.6), we deduce that $\underline{p} \equiv \overline{p} \not\equiv 0$ and therefore, $u(x,t) \to p$ pointwise in Ω, where p denotes the unique non-trivial solution of (1.6).

In the other case, when $\lambda_p \geqslant 0$ we argue as follows. As above, we have $0 \leqslant u(t,x) \leqslant \overline{\Psi}(t,x)$ and $\overline{\Psi}$ converges pointwise to $\overline{p}$ a solution of (1.6). By Theorem 1.6 in this situation we have $\overline{p} \equiv 0$, hence $u(x,t) \to 0$ pointwise in Ω. $\square$

Remark 7.1. Note that the above analyse will hold for more general kernel non-negative kernel $k(x,y)$ that satisfies $(\tilde{H}2)$, i.e.

$$\exists c_0 > 0,\ \epsilon_0 > 0 \quad \text{such that} \quad \min_{x\in\Omega}\Big(\min_{y\in B(x,\epsilon_0)} k(x,y)\Big) > c_0.$$

Appendix A

In this appendix, we first prove Proposition 1.1. Then we recall the method of sub- and super-solution to obtain solution of the semilinear problem:

$$\mathcal{M}_{\Omega}[u] = f(x,u) \quad \text{in } \Omega. \tag{A.1}$$

Before going to the proof of Proposition 1.1, let us show that $\lambda_p(\mathcal{L}_{\Omega} + a(x))$ is well defined. Let us first show that the set $\Lambda := \{\lambda \mid \exists\phi \in C(\Omega),\ \phi > 0 \text{ such that } \mathcal{L}_{\Omega}[\phi] + \lambda\phi \leqslant 0\}$ is non-empty. Indeed, as observed in [18] (Theorem 1.8), for Ω, J, g and a satisfying the assumptions $(H1)$–$(H4)$ there exists a continuous positive function ψ satisfying

$$\int_{\Omega} J\left(\frac{x-y}{g(y)}\right)\frac{\psi(y)}{g^n(y)}\,dy = c(x)\psi(x),$$

where $c(x)$ is defined by

$$c(x) := \begin{cases} 1 & \text{if } x \in \{x \in \overline{\Omega} \mid g(x) = 0\}, \\ \int_\Omega J(\frac{y-x}{g(x)}) \frac{dy}{g^n(x)} & \text{otherwise.} \end{cases}$$

Obviously $c(x) \in L^\infty$ and for any $\lambda \leqslant (|a\|_\infty + \|c\|_\infty)$ we have

$$\begin{aligned} \mathcal{L}_\Omega[\psi] + \big(a(x) + \lambda\big)\psi &= \big(a(x) + c(x) + \lambda\big)\psi \\ &\leqslant \big(a(x) + c(x) - \|a\|_\infty - \|c\|_\infty\big)\psi \leqslant 0. \end{aligned}$$

Therefore, the set Λ is non-empty.

Observe now that since J, g are non-negative functions and $a(x) \in L^\infty$, for any continuous positive function ϕ we have

$$\mathcal{L}_\Omega[\phi] + \big(a(x) + \big\|a(x)\big\|_\infty\big)\phi \geqslant 0.$$

Therefore, the set Λ has an upper bound and λ_p is well defined.

Let us now prove Proposition 1.1.

Proof of Proposition 1.1. (i) easily follows from the definition of λ_p. First, let us observe that to obtain

$$\lambda_p\big(\mathcal{L}_{\Omega_2} + a(x)\big) \leqslant \lambda_p\big(\mathcal{L}_{\Omega_1} + a(x)\big)$$

it is sufficient to prove the inequality

$$\lambda \leqslant \lambda_p\big(\mathcal{L}_{\Omega_1} + a(x)\big)$$

for any $\lambda < \lambda_p(\mathcal{L}_{\Omega_2} + a(x))$.

Let us fix $\lambda < \lambda_p(\mathcal{L}_{\Omega_2} + a(x))$. Then by definition of $\lambda_p(\mathcal{L}_{\Omega_2} + a(x))$ there exists a positive function $\phi \in C(\Omega_2)$ such that

$$\mathcal{L}_{\Omega_2}[\phi] + \big(a(x) + \lambda\big)\phi \leqslant 0.$$

Since $\Omega_1 \subset \Omega_2$, an easy computation shows that

$$\mathcal{L}_{\Omega_1}[\phi] + \big(a(x) + \lambda\big)\phi \leqslant \mathcal{L}_{\Omega_2}[\phi] + \big(a(x) + \lambda\big)\phi \leqslant 0.$$

Therefore, by definition of $\lambda_p(\mathcal{L}_{\Omega_1} + a(x))$ we have $\lambda \leqslant \lambda_p(\mathcal{L}_{\Omega_1} + a(x))$. Hence, $\lambda_p(\mathcal{L}_{\Omega_2} + a(x)) \leqslant \lambda_p(\mathcal{L}_{\Omega_1} + a(x))$.

To show (ii), we argue as above. By definition of $\lambda_p(\mathcal{L}_\Omega + a_1(x))$ for any $\lambda < \lambda_p(\mathcal{L}_\Omega + a_1(x))$ there exists a positive $\phi \in C(\Omega)$ such that

$$\mathcal{L}_\Omega[\phi] + \big(a_1(x) + \lambda\big)\phi \leqslant 0$$

and we have

$$\mathcal{L}_\Omega[\phi] + \big(a_2(x) + \lambda\big)\phi \leqslant \mathcal{L}_\Omega[\phi] + \big(a_1(x) + \lambda\big)\phi \leqslant 0.$$

Therefore $\lambda \leqslant \lambda_p(\mathcal{L}_\Omega + a_2(x))$. Hence (ii) holds true.

Let us now prove (iii). Again we fix $\lambda < \lambda_p(\mathcal{L}_\Omega + a(x))$. For this λ, there exists $\phi \in C(\Omega)$, $\phi > 0$ such that

$$\mathcal{L}_\Omega[\phi] + \big(a(x) + \lambda\big)\phi \leqslant 0. \tag{A.2}$$

An easy computation shows that we rewrite the above equation as follows

$$\begin{aligned}\mathcal{L}_\Omega[\phi] + \big(a(x) + \lambda\big)\phi &= \mathcal{L}_\Omega[\phi] + \big(b(x) + \lambda\big)\phi + \big(a(x) - b(x)\big)\phi \\ &\geqslant \mathcal{L}_\Omega[\phi] + \big(b(x) + \lambda - \big\|a(x) - b(x)\big\|_\infty\big)\phi.\end{aligned}$$

Using that (λ, ϕ) satisfies (A.2), it follows that

$$\mathcal{L}_\Omega[\phi] + \big(b(x) + \lambda - \big\|a(x) - b(x)\big\|_\infty\big)\phi \leqslant 0.$$

Therefore, $\lambda - \|a(x) - b(x)\|_\infty \leqslant \lambda_p(\mathcal{L}_\Omega + b(x))$ and we have

$$\lambda \leqslant \lambda_p\big(\mathcal{L}_\Omega + b(x)\big) + \big\|a(x) - b(x)\big\|_\infty.$$

The above computation being valid for any $\lambda < \lambda_p(\mathcal{L}_\Omega + a(x))$, we end up with

$$\lambda_p\big(\mathcal{L}_\Omega + a(x)\big) - \lambda_p\big(\mathcal{L}_\Omega + b(x)\big) \leqslant \big\|a(x) - b(x)\big\|_\infty.$$

Note that the role of $a(x)$ and $b(x)$ can be interchanged in the above argumentation. So, we also have

$$\lambda_p\big(\mathcal{L}_\Omega + b(x)\big) - \lambda_p\big(\mathcal{L}_\Omega + a(x)\big) \leqslant \big\|a(x) - b(x)\big\|_\infty.$$

Hence

$$\big|\lambda_p\big(\mathcal{L}_\Omega + a(x)\big) - \lambda_p\big(\mathcal{L}_\Omega + b(x)\big)\big| \leqslant \big\|a(x) - b(x)\big\|_\infty,$$

which proves (iii).

The proof of (iv) being similar to the proof of (ii), it will be omitted. □

Before recalling the sub/super-solution method, let us introduce some definitions and notations. We call a bounded continuous function $\bar{u}$ (resp. $\underline{u}$) a super-solution (resp. a sub-solution) if $\bar{u}$ (resp. $\underline{u}$) satisfies the following inequalities:

$$\mathcal{M}_\Omega[u] \leqslant (\geqslant)\, f(x, u) \quad \text{in } \Omega. \tag{A.3}$$

Let us now state the theorem.

Theorem A.1. *Assume $f(x, .)$ is a Lipschitz function uniformly in x and let $\bar{u}$ and $\underline{u}$ be respectively a super-solution and a sub-solution of* (A.1) *continuous up to the boundary. Assume further that $\underline{u} \leqslant \bar{u}$. Then there exists a solution $u \in C(\overline{\Omega})$ solution of* (A.1) *satisfying $\underline{u} \leqslant u \leqslant \bar{u}$.*

Proof. Let us first choose $k > |\lambda_p(\mathcal{M}_\Omega)|$ big enough such that the function $-ks + f(x,s)$ is a decreasing function of s uniformly in x. We can increase further k if necessary to ensure that $k \in \rho(\mathcal{M}_\Omega)$, where $\rho(\mathcal{M}_\Omega)$ denotes the resolvent of the operator $\mathcal{M}_\Omega$.

Note that by this choice of k, by Theorem 1.5 the operator $\mathcal{M}_\Omega - k$ satisfies a comparison principle.

Now, let u_1 be the solution of the following linear problem

$$\mathcal{M}_\Omega[u_1] - ku_1 = -k\underline{u} + f(x,\underline{u}) \quad \text{in } \Omega. \tag{A.4}$$

u_1 always exists, since by construction the continuous operator $\mathcal{M}_\Omega - k$ is invertible. We claim that $\underline{u} \leqslant u_1 \leqslant \bar{u}$. Indeed, since $\underline{u}$ and $\bar{u}$ are respectively a sub- and super-solution of (A.1), we have

$$\mathcal{M}_\Omega[u_1 - \underline{u}] - k(u_1 - \underline{u}) \leqslant 0 \quad \text{in } \Omega,$$

$$\mathcal{M}_\Omega[u_1 - \bar{u}] - k(u_1 - \bar{u}) \geqslant -k(\underline{u} - \bar{u}) + f(x,\underline{u}) - f(x,\bar{u}) \geqslant 0 \quad \text{in } \Omega.$$

So, the inequality $\underline{u} \leqslant u_1 \leqslant \bar{u}$ follows from the comparison principle satisfied by the operator $\mathcal{M}_\Omega - k$. Now let u_2 be the solution of (A.4) with u_1 instead of $\underline{u}$. From the monotonicity of $-ks + f(x,s)$ and using the comparison principle, we have $\underline{u} \leqslant u_1 \leqslant u_2 \leqslant \bar{u}$. By induction, we can construct an increasing sequence of function $(u_n)_{n\in\mathbb{N}}$ satisfying $\underline{u} \leqslant u_n \leqslant \bar{u}$ and

$$\mathcal{M}_\Omega[u_{n+1}] - ku_{n+1} = -ku_n + f(x,u_n) \quad \text{in } \Omega. \tag{A.5}$$

Since the sequence is increasing and bounded, $u^-(x) := \sup_{n\in\mathbb{N}} u_n(x)$ is well defined. Moreover, passing to the limit in Eq. (A.5) using Lebesgue's theorem it follows that u^- is a solution of (A.1). $\square$

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