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# Regularity, monotonicity and symmetry of positive solutions of $m$-Laplace equations ☆

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**Abstract**

We consider the Dirichlet problem for positive solutions of the equation $-\Delta_m(u) = f(u)$ in a bounded smooth domain $\Omega$, with $f$ locally Lipschitz continuous, and prove some regularity results for weak $C^1(\bar{\Omega})$ solutions. In particular when $f(s)>0$ for $s>0$ we prove summability properties of $\frac{1}{|Du|}$, and Sobolev's and Poincaré type inequalities in weighted Sobolev spaces with weight $|Du|^{m-2}$. The point of view of considering $|Du|^{m-2}$ as a weight is particularly useful when studying qualitative properties of a fixed solution. In particular, exploiting these new regularity results we can prove a weak comparison principle for the solutions and, using the well known Alexandrov–Serrin moving plane method, we then prove a general monotonicity (and symmetry) theorem for positive solutions $u$ of the Dirichlet problem in bounded (and symmetric in one direction) domains when $f(s)>0$ for $s>0$ and $m>2$. Previously, results of this type in general bounded (and symmetric) domains had been proved only in the case $1<m<2$.


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## 1. Introduction and statement of the results

Let us consider weak $C^1(\bar{\Omega})$ solutions of the problem

$$\begin{cases} -\Delta_m(u) = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $\Delta_m(u) = \operatorname{div}(|Du|^{m-2}Du)$ is the $m$-Laplace operator, $1 < m < \infty$, and we have the following hypotheses on $f$:

(*) $f : [0, \infty) \to \mathbb{R}$ is a continuous function which is locally Lipschitz continuous in $(0, \infty)$.

It is well known that, since the $m$-Laplace operator is singular or degenerate elliptic (respectively if $1 < m < 2$ or $m > 2$), solutions of (1.1) belong generally to the class $C^{1,\tau}$ with $\tau < 1$, and solve (1.1) only in the weak sense. Moreover, there are no general comparison theorems for the solutions as in the case when $m = 2$ or more generally when uniformly elliptic operators are considered.

In this paper we prove some regularity properties of positive solutions of (1.1), such as summability properties of $\frac{1}{|Du|}$, where $Du$ is the gradient of $u$, and Sobolev and Poincaré type inequalities in weighted Sobolev spaces with weight $|Du|^{m-2}$.

Using these regularity results we prove a weak comparison theorem for solutions of differential inequalities involving the $m$-Laplace operator. Exploiting all these results, together with the Alexandrov–Serrin moving plane method [21], we finally prove that the solutions of (1.1) in one direction in domains which are convex (and symmetric) in one direction. Since the case $1 < m < 2$ has been fully considered in [7,8], this will conclude the analysis for the case of positive Lipschitz continuous nonlinearities $f(u)$. We also observe that if $m > 2$ and $f$ changes sign there are counterexamples to the symmetry of the solutions in symmetric domains (see [4,14]). Let us explain our results in details.

In Section 2 we study the linearized operator $L_u$ (see Section 2 for the precise statement) associated to problem (1.1). In particular, we first prove that if $\varphi \in W^{1,2}(\Omega)$ has compact support then

$$L_u(u_{x_i}, \varphi) \equiv \int_{\Omega \setminus Z} [|Du|^{m-2}(Du_{x_i}, D\varphi) + (m-2)|Du|^{m-4}(Du, Du_{x_i})(Du, D\varphi) - f'(u)u_{x_i}\varphi]\, dx$$

is well defined and the following equation holds:

$$L_u(u_{x_i}, \varphi) = 0 \quad \forall \varphi \in W^{1,2}(\Omega),\ \operatorname{supp}(\varphi) \subset \Omega. \tag{1.2}$$

The proofs of our regularity results will be based both on Eqs. (1.1) and (1.2). Let us state some of these results in the following:

**Theorem 1.1.** *Let $u \in C^1(\Omega)$ be a weak solution of* (1.1) *with $f$ satisfying* $(*)$, $1 < m < \infty$. *Then, for any $E \subset\subset \Omega$ and for every $i, j = 1, \dots, N$, we have, for every $x \in \Omega$,*

$$\int_{E \setminus \{u_{x_i} = 0\}} \frac{|Du|^{m-2}}{|u_{x_i}|^{\beta} |x-y|^{\gamma}} |Du_{x_i}|^2 \, dy < C,$$

*where $\beta < 1$, $\gamma < N - 2$ if $N \geqslant 3$, $\gamma = 0$ if $N = 2$ and $C$ depends on $\gamma$, $\beta$, $E$ and on the solution $u$, but does not depend on $x \in \Omega$. Moreover*

$$\int_{E \setminus Z} \frac{|Du|^{m-2-\beta}}{|x-y|^{\gamma}} ||D^2 u||^2 \, dy < C,$$

*where $Z = \{x \in \Omega : Du(x) = 0\}$ is the critical set of the solution.*

*Finally, if $\Omega$ is smooth, $u \in C^1(\bar{\Omega})$ and $f(s) > 0$ for $s > 0$, then $|Z| = 0$ (see* [18]*) and, for any $x \in \Omega$ and for every $r < 1$, we have*

$$\int_{\Omega} \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^{\gamma}} \, dy \leqslant C,$$

*where $C$ does not depend on $x$, $\gamma < N - 2$ if $N \geqslant 3$ and $\gamma = 0$ if $N = 2$.*

As a corollary we also prove that $|Du|^{m-2} Du \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and the derivatives $u_{x_i}$ belong to the weighted Sobolev space $H^{1,2}_{\rho}(\Omega)$.

Let us remark that in a recent paper Lou [18] proved that, if $u \in W^{1,m}_{\mathrm{loc}}(\Omega)$ is a weak solution of the equation

$$-\mathrm{div}(|Du|^{m-2} Du) = f(x) \quad \text{in} \quad \Omega \tag{1.3}$$

with $f \in L^q(\Omega)$, $q > \frac{N}{m}$, $q \geqslant 2$, then $|Du|^{m-1} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $f(x) = 0$ a.e on the critical set $Z = \{x \in \Omega : Du(x) = 0\}$ of the solution, so that $|Z| = 0$ if $f(x) \neq 0$ a.e in $\Omega$.

The lack of regularity of the solutions of (1.1) is one of the greatest difficulty in the applications. In [1] the case when $\Omega$ is a ball is considered. In this case the solutions are radial (see [5,7]) and the authors study the Morse index of a fixed solution in the weighted Sobolev space of *radial* functions in $H^{1,2}_{0,\rho}(\Omega)$ with $\rho = |Du|^{m-2}$.

Here, as in [19,26], if $\rho \in L^1(\Omega)$, the space $H^{1,p}_{\rho}(\Omega)$ is defined as the completion of $C^1(\bar{\Omega})$ (or $C^{\infty}(\bar{\Omega})$) under the norm

$$||v||_{H^{1,p}_{\rho}} = ||v||_{L^p(\Omega)} + ||Dv||_{L^p(\Omega,\rho)} \tag{1.4}$$

and $||Dv||^p_{L^p(\Omega,\rho)} = \int_{\Omega} |Dv|^p \rho \, dx$. In this way $H^{1,p}_{\rho}(\Omega)$ is a Banach space and $H^{1,2}_{\rho}(\Omega)$ is a Hilbert space. In [1] the authors also overcame the lack of regularity of the solutions because in the case when $u$ is a radial solution in a ball $B_r(0)$ and $f$

satisfies some hypotheses (e.g. $f(s)>0$ for $s>0$) then the only critical point of a solution is the origin, and a precise behavior of $u$ near the origin can be obtained using the l' Hospital rule as in [23], namely $|Du(x)|\approx|x|^{\frac{1}{p-1}}$ and $||D^2u(x)||\approx|x|^{\frac{2-p}{p-1}}$ as $x\to 0$.

In general, if we consider solutions of (1.1) in a general bounded smooth domain then the critical set $Z$ may be very irregular and estimates of this kind are not available. However we will show that we can efficiently work in the weighted Sobolev space $H^{1,2}_{0,\rho}(\Omega)$ using only the estimates proved in Theorem 1.1.

In particular, we will prove that if $f(s)>0$ for $s>0$ and $u$ is a solution of (1.1) with $m\geqslant 2$, considering the weight $\rho=|Du|^{m-2}$, for every $p\geqslant 2$ and $v\in H^{1,p}_{0,\rho}(\Omega)$ a weighted Poincaré 's inequality holds , i.e.

$$||v||_{L^p(\Omega)}\leqslant C(|\Omega|)||Dv||_{L^p(\Omega,\rho)}, \tag{1.5}$$

where $C(|\Omega|)\to 0$ if $|\Omega|\to 0$.

In [19,26] Eq. (1.5) is proved by assuming that

$$\rho\in L^1(\Omega),\quad \frac{1}{\rho}\in L^t(\Omega) \tag{1.6}$$

with $t>\frac{N}{p}$ and $p>1+\frac{1}{t}$. In the radial case, if we consider $\rho=|Du|^{m-2}$, $m\geqslant 2$ (or more generally $m>\frac{N+2}{N+1}$ which guarantees the belonging of $\rho$ to $L^1(\Omega)$) and $p=2$, these conditions are satisfied, as shown by the above estimates.

In a general domain, as a corollary of Theorem 1.1, we get that for $m\geqslant 2$, $\frac{1}{|Du|^{(m-1)r}}\in L^1(\Omega)$ for any $r<1$, which implies (1.6) with $p=2$ if $N=2$ or $N\geqslant 3$ and $m<\frac{2N-2}{N-2}$.

In order to avoid this restriction in the applications, in Section 3 we will prove that a weighted Poincaré' s inequality in the space $H^{1,p}_{0,\rho}(\Omega)$ can be obtained using classical potential estimates, similarly to those in [19,26] and assuming that we have the following estimate for the weight $\rho$:

$$\int_\Omega \frac{1}{\rho^t}\frac{1}{|x-y|^\gamma}\,dy\leqslant C, \tag{1.7}$$

where $C$ does not depend on $x\in\Omega$, $\gamma<N$, $t>\frac{N-\gamma}{p}$ and $p>1+\frac{1}{t}$. We will also prove a weighted Sobolev inequality of the same type.

In the case when $u$ is a solution of (1.1) with $m\geqslant 2$ and $\rho=|Du|^{m-2}$, by Theorem 1.1 the previous estimate is satisfied for any $\gamma<N-2$, $t<\frac{m-1}{m-2}$. So, using the regularity results in Theorem 1.1 together with these abstract results, we can prove the following Poincaré type inequality for solutions of (1.1).

**Theorem 1.2.** *Let $u \in C^1(\bar{\Omega})$ be a weak solution of* (1.1) *where $f$ satisfies* ($*$) *and $f(s)>0$ for $s>0$, $m \geqslant 2$. Then, if we consider $\rho = |Du|^{m-2}$ we get, for every $p \geqslant 2$*

$$||v||_{L^p(\Omega)} \leqslant C(|\Omega|)||Dv||_{L^p(\Omega,\rho)} \quad \text{for every } v \in H^{1,p}_{0,\rho}(\Omega), \tag{1.8}$$

*where $C(|\Omega|) \to 0$ if $|\Omega| \to 0$.*

*In particular* (1.8) *holds for every $v \in H^{1,2}_{0,\rho}(\Omega)$.*

**Remark 1.1.** The previous regularity results hold for any $1<m<\infty$, but the weighted Poincaré type inequality holds in this form in the case $m \geqslant 2$. In the case $1<m<2$ Poincaré' s inequalities without weight are often sufficient in the applications, provided the solutions belong to the class $C^1(\bar{\Omega})$ (see e.g. [6], where comparison theorems are proved using them).

We then use the weighted Poincaré type inequality obtained in Theorem 1.2 to prove the following:

**Theorem 1.3** (Weak Comparison Principle). *Suppose that either $1<m<2$ and $u, v \in W^{1,\infty}(\Omega)$; or $m \geqslant 2$, $u, v \in W^{1,m}(\Omega) \cap L^\infty(\Omega)$, where either $\rho \equiv |Du|^{m-2}$ or $\rho \equiv |Dv|^{m-2}$ satisfy condition* (1.7), *namely*

$$\int_\Omega \frac{1}{\rho^t} \frac{1}{|x-y|^\gamma} \, dy \leqslant C,$$

*where $C$ does not depend on $x \in \Omega$, $\gamma < N$, $t>1$ and $t > \frac{N-\gamma}{2}$.*

*Suppose that $u$, $v$ weakly solve*

$$-\text{div}(|Du|^{m-2}Du) + g(x,u) - \Lambda u \leqslant -\text{div}(|Dv|^{m-2}Dv) + g(x,v) - \Lambda v \text{ in } \Omega, \tag{1.9}$$

*where $\Lambda \geqslant 0$ and $g \in C(\bar{\Omega} \times \mathbb{R})$ is such that for every $x \in \Omega$, $g(x,s)$ is nondecreasing for $|s| \leqslant \max\{||u||_{L^\infty}, ||v||_{L^\infty}\}$.*

*Let $\Omega' \subseteq \Omega$ be open and suppose $u \leqslant v$ on $\partial\Omega'$, then there exists $\delta > 0$ such that, if $|\Omega'| \leqslant \delta$, then $u \leqslant v$ in $\Omega'$. If $\Lambda = 0$ the thesis is true for every $\Omega' \subseteq \Omega$.*

*In particular the result holds if either $u$ or $v$ is a weak solutions of* (1.1) *with $f$ satisfying* ($*$) *and $f(s)>0$ for $s>0$.*

The point of view of considering $\rho = |Du|^{m-2}$ as a weight and working in the weighted Sobolev space $H^{1,2}_{0,\rho}(\Omega)$, which is a Hilbert space, is particularly useful when studying qualitative properties of a solution of (1.1), as done e.g. in [1] in studying Morse index and uniqueness questions for *radial* solutions of (1.1).

In this paper, exploiting all the new regularity results together with the well known Alexandrov–Serrin moving plane method, we study monotonicity and symmetry properties of the solutions. In particular, when degenerate operators are considered, to apply the moving plane method, we have to take care of local symmetry phenomena (see [7,8]). We will overcome this difficulty proving a property of the critical set $Z$ of the solution, which is interesting in itself:

**Theorem 1.4.** *Let $u\in C^1(\bar{\Omega})$ be a weak solution of* (1.1) *where $\Omega$ is a general bounded domain, and suppose that $f$ satisfies* $(*)$ *and $f(s)>0$ if $s>0$. Then $\Omega\backslash Z$ does not contain any connected component $C$ such that $\bar{C}\subset\Omega$. Moreover, if we assume that $\Omega$ is a smooth bounded domain with connected boundary, it follows that $\Omega\backslash Z$ is connected.*

To state our monotonicity and symmetry result we need some notations.

Let $v$ be a direction in $\mathbb{R}^N$. For a real number $\lambda$ we define

$$T_\lambda^v = \{x\in\mathbb{R} : x\cdot v = \lambda\}, \tag{1.10}$$

$$\Omega_\lambda^v = \{x\in\Omega : x\cdot v<\lambda\}, \tag{1.11}$$

$$x_\lambda^v = R_\lambda^v(x) = x + 2(\lambda - x\cdot v)v, \quad x\in\mathbb{R}^N \tag{1.12}$$

and

$$a(v) = \inf_{x\in\Omega} x\cdot v. \tag{1.13}$$

If $\lambda>a(v)$ then $\Omega_\lambda^v$ is nonempty, thus we set

$$(\Omega_\lambda^v)' = R_\lambda^v(\Omega_\lambda^v). \tag{1.14}$$

Following [11,21] we observe that for $\lambda - a(v)$ small then $(\Omega_\lambda^v)'$ is contained in $\Omega$ and will remain in it, at least until one of the following occurs:

(i) $(\Omega_\lambda^v)'$ becomes internally tangent to $\partial\Omega$ .
(ii) $T_\lambda^v$ is orthogonal to $\partial\Omega$ .

Let $\Lambda_1(v)$ be the set of those $\lambda>a(v)$ such that for each $\mu<\lambda$ none of conditions (i) and (ii) holds and define

$$\lambda_1(v) = \sup \Lambda_1(v). \tag{1.15}$$

Moreover, let

$$\Lambda_2(v) = \{\lambda>a(v) : (\Omega_\mu^v)'\subseteq\Omega \quad \forall\mu\in(a(v),\lambda]\} \tag{1.16}$$

and

$$\lambda_2(v) = \sup \Lambda_2(v). \tag{1.17}$$

Note that since $\Omega$ is supposed to be smooth neither $\Lambda_1(v)$ nor $\Lambda_2(v)$ are empty, and $\Lambda_1(v)\subseteq\Lambda_2(v)$ so that $\lambda_1(v)\leqslant\lambda_2(v)$ (in the terminology of [11] $\Omega_{\lambda_1(v)}^v$ and $\Omega_{\lambda_2(v)}^v$

correspond to the 'maximal cap', respectively to the 'optimal cap'). Finally define

$$\Lambda_0(v) = \{\lambda > a(v) : u \leqslant u_\lambda^v \quad \forall \mu \in (a(v), \lambda]\} \tag{1.18}$$

and

$$\lambda_0(v) = \sup \Lambda_0(v). \tag{1.19}$$

**Theorem 1.5.** *Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, $1 < m < \infty$, $f : [0, \infty) \to \mathbb{R}$ a continuous function which is strictly positive and locally Lipschitz continuous in $(0, \infty)$, and $u \in C^1(\bar{\Omega})$ a weak solution of* (1.1).

*For any direction $v$ and for $\lambda$ in the interval $(a(v), \lambda_1(v)]$ we have*

$$u(x) \leqslant u(x_\lambda^v) \quad \forall x \in \Omega_\lambda^v. \tag{1.20}$$

*Moreover, for any $\lambda$ with $a(v) < \lambda < \lambda_1(v)$ we have*

$$u(x) < u(x_\lambda^v) \quad \forall x \in \Omega_\lambda^v \backslash Z_\lambda^v, \tag{1.21}$$

*where $Z_\lambda^v \equiv \{x \in \Omega_\lambda^v : Du(x) = Du_\lambda^v(x) = 0\}$. Finally,*

$$\frac{\partial u}{\partial v}(x) > 0 \quad \forall x \in \Omega_{\lambda_1(v)}^v \backslash Z, \tag{1.22}$$

*where $Z = \{x \in \Omega : Du(x) = 0\}$.*

*If $f$ is locally Lipschitz continuous in the closed interval $[0, \infty)$ then* (1.20) *and* (1.21) *hold for any $\lambda$ in the interval $(a(v), \lambda_2(v))$ and* (1.22) *holds for any $x \in \Omega_{\lambda_2(v)}^v \backslash Z$.*

**Corollary 1.1.** *If $f$ is locally Lipschitz continuous in the closed interval $[0, \infty)$ and strictly positive in $(0, \infty)$, and the domain $\Omega$ is convex with respect to a direction $v$ and symmetric with respect to the hyperplane $T_0^v = \{x \in \mathbb{R}^N : x \cdot v = 0\}$, then $u$ is symmetric, i.e. $u(x) = u(x_0^v)$, and nondecreasing in the $v$–direction in $\Omega_0^v$ with $\frac{\partial u}{\partial v}(x) > 0$ in $\Omega_0^v \backslash Z$.*

*In particular if $\Omega$ is a ball then $u$ is radially symmetric and $\frac{\partial u}{\partial r} < 0$, where $\frac{\partial u}{\partial r}$ is the derivative in the radial direction.*

**Remark 1.2.** The strength of our approach consists in the fact that it allows to consider the case $m > 2$, in general smooth domains, without any a priori assumption on the critical set of the solution $u$. If $1 < m < 2$ the previous monotonicity and symmetry result had been proved in [7,8] for a function $f$, not necessarily positive, which is either locally Lipschitz continuous in $[0, \infty)$ or locally Lipschitz continuous in $(0, \infty)$ and satisfies some weak positivity assumption close to 0. Anyway, in the case when $f$ is locally Lipschitz continuous in the closed interval $[0, \infty)$ and $f(s) > 0$ for $s > 0$ we slightly extend the result also in the case $1 < m < 2$, because (1.20) is true

$\forall\lambda\in(a(\nu),\lambda_2(\nu)]$ and not only $\lambda\in(a(\nu),\lambda_1(\nu)]$, and $\lambda_2(\nu)\geqslant\lambda_1(\nu)$ and can be strictly greater (consider e.g. a smoothed rectangle). We also simplify considerably the proof in [7,8] (where to exclude local symmetry phenomena a long technical device is needed).

**Remark 1.3.** Let us observe that in the case when $f\geqslant 0$ every nontrivial nonnegative solution of the equation $-\Delta_m u = f(u)$ is in fact positive, by the strong maximum principle (see Theorem 2.1 in Section 2), and all the results we prove apply to *nonnegative* solutions.

Let us recall some other works in the literature dealing with the problem of symmetry and monotonicity of solutions of (1.1). When $\Omega$ is a ball in [2] the symmetry is obtained by assuming that the gradient vanishes only at the center. A different approach is used in [15] where the case of $f$ continuous and positive is considered when $\Omega$ is a ball and $p = N$. In [4,5], with the aid of the so called "Continuous Steiner Symmetrization", the author prove that solutions of (1.1), in the ball, are radially symmetric under fairly weak assumption on the nonlinearity $f$.

Let us remark that the monotonicity results of Theorem 1.5 are important also in the case of general (i.e. not symmetric) bounded domains. For example in the case of strictly convex domains they show that there cannot be a concentration of maxima of family of solutions approaching the boundary, and this is very important when dealing with blow-up analysis and a priori estimates.

Let us finally remark that in the case of ground states of quasilinear elliptic equations in the whole space, radial symmetry results were obtained in [9,22,4,5].

The paper is organized as follows. In Section 2 we prove Theorem 1.1 and some related regularity results. In Section 3 we state sufficient conditions to get general weighted Sobolev and Poincaré's inequality and then we exploit them together with Theorem 1.1 to prove Theorem 1.2. Moreover we exploit the weighted Poincaré's inequality obtained and we prove Theorem 1.3. Finally in Section 4 we prove our monotonicity and symmetry results.

## 2. Regularity results

In this section we prove all the statements of Theorem 1.1 and some other related results.

Let us first recall a particular version of the Strong Maximum Principle and of the Hopf's Lemma [13] for the $m$-laplacian (see [27] for the case of the $m$-laplacian and [20] for general quasilinear elliptic operators).

**Theorem 2.1** (Strong Maximum Principle and Hopf's Lemma). *Let $\Omega$ be a domain in $\mathbb{R}^N$ and suppose that $u\in C^1(\Omega)$, $u\geqslant 0$ in $\Omega$, weakly solves*

$$-\Delta_m u + cu^q = g\geqslant 0 \quad \textit{in} \quad \Omega$$

*with* $1<m<\infty$, $q\geqslant m-1$, $c\geqslant 0$ *and* $g\in L^{\infty}_{\mathrm{loc}}(\Omega)$. *If* $u\neq 0$ *then* $u>0$ *in* $\Omega$. *Moreover for any point* $x_0\in\partial\Omega$ *where the interior sphere condition is satisfied, and such that* $u\in C^1(\Omega\cup\{x_0\})$ *and* $u(x_0)=0$ *we have that* $\frac{\partial u}{\partial s}>0$ *for any inward directional derivative* (*this means that if* $y$ *approaches* $x_0$ *in a ball* $B\subseteq\Omega$ *that has* $x_0$ *on its boundary, then* $\lim_{y\to x_0}\frac{u(y)-u(x_0)}{|y-x_0|}>0$).

**Remark 2.1.** By standard elliptic regularity, a $C^1(\Omega)$ solution $u$ of (1.1) with $f$ satisfying ($^*$) belongs to the class $C^2(\Omega\backslash Z)$, where $Z=\{x\in\Omega: Du(x)=0\}$ is the critical set of the solution (see [10,12,16,24]). Therefore the generalized derivatives of $|Du|^{m-2}u_{x_i}$, coincide there with the classical ones.

Let us put

$$\tilde{u}_{ij}=\begin{cases} u_{x_ix_j}, & \text{in} \quad \Omega\backslash Z,\\ 0, & \text{in} \quad Z,\end{cases} \tag{2.1}$$

we will also use the notation $\tilde{D}u_i$ for the "gradient" $(\tilde{u}_{i1},\dots,\tilde{u}_{iN})$.

We will prove later that $|Du|^{m-2}u_{x_i}$ belong to the Sobolev space $W^{1,2}(\Omega)$, so that by Stampacchia's Theorem (see e.g. [25, Theorem 1.56, p. 79]), the generalized derivatives of $|Du|^{m-2}u_{x_i}$ are zero almost everywhere in $Z$, and we will get

$$\frac{\partial}{\partial x_j}(|Du|^{m-2}u_{x_i})\equiv(|Du|^{m-2}\tilde{u}_{ij}+(m-2)|Du|^{m-4}(Du,\tilde{D}u_j)u_{x_i}),$$

where $\frac{\partial}{\partial x_j}$ stands for the distributional derivative.

**Definition 2.1.** If $\rho\in L^1(\Omega)$, let us define as in [19,26], the space $H^{1,p}_\rho(\Omega)$, as the completion of $C^1(\bar{\Omega})$ (or $C^\infty(\bar{\Omega})$) under the norm

$$||v||_{H^{1,p}_\rho}=||v||_{L^p(\Omega)}+||Dv||_{L^p(\Omega,\rho)}, \tag{2.2}$$

where $||Dv||^p_{L^p(\Omega,\rho)}=\int_\Omega|Dv|^p\rho\,dx$. In this way $H^{1,p}_\rho(\Omega)$ is a Banach space and $H^{1,2}_\rho(\Omega)$ is a Hilbert space. Moreover, we define $H^{1,p}_{0,\rho}(\Omega)$ as the closure of $C^1_c(\Omega)$ (or $C^\infty_c(\Omega)$) in $H^{1,p}_\rho(\Omega)$.

Observe that if $\rho\in L^\infty(\Omega)$ then $W^{1,p}(\Omega)$ has a continuous embedding in $H^{1,p}_\rho(\Omega)$. Let us also observe that if $u\in W^{1,m}(\Omega)$, $m\geqslant 2$ and $\rho=|Du|^{m-2}$, then by Hölder's inequality $W^{1,m}(\Omega)$ has a continuous embedding in the Hilbert space $H^{1,2}_\rho(\Omega)$.

Let us recall the definition of the linearized operator

$$L_u(g,\varphi) \equiv \int_\Omega [|Du|^{m-2}(Dg, D\varphi) + (m-2)|Du|^{m-4}(Du, Dg)(Du, D\varphi) - f'(u)g\varphi]\, dx.$$

The linearized operator is well defined if $g, \varphi \in H^{1,2}_\rho(\Omega)$, $\rho = |Du|^{m-2}$, or if $g \in L^2(\Omega, \mathbb{R})$, $|Du|^{m-2}\, Dg \in L^2(\Omega, \mathbb{R}^N)$ and $\varphi \in W^{1,2}(\Omega)$.

We will prove later that $|Du|^{m-2}\, Du \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ if $u \in C^1(\Omega)$ is a solution of (1.1), so that if $\varphi \in W^{1,2}(\Omega)$ has compact support we can also define

$$L_u(u_{x_i},\varphi) \equiv \int_\Omega [|Du|^{m-2}(\tilde{D}u_i, D\varphi) + (m-2)|Du|^{m-4}(Du, \tilde{D}u_i)(Du, D\varphi) - f'(u)u_{x_i}\varphi]\, dx.$$

For the time being we use the definition of the linearized operator at the fixed solution $u$ only with test function $\varphi \in W^{1,2}(\Omega)$ *with compact support in* $\Omega \backslash Z$, where $Z = \{x \in \Omega : Du(x) = 0\}$ is the critical set of the solution $u$, and prove the following:

**Lemma 2.1.** *Let $u \in C^1(\Omega)$ be a weak solution of* (1.1), *with $f$ satisfying* $(*)$. *Then we have $L_u(u_{x_i}, \varphi) = 0$ for every $\varphi \in W^{1,2}(\Omega)$ with compact support in $\Omega \backslash Z$.*

**Proof.** Let $\varphi \in C^\infty_c(\Omega \backslash Z)$, and set $\psi \equiv \frac{\partial \varphi}{\partial x_i} \in C^\infty_c(\Omega)$. Using $\psi$ as test function in (1.1) we get

$$\int_\Omega \left( |Du|^{m-2} Du, D\frac{\partial \varphi}{\partial x_i} \right) dx = \int_\Omega f(u) \left( \frac{\partial \varphi}{\partial x_i} \right) dx.$$

Since the domain of integration is a subset of $\Omega \backslash Z$, where $u \in C^2$, $|Du|^{m-2} u_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega \backslash Z)$, and since $f$ is locally Lipschitz continuous in $(0, \infty)$ and $u$ is positive in $\Omega$, $f(u) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ we can integrate by parts obtaining

$$\int_\Omega \left( \frac{\partial}{\partial x_i} (|Du|^{m-2} Du), D\varphi \right) dx = \int_\Omega f'(u) u_{x_i} \varphi\, dx.$$

and get

$$\begin{aligned} &\int_\Omega [|Du|^{m-2}(\tilde{D}u_i, D\varphi) + (m-2)|Du|^{m-4}(Du, \tilde{D}u_i)(Du, D\varphi)]\, dx \\ &\quad - \int_\Omega [fy'(u) u_{x_i} \varphi]\, dx = 0 \end{aligned} \tag{2.3}$$

i.e.

$$L_u(u_{x_i}, \varphi) = 0.$$

By density we get the general case of $\varphi \in W^{1,2}(\Omega)$ with compact support in $\Omega \backslash Z$. $\square$

We can now prove

**Theorem 2.2.** *Let $u \in C^1(\Omega)$ be a weak solution of* (1.1), *with $f$ satisfying* $(*)$ $1 < m < \infty$. *Then for any $E \subset\subset \Omega$ and for every $i, j = 1, \ldots, N$, we have*

$$\sup_{x \in \Omega} \int_{E \backslash \{u_{x_i} = 0\}} \frac{|Du|^{m-2}}{|u_{x_i}|^{\beta} |x-y|^{\gamma}} |\tilde{D}u_i|^2 \, dy < C,$$

*where $\beta < 1$, $\gamma < N-2$ if $N \geqslant 3$, $\gamma = 0$ if $N = 2$ and $C = C(\beta, \gamma, E)$. Moreover*

$$\sup_{x \in \Omega} \int_{E \backslash Z} \frac{|Du|^{m-2-\beta}}{|x-y|^{\gamma}} ||D^2 u||^2 \, dy < C,$$

*where $Z = \{x \in \Omega : Du(x) = 0\}$ is the critical set of the solution.*

**Proof.** Let us observe that we can suppose that $x \in E$ without loss of generality. In fact, suppose that we prove that for every measurable set $E \subset\subset \Omega$ we have

$$\sup_{x \in E} \int_{E \backslash \{u_{x_i} = 0\}} \frac{|Du|^{m-2}}{|u_{x_i}|^{\beta} |x-y|^{\gamma}} |\tilde{D}u_i|^2 \, dy \leqslant K(\beta, \gamma, E).$$

Then if $0 < \delta \leqslant \frac{1}{2} \operatorname{dist}(E, \partial\Omega)$ and $E_\delta = \{x \in \Omega : \operatorname{dist}(x, E) \leqslant \delta\}$ considering the two cases $x \in E_\delta$ and $x \in \Omega \backslash E_\delta$, it follows that

$$\sup_{x \in \Omega} \int_{E \backslash \{u_{x_i} = 0\}} \frac{|Du|^{m-2}}{|u_{x_i}|^{\beta} |x-y|^{\gamma}} |\tilde{D}u_i|^2 \, dy \leqslant K(\beta, \gamma, E_\delta) + \frac{1}{\delta^{\gamma}} K(\beta, 0, E).$$

Let $E \subset\subset \Omega$, $x \in E$, and consider a cut-off function $\varphi \in C_c^{\infty}(\Omega)$ such that $\varphi \geqslant 0$ in $\Omega$, and $\varphi \equiv 1$ in $E_\delta = \{x \in \Omega \mid \operatorname{dist}(x, E) \leqslant \delta\}$ where $0 < \delta \leqslant \frac{1}{2} \operatorname{dist}(E, \partial\Omega)$.

Let $G_\varepsilon$ be defined by

$$\begin{cases} G_\varepsilon(s) = 0 & \text{if } |s| \leqslant \varepsilon, \\ G_\varepsilon(s) = 2s - 2\varepsilon & \text{if } \varepsilon \leqslant |s| \leqslant 2\varepsilon, \\ G_\varepsilon(s) = s & \text{if } |s| \geqslant 2\varepsilon, \end{cases}$$

so that $G_\varepsilon$ is a Lipschitz continuous function and $0 \leqslant G'_\varepsilon \leqslant 2$. To obtain our result we will consider the case $x \in E \cap Z$ and $x \in E \backslash Z$ separately.

*Case* 1: Suppose first that $x \in E \cap Z$. In this case define $\psi_{\varepsilon,x}(y) = \frac{G_\varepsilon(u_{x_i})(y)}{|u_{x_i}(y)|^\beta}\frac{\varphi(y)}{|x-y|^\gamma}$ with $\beta < 1$, $\gamma < N-2$ and $N \geqslant 3$. If $N = 2$ we use $\psi_{\varepsilon,x} = \frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta}\varphi$. Since $G_\varepsilon(u_{x_i})$ vanishes in a neighborhood of each critical point, in particular in a neighborhood of $y = x$, we can use $\psi_{\varepsilon,x}$ as a test function in (1.2) and get

$$\begin{aligned}
&\int_\Omega \frac{|Du|^{m-2}}{|u_{x_i}|^\beta}\frac{|\tilde{D}u_i|^2}{|x-y|^\gamma}\left(G'_\varepsilon(u_{x_i}) - \beta\frac{G_\varepsilon(u_{x_i})}{u_{x_i}}\right)\varphi\, dy\\
&\quad + (m-2)\int_\Omega \frac{|Du|^{m-4}(Du,\tilde{D}u_i)^2}{|u_{x_i}|^\beta|x-y|^\gamma}\left(G'_\varepsilon(u_{x_i}) - \beta\frac{G_\varepsilon(u_{x_i})}{u_{x_i}}\right)\varphi\, dy\\
&\quad + \int_{\Omega\setminus E_\delta} |Du|^{m-2}(\tilde{D}u_i, D\varphi)\frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta}\frac{1}{|x-y|^\gamma}\, dy\\
&\quad + (m-2)\int_{\Omega\setminus E_\delta} |Du|^{m-4}(Du,\tilde{D}u_i)(Du,D\varphi)\frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta}\frac{1}{|x-y|^\gamma}\, dy\\
&\quad + \int_\Omega |Du|^{m-2}\left(\tilde{D}u_i, D_y\left(\frac{1}{|x-y|^\gamma}\right)\right)\frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta}\varphi\, dy\\
&\quad + (m-2)\int_\Omega |Du|^{m-4}(Du,\tilde{D}u_i)\left(Du, D_y\left(\frac{1}{|x-y|^\gamma}\right)\right)\frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta}\varphi\, dy\\
&\quad = \int_\Omega f'(u)u_{x_i}\frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta}\frac{1}{|x-y|^\gamma}\varphi\, dy.
\end{aligned}$$

By the definition of $G_\varepsilon$ it follows that $(G'_\varepsilon(u_{x_i}) - \beta\frac{G_\varepsilon(u_{x_i})}{u_{x_i}}) \geqslant 0$ in $\Omega$. Therefore we get

$$\begin{aligned}
&\int_\Omega \frac{|Du|^{m-2}|\tilde{D}u_i|^2}{|u_{x_i}|^\beta|x-y|^\gamma}\left(G'_\varepsilon(u_{x_i}) - \beta\frac{G_\varepsilon(u_{x_i})}{u_{x_i}}\right)\varphi\, dy\\
&\quad \leqslant (m-1)\int_{\Omega\setminus E_\delta}\frac{|Du|^{m-2}|\tilde{D}u_i||D\varphi|}{|x-y|^\gamma}\frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta}\, dy\\
&\qquad + \gamma(m-1)\int_\Omega \frac{|Du|^{m-2}|\tilde{D}u_i|}{|x-y|^{\gamma+1}}\frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta}\varphi\, dy\\
&\qquad + \int_\Omega \frac{|f'(u)||u_{x_i}|^{2-\beta}}{|x-y|^\gamma}\varphi\, dy.
\end{aligned}$$

By the definition of $E_\delta$, since $x \in E$, we know that $\sup_{y\in\Omega\setminus E_\delta}\frac{1}{|x-y|^\gamma} \leqslant \frac{1}{\delta^\gamma}$ and, using the fact that $|Du|^{m-2}|\tilde{D}u_i| \in L^2_{\mathrm{loc}}(\Omega)$, since $\varphi$ has compact support in $\Omega$, we get

$$\int_{\Omega\setminus E_\delta}\frac{|Du|^{m-2}|\tilde{D}u_i||D\varphi|}{|x-y|^\gamma}\frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta}\, dy \leqslant C_1,$$

where $C_1$ does not depend on $x$. Since $\Omega$ is bounded, then $\int_\Omega \frac{1}{|x-y|^s\,dx}$ is uniformly bounded for any fixed $s<n$ and, using the fact that $u\in C^1(\Omega)$ and $|f'(u)|$ is bounded in $\mathrm{supp}(\varphi)$, we get

$$\int_\Omega \frac{|f'(u)||u_{x_i}|^{2-\beta}}{|x-y|^\gamma}\varphi\,dy \leqslant C_2,$$

where $C_2$ does not depend on $x$. Here we have used that $u\in C^1(\Omega)$ and $\frac{|G_\varepsilon(u_{x_i})|}{|u_{x_i}|^\beta}\leqslant |u_{x_i}|^{1-\beta}\leqslant C_0$. Therefore

$$\begin{aligned}&\int_\Omega \frac{|Du|^{m-2}|\tilde{D}u_i|^2}{|u_{x_i}|^\beta|x-y|^\gamma}\left(G'_\varepsilon(u_{x_i})-\beta\frac{G_\varepsilon(u_{x_i})}{u_{x_i}}\right)\varphi\,dy\leqslant C_3\\ &\quad + C_4\int_\Omega \frac{|Du|^{\frac{m-2}{2}}|\tilde{D}u_i|}{|u_{x_i}|^{\frac{\beta}{2}}|x-y|^{\frac{\gamma}{2}}}\left(\frac{|G_\varepsilon(u_{x_i})|}{|u_{x_i}|}\varphi\right)^{\frac12}\frac{|Du|^{\frac{m-2}{2}}(|G_\varepsilon(u_{x_i})|)^{\frac12}}{|x-y|^{\frac{\gamma}{2}+1}}\varphi^{\frac12}|u_{x_i}|^{\frac12-\frac{\beta}{2}}\,dy.\end{aligned}$$

By Young's inequality ($ab\leqslant \sigma a^2+b^2\backslash 4\sigma$), if $\sigma>0$ we get

$$\begin{aligned}&\int_\Omega \frac{|Du|^{\frac{m-2}{2}}|\tilde{D}u_i|}{|u_{x_i}|^{\frac{\beta}{2}}|x-y|^{\frac{\gamma}{2}}}\left(\frac{|G_\varepsilon(u_{x_i})|}{|u_{x_i}|}\varphi\right)^{\frac12}\frac{|Du|^{\frac{m-2}{2}}(|G_\varepsilon(u_{x_i})|)^{\frac12}}{|x-y|^{\frac{\gamma}{2}+1}}\varphi^{\frac12}|u_{x_i}|^{\frac12-\frac{\beta}{2}}\,dy\\ &\quad\leqslant \sigma\int_\Omega \frac{|Du|^{m-2}|\tilde{D}u_i|^2}{|u_{x_i}|^\beta|x-y|^\gamma}\frac{|G_\varepsilon(u_{x_i})|}{|u_{x_i}|}\varphi\,dy\\ &\qquad+\frac{1}{4\sigma}\int_\Omega \frac{|Du|^{m-2}|G_\varepsilon(u_{x_i})|}{|x-y|^{\gamma+2}}\varphi|u_{x_i}|^{1-\beta}\,dy.\end{aligned}$$

Since $\frac{|G_\varepsilon(u_{x_i})|}{|u_{x_i}|}\equiv\frac{G_\varepsilon(u_{x_i})}{u_{x_i}}$, we can take $\sigma>0$ such that $(1-\beta-\sigma)>0$ and

$$\begin{aligned}&\int_\Omega \frac{|Du|^{m-2}|\tilde{D}u_i|^2}{|u_{x_i}|^\beta|x-y|^\gamma}\left(G'_\varepsilon(u_{x_i})-(\beta+\sigma)\frac{G_\varepsilon(u_{x_i})}{u_{x_i}}\right)\varphi\,dy\\ &\quad\leqslant C_3+C_5\int_\Omega \frac{1}{|x-y|^{\gamma+2}}\,dy\leqslant C_6,\end{aligned}$$

where $C_6$ does not depend on $x$. Let us note that, by definition, $(G'_\varepsilon(u_{x_i})-(\beta+\sigma)\frac{G_\varepsilon(u_{x_i})}{u_{x_i}})\geqslant 0$ and $(G'_\varepsilon(u_{x_i})-(\beta+\sigma)\frac{G_\varepsilon(u_{x_i})}{u_{x_i}})\to 1-(\beta+\sigma)$ in $\{u_{x_i}\neq 0\}$. Therefore, by Fatou's Lemma, we get

$$\int_{\Omega\backslash\{u_{x_i}=0\}}\frac{|Du|^{m-2}|\tilde{D}u_i|^2}{|u_{x_i}|^\beta|x-y|^\gamma}\varphi\,dy\leqslant C,$$

where $C$ does not depend on $x \in E \cap Z$. In particular,

$$\int_{\Omega \setminus \{u_{x_i}=0\}} \frac{|Du|^{m-2-\beta}|\tilde{D}u_i|^2}{|x-y|^\gamma} \varphi \, dy \leqslant C$$

and, since $u_{x_i x_j} = 0$ a.e. in $\{u_{x_i} = 0\} \setminus Z$, it follows, for any $i = 1, \dots, N$, that

$$\int_{\Omega \setminus Z} \frac{|Du|^{m-2-\beta}|\tilde{D}u_i|^2}{|x-y|^\gamma} \varphi \, dy \leqslant C,$$

where $x \in Z$ and $C$ does not depend on $x$. Moreover, since $\varphi \equiv 1$ in $E$, we get

$$\int_{E \setminus Z} \frac{|Du|^{m-2-\beta}||D^2 u||^2}{|x-y|^\gamma} \, dy \leqslant C.$$

*Case* 2: Suppose now that $x \in E \setminus Z$. In this case consider $E$ and $E_\delta$ as above, and for $\varepsilon > 0$ small consider a cut-off function $\varphi_{\varepsilon,x} \in C_c^\infty(\Omega)$ such that $\varphi_{\varepsilon,x} \geqslant 0$ in $\Omega$, $\varphi_{\varepsilon,x} \equiv 0$ in $B_\varepsilon(x)$, $\varphi_{\varepsilon,x} \equiv 1$ in $E_\delta \setminus B_{2\varepsilon}(x)$, $|D\varphi_{\varepsilon,x}| \leqslant \frac{C}{\varepsilon}$ in $B_{2\varepsilon}(x) \setminus B_\varepsilon(x)$ and $|D\varphi_{\varepsilon,x}| \leqslant c_1$ outside $B_{2\varepsilon}(x)$. Moreover suppose that there exists a set $A \subset\subset \Omega$ such that $\operatorname{supp}(\varphi_{\varepsilon,x}) \subset A$ for every $\varepsilon$ and $x \in E$.

Using $\psi_{\varepsilon,x} = \frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta} \frac{1}{|x-y|^\gamma} \varphi_{\varepsilon,x}$ as a test function in (1.2), by the same estimates we have used before, it follows

$$\begin{aligned} &\int_\Omega \frac{|Du|^{m-2}|\tilde{D}u_i|^2}{|u_{x_i}|^\beta |x-y|^\gamma} \left( G'_\varepsilon(u_{x_i}) - (\beta+\sigma) \frac{G_\varepsilon(u_{x_i})}{u_{x_i}} \right) \varphi_{\varepsilon,x} \, dy \\ &\quad \leqslant C_7 + C_8 \int_{B_{2\varepsilon}(x) \setminus B_\varepsilon(x)} \frac{|Du|^{m-2}|\tilde{D}u_i|}{|x-y|^\gamma} |D\varphi_{\varepsilon,x}| \frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta} \, dy. \end{aligned}$$

Since $x \in E \setminus Z$, by standard elliptic estimates, we have that $u$ is regular near $x$, and for $\varepsilon$ sufficiently small, there exists a constant $C_9(\varepsilon, x)$ depending on $\varepsilon$ and on $x$ such that $|Du|^{m-2}|\tilde{D}u_i| \leqslant C_9(\varepsilon, x)$ in $B_{2\varepsilon}(x)$. Moreover, if $x$ is fixed and $\varepsilon$ is small, we can suppose that $C_9(\varepsilon, x)$ does not depend on $\varepsilon$. Therefore, if $\varepsilon$ is sufficiently small, we get

$$C_8 \int_{B_{2\varepsilon}(y) \setminus B_\varepsilon(y)} \frac{|Du|^{m-2}|\tilde{D}u_i|}{|x-y|^\gamma} |D\varphi_{\varepsilon,y}| \frac{G_\varepsilon(u_{x_i})}{|u_{x_i}|^\beta} \, dy \leqslant C_9(x) \frac{\varepsilon^N}{\varepsilon^{\gamma+1}}.$$

Since $\gamma < N-2$, then $C_9(x) \frac{\varepsilon^N}{\varepsilon^{\gamma+1}} \to 0$ if $\varepsilon \to 0$, and, for $\varepsilon$ sufficiently small, we have

$$\int_\Omega \frac{|Du|^{m-2}|\tilde{D}u_i|^2}{|u_{x_i}|^\beta |x-y|^\gamma} \left( G'_\varepsilon(u_{x_i}) - (\beta+\sigma) \frac{G_\varepsilon(u_{x_i})}{u_{x_i}} \right) \varphi_{\varepsilon,x} \, dy \leqslant C_7 + 1,$$

where $C_7$ does not depend on $x$. Using Fatou's Lemma we get the thesis also for the case $x \in E \backslash Z$. Note that, to get the estimates above, the choice of $\varepsilon$ depends on $x$. In spite of this, exploiting Fatou's Lemma, we get estimates which do not depend on $x$.

Finally, taking the greatest constant between the ones obtained in the two cases, we prove the theorem. $\square$

To extend Theorem 2.2 up to the boundary we need some informations on the regularity of the solution on the boundary, which would be implied by assuming e.g. $\Omega$ smooth and $f$ sufficiently smooth and nonnegative (so that Hopf's lemma holds at the boundary). Since here we do not need to extend Theorem 2.2 up to the boundary, we will only note that, if we consider $x$ fixed, then we have the following.

**Corollary 2.1.** *Let $\Omega$ be a smooth domain, $u \in C^1(\bar{\Omega})$ be a weak solution of* (1.1) *with $f$ locally Lipschitz continuous in $[0, \infty)$ and $f(s) > 0$ for $s > 0$, $1 < m < +\infty$. Then, $|Z| = 0$ and, for every fixed $x \in \Omega$,*

$$\int_{\Omega} \frac{|Du|^{m-2-\beta} |u_{x_i x_j}|^2}{|x-y|^\gamma} \, dy \leqslant C,$$

*where $\beta < 1$, $\gamma < N-2$ if $N \geqslant 3$ and $\gamma = 0$ if $N = 2$. In particular*

$$\int_{\Omega} |Du|^{m-2-\beta} |u_{x_i x_j}|^2 \, dx \leqslant C.$$

As a consequence of the previous estimates we can prove

**Corollary 2.2.** *Let $u \in C^1(\Omega)$ be a weak solution of* (1.1) *with $f$ satisfying* $(*)$, *$1 < m < \infty$. Then $u \in C^2(\Omega \backslash Z)$, where $Z = \{x \in \Omega : Du(x) = 0\}$ is the critical set of the solution, $|Du|^{m-2} Du \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, therefore $|Du|^{m-1} \in W^{1,2}_{\mathrm{loc}}(\Omega)$.*

*If moreover $\Omega$ is smooth, $u \in C^1(\bar{\Omega})$ and $f$ is nonnegative and locally Lipschitz continuous in the closed interval $[0, \infty)$, then $Z \cap \partial\Omega = \emptyset$, $u \in C^2(\bar{\Omega} \backslash Z)$, $|Du|^{m-2} Du \in W^{1,2}(\Omega, \mathbb{R}^N)$ and $|Du|^{m-1} \in W^{1,2}(\Omega)$.*

**Proof.** Since $u \in C^1(\Omega)$ is positive in $\Omega$ and $f$ is locally Lipschitz continuous in $(0, \infty)$, we have that $f(u)$ is locally Lipschitz continuous in $\Omega$ and by elliptic regularity $u \in C^2(\Omega \backslash Z)$, since it satisfies an uniformly elliptic equation in a neighborhood of each regular point $x \in \Omega \backslash Z$. Recall that in Theorem 2.2 (where we have used test function with compact support in $\Omega \backslash Z$ only) we obtain that

$$\int_{E \backslash Z} |Du|^{m-2-\beta} ||D^2 u||^2 \, dx < C, \tag{2.4}$$

where $\beta < 1$, $Z = \{x \in \Omega : Du(x) = 0\}$ is the critical set of the solution, and $E$ is any compact set contained in $\Omega$.

Let us now set

$$\phi_n \equiv G_{\frac{1}{n}}(|Du|^{m-2} u_{x_i}),$$

where $G_{\frac{1}{n}}$ is defined as in Theorem 2.2, $n \in \mathbb{N}$ and $i \in \{1, \dots, N\}$. By the definition of $G_{\frac{1}{n}}$ we get that $\phi_n \in W^{1,2}(E)$ and

$$\frac{\partial}{\partial x_j} \phi_n = G_{\frac{1}{n}}'(|Du|^{m-2} u_{x_i}) \frac{\partial}{\partial x_j} (|Du|^{m-2} u_{x_i}). \tag{2.5}$$

Therefore, taking into account Remark 2.1 and exploiting (2.4), we get

$$||\phi_n||_{W^{1,2}(E)} \leqslant K \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Since $W^{1,2}(E)$ has a compact embedding in $L^2(E)$, up to subsequences there exists $h \in W^{1,2}(E)$ such that

$$\phi_n \to h \text{ strongly in } L^2(E),$$

as $n$ tends to infinity and

$$\phi_n \to h \text{ almost everywhere in } E.$$

Since $\phi_n \to |Du|^{m-2} u_{x_i}$ almost everywhere in $E$, we get

$$|Du|^{m-2} u_{x_i} \equiv h \in W^{1,2}(E). \tag{2.7}$$

Since $i \in \{1, \dots, N\}$ is arbitrary the thesis follows and $|Du|^{m-2} Du \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

If moreover $\Omega$ is smooth, $u \in C^1(\bar{\Omega})$ and $f$ is nonnegative and locally Lipschitz continuous in the closed interval $[0, \infty)$, then $f(u)$ is Lipschitz continuous in $\bar{\Omega}$ and $Z \cap \partial\Omega = \emptyset$ by the Hopf's lemma. By standard elliptic regularity it follows that $u$ belongs to the class $C^2$ in a neighborhood of the boundary, so that $u \in C^2(\bar{\Omega} \backslash Z)$ and $|Du|^{m-2} Du \in W^{1,2}(\Omega, \mathbb{R}^N)$. $\square$

Let us remark that in a recent paper Lou [18] proved that if $u \in W^{1,m}_{\mathrm{loc}}(\Omega)$ is a weak solution of the equation

$$-\mathrm{div}(|Du|^{m-2} Du) = f(x) \quad \text{in} \quad \Omega$$

with $f \in L^q(\Omega)$, $q > \frac{N}{m}$, $q \geqslant 2$, $1 < m < \infty$, then $|Du|^{m-1} \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

**Remark 2.2.** We recall that under the assumptions on the boundary of Corollary 2.2, by the regularity results up to the boundary of Lieberman [17], it follows that any solution $u$ of (1.1) belongs to the class $C^{1,\tau}(\bar{\Omega})$.

**Remark 2.3.** Since a $C^1(\Omega)$ solution $u$ of (1.1) with $f$ satisfying $(^*)$ is regular in $\Omega \backslash Z$, the generalized derivatives of $|Du|^{m-2}u_{x_i}$, coincide there with the classical ones. Moreover in $\{u_{x_i}=0\}$, by Stampacchia's Theorem(see e.g. [25, Theorem 1.56, p. 79]), the generalized derivatives of $|Du|^{m-2}u_{x_i}$ are zero almost everywhere. From now on we will do all computations taking into account this fact. In particular, we get

$$\frac{\partial}{\partial x_j}(|Du|^{m-2}u_{x_i}) \equiv (|Du|^{m-2}\tilde{u}_{ij} + (m-2)|Du|^{m-4}(Du, \tilde{D}u_j)u_{x_i}),$$

where $\frac{\partial}{\partial x_j}$ stands for the distributional derivative and $\tilde{u}_{ij}$ are defined as in Remark 2.1 by

$$\tilde{u}_{ij} = \begin{cases} u_{x_i x_j} & \text{in} \quad \Omega \backslash Z, \\ 0 & \text{in} \quad Z \end{cases} \tag{2.8}$$

and $\tilde{D}u_i$ stands for the "gradient" $(\tilde{u}_{i1}, \dots, \tilde{u}_{iN})$.

Let us now prove an elementary consequence of Corollary 2.2.

**Lemma 2.2.** *Let* $u \in C^1(\Omega)$ *be a weak solution of* (1.1). *Then we have* $|Du|^{m-2}\tilde{u}_{ij} \in L^2_{\mathrm{loc}}(\Omega)$.

**Proof.** We have already shown that $(|Du|^{m-2}u_{x_i})_{x_j} \in L^2_{\mathrm{loc}}(\Omega)$. With the aid of Remark 2.3 we can write

$$(|Du|^{m-2}u_{x_i})_{x_j} = (|Du|^{m-2})\tilde{u}_{ij} + (m-2)|Du|^{m-4}(Du, \tilde{D}u_j) \cdot u_{x_i}. \tag{2.9}$$

Since $|Du|^{m-1} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ we also know that

$$(m-1)|Du|^{m-3}(Du, \tilde{D}u_j) \in L^2_{\mathrm{loc}}(\Omega). \tag{2.10}$$

So we get

$$(m-2)|Du|^{m-4}(Du, \tilde{D}u_j) \cdot u_{x_i} \in L^2_{\mathrm{loc}}(\Omega). \tag{2.11}$$

Therefore $|Du|^{m-2}\tilde{u}_{ij} \in L^2_{\mathrm{loc}}(\Omega)$ since it is a linear combination of elements of $L^2_{\mathrm{loc}}(\Omega)$. $\square$

**Remark 2.4.** Let us observe that in the case when $f \geqslant 0$ every nontrivial nonnegative solution of the equation $-\Delta_m u = f(u)$ is in fact positive, by the strong maximum principle, and all the results we prove apply to *nonnegative* solutions.

In the case when $f$ is positive then $|Z| = 0$ by Lou's result [18] (see next theorem where we prove with our techniques a stronger result).

Therefore, since $u$ is regular in $\Omega \backslash Z$, the classical second derivatives $u_{x_i x_j}$ are defined almost everywhere, and coincide with $\tilde{u}_{ij}$. Since, from now on, in this section we consider the case of positive nonlinearities, in order to simplify the statements, we will use $u_{x_i x_j}$ instead of $\tilde{u}_{ij}$.

Moreover, since we have assumed $\Omega$ to be smooth, in the case of $f$ positive, Hopf's Lemma applies and shows that, in a neighborhood of $\partial\Omega$, there are not points where the gradient of $u$ vanishes.

Consequently all regularity results, which we have proved, except for Theorem 2.2, can be extended *up to the boundary*.

**Lemma 2.3.** *Let $u \in C^1(\Omega)$ be a weak solution of* (1.1), *with $f$ satisfying* $(*)$. *Then we have $L_u(u_{x_i}, \varphi) = 0$ for every $\varphi \in W^{1,2}(\Omega)$ with compact support in $\Omega$. If moreover $\Omega$ is smooth, $u \in C^1(\bar{\Omega})$ and $f$ is locally Lipschitz continuous and nonnegative in the closed interval* $[0, \infty)$, *then $L_u(u_{x_i}, \varphi) = 0$ for every $\varphi \in W_0^{1,2}(\Omega)$.*

**Proof.** By Corollary 2.2, $|Du|^{m-2} u_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega)$, so that we can proceed as in Lemma 2.1 integrating by parts and, if $\varphi \in C_c^\infty(\Omega)$, we get

$$\int_\Omega [|Du|^{m-2}(\tilde{D}u_i, D\varphi) + (m-2)|Du|^{m-4}(Du, \tilde{D}u_i)(Du, D\varphi)]\, dx \\ - \int_\Omega [f'(u) u_{x_i} \varphi]\, dx = 0$$

i.e.

$$L_u(u_{x_i}, \varphi) = 0.$$

By density we get the general case of $\varphi \in W^{1,2}(\Omega)$ with compact support.

If moreover $\Omega$ is smooth, $u \in C^1(\bar{\Omega})$ and $f$ is locally Lipschitz continuous and nonnegative in the closed interval $[0, \infty)$, then again by Corollary 2.2, $|Du|^{m-2} u_{x_i} \in W^{1,2}(\Omega)$, and $f(u) \in W^{1,2}(\Omega)$, so by density we can consider $\varphi \in W_0^{1,2}(\Omega)$. $\square$

The results proved in this section allow us finally to get the summability properties of the inverse of the weight $\rho = |Du|^{m-2}$ stated in the introduction.

**Theorem 2.3.** *Let $\Omega$ be a smooth domain in $\mathbb{R}^N$ $u \in C^1(\bar{\Omega})$ be a weak solution of* (1.1) *with $f$ satisfying* $(*)$ *and $f(s) > 0$ for $s > 0$, $1 < m < +\infty$. Then, for any $x \in \Omega$ and for* *every $r<1$, we have that* ($|Z|=0$ *and*)

$$\int_{\Omega} \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^{\gamma}}\, dy \leqslant C,$$

*where $C$ does not depend on $x$, $\gamma < N-2$ if $N \geqslant 3$ and $\gamma = 0$ if $N = 2$.*

**Proof.** Since $f$ is positive, by Hopf's Lemma, there exists $E$ such that $Z \subset\subset E \subset\subset \Omega$. Moreover we can suppose $\operatorname{dist}(Z, \partial E) > 0$. Since $(\Omega \backslash E) \cap Z = \emptyset$, it follows that

$$\int_{\Omega\backslash E} \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^{\gamma}}\, dy \leqslant \frac{1}{\min_{\Omega\backslash E} |Du|^{(m-1)r}} \int_{\Omega\backslash E} \frac{1}{|x-y|^{\gamma}}\, dy \leqslant C$$

and therefore to prove the theorem it is sufficient to show that for every $x \in \Omega$ we have that

$$\int_{E} \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^{\gamma}}\, dy \leqslant C,$$

where $C$ does not depend on $x$. Finally the same arguments in the proof of Theorem 2.2 allow to reduce to proving that, considering only $x \in E$,

$$\int_{E} \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^{\gamma}}\, dy \leqslant C,$$

where $C$ does not depend on $x \in E$.

Let now $\varphi_{\varepsilon,x}$ be defined as in Theorem 2.2 and define

$$\psi_{\varepsilon,x} = \frac{1}{(|Du|^{m-1} + \varepsilon)^r} \frac{\varphi_{\varepsilon,x}}{|x-y|^{\gamma}}.$$

Since $|Du|^{m-1} \in W^{1,2}(\Omega)$, its gradient vanishes a.e. in the critical set $Z$ and $\psi_{\varepsilon,x}$ can be used as test function in (1.1). Since $u \geqslant n > 0$ in $E$, by the positivity hypothesis on $f$, we have $f(u(y)) \geqslant \frac{1}{C_1} > 0$ for any $y \in E$, so that we get

$$\begin{aligned}
\int_E \psi_{\varepsilon,x}\, dy &\leqslant C_1 \int_E \psi_{\varepsilon,x} f(u)\, dy \leqslant C_1 \int_\Omega \psi_{\varepsilon,x} f(u)\, dy \\
&\leqslant C_1 \int_\Omega |Du|^{m-2}(Du, D\psi_{\varepsilon,x})\, dy \leqslant \int_{\Omega\setminus E_\delta} \frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^r} \frac{|D\varphi_{\varepsilon,x}|}{|x-y|^\gamma}\, dy \\
&\quad + \int_{B_{2\varepsilon}(x)\setminus B_\varepsilon(x)} \frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^r} \frac{|D\varphi_{\varepsilon,x}|}{|x-y|^\gamma}\, dy \\
&\quad + C_2 \int_\Omega \frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^r} \frac{\varphi_{\varepsilon,x}}{|x-y|^{\gamma+1}}\, dy \\
&\quad + C_2 \int_{E\setminus Z} \frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^{r+1}} \frac{|Du|^{m-2}||D^2u||}{|x-y|^\gamma}\, \varphi_{\varepsilon,x}\, dy \\
&\quad + C_2 \int_{\Omega\setminus E} \frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^{r+1}} \frac{|Du|^{m-2}||D^2u||}{|x-y|^\gamma}\, \varphi_{\varepsilon,x}\, dy.
\end{aligned}$$

Since $r<1$, we have $\frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^r} \leqslant c$ in $\Omega$ and, since we are supposing $x \in E$, we have $||\,|D\varphi_{\varepsilon,x}|\,||_{L^\infty(\Omega\setminus E_\delta)} < \infty$. Since $u$ is regular in $\Omega\setminus E$ and $\operatorname{dist}(Z, \partial E) > 0$, we have $||\,|Du|^{m-2}||D^2u||\,||_{L^\infty((\Omega\setminus E)\cap A)} < \infty$, where $A$ is such that $\operatorname{supp}(\varphi_{\varepsilon,x}) \subset A \subset\subset \Omega$ for every $\varepsilon$ and $x$. Moreover, since $Z \subset E$ and $\operatorname{dist}(Z, \partial E) > 0$, then $\frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^{r+1}} \leqslant c_2$ in $\Omega\setminus E$. Therefore

$$\begin{aligned}
&\int_{\Omega\setminus E} \frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^r} \frac{|D\varphi_{\varepsilon,x}|}{|x-y|^\gamma}\, dy \\
&\quad + \int_{\Omega\setminus E} \frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^{r+1}} \frac{|Du|^{m-2}||D^2u||}{|x-y|^\gamma}\, \varphi_{\varepsilon,x}\, dy \\
&\quad \leqslant c_3 \int_{\Omega\setminus E} \frac{1}{|x-y|^\gamma}\, dy \leqslant c_4,
\end{aligned}$$

where $c_4$ does not depend on $x$. In the same way

$$\begin{aligned}
&\int_\Omega \frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^r} \frac{\varphi_{\varepsilon,x}}{|x-y|^{\gamma+1}}\, dy \\
&\quad \leqslant c_5 \int_\Omega \frac{1}{|x-y|^{\gamma+1}}\, dy \leqslant c_6,
\end{aligned}$$

where $c_6$ does not depend on $x$. As in Theorem 2.2 we also get

$$\int_{B_{2\varepsilon}(x)\setminus B_\varepsilon(x)} \frac{|Du|^{m-1}}{(|Du|^{m-1}+\varepsilon)^r} \frac{|D\varphi_{\varepsilon,x}|}{|x-y|^\gamma}\, dy \leqslant c_7(x) \frac{\varepsilon^N}{\varepsilon^{\gamma+1}}.$$

Therefore, for $\varepsilon$ sufficiently small, we can write for any $\beta < 1$

$$\int_E \frac{\varphi_{\varepsilon,x}}{(|Du|^{m-1} + \varepsilon)^r |x-y|^\gamma}\, dy \leqslant C_3 + C_4 \int_{E\setminus Z} \frac{|Du|^{\frac{m-2-\beta}{2}} ||D^2 u||}{|x-y|^{\frac{\gamma}{2}}} (\varphi_{\varepsilon,x})^{\frac{1}{2}} \times \frac{|Du|^{\frac{m-2+\beta}{2}}}{|Du|^{\frac{(m-1)r}{2}}} \frac{(\varphi_{\varepsilon,x})^{\frac{1}{2}}}{(|Du|^{m-1}+\varepsilon)^{\frac{r}{2}}} \frac{1}{|x-y|^{\frac{\gamma}{2}}}\, dy.$$

Note that here we do not need to consider the case $x \in Z$ and $x \in E\setminus Z$ separately. If now we choose $\beta < 1$ such that $r = \frac{m-2+\beta}{m-1} < 1$, using Young's inequality as in Theorem 2.2, we can choose $\sigma$ small such that

$$(1-\sigma) \int_E \frac{\varphi_{\varepsilon,x}}{(|Du|^{m-1} + \varepsilon)^r} \frac{1}{|x-y|^\gamma}\, dy \leqslant C_3 + C_5 \int_{E\setminus Z} \frac{|Du|^{m-2-\beta} ||D^2 u||^2}{|x-y|^\gamma}\, dy.$$

Therefore, by Theorem 2.2,

$$\int_E \frac{\varphi_{\varepsilon,x}}{(|Du|^{m-1} + \varepsilon)^r} \frac{1}{|x-y|^\gamma}\, dy \leqslant C,$$

where $C$ does not depend on $x$.

Since $\frac{\varphi_{\varepsilon,x}}{(|Du|^{m-1}+\varepsilon)^r} \frac{1}{|x-y|^\gamma} \xrightarrow{\varepsilon \to 0} \frac{1}{|Du|^{(m-1)r}} \frac{1}{|x-y|^\gamma}$ almost everywhere in $E\setminus Z$, while it tends to $+\infty$ in $Z$, by Fatou's Lemma we get that $|Z| = 0$ and the thesis. $\square$

As a consequence we get the following summability result for the a.e. defined second derivatives of $u$.

**Proposition 2.1.** *Let $\Omega$ be a smooth domain, $u \in C^1(\bar{\Omega})$ be a weak solution of* (1.1), *and suppose that $f$ is locally Lipschitz continuous in the closed interval* $[0, \infty)$ *and $f(s) > 0$ for $s > 0$. Then $|u_{x_i x_j}| \in L^2(\Omega)$ if $1 < m < 3$. If otherwise $m \geqslant 3$, then $|u_{x_i x_j}| \in L^p(\Omega)$ with $p < \frac{m-1}{m-2}$.*

**Proof.** By Corollary 2.1, $|Du|^{\frac{m-2-\beta}{2}} u_{x_i x_j} \in L^2(\Omega)$ for every $\beta < 1$, proving the thesis for $1 < m < 3$. Moreover by Theorem 2.3 we know that $\frac{1}{|Du|^{(m-1)r}} \in L^1(\Omega)$ for every $r < 1$. Consider $|u_{x_i x_j}|^p$ as product of two functions in the following way:

$$|u_{x_i x_j}|^p \equiv |u_{x_i x_j}|^p |Du|^{\frac{m-2-\beta}{2} p} \frac{1}{|Du|^{\frac{m-2-\beta}{2} p}}.$$

If $m \geqslant 3$ and $p < \frac{m-1}{m-2}$ then $p < 2$. Therefore we can choose $\beta < 1$ such that $\frac{m-2-\beta}{2} p (\frac{2}{2-p}) < m-1$ (because for $m \geqslant 3$ and $\beta = 1$ we have that $\frac{m-2-\beta}{2} p (\frac{2}{2-p}) < m-1$

iff $p<\frac{m-1}{m-2}$), and we get that $|u_{x_ix_j}|^p|Du|^{\frac{m-2-\beta}{2}p}\in L^{\frac{2}{p}}$, and $\frac{1}{|Du|^{\frac{m-2-\beta}{2}p}}\in L^{\frac{2}{2-p}}$. By Hölder's inequality we get the thesis. $\square$

We can now easily prove that $u_{x_i}\in H^{1,2}_\rho(\Omega)$. To this end we have to show that the distributional derivatives of $u_{x_i}$ are measurable functions. More generally let us prove the following:

**Proposition 2.2.** *Let $\Omega$ be a smooth domain, $u\in C^1(\bar{\Omega})$ be a weak solution of* (1.1), *and suppose that $f$ is locally Lipschitz continuous in $[0,\infty)$ and $f(s)>0$ for $s>0$. Then if $1<m<3$, $u_{x_i}\in W^{1,2}(\Omega)$, while if $m\geqslant 3$ then $u_{x_i}\in W^{1,p}(\Omega)$, $\forall i=1,\dots,N$ for every $p<\frac{m-1}{m-2}$. Moreover the generalized derivatives of $u_{x_i}$ coincide with the classical ones, both denoted with $u_{x_ix_j}$, almost everywhere in $\Omega$.*

*Finally $u_{x_i}\in H^{1,2}_\rho(\Omega)$.*

**Proof.** Let $G_\varepsilon$ be defined as in Theorem 2.2. Integrating by parts we get

$$\int_\Omega G'_\varepsilon(u_{x_i})\tilde{u}_{ij}\varphi\,dx=-\int_\Omega G_\varepsilon(u_{x_i})\varphi_{x_j}\,dx\quad\forall\varphi\in C^\infty_c(\Omega).$$

For $\varepsilon\to 0$, since $\tilde{u}_{ij}\in L^1(\Omega)$ and $G'_\varepsilon$ is bounded, we can use Lebesgue's Dominated Convergence Theorem and get

$$\int_\Omega \tilde{u}_{ij}\varphi\,dx=-\int_\Omega u_{x_i}\varphi_{x_j}\,dx\quad\forall\varphi\in C^\infty_c(\Omega)$$

which shows that $\tilde{u}_{ij}$ are the second distributional derivatives. In the case of $f$ positive we know that $|Z|=0$, so that $u_{x_ix_j}\equiv\tilde{u}_{ij}$ a.e. (more precisely in $\Omega\backslash Z$). Finally all the integrability properties have been already proved. $\square$

## 3. Weighted Poincaré type inequality and weak comparison principle

In this section we prove a weighted Poincaré type inequality, and then we use it to prove a weak comparison principle in small domains. Let us start by recalling some known results about the potential of a function. If $f\in L^a(\Omega)$, $a\geqslant 1$, and $0<\alpha<N$ then the potential of order $\alpha$ generated by $f$ is defined by

$$U_\alpha[f](x)=\int_\Omega f(y)|x-y|^{\alpha-N}\,dy.$$

If $1<a<\frac{N}{\alpha}$ denoting by $b$ the number defined by $\frac{1}{b}=\frac{1}{a}-\frac{\alpha}{N}$, one can show that the linear map $f\in L^a(\Omega)\to L^b(\Omega)\ni U_\alpha[f]$ is continuous.

More precisely there is a constant $C = C(N, \alpha, a) > 0$ such that for any $q$, $1 \leqslant q \leqslant b$,

$$||U_\alpha[f]||_q \leqslant C|\Omega|^{\frac{1}{q}-\frac{1}{b}}||f||_a. \tag{3.1}$$

If instead $a > \frac{N}{\alpha}$ then (3.1) holds for any $q \leqslant \infty$ (and $\frac{1}{b} = \frac{1}{a} - \frac{\alpha}{N}$ negative in this case), while if $a = \frac{N}{\alpha}$ then (3.1) holds for every $q < b = +\infty$ with $C = C_q$ depending on $q$ in this case.

Suppose now that $\rho \in L^1(\Omega)$, $\frac{1}{\rho} \in L^t(\Omega)$ with $t > \frac{N}{p}$, $t > 1$ and $1 + \frac{1}{t} < p < N(1 + \frac{1}{t})$. Let now $p^\sharp$ be defined as

$$\frac{1}{p^\sharp} = \frac{1}{p}\left(1 + \frac{1}{t}\right) - \frac{1}{N}. \tag{3.2}$$

Using the above estimates, in [19,26], the following Sobolev inequality is proved for any function $u$ in the weighted Sobolev space $H^{1,p}_{0,\rho}(\Omega)$ (see Definition 2.1)

$$||u||_{L^{p^\sharp}(\Omega)} \leqslant C(|\Omega|)||Du||_{L^p(\Omega,\rho)}, \tag{3.3}$$

where $C(|\Omega|) \to 0$ if $|\Omega| \to 0$.

If $t > N/p$, then $p^\sharp > p$, and by Hölder's inequality we get a weighted Poincaré's inequality

$$||u||_{L^p(\Omega)} \leqslant \tilde{C}(|\Omega|)||Du||_{L^p(\Omega,\rho)}, \tag{3.4}$$

where $\tilde{C}(|\Omega|) \to 0$ if $|\Omega| \to 0$. The same inequality holds if $p \geqslant N(1 + \frac{1}{t})$, provided $t > N/p$.

In the case of problem (1.1) the weighted space which is naturally associated to this equation, is $H^{1,2}_\rho(\Omega)$ with $\rho \equiv |Du|^{m-2}$. If $\Omega$ is a ball, then under suitable hypothesis (see [5,7]) every solution is radial and, as shown in [1] it follows that the gradient of $u$ vanishes only at a point , e.g. in 0, and $|Du| \approx |x|^{\frac{1}{m-1}}$. This implies that the condition $\frac{1}{\rho} \in L^t(\Omega)$ with $t > \frac{N}{2}$ is satisfied in the case $m \geqslant 2$ (while if $1 < m < 2$ the condition $\rho \in L^1(\Omega)$ is satisfied if $m > \frac{N+2}{N+1}$).

In a general domain, having proved that $\frac{1}{|Du|^{(m-1)r}} \in L^1(\Omega)$ for every $r < 1$, we get that $\frac{1}{\rho} \in L^t(\Omega)$ with $t > \frac{N}{2}$ if $N = 2$ or $N \geqslant 3$ and $m < \frac{2N-2}{N-2}$ which allows to obtain (3.4) in this case.

In order to avoid such restrictions on $m$, in what follows we will use the estimates proved in Section 2 to handle the general case.

We begin by proving general Sobolev and Poincaré type inequalities, using potential estimates as in [19,26].

**Theorem 3.1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ and let $\rho \in L^1(\Omega)$ be a positive weight function such that*

$$\int_\Omega \frac{1}{\rho^t |x-y|^\gamma}\, dy \leqslant C \quad \forall x \in \Omega,$$

*where $t > 1$, $0 \leqslant \gamma < N$ and $C$ does not depend on $x$.*

*Assume also that $p > 1$ satisfy $1 + \frac{1}{t} < p$ and $t > \frac{N-\gamma}{p}$.*

*If $p < N(1+\frac{1}{t}) - \frac{\gamma}{t}$ then there exists a constant $c_0 = c_0(N, p, \rho, t, \gamma)$ such that the following weighted Sobolev's inequality holds for any $u \in H^{1,p}_{0,\rho}(\Omega)$:*

$$||u||_{L^{p^*}} \leqslant c_0 ||Du||_{L^p(\Omega,\rho)}, \tag{3.5}$$

*where $p^*$ is defined by*

$$\frac{1}{p^*} = \frac{1}{p}\left(1 + \frac{1}{t}\right) - \frac{1}{N} - \frac{\gamma}{N(pt)}.$$

*If $(1 + \frac{1}{t} \langle p, t \rangle \frac{N-\gamma}{p}$ and) $p = N(1+\frac{1}{t}) - \frac{\gamma}{t}$ then*

$$||u||_{L^q} \leqslant c_q ||Du||_{L^p(\Omega,\rho)}, \tag{3.6}$$

*for any $u \in H^{1,p}_{0,\rho}(\Omega)$ and for every $q > 1$.*

*If instead $(1 + \frac{1}{t} \langle p, t \rangle \frac{N-\gamma}{p}$ and) $p > N(1+\frac{1}{t}) - \frac{\gamma}{t}$ then we get*

$$||u||_{L^\infty} \leqslant c_0 ||Du||_{L^p(\Omega,\rho)} \tag{3.7}$$

*for any $u \in H^{1,p}_{0,\rho}(\Omega)$.*

*Finally for any $p$ such that $p > 1 + \frac{1}{t}$ and $t > \frac{N-\gamma}{p}$ we get the following weighted Poincaré's inequality for any $u \in H^{1,p}_{0,\rho}(\Omega)$:*

$$||u||_{L^p(\Omega)} \leqslant C(|\Omega|) ||Du||_{L^p(\Omega,\rho)}, \tag{3.8}$$

*where $C(|\Omega|) \to 0$ if $|\Omega| \to 0$.*

**Proof.** By density arguments we may suppose $u \in C^1_c(\Omega)$, so that there exists a constant $C_N$, depending only on $N$, such that, for every $x \in \Omega$, we get

$$\begin{aligned} |u(x)| &\leqslant C_N \int_\Omega \frac{|Du(y)|}{|x-y|^{N-1}}\, dy \leqslant C_N \int_\Omega \frac{|Du(y)|\rho^{\frac{1}{p}}}{|x-y|^{N-1-\frac{\gamma}{pt}}} \frac{1}{\rho^{\frac{1}{p}} |x-y|^{\frac{\gamma}{pt}}}\, dy \\ &\leqslant C_N \left( \int_\Omega \frac{1}{\rho^t |x-y|^\gamma}\, dy \right)^{\frac{1}{pt}} \left|\left| \frac{|Du(y)|\rho^{\frac{1}{p}}}{|x-y|^{N-1-\frac{\gamma}{pt}}} \right|\right|_{L^{(pt)'}(\Omega)} \end{aligned} \tag{3.9}$$

Let us set

$$f(y) = (|Du(y)|\rho^{\frac{1}{p}})^{(pt)'}.$$

If $N - 1 - \frac{\gamma}{pt} \leqslant 0$, then by (3.9), since $(pt)' < p$ by the hypothesis $1 + \frac{1}{t} < p$, we get immediately

$$||u||_{L^\infty} \leqslant K_1 |||Du|\rho^{\frac{1}{p}}||_{L^{(pt)'}} \leqslant K_2 |||Du|\rho^{\frac{1}{p}}||_{L^p} = K_2 ||Du||_{L^p(\Omega,\rho)}.$$

If instead $(N - 1 - \frac{\gamma}{pt}) > 0$, let us set $N - \alpha = (N - 1 - \frac{\gamma}{pt})(pt)'$ and get

$$|u(x)| \leqslant C_N C^{\frac{1}{pt}} |U_\alpha[(|Du(y)|\rho^{\frac{1}{p}})^{(pt)'}]|^{\frac{1}{(pt)'}}.$$

Note that we use the fact that $t > \frac{N-\gamma}{p}$ to get $\alpha = N - (N - 1 - \frac{\gamma}{pt})(pt)' > 0$.

Moreover, since $|Du|\rho^{\frac{1}{p}} \in L^p(\Omega)$ we get $f \in L^{\frac{p}{(pt)'}}$ where $\frac{p}{(pt)'} > 1$ by the assumption $p > 1 + \frac{1}{t}$.

Let us consider first the case $p < N(1 + \frac{1}{t}) - \frac{\gamma}{t}$. In this case (it is easy to see that $(N - 1 - \frac{\gamma}{pt}) > 0$, and ) $\frac{(pt)'}{p} > \frac{\alpha}{N}$, so that we can set $b > 1$ such that

$$\frac{1}{b} = \frac{(pt)'}{p} - \frac{\alpha}{N}.$$

Therefore $U_\alpha[f] \in L^b(\Omega)$ and, for every $\theta \geqslant (pt)'$ we have

$$||u||_{L^\theta} \leqslant C_N C^{\frac{1}{pt}} |||U_\alpha[f]|^{\frac{1}{(pt)'}}||_{L^\theta} = C_N C^{\frac{1}{pt}} ||U_\alpha[f]||^{\frac{1}{(pt)'}}_{L^{\frac{\theta}{(pt)'}}}. \tag{3.10}$$

Taking $\theta = b(pt)'$, by (3.1), we get

$$||u||_{L^{b(pt)'}} \leqslant C(C_N)^{\frac{1}{pt}} C_2 ||(|Du|\rho^{\frac{1}{p}})^{(pt)'}||^{\frac{1}{(pt)'}}_{\frac{p}{(pt)'}} \leqslant c_0 ||Du||_{L^p(\Omega,\rho)}. \tag{3.11}$$

Since $b(pt)' = p^*$ we get (3.5).

If $p = N(1 + \frac{1}{t}) - \frac{\gamma}{t}$ we get (3.10) and (3.11) for every $\theta > (pt)'$ and therefore we prove (3.6). If otherwise $p > N(1 + \frac{1}{t}) - \frac{\gamma}{t}$ we also get $\theta = +\infty$ in (3.10) and (3.7) follows.

Finally, let us note that if $p \geqslant N(1 + \frac{1}{t}) - \frac{\gamma}{t}$ and $t > \frac{N-\gamma}{p}$, then Poincaré's inequality (3.8) follows immediately by (3.6) and (3.7). Otherwise, if $p < N(1 + \frac{1}{t}) - \frac{\gamma}{t}$, by the assumption $t > \frac{N-\gamma}{p}$, we get $p^* > p$ and, by Hölder's inequality

$$||u||_{L^p(\Omega)} \leqslant ||u||_{L^{p^*}(\Omega)} |\Omega|^{\frac{1}{p}-\frac{1}{p^*}} \leqslant c_0 |\Omega|^{\frac{1}{p}-\frac{1}{p^*}} ||Du||_{L^p(\Omega,\rho)}$$

which proves (3.8). $\square$

We will now apply this result to the case $\rho = |Du|^{m-2}$, $m \geqslant 2$ and $u$ is a weak solution of (1.1).

**Theorem 3.2.** *Let $u \in C^1(\bar{\Omega})$ be a weak solution of* (1.1) *with $f$ satisfying* $(*)$ *and $f(s) > 0$ for $s > 0$, $m \geqslant 2$. Then, if we consider $\rho = |Du|^{m-2}$ we get, for every $p \geqslant 2$*

$$||v||_{L^p(\Omega)} \leqslant C(|\Omega|) ||Dv||_{L^p(\Omega,\rho)} \quad \textit{for every } v \in H^{1,p}_{0,\rho}(\Omega). \tag{3.12}$$

*where $C(|\Omega|) \to 0$ if $|\Omega| \to 0$.*

*In particular* (3.12) *holds for every $v \in H^{1,2}_{0,\rho}(\Omega)$.*

**Proof.** Since $u \in C^1(\bar{\Omega})$ and $m \geqslant 2$, obviously $\rho = |Du|^{m-2} \in L^1(\Omega)$. By Theorem 2.3 we have

$$\int_\Omega \frac{1}{\rho^t |x-y|^\gamma} \, dy \leqslant C,$$

where $C$ does not depend on $x$, $t < \frac{m-1}{m-2}$ and $\gamma < N-2$. Thus we have that $t > \frac{N-\gamma}{p}$ if $\frac{m-1}{m-2} > \frac{2}{p}$ and $\gamma$ is sufficiently close to $N-2$. Therefore, for $p \geqslant 2$ and $m \geqslant 2$, the condition $t > \frac{N-\gamma}{p}$ is always verified.

Moreover we have $p > 1 + \frac{1}{t}$ since $t > 1$. Therefore we can apply Theorem 2.3, to get the thesis for $v \in H^{1,p}_{0,\rho}(\Omega)$. $\square$

Note that usually the case $p = 2$, which gives a Hilbert space $H^{1,2}_{0,\rho}(\Omega)$, is considered. Therefore the condition $p \geqslant 2$ is not restrictive.

Moreover if $m \geqslant 2$, $p \geqslant 2$ and $v \in W^{1,p}_0(\Omega)$, the same conclusion holds. In fact, being $u \in C^1(\bar{\Omega})$, and $m \geqslant 2$, $\rho = |Du|^{m-2}$ is bounded, so that $W^{1,p}_0(\Omega) \hookrightarrow H^{1,p}_{0,\rho}(\Omega)$.

The previous inequality allows us to prove the following:

**Theorem 3.3** (Weak Comparison Principle). *Suppose that either $1 < m < 2$ and $u, v \in W^{1,\infty}(\Omega)$; or $m \geqslant 2$, $u, v \in W^{1,m}(\Omega) \cap L^\infty(\Omega)$, where either $\rho \equiv |Du|^{m-2}$ or*

$\rho \equiv |Dv|^{m-2}$ *satisfy condition* (1.7), *namely*

$$\int_\Omega \frac{1}{\rho^t}\frac{1}{|x-y|^\gamma}\,dy \leqslant C,$$

*where C does not depend on* $x \in \Omega$, $\gamma \langle N, t \rangle 1$ *and* $t > \frac{N-\gamma}{2}$.

*Suppose that u, v weakly solve*

$$-\mathrm{div}\ (|Du|^{m-2}Du) + g(x,u) - \Lambda u \leqslant -\mathrm{div}\ (|Dv|^{m-2}Dv) + g(x,v) - \Lambda v \ \textit{in}\ \Omega, \tag{3.13}$$

*where* $\Lambda \geqslant 0$ *and* $g \in C(\bar{\Omega} \times \mathbb{R})$ *is such that for every* $x \in \Omega$, $g(x,s)$ *is nondecreasing for* $|s| \leqslant \max\{||u||_{L^\infty}, ||v||_{L^\infty}\}$.

*Let* $\Omega' \subseteq \Omega$ *be open and suppose* $u \leqslant v$ *on* $\partial\Omega'$, *then there exists* $\delta > 0$ *such that, if* $|\Omega'| \leqslant \delta$, *then* $u \leqslant v$ *in* $\Omega'$. *If* $\Lambda = 0$ *the thesis is true for every* $\Omega' \subseteq \Omega$.

*In particular the result holds if either u or v is a* $C^1(\bar{\Omega})$ *weak solutions of* (1.1) *with f satisfying* $(*)$ *and* $f(s) > 0$ *for* $s > 0$.

**Proof.** The case $1 < m < 2$ has been considered in [6] and from now we suppose $m > 2$. Let us consider in $\Omega'$ the function $(u-v)^+$. It is bounded, it vanishes on $\partial\Omega'$ and it belongs to $W_0^{1,m}(\Omega)$, so that (see Definition 2.1) it belongs to $H_{0,\rho}^{1,2}(\Omega') \cap L^\infty(\Omega')$ and can be used as test function in (3.13), obtaining

$$\begin{aligned}&\int_{[u\geqslant v]} (|Du|^{m-2}Du - |Dv|^{m-2}Dv)(Du - Dv)dx \\ &\quad + \int_{[u\geqslant v]} [g(x,u) - g(x,v)](u-v)\,dx - \Lambda \int_{[u\geqslant v]} (u-v)^2\,dx \leqslant 0,\end{aligned} \tag{3.14}$$

where $[u \geqslant v] = \{x \in \Omega' : u(x) \leqslant v(x)\}$. Moreover $g(x,u) \leqslant g(x,v)$ if $u \leqslant v$, so that

$$\int_{[u\geqslant v]} (|Du|^{m-2}Du - |Dv|^{m-2}Dv)(Du - Dv)\,dx \leqslant \Lambda \int_{[u\geqslant v]} (u-v)^2\,dx. \tag{3.15}$$

By standard estimates (see e.g. [6, Lemma 2.1],), the following inequality follows

$$\int_{\Omega'} (|Du|^{m-2} + |Dv|^{m-2})|D(u-v)^+|^2\,dx \leqslant C_m\,\Lambda \int_{\Omega'} [(u-v)^+]^2\,dx, \tag{3.16}$$

where $C_m$ depends on $m$, so that

$$\int_{\Omega'} |D(u-v)^+|^2 \rho\,dx \leqslant C_m\,\Lambda \int_{\Omega'} [(u-v)^+]^2\,dx, \tag{3.17}$$

where we can take $\rho \equiv |Du|^{m-2}$ or $\rho \equiv |Dv|^{m-2}$. By Poincarè's inequality with weight Theorem 3.2, we get

$$\int_{\Omega'} |D(u-v)^+|^2\,\rho\,dx \leqslant C_m\,\Lambda\,C(|\Omega'|) \int_{\Omega'} |D(u-v)^+|^2\,\rho\,dx. \tag{3.18}$$

A contradiction occurs if $C_m\, \Lambda\, C(|\Omega'|) < 1$, unless $(u-v)^+ = 0$ in $\Omega'$, i.e. $u \leqslant v$ in $\Omega'$. (Let us recall that the integral in the last inequality define a norm). If $\Lambda = 0$, the same arguments prove the result for every $\Omega' \subseteq \Omega$. $\square$

**Remark 3.1.** Let us point out that the parameters in the previous result may depend only on $u$. This will be useful in the study of symmetry where $v \equiv u_\lambda$ is not fixed.

We end the section by recalling the following result, which we will use in Section 4 (see [6]).

**Theorem 3.4** (Strong Comparison Principle). *Let $1 < m < \infty$, and $u, v \in C^1(\Omega)$ satisfy*

$$-\operatorname{div}(|Du|^{m-2}Du) + \Lambda u \leqslant -\operatorname{div}(|Dv|^{m-2}Dv) + \Lambda v, \quad u \leqslant v \ \text{in}\ \Omega. \tag{3.19}$$

*Define $Z_{u,v} = \{x \in \Omega : |Du(x)| + |Dv(x)| = 0\}$ if $m \neq 2$, $Z_{u,v} = \emptyset$ if $m = 2$. If $x_0 \in \Omega \backslash Z_{u,v}$ and $u_{x_0} = v_{x_0}$ then $u \equiv v$ in the connected component of $\Omega \backslash Z_{u,v}$ containing $x_o$.*

**Remark 3.2.** Theorems 3.3 and 3.4 apply for solutions $u$ of (1.1) once we note that a function $f : I \to \mathbb{R}$ is locally Lipschitz continuous in an interval $I$ if and only if for each compact subinterval $[a,b] \subset I$ there exist two positive costants $C_1$ and $C_2$ such that

(i) $f_1(s) = f(s) - C_1 s$ is nonincreasing in $[a,b]$.
(ii) $f_2(s) = f(s) + C_2 s$ is nondecreasing in $[a,b]$.

## 4. Qualitative properties of the solutions

In this section we will study some properties of the critical set and some qualitative properties, such as monotonicity and symmetry in some directions, of solutions of (1.1).

Properties of the critical set $Z$ are very important in the study of solutions of (1.1). In particular, as we will see in Theorem 4.2, it is very useful to know whether $\Omega \backslash Z$ is connected or not. We are able to give a positive answer in the case when $f$ is positive.

**Theorem 4.1.** *Let $u \in C^1(\bar{\Omega})$ be a weak solution of (1.1) where $\Omega$ is a general bounded domain, and suppose that $f(s) > 0$ if $s > 0$. Then $\Omega \backslash Z$ does not contain any connected component $C$ such that $\bar{C} \subset \Omega$. Moreover, if we assume that $\Omega$ is a smooth bounded domain with connected boundary, it follows that $\Omega \backslash Z$ is connected.*

**Proof.** Let $C$ be a connected component of $\Omega \backslash Z$ such that $C \subset\subset \Omega$. Then

$$Du(x) = 0 \quad \forall x \in \partial C. \tag{4.1}$$

By Corollary 2.2, since $|Du|^{m-2}Du$ is continuous and identically zero on $\partial C$, we get $|Du|^{m-2}Du \in W_0^{1,2}(C, \mathbb{R}^N)$. Then there exists a vector field $A_n \in C_0^\infty(C, \mathbb{R}^N)$ which approximates $|Du|^{m-2}Du$ in the norm of $W_0^{1,2}(C, \mathbb{R}^N)$. If now $E \subset C$ is a smooth subset such that

$$\mathrm{supp}(A_n) \subset\subset E \subset\subset C$$

by the Divergence Theorem applied to $A_n$ in $E$, it follows, for every $\phi \in W^{1,2}$

$$\begin{aligned}\int_C \mathrm{div}(A_n)\phi + (A_n, D\phi)\,dx &= \int_E \mathrm{div}(A_n)\phi + (A_n, D\phi)\,dx \\ &= \int_{\partial E} \phi(A_n, \eta)\,d\sigma = 0. \end{aligned}\tag{4.2}$$

Moreover, since when $f$ is positive $|Z| = 0$, by (1.1) we get

$$-\mathrm{div}(|Du|^{m-2}Du) = f(u) \quad \text{almost everywhere in } C.$$

If now we choose $\phi \equiv k \neq 0$ then we get

$$\begin{aligned}\int_C f(u)\,\phi\,dx &= \int_C -\mathrm{div}(|Du|^{m-2}Du)\,\phi\,dx \\ &= \lim_{n\to\infty}\int_C -\mathrm{div}(A_n)\,\phi\,dx = 0\end{aligned}\tag{4.3}$$

and by (4.3)

$$\int_C f(u)\,dx = 0 \tag{4.4}$$

which is impossible when $f$ is positive.

If $\Omega$ is smooth, since $f$ is positive, by Hopf's Lemma a neighborhood of the boundary belongs to a component $C$ of $\Omega \backslash Z$. A second component $C'$ would be compactly contained in $\Omega$, which is impossible by what we have just proved. So $\Omega \backslash Z$ is connected. □

**Remark 4.1.** The proof of Theorem 4.1 shows that the same conclusion holds if $u \in W^{1,m}(\Omega)$ is a weak solution of equation

$$-\mathrm{div}(|Du|^{m-2}Du) = f(x) \quad \text{in} \quad \Omega$$

with $|Du|^{m-2}Du \in W^{1,2}(\Omega, \mathbb{R}^N)$, $f \in L^q(\Omega)$, $q > \frac{N}{m}$, $q \geqslant 2$, $1 < m < \infty$, and $f \geqslant 0$ does not vanish identically in any open subset of $\Omega$.

Now we want to prove some monotonicity and symmetry properties for solution $u$ of (1.1) with positive nonlinearities in general smooth domains. If $1 < m < 2$, this problem has been studied in [7,8] where the case of $f$ locally Lipschitz continuous but not necessarily positive is considered. In the case when $f$ is positive we extend the result to the case $m>2$ using all the regularity results in previous sections and the Alexandrov–Serrin moving planes method, following the approach of Berestycki and Nirenberg in [3].

Moreover in the case of a positive nonlinearity $f$, even in the case $1<m<2$ we simplify considerably the proof of the same result in [7], using Theorem 4.1 to exclude local symmetry phenomena, avoiding the long and technical analysis in [7]. We also extend the result to a more general class of domains (see Remark 1.2).

We can now prove the following result (see Section 1 for notations).

**Theorem 4.2.** *Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$, $N\geqslant 2$, $1<m<\infty$, $f:[0,\infty)\to\mathbb{R}$ a continuous function which is strictly positive and locally Lipschitz continuous in $(0,\infty)$, and $u\in C^1(\bar{\Omega})$ a weak solution of* (1.1).

*For any direction $v$ and for $\lambda$ in the interval $(a(v),\lambda_1(v)]$ we have*

$$u(x)\leqslant u(x_\lambda^v) \quad \forall x\in\Omega_\lambda^v. \tag{4.5}$$

*Moreover, for any $\lambda$ with $a(v)<\lambda<\lambda_1(v)$ we have*

$$u(x)<u(x_\lambda^v) \quad \forall x\in\Omega_\lambda^v\backslash Z_\lambda^v, \tag{4.6}$$

*where $Z_\lambda^v\equiv\{x\in\Omega_\lambda^v: Du(x)=Du_\lambda^v(x)=0\}$. Finally*

$$\frac{\partial u}{\partial v}(x)>0 \quad \forall x\in\Omega_{\lambda_1(v)}^v\backslash Z, \tag{4.7}$$

*where $Z=\{x\in\Omega: Du(x)=0\}$.*

*If $f$ is locally Lipschitz continuous in the closed interval $[0,\infty)$ then* (4.5) *hold for any $\lambda$ in the interval $(a(v),\lambda_2(v))$ and* (4.7) *holds for any $x\in\Omega_{\lambda_2(v)}^v\backslash Z$.*

**Proof.** Let us first suppose that $f$ is locally Lipschitz continuous in the closed interval $[0,\infty)$. Since $\Omega$ is smooth $\Lambda_2(v)$ is nonempty for any direction $v$. For $a(v)<\lambda<\lambda_2(v)$ we can compare $u$ and $u_\lambda^v\equiv u(x_\lambda^v)$, using Theorems 3.3 and 3.4 taking into account Remark 3.2, since $u_\lambda^v$ satisfies the same equation $-\Delta_m(u_\lambda^v)=f(u_\lambda^v)$ in $\Omega_\lambda^v$.

In particular if $\lambda-a(v)$ is small, then $|\Omega_\lambda^v|$ is small.

Hence, by the Weak Comparison Principle in small domains (see Theorem 3.3), since $u\leqslant u_\lambda^v$ on $\partial\Omega_\lambda^v$, it follows that $u\leqslant u_\lambda^v$ in $\Omega_\lambda^v$ if $\lambda-a(v)$ is small, so that $\Lambda_0(v)\neq\emptyset$ (recall that we put $\Lambda_0(v)=\{\lambda>a(v): u\leqslant u_\lambda^v\ \forall\mu\in(a(v),\lambda]\}$ and $\lambda_0(v)=\sup\Lambda_0(v)$).

Suppose now by contradiction that $\lambda_0(v)<\lambda_2(v)$. By continuity it follows $u_{\lambda_0(v)}^v\geqslant u$ in $\Omega_{\lambda_0(v)}^v$. By the Strong Comparison Principle (see Theorem 3.4) if $C$ is a connected component of $\Omega_{\lambda_0(v)}^v\backslash Z$, then $u_{\lambda_0}^v>u$ unless $u_{\lambda_0(v)}^v\equiv u$ in $C$.

Suppose that $\bar{C}$ is a connected component of $\Omega^{\nu}_{\lambda_0(\nu)}\backslash Z$ and that $u^{\nu}_{\lambda_0(\nu)} \equiv u$ in $\bar{C}$. Since $Z \cap \partial\Omega = \emptyset$ by the Hopf's Lemma, we get that $\partial C \backslash T^{\nu}_{\lambda_0(\nu)} \subset Z$. Moreover, by the local symmetry, we get that $\partial C \backslash T^{\nu}_{\lambda_0(\nu)} \cup R^{\nu}_{\lambda_0(\nu)}(\partial C \backslash T^{\nu}_{\lambda_0(\nu)}) \subset Z$, showing that $\Omega \backslash Z$ would be not connected. Since $\Omega \backslash Z$ is connected by Theorem 4.1, a contradiction occurs, showing that $u^{\nu}_{\lambda_0(\nu)} > u$ in any connected component of $\Omega^{\nu}_{\lambda_0(\nu)}\backslash Z$.

Let now $A$ be an open set such that $Z \cap \Omega^{\nu}_{\lambda_0(\nu)} \subset A \subset \Omega^{\nu}_{\lambda_0(\nu)}$. Since $|Z| = 0$ we can take $A$ of arbitrarily small measure. Consider a compact set $K$ in $\Omega^{\nu}_{\lambda_0(\nu)}$ such that $|\Omega^{\nu}_{\lambda_0(\nu)}\backslash K|$ is sufficiently small in order to guarantee the applicability of Theorem 3.3 (see Remark 3.1). By what we proved before, $u^{\nu}_{\lambda_0(\nu)} - u$ is positive in $K \backslash A$ which is compact. Thus $\min_{K\backslash A}(u^{\nu}_{\lambda_0(\nu)} - u) = m > 0$. By continuity there exists $\varepsilon > 0$ such that, $\lambda_0(\nu) + \varepsilon < \lambda_2(\nu)$ and for $\lambda_0(\nu) < \lambda < \lambda_0(\nu) + \varepsilon$ we have that $|\Omega^{\nu}_{\lambda}\backslash K|$ is still sufficiently small as before and $u^{\nu}_{\lambda} - u > m/2 > 0$ in $K \backslash A$. In particular $u^{\nu}_{\lambda} - u > 0$ on $\partial(K \backslash A)$. Moreover for such values of $\lambda$ we have that $u \leqslant u^{\nu}_{\lambda}$ on $\partial(\Omega^{\nu}_{\lambda}\backslash(K\backslash A))$. By the Weak Comparison Principle applied in $\Omega^{\nu}_{\lambda}\backslash(K\backslash A)$, which has small measure, we get that $u \leqslant u^{\nu}_{\lambda}$ in $\Omega^{\nu}_{\lambda}$, which contradicts the assumption $\lambda_0(\nu) < \lambda_2(\nu)$.

Therefore $\lambda_0(\nu) \equiv \lambda_2(\nu)$ and the thesis is proved.

The proof of (4.6) follows immediately by Theorem 3.4 and the first part of this Theorem. In fact if (4.6) were not true, by the Strong Comparison Principle, there would exist a component of local symmetry, against what we have just proved. Finally, to prove (4.7) let us note that, by the linearity of $L_u$, we get that $\frac{\partial u}{\partial \nu}$ weakly solves (1.2). Therefore, by the strong maximum principle for uniformly elliptic operators, we have that (4.7) holds unless $\frac{\partial u}{\partial \nu} \equiv 0$. Since this is not possible by (4.6) the thesis follows.

When $f$ is not Lipschitz up to 0, Lemma 2.2, p. 1187 in [8] works as it is in our context and shows that for any direction $\nu$ and $\lambda'$ in the interval $(a(\nu), \lambda_1(\nu)]$ there exist neighborhoods $I$ of $\partial\Omega$ and $J$ of $\lambda'$ such that we have $u(x) \leqslant u(x^{\nu}_{\lambda})$ for any $x \in \Omega^{\nu}_{\lambda} \cap I$, $\lambda \in J$.

Of course this is true only up to $\lambda_1(\nu)$ (which can be strictly lower than $\lambda_2(\nu)$), since the proof exploits the Hopf's lemma and needs that the normal to the boundary is not perpendicular to the direction $\nu$.

Far from the boundary $u$ is positive and $f$ Lipschitz continuous in the range of $u$ and the proof goes through as before using our comparison principles in smaller domains. $\square$

An immediate consequence is the following.

**Corollary 4.1.** *If $f$ is locally Lipschitz continuous in the closed interval $[0, \infty)$ and strictly positive in $(0, \infty)$, and the domain $\Omega$ is convex with respect to a direction $\nu$ and symmetric with respect to the hyperplane $T^{\nu}_0 = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}$, then $u$ is symmetric, i.e. $u(x) = u(x^{\nu}_0)$, and nondecreasing in the $\nu$–direction in $\Omega^{\nu}_0$ with $\frac{\partial u}{\partial \nu}(x) > 0$ in $\Omega^{\nu}_0\backslash Z$.*

*In particular if $\Omega$ is a ball then u is radially symmetric and $\frac{\partial u}{\partial r}<0$, where $\frac{\partial u}{\partial r}$ is the derivative in the radial direction.*

**Proof.** It is immediate from the previous theorem. Let us only note that in the case of a ball, since the level sets of the solutions are spheres, an application of Hopf's Lemma (recall that $f$ is positive) shows that 0 is the only critical point and that the derivative in the radial direction is negative in all the other points. $\square$

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## Note added in the proof

It was pointed out to us that Serrin and Zou, in their celebrated paper [28], state in the case $1<m<2$ the solution $\mu$ belongs to the Sobolev space $W^{1,2}_{\mathrm{loc}}(\Omega)$, among other regularity results for solutions of quasilinear elliptic equations.

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