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On the continuity of left translations in the *LUC*-compactification

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Abstract

Consider the continuity of left translations in the LUC-compactification G^{LUC} of a locally compact group G. For every $X \subseteq G$, let $\kappa(X)$ be the minimal cardinality of a compact covering of X in G. Let $\mathcal{U}(G)$ be the points in G^{LUC} that are not in the closure of any $X \subseteq G$ with $\kappa(X) < \kappa(G)$. We show that the points at which no left translation in $\mathcal{U}(G)$ is continuous are dense in $\mathcal{U}(G)$. This result is a generalization of a theorem by van Douwen concerning discrete groups. We obtain a new proof for the fact that the topological center of $G^{LUC} \setminus G$ is empty.

MSC: 22A15; 54D35

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1. Introduction

This paper extends and unifies the known results concerning the continuity of left translations in the LUC-compactification G^{LUC} of a locally compact group G. The most important result in this area is due to Lau and Pym [6]: the topological center of G^{LUC} is G; that is, no left translation by a member of $G^* = G^{LUC} \setminus G$ is continuous on the whole G^{LUC}. Already Ruppert [10] proved this result for significant classes of groups, including

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connected groups. In their proof, Lau and Pym used Ruppert's work to reduce the question to totally disconnected groups.

Recently, Protasov and Pym [8] have shown that the topological center of G^* is empty. In the case of σ-compact groups, they showed a far stronger result: there exist points in G^* at which *no* left translation in the semigroup G^* is continuous, and moreover, these points are dense in G^*. The main theorem of this paper extends this result to all locally compact groups, but the formulation of this extension is not the most obvious one, obtained simply by discarding the assumption of σ-compactness. In fact, Protasov [7] has shown that the existence of measurable cardinals would imply that the set of points at which no left translation is continuous is not always dense in G^*. Instead of considering the whole G^*, we shall restrict ourselves to the "uniform points" in G^{LUC}, a set coinciding with G^* when G is σ-compact. The basis for this approach is a theorem by van Douwen. Denote the set of uniform ultrafilters on a discrete group G by $\mathcal{U}(G)$, which is a closed subsemigroup of the Stone–Čech compactification βG of G. Then van Douwen's result [4, Theorem 9.7] says that points at which no left translation in $\mathcal{U}(G)$ is continuous are dense in $\mathcal{U}(G)$. (The paper [4] was published after van Douwen's death. Van Douwen announced the result [4, Theorem 9.7] in a letter to Neil Hindman but without a proof; the proof in [4] is due to Neil Hindman and Jan Pelant.)

We shall generalize van Douwen's result to all locally compact groups by replacing the Stone–Čech compactification with the *LUC*-compactification and by generalizing the concept of uniform ultrafilters. Define the compact covering number $\kappa(X)$ of $X \subseteq G$ to be the minimal cardinality of a compact covering of X in G. Let $\mathcal{U}(G)$ be the set of points in G^{LUC} that are not in the closure of any subset of G with a compact covering number less than $\kappa(G)$. Then $\mathcal{U}(G)$ is a closed subsemigroup of G^{LUC}. In the case that G is discrete, $\mathcal{U}(G)$ may be identified with the set of uniform ultrafilters on G. The main theorem of this paper reads exactly as van Douwen's theorem, but now in the context of *LUC*-compactifications of locally compact groups.

Theorem. *There is a dense set in $\mathcal{U}(G)$ consisting of points at which no left translation in the semigroup $\mathcal{U}(G)$ is continuous.*

The phrase "left translation in the semigroup $\mathcal{U}(G)$" means "left translation by a member of $\mathcal{U}(G)$ restricted to $\mathcal{U}(G)$". Besides generalizing van Douwen's result, the main theorem is also an extension of the result by Protasov and Pym because $\mathcal{U}(G) = G^*$ if G is σ-compact. Furthermore, an application of this theorem gives a new proof for the fact that the topological center of G^* is empty.

2. Prerequisites

Next we shall review the formal definitions used in this paper. Throughout the paper, G is a noncompact, locally compact group. A semigroup S that is also a topological space is called *right topological* if all right translations $s \mapsto st$ are continuous on S. The *topological center* of a right topological semigroup S is the set of all t in S such that the left translation $s \mapsto ts$ is continuous. A *semigroup compactification* of G is a compact right topological semigroup S with a continuous homomorphism $\phi : G \to S$ such that $\phi(G)$ is dense in S and is included in the topological center of S. A semigroup compactification (S, ϕ) is said to have the *joint continuity property* if the multiplication is jointly continuous on $\phi(G) \times S$. Now the *LUC-compactification* G^{LUC} of G is the largest semigroup compactification of G that has the joint continuity property ("largest" meaning that every other such semigroup compactification is a quotient of the *LUC*-compactification). Since G is locally compact, Ellis's joint continuity theorem implies that the *LUC*-compactification is, in fact, the largest among all semigroup compactifications of G. The *LUC*-compactification may be realized as the spectrum of the C*-algebra consisting of all left uniformly continuous functions on G; a function $f : G \to \mathbb{C}$ is *left uniformly continuous* if $s \mapsto \ell_s f$, where $\ell_s f(t) = f(st)$, is a continuous map from G to the space of bounded continuous functions on G with the uniform norm. Since the continuous homomorphism associated with the *LUC*-compactification of G is an isomorphism and a homeomorphism, we may identify G with its image in G^{LUC}. See [2] for a comprehensive treatment of semigroup compactifications.

In order to generalize van Douwen's theorem for locally compact groups, we shall generalize the notion of uniform ultrafilter for locally compact groups as done in [5]. For every subset X of G, let $\kappa(X)$ be the minimal cardinality of a compact covering of X (that is, the minimal number of compact sets in G needed to cover X). For simplicity, denote $\kappa(G)$ by κ. Define the "norm" of x in G^{LUC} by

$$\|x\| = \min\{\kappa(X);\ X \subseteq G \text{ and } x \in \overline{X}\}.$$

(Throughout this paper, the overbar denotes the closure in G^{LUC}.) Let $\mathcal{U}(G)$ be the set of all x in G^{LUC} with $\|x\| = \kappa$. If G is discrete, then $\mathcal{U}(G)$ corresponds to the set of uniform ultrafilters on G; if G is σ-compact, then $\mathcal{U}(G)$ is $G^* = G^{LUC} \setminus G$. Using the local structure theorem from [9], it is easy to see that $\mathcal{U}(G)$ is closed in G^{LUC}. Then the continuity of right translations implies that $\mathcal{U}(G)$ is a left ideal in G^{LUC}.

We say that $X \subseteq G$ is *left uniformly discrete* with respect to a neighborhood U of the identity if $Us \cap Ut = \emptyset$ for every $s \neq t$ in X. If X is a left uniformly discrete subset of G, then the closure of X in G^{LUC} is homeomorphic with the Stone–Čech compactification βX of the discrete space X (see, for example, [9]).

3. The results

We shall need the following lemma to prove the main theorem. In particular, this lemma shows that if H is an open subgroup of G, then $H^{LUC} \cong \overline{H}$ is a closed and open subsemigroup of G^{LUC}.

Lemma 1. *If H is an open subgroup of G and $A \subseteq G$, then $\overline{HA}$ is open in G^{LUC}.*

Proof. The characteristic function χ_{HA} is left uniformly continuous, so it has a unique continuous extension to G^{LUC}. Since this extension is the characteristic function of $\overline{HA}$, it follows that $\overline{HA}$ is open in G^{LUC}. □

In the proof of the following generalization of van Douwen's theorem, we shall use some machinery from [4], but our proof is direct and simpler than the one in [4].

Theorem 2. *There is a dense set in $\mathcal{U}(G)$ consisting of points at which no left translation in the semigroup $\mathcal{U}(G)$ is continuous.*

Proof. We split the proof into two cases: $\mathrm{cf}\,\kappa \geqslant \omega_1$ and $\mathrm{cf}\,\kappa = \omega$. (Recall that the cofinality $\mathrm{cf}\,\kappa$ of κ is the minimal cardinality of a cofinal subset of κ.) The σ-compact case is covered by [8] and with minor changes that proof works for any κ with $\mathrm{cf}\,\kappa = \omega$. For completeness, we shall sketch a proof also for this case, but the main interest lies in the case when $\mathrm{cf}\,\kappa \geqslant \omega_1$.

Case $\mathrm{cf}\,\kappa \geqslant \omega_1$. Let N be a closed neighborhood in G^{LUC} of a point in $\mathcal{U}(G)$. Our objective is to find a point x in $N \cap \mathcal{U}(G)$ such that no left translation in the semigroup $\mathcal{U}(G)$ is continuous at x. To this end, we shall construct a "thin" set T in $N \cap G$ and consider the closure $\overline{T} \cong \beta T$.

Let H be an open σ-compact subgroup of G (for example, pick any compact symmetric neighborhood U of the identity and put $H = \bigcup_{n=1}^{\infty} U^n$). The number of right cosets of H is κ because each compact set in G meets only finitely many cosets of H. Form a set $\{u_\alpha\}_{\alpha<\kappa}$ by picking exactly one representative from each right coset of H. Put

$$A_\alpha = \bigcup_{\beta \leqslant \alpha} H u_\beta \quad (\alpha < \kappa),$$

and note that $\kappa(A_\alpha) = \max\{\omega, |\alpha|\}$. Since N is a neighborhood of a point in $\mathcal{U}(G)$, it follows that $\kappa(N \cap G) = \kappa$. Therefore, we can construct by transfinite induction a subset $T = \{t_\alpha\}_{\alpha<\kappa}$ of $N \cap G$ such that

$$A_\alpha t_\alpha \cap A_\beta t_\beta = \emptyset \quad \text{whenever } \alpha \neq \beta \text{ in } \kappa. \tag{1}$$

The set T is left uniformly discrete so the closure of T in G^{LUC} is homeomorphic with βT. Denote the set $\overline{T} \cap \mathcal{U}(G)$ by $\mathcal{U}(T)$, and notice that the identification of $\overline{T}$ with βT associates $\mathcal{U}(T)$ with the set of uniform ultrafilters on the set T. Since N is closed, $\mathcal{U}(T) \subseteq N \cap \mathcal{U}(G)$.

By [4, Lemma 9.6], there is a point x in $\mathcal{U}(T)$ and a (transfinite) sequence $\{x_i\}_{i<\mathrm{cf}\,\kappa}$ in $\mathcal{U}(T)$ such that

$$x_i \to x, \quad \text{but } x \notin \overline{\{x_i\}_{i<j}} \quad \text{for any } j < \mathrm{cf}\,\kappa. \tag{2}$$

(As argued in [4], $\mathcal{U}(T)$ includes a copy of the Gleason space of 2^{2^κ} [3, proof of Corollary 7.15], so we can apply [1, Theorem 3.5] to obtain the sequence $\{x_i\}_{i<\mathrm{cf}\,\kappa}$ and the point x.) To complete this case, it suffices to show that $yx_i \not\to yx$ for any y in $\mathcal{U}(G)$.

Let $\{\lambda_i\}_{i<\mathrm{cf}\,\kappa}$ be an increasing cofinal sequence in κ. Applying (2) and recalling that $\overline{T} \cong \beta T$, we see that for every $j < \mathrm{cf}\,\kappa$ there exists a subset T_j of T such that

$$x \in \overline{T_j} \quad \text{and} \quad \overline{T_j} \cap \{x_i\}_{i<j} = \emptyset.$$

Since $\|x\| = \kappa$, we can assume without loss of generality that for every $j < \mathrm{cf}\,\kappa$

$$T_j \cap \{t_\alpha\}_{\alpha<\lambda_j} = \emptyset. \tag{3}$$

Define $\mu(\alpha) = \min\{i;\ \alpha \leqslant \lambda_i\}$. Since $\alpha \leqslant \lambda_{\mu(\alpha)}$, it follows from (3) that

$$T_{\mu(\alpha)} \cap \{t_\beta\}_{\beta<\alpha} = \emptyset \tag{4}$$

for every $\alpha < \kappa$. Put

$$X = \bigcup_{\alpha<\kappa} Hu_\alpha T_{\mu(\alpha)}.$$

Note that the closure $\overline{X}$ is open in G^{LUC} by Lemma 1. We shall prove that if $y \in \mathcal{U}(G)$, then $yx \in \overline{X}$ but $yx_i \notin \overline{X}$ for any $i < \operatorname{cf}\kappa$.

Now we shall show that $yx \in \overline{X}$ for every y in G^{LUC}. Let $s \in G$. Then s belongs to one of the right cosets of H, so $sT_{\mu(\alpha)} \subseteq X$ for some $\alpha < \kappa$. Since $x \in \overline{T_{\mu(\alpha)}}$, it follows that $sx \in s\overline{T_{\mu(\alpha)}} \subseteq \overline{X}$. Taking $s \to y$, we see that $yx \in \overline{X}$.

To prove that $yx_i \notin \overline{X}$ for any y in $\mathcal{U}(G)$ and $i < \operatorname{cf}\kappa$, we need the inclusion

$$T \cap \{s \in G;\ Hu_\gamma s \cap X \neq \emptyset\} \subseteq T_{\mu(\gamma)} \cup \{t_\delta\}_{\delta<\gamma} \quad (\gamma < \kappa). \tag{5}$$

To verify (5), suppose that $Hu_\gamma t_\alpha \cap X \neq \emptyset$ for some $\alpha < \kappa$. If $\alpha < \gamma$, we are done. Suppose that $\alpha \geqslant \gamma$. By the definition of X,

$$Hu_\gamma t_\alpha \cap Hu_\delta t_\beta \neq \emptyset \tag{6}$$

for some t_β in $T_{\mu(\delta)}$. By (4), $\beta \geqslant \delta$. Now $Hu_\gamma \subseteq A_\alpha$ because $\alpha \geqslant \gamma$, and $Hu_\delta \subseteq A_\beta$ because $\beta \geqslant \delta$. By (1) and (6), $\alpha = \beta$. So $Hu_\gamma \cap Hu_\delta \neq \emptyset$, which implies that $\gamma = \delta$. Hence $t_\alpha = t_\beta$ belongs to $T_{\mu(\delta)} = T_{\mu(\gamma)}$, which confirms (5).

Now we are ready to show that $yx_i \notin \overline{X}$ whenever $y \in \mathcal{U}(G)$ and $i < \operatorname{cf}\kappa$. Fix $\alpha < \kappa$ such that $x_i \notin \overline{T_{\mu(\beta)}}$ for any $\beta > \alpha$ (any α with $\mu(\alpha) > i$ will do). We start by showing that $sx_i \notin \overline{X}$ if $s \in G \setminus A_\alpha$. Suppose to the contrary that there exists s in $G \setminus A_\alpha$ such that $x_i \in s^{-1}\overline{X} = \overline{s^{-1}X}$. For some $\gamma < \kappa$, the point s is in Hu_γ. Note that $\gamma > \alpha$ because $s \notin A_\alpha$. Now $X = s(s^{-1}X) \subseteq Hu_\gamma(s^{-1}X)$, so $Hu_\gamma v \cap X \neq \emptyset$ for every v in $s^{-1}X$. Then it follows from (5) that

$$T \cap s^{-1}X \subseteq T_{\mu(\gamma)} \cup \{t_\delta\}_{\delta<\gamma}. \tag{7}$$

Since $\overline{X}$ is open, $\overline{s^{-1}X}$ is an open neighborhood of x_i in G^{LUC}. Therefore x_i is in the closure of $T \cap \overline{s^{-1}X}$. But X is closed in G because the complement of X is a union of cosets of an open subgroup, and so $G \cap \overline{s^{-1}X} = s^{-1}X$. Hence $T \cap \overline{s^{-1}X} = T \cap s^{-1}X$. It follows from (7) that $x_i \in \overline{T_{\mu(\gamma)}} \cup \overline{\{t_\delta\}_{\delta\leqslant\gamma}}$. But $\|x_i\| = \kappa$, so $x_i \in \overline{T_{\mu(\gamma)}}$, a contradiction with the choice of α. Therefore $sx_i \notin \overline{X}$ whenever $s \in G \setminus A_\alpha$. Taking $s \to y$, which is possible because $\|y\| = \kappa$ and $\kappa(A_\alpha) = \max\{\omega, |\alpha|\}$, we see that $yx_i \notin \overline{X}$ (recall that $\overline{X}$ is open in G^{LUC}).

We have shown that, for any y in $\mathcal{U}(G)$, the set $\overline{X}$ is a neighborhood of yx while $yx_i \notin \overline{X}$ for any $i < \operatorname{cf}\kappa$. Since $x_i \to x$ and $yx_i \not\to yx$, the left translation by y is not continuous at x.

Case $\operatorname{cf}\kappa = \omega$. Let N be a closed neighborhood in G^{LUC} of a point in $\mathcal{U}(G)$. We shall find a point x in $N \cap \mathcal{U}(G)$ such that no left translation in the semigroup $\mathcal{U}(G)$ is continuous at x.

Since $\operatorname{cf}\kappa = \omega$, there is an increasing cofinal sequence $\{\lambda_n\}_{n<\omega}$ in κ. Let U be a symmetric, relatively compact, open neighborhood of the identity in G. Let $\{A_\alpha\}_{\alpha<\kappa}$ be a cover of G such that

$$U \subseteq A_0, \quad \bigcup_{\beta<\alpha} UA_\beta \subseteq A_\alpha, \quad \text{and} \quad \kappa(A_\alpha) \leqslant |\alpha|$$

for every $\alpha < \kappa$. An application of transfinite induction gives a subset $T = \{t_\alpha\}_{\alpha<\kappa}$ of $N \cap G$ such that

$$A_\alpha t_\alpha \cap A_\beta t_\beta = \emptyset \quad \text{whenever } \alpha \neq \beta.$$

Similarly as in the earlier case, the set T is left uniformly discrete and we may identify $\overline{T}$ with βT. Denote $\overline{T} \cap \mathcal{U}(G)$ by $\mathcal{U}(T)$, and notice that $\mathcal{U}(T) \subseteq N \cap \mathcal{U}(G)$.

Let $x \in \mathcal{U}(T)$ such that x is not a P-point in $\mathcal{U}(T)$. Then there exist sets $T = T_1 \supseteq T_2 \supseteq \cdots$ such that, for every $n = 1, 2, \ldots$

- $\overline{T_n} \cap \mathcal{U}(T)$ is a neighborhood of x in $\mathcal{U}(T)$,
- $\bigcap_{k=1}^{\infty} \overline{T_k} \cap \mathcal{U}(T)$ is *not* a neighborhood of x in $\mathcal{U}(T)$,
- $\{t_\alpha\}_{\alpha<\lambda_n} \cap T_n = \emptyset$.

For every $\alpha < \kappa$, there is a unique $n(\alpha)$ such that $t_\alpha \in T_{n(\alpha)} \setminus T_{n(\alpha)+1}$. Note that $\lambda_{n(\alpha)} \leqslant \alpha$ by the third condition imposed on the sets T_n.

Now $A_{\lambda_{n(\alpha)}} t_\alpha \cap (A_\beta \setminus A_{\lambda_{n(\beta)}}) t_\beta = \emptyset$ for every $\alpha, \beta < \kappa$. Put

$$A = \bigcup_{\alpha<\kappa} A_{\lambda_{(n(\alpha)-1)}} t_\alpha \quad \text{and} \quad B = \bigcup_{\beta<\kappa} (A_\beta \setminus A_{\lambda_{n(\beta)}}) t_\beta,$$

so that $UA \cap B = \emptyset$ (recall that $UA_\alpha \subseteq A_{\alpha+1}$). Then there exists a left uniformly continuous function f on G such that $f = 1$ on A and $f = 0$ on B. Denote the continuous extension of f to G^{LUC} by $\tilde{f}$.

Let $y \in \mathcal{U}(G)$. Similarly as in [8], we see that $\tilde{f}(yx) = 1$ and that, for every $E \subseteq T$ with $x \in \overline{E}$, there exists $x_E \in \overline{E} \cap \mathcal{U}(T)$ such that $\tilde{f}(yx_E) = 0$. So $\tilde{f}(yx_E) \not\to \tilde{f}(yx)$ and therefore $yx_E \not\to yx$, although $x_E \to x$. □

The following theorem is due to Protasov and Pym [8], but with the preceding theorem we obtain a new proof.

Theorem 3. *The topological center of G^* is empty.*

Proof. Let $y \in G^*$. Let $E \subseteq G$ such that $y \in \overline{E}$ and $\kappa(E) = \|y\|$. If H denotes the subgroup of G generated by E, then $\kappa(H) = \|y\|$. The closure of H in G^{LUC} may be identified with H^{LUC}, and $y \in \mathcal{U}(H)$. Pick x from $\mathcal{U}(H) \subseteq G^*$ such that the left translation by y is not continuous at x (Theorem 2). □

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