The Countable Extension Basis Theorem and its applications

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Abstract

We prove the Countable Extension Basis (CEB) Theorem for noncompact metric spaces. As an application of the CEB Theorem we give alternative proofs of some classical results in the area. In particular, we construct a universal complete metric space $X_K$ in the class of spaces with $e$-dim $X \leq K$ together with a $K$-soft map onto the Hilbert cube. As a corollary we obtain Olszewski’s completion theorem. Also we apply the CEB Theorem to give an alternative proof of the Splitting Theorem in extension theory for noncompact spaces. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

One of the main problems topology deals with is the extension problem. By an extension problem we mean the problem of completing the following diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & K \\
\downarrow & & \downarrow \\
X & \downarrow & \\
\end{array}
$$

where $A$ is a closed subset of $X$. Usually this problem is considered for a relatively nice space $K$ such as a polyhedron or CW complex. When $(X, A)$ is a CW-complex pair and $K$ is homotopy simple or nilpotent, classical Obstruction Theory takes care of this problem.
The problem can be solved if and only if all sets \( O_i \subset H^{i+1}(X,a;\pi_i(K)) \) in a certain inductively constructed sequence of so called obstruction sets contain zero. When \( X \) is a finite-dimensional compactum, a corresponding refinement of the Obstruction Theory still works.

The theory of topological spaces which admit a solution of any extension problem with respect to a given complex \( K \) we call \( K \)-Extension Theory or just Extension Theory. Kuratowski introduced [12] the notation \( X \tau K \) for a space having this property. This condition also can be denoted as \( K \in \text{AE}(X) \), i.e., a complex \( K \) is an absolute extensor for the class of spaces which consists of one space \( X \). Though this notation is self-explanatory, Kuratowski’s notation is preferable in the cases when we want to place the stress on the space \( X \) not on \( K \). Classical examples of this situation are presented by dimension theory and cohomological dimension theory:

\[
X \tau S^n \Leftrightarrow \dim X \leq n \quad \text{and} \quad X \tau K(G,n) \Leftrightarrow \dim_G X \leq n.
\]

These examples lead to the notation \( \text{e-dim} X \leq K \) for the property \( X \tau K \) which can be read ‘the extensional dimension of \( X \) does not exceed \( K \)’. This phrase can be given a precise meaning [3,8]. Many of the classical theorems of dimension theory have analogs which hold for extension theory.

It turns out that the same theorems in the case of compact metric spaces have simpler proofs than in the case of general separable metric spaces (compare, for example, the proofs of the Splitting Theorem [5,9]). The main reason for this, as we understand it, is the existence of a countable basis of extension problems in the compact case. Precisely, for every compact metric space \( X \) and a countable complex \( K \) there is a countable family \( \{ f_i : C_i \to K \} \) of mappings of closed subsets \( C_i \subset X \) such that for any other continuous map \( f : B \to K \) given on a closed subset \( B \subset X \) there is an \( i \) such that \( B \subset C_i \) and the restriction \( f_i|_B \) is homotopic to \( f \). The following is obvious.

**Countable Extension Basis Theorem** (compact case). If all problems \( \{ f_i : C_i \to K \} \) are solvable, then \( X \tau K \).

The Countable Extension Basis Theorem is especially useful for constructing various spaces \( X \) with \( X \tau K \) as limits of inverse sequences (see, for example, Theorem 2.4 of [10]).

It is clear that there cannot be an obvious analog of the Countable Extension Basis Theorem for non-compact spaces. Nevertheless in Section 2 we formulate (and prove) a version of the Countable Extension Basis Theorem in the non-compact case which allows us to construct noncompact metric spaces with \( X \tau K \) and different other properties. In particular using the CEB Theorem one can construct a universal complete separable metric space \( Y_K \) for the class of spaces with \( \text{e-dim} X \leq K \) for a countable complex \( K \) such that \( \text{e-dim} Y_K \leq K \) and \( Y_K \) is an absolute extensor for the class of all spaces \( X \) with \( \text{e-dim} X \leq K \). For finite complexes \( K \) such spaces can be constructed more or less easily. If \( K = S^n \), then the Nöbeling space \( \nu^n \) can be taken as \( Y_K \). The first example of a universal space \( Y_K \) for an infinite complex \( K \) was constructed in [16]. In [1] it was done for all countable complexes \( K \).

Also, the Countable Extension Basis Theorem allows us to give alternative proofs of several other fundamental theorems in the theory.
2. Countable Extension Basis Theorem

An extension problem on $X$ with respect to $K$ is a homotopy class of maps $[\phi : A \to K] \in [A, K]$ where $A \subset X$ is a closed subset. A solution of an extension problem $[\phi : A \to K]$ is a map $f : X \to K$ with $f|A \in [\phi : A \to K]$. An extension problem is called solvable if there is a solution of it.

Suppose that $X \subset Q$ and $P \subset Q$ with $P$ closed in $X \cup P$. Let $f : P \to K$. We say that $[f : P \to K]$ is an extension problem relative to $X$ if it is an extension problem with respect to $X \cup P$ in the preceding definition. We say that $\tilde{f} : X \cup P \to K$ is a solution relative to $X$ if $\tilde{f}|P \in [f : P \to K]$. When a solution $\tilde{f}$ exists, we say that $[f : P \to K]$ is solvable relative to $X$.

Suppose that $X \subset Q$ and $A, B \subset Q$. The sum $[\phi : A \to K] + [\psi : B \to K]$ of two extension problems relative to $X$ is defined when $A$ and $B$ are both closed in $A \cup B$ and $A$ is closed in $X \cup A$ and $B$ is closed in $X \cup B$. In this case the sum $[\phi : A \to K] + [\psi : B \to K]$ is an extension problem $[f : A \cup B \to K]$ such that $f|A \sim \phi$ and $f|B \sim \psi$. Note that by the assumptions $A \cup B$ is closed in $X \cup A \cup B$. The sum is defined if and only if $\phi|A \cap B \sim \psi|A \cap B$. The difference $[\phi : A \to K] - [\psi : B \to K]$ is defined whenever $A$ is closed in $X \cup A$ and is given by $[\phi|\text{Cl}_A(A \setminus B) : \text{Cl}_A(A \setminus B) \to K]$.

Let $\mathcal{P}$ be a family of solvable extension problems relative to $X$ with respect to $K$. We denote by $S(\mathcal{P})$ a set of solutions relative to $X$ for $\mathcal{P}$, one for each problem.

**Definition.** Let $X \subset Q$ and let $\mathcal{B}$ be a countable collection of closed sets whose interiors form a basis for the topology of $Q$. Let $K$ be a countable complex and let $\mathcal{C}$ be the set of all homotopy classes of maps to $K$ with domains from $\mathcal{B}$. Think of $\mathcal{C}$ as a set of extension problems relative to $X$ with respect to this $K$. We say that $\mathcal{C}$ admits a tower of solvable extension problems, if the elements of $\mathcal{C}$ are solvable relative to $X$ and if there is a sequence of families of solvable extension problems relative to $X$, $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \mathcal{P}_k \subset \cdots$ with $\mathcal{P}_0 = \mathcal{C}$ and $\mathcal{P}_{k+1} = (S(\mathcal{P}_k) - \mathcal{C}) + \mathcal{C}$ for $k = 0, 1, 2, \ldots$.

A family of compact sets $\mathcal{F}$ is said to be additively closed if $C \cup D \in \mathcal{F}$ and $\text{Cl}(C \setminus D) \in \mathcal{F}$ for all $C$ and $D$ in $\mathcal{F}$. Let $\mathcal{B}$ be an additively closed countable collection of closed sets in the Hilbert cube $Q$ which form a basis of the topology of $Q$. Denote by $\mathcal{C}$ the set of all extension problems with respect to $K$ with domains from $\mathcal{B}$. Note that $\mathcal{C}$ is countable.

**Countable Extension Basis Theorem** (general case). Let $X \subset Q$ and let $\mathcal{C}$ be the set of all extension problems relative to $X$ with respect to $K$ defined on some countable additively closed basis $\mathcal{B}$ in $Q$. Then the following are equivalent.

1. $\mathcal{C}$ admits a tower of solvable extension problems relative to $X$.
2. $X \tau K$.

**Proof.** (2) ⇒ (1) Assume that $K \tau K$ and that the other hypotheses of the theorem hold. Now $\mathcal{P}_0 = \mathcal{C}$. We need to show that the elements of $\mathcal{C}$ have solutions relative to $X$. Let $[f : C \to K] \in \mathcal{C}$ with $C \in \mathcal{B}$. Let $C'$ be a compact set in $Q$ such that $C$ is in the interior of
Thus, the elements of $C$ are solvable relative to $X$ as required.

Now suppose that $P_k$ is defined for $k \geq 0$. We want to show that $P_k \subset P_{k+1}$ with $P_{k+1} = (S(P_k) - C) + C$ and with $P_{k+1}$ also a collection of solvable extension problems. Since the elements of $P_k$ are solvable relative to $X$ by assumption, the set $P_{k+1}$ is defined. We need to show that these extension problems relative to $X$ are solvable relative to $X$. Let $[f : A \rightarrow K] \in P_{k+1}$. By induction one can show that $A = \text{Cl}(X \setminus C_1) \cup C_2$ for some $C_1$ and $C_2$ in $B$. By the Walsh Lemma [11, Lemma 4.2] we may assume that there is an $O$ open in $Q$ with $A \subset O$ such that $f$ has an extension to $f' : O \rightarrow K$. Let $C_3$ be a compact set in the interior of $C_3$ and with $C_3 \subset O$. Let $g$ be an extension of $f \mid_{C_3 \cap X}$ to all of $X$ homotopic to $f'$ on $X$. Then $\tilde{f} : X \cup A \rightarrow K$ be defined by $\tilde{f}(x) = g(x)$ for $x \in X$ and $\tilde{f}(x) = f(x)$ for $x \in C_2$. Then $\tilde{f} : X \cup A \rightarrow K$ is a solution relative to $X$ to the extension problem $[f : A \rightarrow K]$. This shows that $P_{k+1}$ consists of solvable extension problems relative to $X$.

We now need to show that if $k \geq 1$, then $P_k \subset P_{k+1}$. Let $[f : A \rightarrow K] \in P_k$ and let $\tilde{f} : X \cup A \rightarrow K$ be in $S(P_k)$. As already noted, $A = \text{Cl}(X \setminus C_1) \cup C_2$ for some $C_1$ and $C_2$ in $B$. Let $g = \tilde{f} \mid_{\text{Cl}(X \cup A \setminus C_1)}$. Then $f$ is in the homotopy class of $[g : \text{Cl}(X \cup A \setminus C_1) \rightarrow K] + [f \mid_{C_2} : C_2 \rightarrow K]$. Thus, $[f : A \rightarrow K] \in P_{k+1}$. This completes the proof that (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Let $f : A \rightarrow K$ be an extension problem. By the Walsh Lemma [11, Lemma 4.2], we may assume that there exists an $O$ open in $Q$ with $A \subset O$ such that $f$ has an extension $f' : O \rightarrow K$. There is a sequence of compact sets $(C_i)_{i=1}^{\infty}$ which are the support of problems from $C$ such that

$$\text{Cl}(Q) \cap O \subset \bigcup_{i=1}^{\infty} C_i \subset O$$

and $C_1 \subset N_{1/1}(A)$. To obtain that, consider a presentation of the locally compact space $Y = \text{Cl}(Q) \cap O$ as the union of compact sets $\bigcup_{i=1}^{\infty} X_i$ with $X_i \subset \text{int} X_{i+1}$. Then take a basic neighborhood $C_i$ of $X_i$ in $O \cap N_{1/1}(A)$ with $C_i \cap Y \subset \text{int} X_{i+1}$.

By induction we define $V_i \subset \text{cl} X$, $f_i : V_i \rightarrow K$, and $\varepsilon_i > 0$ such that $V_i \subset \text{int} X V_i+1$ and $f_i \mid_{V_i} = f_i$.

Let $\varepsilon_1 = 1$, $V_1 = \text{Cl}(X \setminus W_1) \cup C_1$ where $W_1$ is the support of a problem in $C$ having the property that $\bigcup_{i>1} C_i \setminus C_1 \subset \text{int} W_1 \subset W_1 \subset N_{1/2}(\bigcup_{i>1} C_i)$. Define $V_1 = V_1 \setminus X$. Let $f_1$ be a solution to $[f' \mid_{C_1} : C_1 \rightarrow K]$ restricted to $V_1$. Let $f_1 = f_1 \mid_{V_1}$.

Consider $[f_1 : V_1 \rightarrow K]$ and $[f' \mid_{C_2 \setminus C_1} : C_2 \setminus C_1 \rightarrow K]$. Now

$$\begin{align*}
C_2 \setminus C_1 \cap V_1 &= C_2 \setminus C_1 \cap (\text{Cl}(X \setminus W_1) \cup C_1) \\
&= (C_2 \setminus C_1 \cap \text{Cl}(X \setminus W_1)) \cup (C_2 \setminus C_1 \cap C_1) \\
&= C_2 \setminus C_1 \cap C_1 \\
&= C_2 \cap \partial C_1.
\end{align*}$$
The above functions agree on the intersection. Consider

\[ \varphi_2 : \tilde{V}_1 \cup C_2 \to K = [\tilde{f}_1 : \tilde{V}_1 \to K] + [f'_2|_{\bar{C}_2 \setminus C_1} : \bar{C}_2 \setminus C_1 \to K] \]

such that \( \varphi_2|_{\tilde{V}_1} = \tilde{f}_1 \). Since \( \varphi_2 : \tilde{V}_1 \cup C_2 \to K \in (S(C) - C) + C \), there is an extension \( \varphi_2 \) over \( X \). We define \( \varepsilon_2 = \text{dist}(\bigcup_{i > 2} C_i \setminus C_2, (Q \setminus \text{int} W_1) \cup C_1) \). Define \( W_2 \in C \) such that

\[ \bigcup_{i > 2} C_i \setminus C_2 \subset \text{int} W_2 \subset W_2 \subset N_{\varepsilon_2/2}(\bigcup_{i > 2} C_i \setminus C_2) \].

Let \( \tilde{V}_2 = Cl_X(X \setminus W_2) \cup C_2 \) and \( \tilde{f}_2 = \tilde{\varphi}_2|_{\tilde{V}_2} \) and \( V_2 = \tilde{V}_2 \cap X \) and \( f_2 = \tilde{f}_2|_{V_2} \).

We now show that \( V_1 \subset \text{int}_X(V_2) \). Now \( \text{int}_X(V_2) = (X \setminus W_2) \cup ((\text{int} C_2) \cap X) \). If \( x \in V_1 \), then \( x \in Cl_X(X \setminus W_1) \) or \( x \in C_1 \). If the former is the case, then \( x \in X \setminus \text{int} W_1 \) which implies that \( N_{\varepsilon_2/2}(x) \subset X \setminus N_{\varepsilon_2/2}(\bigcup_{i > 2} C_i \setminus C_2) \subset X \setminus W_2 \subset \text{int} V_2 \).

If \( x \in C_2 \), then it is also the case that \( N_{\varepsilon_2/2}(x) \subset X \setminus W_2 \).

It is clear that we can continue this construction inductively so that our sequences \( V_i \), \( f_i \), and \( \varepsilon_i \) exist. It is also clear that the following claim holds because of the construction.

**Claim.** If we let \( \tilde{f} = \bigcup_{i = 1}^{\infty} f_i : X \to K \), then \( \tilde{f} \) will be continuous.

The existence of the map \( \tilde{f} \) shows that \( X \tau K \). \( \square \)

### 3. Universal spaces

A map \( f : X \to Y \) is called \( K \)-soft for every commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & X \\
\downarrow^{f} & & \downarrow^{f} \\
Z & \xrightarrow{\psi} & Y
\end{array}
\]

with \( \text{e-dim} Z \leq K \) and an inclusion of a closed subset \( i : A \to Z \) there is a lift \( \tilde{\psi} : Z \to X \) such that \( \tilde{\psi}|_A = \phi \). The class of absolute extensors for the spaces with \( \text{e-dim} X \leq K \) we
denote as \( AE([K]) \). The notation is borrowed from [2]. It is based on analogy with \( AE(n) \), absolute extensors in the dimension \( n \).

The following is obvious.

**Proposition 1.**

1. If \( f' \) is a map, parallel to a \( K \)-soft map \( f \) in a pullback diagram, then \( f' \) is \( K \)-soft.
2. If \( f : X \to Y \) is \( K \)-soft and \( Y \in AE([K]) \), then \( X \in AE([K]) \).

Let \( K \) be a countable locally finite complex and let \( v : K \times [0, 1] \to cone(K) \) be the quotient map. We note that generally the map \( v \) is not \( K \)-soft. We describe a slight modification of \( v \) into \( \tilde{\xi} : \tilde{K} \to cone(K) \) with the same range and the domain homotopy equivalent to \( K \). We define

\[
T = \{(v(x, t), x) \in cone(K) \times K \mid t < 1\}.
\]

Let \( U \) be an open PL neighborhood of the vertex of the \( cone(K) \) and let \( \overline{U} \) be its closure. We define \( \tilde{K} = (U \times K) \cup T \subset cone(K) \times K \). Let \( \pi_1 \) and \( \pi_2 \) be projections of the product \( cone(K) \times K \) to the first and to the second factor, respectively. Let \( \tilde{\xi} : \tilde{K} \to cone(K) \) be the restriction of \( \pi_1 \).

**Proposition 2.** The map \( \tilde{\xi} : \tilde{K} \to cone(K) \) is \( K \)-soft.

**Proof.** Let

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & \tilde{K} \\
\downarrow{\iota} & & \downarrow{\tilde{\xi}} \\
Z & \xrightarrow{\psi} & cone(K)
\end{array}
\]

be a commutative diagram with e-dim \( Z \leq K \). Since the map \( \tilde{\xi} \) is one-to-one over \( cone(K) \setminus U \), there is a unique lift \( \tilde{\psi}_1 : Z \setminus \psi^{-1}(U) \to \tilde{K} \). Let \( X = A \cap \psi^{-1}(\overline{U}) \cup \psi^{-1}(\partial U) \). Let \( \psi' \) be the union of \( \phi \) and \( \tilde{\psi}_1 \) restricted to \( X \). Since e-dim \( \psi^{-1}(\overline{U}) \leq K \), there is an extension \( \tilde{\phi}' : \psi^{-1}(\overline{U}) \to K \) of the map \( \phi' = \pi_2 \circ \psi' \). We define \( \tilde{\psi}_2 : \psi^{-1}(\overline{U}) \to cone(K) \times K \) as the diagonal product of \( \psi|\psi^{-1}(\overline{U}) \) and \( \tilde{\phi}' \). It is easy to check that \( \tilde{\psi}_2|\psi^{-1}(\partial U) = \tilde{\psi}_1|\psi^{-1}(\partial U) \). Then \( \tilde{\psi} = \tilde{\psi}_1 \cup \tilde{\psi}_2 \) is a lift of \( \psi \) with \( \tilde{\psi}|A = \phi \). □

Let \( g : Y \to X \) be a map and let \( [f : A \to K] \) be an extension problem on \( X \). We call the problem \( g^{-1}(\{f\}) = [f \circ g|_{g^{-1}(A)} : g^{-1}(A) \to K] \) a lift of the extension problem \([f]\). We say a map \( g : Y \to X \) resolves an extension problem \([f : A \to K]\) on \( X \) if the lift \( g^{-1}(\{f\}) \) is solvable.

**Lemma 1.** Let \( [f : A \to K] \) be an extension problem on a locally compact space \( X \) with respect to a countable locally finite complex \( K \). Then there is a \( K \)-soft map \( g : Y \to X \) of a locally compact space \( Y \) which resolves \([f]\).
Proof. Extend $f$ to $\tilde{f}: X \to cone(K)$ and consider the pull-back diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\tilde{j}'} & \tilde{K} \\
\downarrow \xi & & \downarrow \\
X & \xrightarrow{\tilde{j}} & cone(K)
\end{array}
$$

Note that $g' = \pi_2 \circ \tilde{j}'$ restricted to $g^{-1}(A)$ coincides with $f \circ g|_{\tilde{K}}^{-1}(A)$, i.e., $g$ resolves $[f]$. By Proposition 1 $g$ is $K$-soft. The space $Y$ is locally compact, since it can be presented as the union of a locally compact space, homeomorphic to $W \times K$, and a compact space. \(\Box\)

The following theorem and its corollaries were first proved in [1] (see also [16]).

Theorem 1. For every countable complex $K$ there is a $K$-soft map $\xi_K: Y_K \to Q$ of a complete separable metric space with $\varepsilon$-dim $X \leq K$ onto the Hilbert cube.

Proof. Let $Q = Q_1 \times Q_2 \times \cdots$ be an infinite product of Hilbert cubes. Let $C'_k$ be a countable basis of extension problems on $Q_1 \times \cdots \times Q_k$ defined on an additively closed basis. Let

$$
\pi_k^{k+i}: Q_1 \times \cdots \times Q_{k+i} \to Q_1 \times \cdots \times Q_k
$$

denote the projection. Then the union of $(\pi_k^K)^{-1}(C'_k)$ is a countable additively closed basis on $Q_1 \times \cdots \times Q_k \times \cdots$.

By induction on $k$ we construct locally compact spaces $X_k \subset Q_1 \times \cdots \times Q_k$ and $K$-soft bonding maps $q_k^{k-1}: X_k \to X_{k-1}$ such that $q_k^{k-1} = \pi_k^{k-1}|_{X_k}$. Also we construct a sequence of countable families $B_k$ of extension problems on $X_k$ with respect to $K$ together with an enumeration of elements of $B = \bigcup_{i=1}^{\infty} B_i$ in such a way that $q_k^{k-1}$ resolves a lift of the problem with number $k$. Additionally we require $C'_k \subset B_k$.

Let $\{p_i\}$ be the sequence of all prime numbers, thus $p_1 = 2$. We define $B_1 = C'_1$ and enumerate elements of $B_1$ by powers of two: $1, 2, 2^2, \ldots$. Define $X_1 = Q_1$.

Assume that all this is done for all $i \leq k$, i.e., we have $K$-soft maps $q_i^i: X_i \to X_{i-1}$ and elements of $B_i$ are enumerated by all natural numbers which are not divisible by $p_j$ for $j > i$. Note that $k$ is not divisible by $p_j$ for $j \geq k$. Hence there is an extension problem with the number $k$ in $B_i$ for some $i < k$. According to Lemma 1 there is a $K$-soft map $q_k^{k+i+1}$ which resolves the lift to $X_k$ of the problem with the number $k$. We imbed $X_{k+1}$ in $(Q_1 \times \cdots \times Q_k) \times Q_{k+1}$ as the graph of $q_k^{k+1}$.

We assume that $C'_{k+1}$ contains the lifts by $\pi_k^{k+1}$ of all the problems $C'_k$. Let $\phi_i: X_{i+1} \to K$ be a solution to the lift of the problem with number $i$. Let $S_k = (\phi_i \circ q_k^{k+1})_{i=1}^k$. Denote $B_{k+1} = (S_k - C'_{k+1}) + C'_{k+1}$. Enumerate these problems by all numbers divisible by $p_{k+1}$ and not divisible by higher prime numbers.

As the result we obtain the limit space $Y_K = \lim X_k$ imbedded in $Q$ as a $G_k$. The map $\xi_K = q_1^K: Y_K \to Q$ is $K$-soft. Hence Proposition 1 implies that $Y_K \in AE([K])$.

Now we show that $C$ admits a tower of solvable extension problems. First, the lift to $Y_K$ of all problems $C'_k$ are solvable by the construction. Therefore the ground flow $P_0 = C$ is...
solvable. The set of solutions $S(C)$ is represented by lifts to $Y_K$ of corresponding maps $\phi$. For every $f \in S(C)$ every element in $(\{ f \} - C) + C$ is counted and hence resolved on some level. Thus we have solvable first flow $P_1$ and so on.

The Countable Extension Basis Theorem implies that $\text{e-dim } Y_K \leq K$.

**Corollary 1.** For any countable complex $K$ there is a Polish AE([K])-space $Y_K$ with $\text{e-dim } Y_K \leq K$ which is universal in the class of separable metric spaces with $\text{e-dim } Y_K \leq K$.

**Proof.** Since $\xi_K$ is $K$-soft, Proposition 1 implies that $Y_K \in \text{AE}([K])$. 

If the complex $K$ is finite, then the space $Y_K$ can be chosen compact. For $K = K(\mathbb{Z}, n)$, $n > 1$ there is no compact $Y_K$ (see [16]).

**Corollary 2** (Completion Theorem [13]). For any countable complex $K$ and a separable metric space $X$ with $\text{e-dim } X \leq K$ there is a completion $X'$ with $\text{e-dim } X' \leq K$.

**Proof.** Consider the closure of $X$ in the universal space $Y_K$. 

4. Other applications

The proof of the following theorem can be extracted from Proposition 6.8 of [1].

**Theorem 2.** Let $K$ be a countable complex and $Y \subset X$ and $\text{e-dim } X \leq K$. Then $\text{e-dim } Y \leq K$.

**Proof.** We assume that $X \subset Q$. Take an additively closed basis of extension problems $C$ on $Q$. Since $X \tau K$, there is a tower of solvable extension problems $P_1 \subset P_2 \subset \cdots \subset P_k \subset \cdots$ generated by $P_0 = C$. The restriction of this tower on $Y$ defines a tower of solvable problems on $Y$ generated by $C$. The Countable Extension Basis Theorem implies that $\text{e-dim } Y \leq K$.

Let $K * L$ denote the join product, i.e., the quotient space $K \times L \times [-1, 1]/(x, y, -1) \sim (x', y, -1); (x, y, 1) \sim (x, y', 1)$. There are natural imbeddings $K \subset K * L$ and $L \subset K * L$. Denote by $\pi_K : K * L \rightarrow K$ and $\pi_L : K * L \rightarrow L$ the natural projections.

**Lemma 2** [5]. Let $K$ and $L$ be countable complexes. Let a metric space $X = A \cup B$ be the union of closed subsets and let $f : A \rightarrow K$ and $g : B \rightarrow L$ be maps. Then there exists a map $\psi : X \rightarrow K * L$ with the properties $\pi_K \circ \psi|_A = f$ and $\pi_L \circ \psi|_B = g$.

The following two theorems generalizes the Eilenberg–Borsuk theorem and the Urysohn splitting theorem. They were proved for compact spaces in [5] and for separable metric spaces in [9].
Theorem 3. Let $K$ and $L$ be countable complexes and let $g : A \to K$ be a continuous map of a closed subset $A \subset X$ of a separable metric space $X$ with e-dim $X \leq K * L$. Then there are a closed subset $Y \subset X$ with e-dim $Y \leq L$ and a continuous extension $\tilde{g} : X \setminus Y \to K$ of the map $g$.

Proof. We assume that $X$ lies in the Hilbert cube $Q$. Let $C$ be a countable additively closed basis of extension problems on $Q$ with respect to $L$. By induction we construct a sequence of maps $g_i : A_i \to K$ with $A_i \subset \text{int}(A_{i+1})$, $g_{i+1}|_{A_i} = g_i$, $A_1 = A$, $g_1 = g$ and a tower of extension problems with respect to $L$, generated by $C$, with an enumeration $\{\phi_i : B_i \to L\}$ such that $\phi_i$ is extendable over $X \setminus A_{i+1}$. Then the union $\bigcup g_i$ gives an extension of $g$ over $\bigcup A_i$. We define $Y = X \setminus \bigcup A_i$. Since all problems $\phi_i$ are solvable on $Y$, we have a solvable tower of extension problems on $Y$ generated by $C$. The Countable Extension Basis Theorem implies that e-dim $Y \leq L$.

Construction. We enumerate prime numbers $\{p_i\}$. Additionally for every $i$ we construct a countable family $B_i$ of extension problems on $X$ such that the union $\bigcup B_i$ forms the required tower.

$i = 1$. We define $B_1 = C$ and enumerate all the problems in $B_1$ by powers of 2 $= p_1$. We take $A_1 = A$, $g_1 = g$. We define $\phi_1$ to be the problem number $1 = 2^0$ in $B_1$.

Assume that $A_n$, $g_n$, $\phi_n$ and $B_n$ are constructed and enumerated by all numbers which are not divisible by $p_j$ for $j > n$. By Lemma 2 there is a map $\psi_n : A_n \cup B_n \to K * L$. Consider an extension $\tilde{\psi}_n|_X : X \to K * L$. Let $U_n \supset \tilde{\psi}_n^{-1}(L)$ be an open neighborhood in $\tilde{\psi}_n^{-1}(K * L \setminus K)$ such that $\overline{U}_n \subset X \setminus A_n$. We define $A_{n+1} = X \setminus U_n$ and $g_{n+1} = \pi_K \circ \tilde{\psi}_n|_{A_{n+1}}$. Then $\phi_n$ is extendable over $\tilde{\psi}_n^{-1}(K * L \setminus K)$ and hence, over $U_n = X \setminus A_{n+1}$. Let $\phi_n$ be an extension. We define $A_n = (\phi_n) - C$ and $B_{n+1} = B_n \cup A_n$. Enumerate elements of $A_n$ by all numbers divisible by $p_{n+1}$ and not divisible by higher prime numbers. Define $\phi_{n+1}$ as the problem from $B_{n+1}$ with the number $n + 1$. $\square$

Theorem 4. Let $K$ and $L$ be countable complexes. Assume that a separable metric space $X$ has the property e-dim $X \leq K * L$. Then there is a $G_δ$ set $Y \subset X$ with e-dim $Y \leq K$ and e-dim$(X \setminus Y) \leq L$.

Proof. Let $X \subset Q$ and let $C$ be a countable additively closed basis of extension problems on $Q$ with respect to $K$. By Theorem 3 there is a countable family of closed sets $Z^0_i$ such that all problems $\mathcal{C} = \mathcal{P}_0$ are solvable over $X \setminus \bigcup Z^0_i$. Let $S(\mathcal{P}_0)$ be a set of solutions, one for each problem. We define $\mathcal{P}_1 = (S(\mathcal{P}_0) - C) + C$ and again apply Theorem 3 to obtain a countable family of closed sets $Z^1_i$ such that all problems in $\mathcal{P}_1$ are solvable in $X \setminus \bigcup Z^1_i$. By induction we construct countable families $\{Z^k_i\}$ of closed subsets in $X$ and a tower of extension problems, generated by $C$ and solvable over $X \setminus \bigcup_{k}(\bigcup Z^k_i)$. We define $Y = X \setminus \bigcup_{k}(\bigcup Z^k_i)$, then by the Countable Extension Basis Theorem e-dim $Y \leq K$. The Countable Union theorem (see, for example, [3]) implies that $X \setminus Y = \bigcup_{k}(\bigcup Z^k_i)$ is at most $L$-dimensional. $\square$
References