Let $V$ be a vector space of dimension $n$ over an algebraically closed field $K$. We consider the fixed point set of a unipotent transformation $u$ on the flag variety $F = F(V)$. We prove in particular that its irreducible components have all the same dimension and are in bijective correspondence with standard tableaux associated to $u$.

Consider a Young diagram of size $n$. Let $S$ be the set of standard tableaux associated to it [1, p. 20, 71]. For every $\sigma \in S$, let $\alpha_i$ be the number of the column in which $i$ lies (for $\sigma$). Then $\sigma$ is completely described by the sequence $(\alpha_1, \ldots, \alpha_n)$. For $\sigma, \tau \in S$, say $\sigma < \tau$ if for some $i$ ($1 < i < n$), $\alpha_i < \tau_i$ and for $i < j < n$, $\alpha_j = \tau_j$. This defines a total order on $S$.

For $\sigma \in S$, we shall write $\sigma'$ for the standard tableau of dimension $(n-1)$ obtained from $\sigma$ by removing $n$.

Now let $u$ be an unipotent element of $GL(V)$ and let $X = F_u$. For every integer $i > 1$, let $c_i$ be the number of Jordan blocks of $u$ of dimension $i$. We associate to $u$ the Young diagram (of dimension $n$) with $c_i$ lines of length $i$ and the ordered set $S$ of standard tableaux corresponding to this diagram. Let $d_i$ be the length of the $i$th column.

We now define an application $\pi: X \to S$. Consider $F = (F_0, \ldots, F_n) \in X$. $u$ induces an unipotent endomorphism $u'$ of $V/F_1$ and $u'$ fixes $F/F_1 = (F_1/F_1, \ldots, F_n/F_1) \in F(V/F_1)$. The Young diagram of $u'$ is just the Young diagram of $u$ minus a corner. By induction on dim $V$, we may assume that a standard tableau $\sigma'$ of dimension $(n-1)$ is associated to $F/F_1$. Then $\pi(F)$ is the standard tableau $\sigma$ obtained from $\sigma'$ by placing $n$ in the remaining corner. Clearly $\alpha_n = i$, say, iff $F_1$ lies in $\text{Ker}(u-1) \cap \text{Im}(u-1)^{i-1}$ but not in $\text{Ker}(u-1) \cap \text{Im}(u-1)^i$.

For $\sigma \in S$ we may now define $X_{\sigma} = \pi^{-1}(\sigma)$. Thus $(X_{\sigma})_{\sigma \in S}$ is a partition of $X$. Similarly define $X_{\sigma'} = X_{\sigma'}(F)$ to be the subset of $F(V/F_1)_{u'}$ corresponding to $\sigma'$.

Choose a flag $W = (W_0, \ldots, W_{d_1}) \in F(\text{Ker}(u-1))$ such that $W_{d_i} = \text{Ker}(u-1) \cap \text{Im}(u-1)^{i-1}$ for every $i > 1$. Define $B_i = \mathcal{P}(W_i) - \mathcal{P}(W_{i-1})$ and $p : X \to \mathcal{P}(V), F \mapsto F_1$. 

Then we have another partition \((X_t)_{1 \leq t \leq d_t}\), where \(X_t = p^{-1}(B_t)\). For every integer \(j \ (1 < j < d_t)\), \(\bigcup_{t<j} X_t\) is closed in \(X\) and \(X_j\) is locally closed in \(X\).

Choose also a line \(L \in B_t\) and let \(X'_t = p^{-1}(L) = \{F \in X' | F_1 = L\}\). \(X'_t\) may be identified with \(\mathcal{F}(V/L)_{u'}\).

**Lemma.** There exists an isomorphism of algebraic varieties

\[ f: X_t \to B_t \times X'_t \]

such that the following diagram commute:

\[ \begin{array}{ccc}
X_t & \xrightarrow{f} & B_t \times X'_t \\
\downarrow & & \downarrow \\
X' & & B'_t
\end{array} \]

**Proof.** We use a Jordan basis of \(V\) (for \(u\)). We shall denote it by \((e^u_{jk}) \ (1 < j < h, \ 1 < k < c_h, \ (u-1)e^u_{jk} = e^u_{j-1,k}\) if \(j \neq 1\), 0 if \(j = 1\)). We may do that in such a way that:

a) \(L\) is generated by a vector \(e^u_{1,0}\) of the basis.

b) \(W_{t-1} = \sum K e^u_{1,k}\), where the sum is taken over the indices \((h, k)\) such that \(h > h_0\) or \(h = h_0\) and \(k > k_0\).

We prove the lemma with \(W_{t-1}\) instead of \(B_t\), using the isomorphism \(W_{t-1} \to B_t, \ w \mapsto K(e^u_{1,0} + w)\).

Consider \(w \in W_{t-1}\). If \(w = \sum a^u_{hk} e^u_{nk}\), define \(w_j\) by \(w_j = \sum a^u_{hk} e^u_{nk} (1 < j < h_0)\) (this makes sense). Let \(g_w\) be the automorphism of \(V\) leaving \(e^u_{1,0}\) fixed if \((h, k) \neq (h_0, k_0)\) and such that \(g_w(e^u_{1,0}) = e^u_{1,0} + w_j \ (1 < j < h_0)\). Then

\[ g: W_{t-1} \times X'_t \to X_t \]

\((w, F') \mapsto g_w(F')\) is well defined and is easily seen to be an isomorphism of algebraic varieties. Then \(f = g^{-1}\) is the required isomorphism.

**Proposition.** a) For every \(\sigma \in S\), \(\bigcup_{\tau \geq \sigma} X_{\tau}\) is closed in \(X\) and \(X_{\sigma}\) is locally closed in \(X\).

b) \(\dim X_{\sigma} = \sum_{\tau \geq \sigma} d_{\sigma}(d_{\sigma}-1)/2\) for every \(\sigma \in S\).

c) For every \(\sigma \in S\), there exists a partition \((Y_j)_{1 \leq j \leq m}\) of \(X_{\sigma}\) such that \(Y_j\) is isomorphic to an affine space and \(\bigcup_{k<j} Y_k\) is closed in \(X_{\sigma}\) \((1 < j < m)\).

**Proof.** (a) Let \(\sigma_n = h\). Then \(\bigcup_{\tau_n \geq h} X_{\tau}\) and \(\bigcup_{\tau_n > h} X_{\tau}\) are closed in \(X\). It is therefore sufficient to prove that

\[ \bigcup_{\tau \geq \sigma} X_{\tau}\] is open in \(\bigcup_{\tau \geq \sigma} X_{\tau}\).
Let $F \in X_\sigma$, $\sigma < \sigma$, $\sigma_n = h$. We use lemma 2 with $i = d_h$ and $L = F_1$. Then $X_i$ is open in $\bigcup_{n=h} X_r$. We may assume by induction that $\bigcup_{r'<\sigma'} X_{r'}$ is open in $X_i$. Then

$$X_i \cap \bigcup_{\tau < \sigma} \bigcup_{r' < \sigma'} X_{r'} \cong B_i \times \bigcup_{r' < \sigma'} X_{r'}$$

is an open neighbourhood of $F$ in

$$\bigcup_{\tau_n = h} X_{\tau}$$

and is contained in $\bigcup_{\tau < \sigma} X_{\tau'}$. This proves (a).

(b) \( p: X_\sigma \to \mathcal{P}(W_{d_h}) - \mathcal{P}(W_{d_h - 1}) \) is surjective and the fibre over any point is isomorphic to $X_{r'}$. By induction, we may assume that

$$\dim X_{r'} = \sum d'_s (d'_s - 1)/2$$

where $d'_s = d_s$ for $s \neq h$ and $d'_h = d_h - 1$. Hence

$$\dim X_\sigma = \dim X_{r'} + \dim \mathcal{P}(W_{d_h}) = \sum d_s (d_s - 1)/2.$$ (c) Write $Z_i = X_\sigma \cap X_i$, $d_{h+1} < i < d_h$. Then $\bigcup_{i \leq j} Z_i$ is closed in $X_\sigma$. So we need only to prove (c) with $X_\sigma$ replaced by $Z_i$. From the isomorphism of lemma 2, $Z_i \cong B_i \times X_{r'}$. $B_i$ is isomorphic to an affine space of dimension $(i - 1)$ and we may assume by induction that $X_{r'}$ has a partition of the required type. This proves the proposition.

**Remark.** The partition of $X$ given by (c) extends to a partition of $\mathcal{F}$ into affine spaces. The proof is similar.

An immediate consequence of the proposition is:

**Corollary.** All the irreducible components of $X$ have the same dimension. They are in bijective correspondence with the elements of $S$ via the application $\sigma \mapsto \overline{X}_\sigma$.

$X_\sigma$ is a smooth irreducible variety, and the above partition $(X_\sigma)_{\sigma \in S}$ of $X$ shows that the singular points of $X$ are those contained in more than one component. But $\overline{X}_\sigma$ needs not to be smooth. Take $K = \mathbb{C}$, dim $V = 6$, and choose $u$ with two Jordan blocks of dimension 1 and two of dimension 2. Suppose that the component $\overline{X}_\sigma$ defined by $(\sigma_1, \ldots, \sigma_6) = (1, 1, 2, 1, 2, 1)$ is smooth. It is easy to see that $F \in \overline{X}_\sigma$ iff

$$\dim (F_4 \cap W_4) > 3, \quad \dim (F_2 \cap W_2) > 1$$

and $(u - 1)(F_4) \subset F_2 \subset W_4$, where $W_2 = \text{Im} (u - 1)$, $W_4 = \text{Ker} (u - 1)$. We get easily $H_2(\overline{X}_\sigma; \mathbb{Z}) \cong \mathbb{Z}$ and $H^{12}(\overline{X}_\sigma; \mathbb{Z}) \cong \mathbb{Z}$, contradicting Poincaré duality.
Let \((e_1, \ldots, e_n)\) be a basis of \(V\) and let \(T\) (resp. \(B, F\)) be the maximal torus of \(GL(V)\) (resp. the Borel subgroup of \(GL(V)\), the maximal flag of \(V\)) corresponding to this basis. Let \(\xi_i\) be the simple root:

\[
\text{diag}(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_i \lambda_{i+1}^{-1} \quad (1 \leq i < n-1).
\]

Let \(P_i\) be the rank - 1 parabolic subgroup of \(GL(V)\) corresponding to the simple root \(\xi_i\). A line of type \(\xi_i\) in \(F\) is a subvariety \(gP_iF\) of \(F\) \((g \in GL(V))\). It is known that any two points of \(X\) may be connected by a sequence of arcs of lines of different types contained in \(X\) [2, p. 132]. In particular \(X\) is a point or a union of lines. It is easy to see that the union of all lines of type \(\xi_i\) contained in \(X\) is closed \((1 \leq i < n-1)\). Say a subvariety \(Y\) of \(F\) is an \(\xi_i\)-variety if it is a union of lines of type \(\xi_i\). This means that the subspace of dimension \(i\) of any flag contained in \(Y\) may be choosen arbitrarily between the subspaces of dimension \((i-1)\) and \((i+1)\).

Similarly consider the parabolic subgroup \(P_I\) corresponding to an arbitrary subset \(I\) of \(\{\xi_1, \ldots, \xi_{n-1}\}\). The subvarieties \(gP_I F\) of \(F\) \((g \in GL(V))\) will be called subspaces of type \(I\) in \(F\). Say \(Y\) is an \(I\)-variety if it is a union of subspaces of type \(I\). This means that \(Y = p^{-1}(p(Y))\), where \(p\) is the projection \(F \rightarrow \prod_{\alpha \in I} G_d(V)\). Then for all \(g \in GL(V), gY\) is also an \(I\)-variety. If \(I = I_1 \cup I_2\), then \(Y\) is an \(I\)-variety if it is an \(I_1\)-variety and an \(I_2\)-variety. In particular, for each \(\sigma \in S\), there is a maximal \(I_\sigma \subseteq \{\xi_1, \ldots, \xi_{n-1}\}\) such that \(\mathcal{X}_\sigma\) is an \(I_\sigma\)-variety.

**Proposition.** a) Every line of type \(\xi_i\) contained in \(X\) is contained in an irreducible component which is an \(\xi_i\)-variety.

b) For every \(\sigma \in S\), \(I_\sigma = \{\xi_i | \sigma_{n-i+1} < \sigma_{n-i}\}\).

c) The singular points of \(\mathcal{X}_\sigma\) form an \(I_\sigma\)-variety.

**Proof.** We may assume that \(u \in B\). Consider the line \(P_i F\). For every \(p \in P_i\), \(pF_j = F_j\) for \(j \neq i\). To prove (b), we may assume that \(\sigma = \pi(F)\). Then \(\sigma_{n-i+1} < \sigma_{n-i}\) implies that \(F_{i+1} F_{i-1} \subseteq \text{Ker}(\pi' - 1)\), where \(\pi'\) is the endomorphism induced by \(\pi\) on \(V/F_{i-1}\). This shows that \(pF\) is fixed by \(\pi\) and that at most one point of \(P_i F\) is not contained in \(\mathcal{X}_\sigma\). This implies that \(\mathcal{X}_\sigma\) is an \(\xi_i\)-variety. If \(\sigma_{n-i+1} > \sigma_{n-i}\), it is easy to find a point \(x\) in \(\mathcal{X}_\sigma\) such that the line of type \(\xi_i\) passing through \(x\) is not contained in \(\mathcal{X}_\sigma\). This proves (b). To prove (a), assume that \(P_i F \subseteq X\). This implies that \(F_{i+1} F_{i-1} \subseteq \text{Ker}(\pi' - 1)\) and it follows that at most one point of \(P_i F\) is not in \(\mathcal{X}_\sigma\), where \(\sigma = \min\ \{\pi(pF)|p \in P_i\}\). Clearly \(\sigma_{n-i+1} < \sigma_{n-i}\). So by (b), \(\mathcal{X}_\sigma\) is an \(\xi_i\)-variety containing \(P_i F\). (c) is obvious from the fact that locally \(\mathcal{F}\) may be identified with the group \(U^-\) of unipotent lower triangular matrices and that the traces of the lines of type \(\xi_i\) on such a neighbourhood are the left cosets of a 1-dimensional subgroup of \(U^\perp\).

For example, for the component \(\mathcal{X}_\sigma\) described above to show the existence of singular points, \(I_\sigma = \{\xi_1, \xi_3, \xi_5\}\). Suppose that \(F \in \mathcal{X}_\sigma\). It is
easy to see that $F$ is a regular point of $X_\sigma$ if $F_2 \neq W_2$ or $F_4 \neq W_4$. The remaining points of $X_\sigma$ form a single subspace of type $I_\sigma$. They are therefore all singular, and the variety of singular points of $X_\sigma$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

**Corollary.** Any subspace of type $I$ contained in $X$ is contained in an irreducible component which is an $I$-variety.

**Proof.** Let $Y \subset X$ be a subspace of type $I$. Let $\sigma \in S$ be such that $Y \cap X_\sigma \neq \emptyset$. Let $\alpha_t \in I$. By the proof of the proposition, $Y \cap X_\tau \neq \emptyset$ and $X_\tau$ is an $\alpha_t$-component, where $\tau = \sigma$ or $\tau$ is obtained from $\sigma$ by permutation of $n-i$ and $n-i+1$. Iterating for various elements of $I$, we eventually get $Y \subset X_\sigma$, where $X_\sigma$ is an $I$-component.

Using the fact that the components of $X$ have all the same dimension, we get:

**Corollary.** Let $X_I = \{gP_1 | g \in GL(V), u \in R_u(gP_1)\}$. Then the irreducible components of $X_I$ have all the same dimension and are in bijective correspondence with the set $S_I = \{\sigma \in S | I \subset I_\sigma\}$.

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**References**