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Free Stochastic Measures via Noncrossing Partitions[1]

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We consider free multiple stochastic measures in the combinatorial framework of the lattice of all diagonals of an n-dimensional space. In this free case, one can restrict the analysis to only the noncrossing diagonals. We give definitions of what

relationships between free stochastic measures of different orders. These allow us to calculate, for example, free Poisson–Charlier polynomials, which are the orthogonal polynomials with respect to the free Poisson measure. © 2000 Academic Press

1. INTRODUCTION

The motivation for this paper is twofold. On the one hand, in [RW97] Rota and Wallstrom show that much of the classical theory of multiple stochastic integrals can be done combinatorially, using the properties of the lattice of all partitions of a set, especially the Möbius inversion formula. As one consequence they get a number of combinatorial formulas describing the properties of orthogonal polynomials. On the other hand, recently Biane and Speicher in [BS98] made major advances in the study of the free Brownian motion, started earlier in [Bia97a, KS92, Spe91, Fag91].

We continue the study of more general free stochastic processes, concentrating especially on the free Poisson process. The starting point of the [RW97] paper is the observation (which has been made before) that the first difficulty in dealing with stochastic measures, as compared with scalar measures, is that multiple stochastic measures should not be taken as simply product measures of one-dimensional ones. Indeed, various diagonals in the n-dimensional space, which have Lebesgue measure 0, have nonzero product stochastic measure. Rota and Wallstrom point out, however, that

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removal of these diagonals from the space, which is how one usually defines multiple stochastic measures, corresponds precisely to the Möbius inversion on the lattice of all partitions of a set, which is the same as the lattice of all diagonals (see Section 2.2). We apply this idea to the study of stochastic measures in free probability. Inspired by [RW97], we define n-dimensional free stochastic measures and, more generally, free stochastic measures depending on a partition π in Section 2.3. Interestingly, in the free case the diagonals corresponding to crossing partitions all have weight 0 to begin with. This is in accordance with the general approach of Speicher that combinatorially, the transition from the classical to the free probability corresponds to the transition from the lattice of all to the lattice of noncrossing partitions.

After the free Brownian motion, the most important process with free increments is the free Poisson process. Using the combinatorial machinery, we can calculate explicitly the multiple stochastic measures for the free Poisson process, and more generally for the free compound Poisson processes. These in turn give us recurrence relations for the orthogonal polynomials with respect to the corresponding (scalar) measures. In particular we calculate the free Poisson–Charlier polynomials. Finally, in the language of the lattice of noncrossing partitions one can easily express the Itô product formula for general free stochastic measures.

The paper is organized as follows. In Section 2 we collect various combinatorial preliminaries, and the main definitions. In Section 3, we look at some general properties of free stochastic measures, and calculate the distributions for their main diagonal measures. These diagonal measures are calculated explicitly for the free Poisson process in Section 4.2 and for the free compound Poisson processes in Section 4.3. Section 5 is devoted to the consideration of product measures, especially in the free Brownian motion and free Poisson cases, and combinatorial formulas that have implications for families of orthogonal polynomials. It also contains the combinatorial Itô product formula. In Section 6.2 we show that at least for a particular scheme, the stochastic measures are always well-defined. Section 7 is devoted to various recursion relations between stochastic measures, and the relation to orthogonal polynomials. Finally, in Section 8 we list a few preliminary facts about the result of integration with respect to free stochastic processes.

2. NOTATION AND DEFINITIONS

2.1. *Partitions.* We will consider the following three lattices of partitions. By $\mathscr{P}(n)$ we will denote the lattice of all partitions of the set $\{1, 2, ..., n\}$.

By $NC(n)$ we will denote the lattice of noncrossing partitions [Kre72]. These are the partitions with the property that

$$i<j<k, \qquad i \overset{\pi}{\sim} k, \qquad j \overset{\pi}{\sim} l, \qquad i \overset{\pi}{\nsim} j \Rightarrow i<l<k.$$

Finally, the third lattice, used mostly for notational convenience, is the lattice $Int(n)$ of interval partitions [vW73]. These are the partitions whose classes are intervals, and $Int(n)$ is isomorphic as a lattice to the lattice of subsets of a set of $(n-1)$ elements.

There is a partial order $\leqslant$ on $\mathscr{P}(n)$ which restricts to the other two lattices. We denote the smallest element in $\mathscr{P}(n)$ by $\hat{0}=\{(1), (2), ..., (n)\}$, and the largest one by $\hat{1}=\{(1, 2, ..., n)\}$. We denote the meet and the join in the lattices by $\wedge$ and $\vee$, respectively.

We need the following operations on partitions. For $\pi \in \mathscr{P}(n)$, we define $\pi^{op} \in \mathscr{P}(n)$ to be π taken in the opposite order, i.e.

$$i \overset{\pi^{op}}{\sim} j \Leftrightarrow (n-i+1) \overset{\pi}{\sim} (n-j+1).$$

We define $\pi^k \in \mathscr{P}(nk)$, the k-thickening of π, by

$$i \overset{\pi^k}{\sim} j \Leftrightarrow [(i-1)/k]+1 \overset{\pi}{\sim} [(j-1)/k]+1,$$

where $[\cdot]$ denotes the integer part of a real number. In words, we expand each point of the set $\{1, 2, ..., n\}$ into k points, and require that those points lie consecutively and in the same class of π^k. For $\pi \in \mathscr{P}(n)$, $\sigma \in \mathscr{P}(k)$, we define $\pi+\sigma \in \mathscr{P}(n+k)$ by

$$i \overset{\pi+\sigma}{\sim} j \Leftrightarrow ((i, j \leqslant n, i \overset{\pi}{\sim} j) \qquad \text{or} \quad (i, j>n, (i-n) \overset{\sigma}{\sim} (j-n))).$$

We will denote $m\pi := \pi+\pi+ \cdots +\pi$ m times.

Following [BLS96], we divide the blocks of a noncrossing partition π into inner and outer: a block $B \in \pi$ is called *inner* if there exist $(i \overset{\pi}{\sim} j$, $i, j \notin B)$ such that $i<k<j$ for some, hence all, $k \in B$. A block that is not inner is called outer.

Finally, for $\pi \in \mathscr{P}(n)$, we define the number of crossings of π, $c(\pi)$, to be

$$c(\pi)=\min(|\sigma|-|\pi| : \sigma \in NC(n), \sigma \leqslant \pi).$$

In words, this is the minimal number of "cuts" in the classes of π required to make it noncrossing. Notice that this number is different from the reduced number of crossings of [Nic95], the number of the restricted crossings of [Bia97b], and $m(\pi)$ of [Mar98].

2.2. *Diagonals.* Fix n, let π be a set partition of n, with $|\pi| = k$ classes $B_1, B_2, ..., B_k$. For a set $S = \{1, 2, ..., N\}$, denote by S^n_π the π-diagonal of S^n, that is the set of n-tuples $(i_1, i_2, ..., i_n) \in S^n$ such that

$$k \overset{\pi}{\sim} l \text{ (i.e. } k \text{ and } l \text{ lie in the same class of } \pi) \Leftrightarrow i_k = i_l.$$

For example, to the partition $\{(1, 5, 8), (2, 7), (3), (4, 6)\}$ of the set of eight elements there corresponds the set $\{(i_1, i_2, i_3, i_4, i_1, i_4, i_2, i_1): i_1 \neq i_2 \neq i_3 \neq i_4\}$. Note that $S^n_\pi \neq \emptyset$ only if $N \geqslant |\pi|$; more generally, $|\{1, 2, ..., N\}^k_\pi| = (N)_{|\pi|}$, where $(N)_m = N(N-1) \cdots (N-m+1)$. Similarly, denote by $S^n_{\geqslant \pi}$ the set of n-tuples $(i_1, i_2, ..., i_n) \in S^n$ such that

$$k \overset{\pi}{\sim} l \Rightarrow i_k = i_l.$$

Note that

$$S^n_{\geqslant \pi} = \bigcup_{\sigma \geqslant \pi} S^n_\sigma. \tag{1}$$

2.3. *Processes with free increments.* Let $(\mathscr{A}, \varphi)$ be a tracial W^*-noncommutative probability space. That is, $\mathscr{A}$ is a finite von Neumann algebra, and φ is a faithful normal trace state on it. We will call the elements of $\mathscr{A}$ noncommutative random variables, or random variables for short.

For definitions of free probabilistic notions that are not given here we refer the reader, for example, to the monograph [VDN92]. Let μ be a freely infinitely divisible distribution with compact support; normalize it so that $\mathrm{Var}(\mu) = 1$. Let $\{\mu_t\}_{t \in [0, \infty)}$ be the corresponding additive free convolution semigroup. Note that in the free case, as opposed to the classical case, both the free normal distribution (the semicircular distribution) and the free Poisson distribution have compact support, so the condition is not as restrictive as it might appear. We will return to the matter of extending the contents of this paper to general freely infinitely divisible distributions elsewhere.

Definition 1. A stationary stochastic process with freely independent increments is a map from the set of finite half-open intervals $I = [a, b) \subset \mathbb{R}$ to the self-adjoint part of $(\mathscr{A}, \varphi)$ (which can be extended to the map on all Borel subsets) $I \mapsto X_I$ with the following three properties:

1. $I_1, I_2, ..., I_n$ disjoint $\Rightarrow \{X_{I_1}, X_{I_2}, ..., X_{I_n}\}$ are freely independent (free increments),
2. $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = J \Rightarrow X_{I_1} + X_{I_2} = X_J$ (additivity),
3. The distribution of X_I is $\mu_{|I|}$ (stationarity).

Substantial study of processes with free increments was started in [GSS92] and [Bia98]. In particular, it was shown in [GSS92] that for any μ as above, there is a realization of such a process. Note that throughout the paper the terms "free stochastic process" and "free stochastic measure" will be used interchangeably.

We now define the product measures $\mathrm{Pr}_\pi(A)$ and the stochastic measures $\mathrm{St}_\pi(A)$, depending on the partition π of k. These will again be additive processes on the real line with free identically distributed increments. Note that this is in contrast with [RW97], where the corresponding objects are processes on a k-dimensional space.

DEFINITION 2. Let A be a union of half-open intervals in $\mathbb{R}$. Denote $X = X_A$, and for an arbitrary N let $X_1^{(N)}, X_2^{(N)}, ..., X_N^{(N)}$ be freely independent, identically distributed, and add up to X. Note that henceforth we will usually omit the explicit dependence on N, to simplify notation. Then

$$\mathrm{St}_\pi(A) = \lim_{N \to \infty} \sum_{\substack{(i_1, i_2, ..., i_k) \\ \in \{1, 2, ..., N\}^k_\pi}} X_{i_1}^{(N)} X_{i_2}^{(N)} \cdots X_{i_k}^{(N)},$$

$$\mathrm{Pr}_\pi(A) = \lim_{N \to \infty} \sum_{\substack{(i_1, i_2, ..., i_k) \\ \in \{1, 2, ..., N\}^k_{\geqslant \pi}}} X_{i_1}^{(N)} X_{i_2}^{(N)} \cdots X_{i_k}^{(N)},$$

where the limit here and, unless noted otherwise, elsewhere are taken in the operator norm. For the discussion of the existence of the limits, see Sections 3 and 6.2.

We call $\psi_k := \mathrm{St}_{\hat{0}}$ *the* stochastic measure of degree k. We also define the k-th diagonal measure of the process by

$$\Delta_k(A) = \mathrm{St}_{\hat{1}} = \mathrm{Pr}_{\hat{1}} = \lim_{N \to \infty} \sum_{i=1}^{N} X_i^k.$$

Note that the second diagonal measure of the process $\Delta_2(A)$ is frequently called the quadratic variation of the process and denoted by $\langle X, X \rangle$.

Throughout most of the paper we will fix the set A and write $X := X_A$, $\mathrm{St}_\pi := \mathrm{St}_\pi(A)$, etc.

2.4. *Multidimensional R-transform and noncrossing cumulants.* This section could have been taken directly from, say, [NS96] and is included for completeness.

Given a family $x_1, x_2, ..., x_k$ in a noncommutative probability space $(\mathcal{A}, \varphi)$, their joint distribution is the collection of their joint moments

$$M(x_{i_1}, x_{i_2}, ..., x_{i_n}) = \varphi(x_{i_1} x_{i_2} \cdots x_{i_n})$$

for $1 \leqslant i_j \leqslant k$, $1 \leqslant j \leqslant n$. For a partition $\pi \in NC(n)$, we define

$$M_\pi(x_{i_1}, x_{i_2}, ..., x_{i_n}) = \prod_{B \in \pi} M(x_{i(j_1)}, x_{i(j_2)}, ..., x_{i(j_{|B|})} : j_1 < j_2 < \cdots < j_{|B|}, \{j_1, j_2, ..., j_{|B|}\} = B).$$

We define the joint free cumulants of $(x_1, x_2, ..., x_k)$, which together comprise the multidimensional R-transform, recursively by

$$M_\pi(x_{i_1}, x_{i_2}, ..., x_{i_n}) = \sum_{\substack{\sigma \in NC(n) \\ \sigma \leqslant \pi}} R_\sigma(x_{i_1}, x_{i_2}, ..., x_{i_n}),$$

where again

$$R_\sigma(x_{i_1}, x_{i_2}, ..., x_{i_n}) = \prod_{B \in \sigma} R(x_{i(j_1)}, x_{i(j_2)}, ..., x_{i(j_{|B|})} : j_1 < j_2 < \cdots < j_{|B|}, \{j_1, j_2, ..., j_{|B|}\} = B).$$

The main property of the R-transform is that

$$(i_j \overset{\pi}{\sim} i_l, \quad x_{i_j} \text{ and } x_{i_l} \text{ freely independent}) \Rightarrow R_\pi(x_{i_1}, x_{i_2}, ..., x_{i_n}) = 0.$$

Denote by K the Kreweras complement map on $NC(n)$ [Kre72, NS96]. This is a certain bijection on the lattice $NC(n)$ and it follows from the main property of the R-transform that for $\{x_1, x_2, ..., x_n\}$ freely independent from $\{y_1, y_2, ..., y_n\}$,

$$\varphi(x_1 y_1 x_2 y_2 \cdots x_n y_n) = \sum_{\pi \in NC(n)} R_{K(\pi)}(x_1, x_2, ..., x_n)\, M_\pi(y_1, y_2, ..., y_n).$$

If $x_{i_1} = x_{i_2} = \cdots = x_{i_n} = x$, we write $R_\pi(x, x, ..., x) := R_\pi(x)$. We denote the individual moments and free cumulants by $m_n(x) = M_{\hat{1}_n}(x)$, $r_n(x) = R_{\hat{1}_n}(x)$. Note that $r_1(x) = m_1(x) = \varphi(x)$.

If x has distribution μ which is freely infinitely divisible, and y has distribution μ_t, then $r_n(y) = t r_n(x)$. Therefore $R_\pi(y) = t^{|\pi|} R_\pi(x)$.

Finally, for an algebra element Z, we denote by Z° the centered version of Z, $Z^\circ = Z - \varphi(Z)$. Note that $r_1(Z^\circ) = 0$, $r_n(Z^\circ) = r_n(Z)$ for $n > 1$.

3. PRELIMINARIES

As defined, the product and stochastic measures depend on the particular triangular array $\{X_i^{(N)}\}_{i=1}^N$, $N \in \mathbb{N}$. We will show that the limits exist for a particular choice of this array in Section 6.2. For now, we make a number of observations which are consequences of the free independence of the increments of the process, and which will hold for any such array.

LEMMA 1.

$$\lim_{N \to \infty} \varphi\Bigg(\sum_{\substack{(i_1, i_2, ..., i_k) \\ \in \{1, 2, ..., N\}^k_\pi}} X_{i_1} X_{i_2} \cdots X_{i_k} \Bigg) = \begin{cases} R_\pi(X) & \text{if } \pi \text{ is noncrossing} \\ 0 & \text{if } \pi \text{ is crossing.} \end{cases}$$

Proof.

$$\begin{aligned} \varphi\Bigg(\sum_{\substack{(i_1, i_2, ..., i_k) \\ \in \{1, 2, ..., N\}^k_\pi}} X_{i_1} X_{i_2} \cdots X_{i_k} \Bigg) &= \sum_{\substack{\sigma \in NC(k) \\ \sigma \leqslant \pi}} \sum_{\substack{(i_1, i_2, ..., i_k) \\ \in \{1, 2, ..., N\}^k_\pi}} R_\sigma(X_{i_1}, X_{i_2}, ..., X_{i_k}) \\ &= (N)_{|\pi|} \sum_{\substack{\sigma \in NC(k) \\ \sigma \leqslant \pi}} N^{-|\sigma|} R_\sigma(X, X, ..., X) \qquad (2) \end{aligned}$$

since $r_j(X_i) = \frac{1}{N} r_j(X)$. If π is noncrossing, then the limit, as $N \to \infty$, of the expression (2) is $R_\pi(X)$. On the other hand, assume that π is crossing. As in [Bia98], the number of elements of $NC(k)$, which is a Catalan number, is less than 4^k and for each σ, $|R_\sigma(X)| \leqslant 4^k \|X\|^k$. Thus the absolute value of the expression (2) is less than $4^{2k} \|X\|^k N^{-c(\pi)}$. In particular, it converges to 0 as $N \to \infty$. ▮

THEOREM 1.

$$\mathrm{St}_\pi = \lim_{N \to \infty} \sum_{\substack{(i_1, i_2, ..., i_k) \\ \in \{1, 2, ..., N\}^k_\pi}} X_{i_1} X_{i_2} \cdots X_{i_k} = 0$$

unless π is noncrossing.

Proof. More generally,

$$\begin{aligned} &\varphi\Bigg(\Bigg(\Bigg(\sum_{\substack{(i_1, i_2, ..., i_k) \\ \in \{1, 2, ..., N\}^k_\pi}} X_{i_1} X_{i_2} \cdots X_{i_k} \Bigg)\Bigg(\sum_{\substack{(i_1, i_2, ..., i_k) \\ \in \{1, 2, ..., N\}^k_\pi}} X_{i_1} X_{i_2} \cdots X_{i_k} \Bigg)^* \Bigg)^n\Bigg) \\ &\quad = \varphi\Bigg(\sum_{\substack{\sigma \in \mathscr{P}(2nk) \\ \sigma \geqslant n(\pi + \pi^{op}) \\ \sigma \wedge 2n\hat{1}_k = n(\pi + \pi^{op})}} \sum_{\substack{(i_1, i_2, ..., i_{2kn}) \\ \in \{1, 2, ..., N\}^{2kn}_\sigma}} X_{i_1} X_{i_2} \cdots X_{i_{2kn}} \Bigg). \qquad (3) \end{aligned}$$

For σ as in equation (3), $c(\sigma) \geqslant 2nc(\pi)$. Applying Lemma 1 and using the estimate in the proof of that Lemma, we see that the above expression (3) is less than $4^{4nk}\,\|X\|^{2nk}\,N^{-2nc(\pi)}\,d_n^{|\pi|}$, where $d_n^m=|\{\sigma\in\mathscr{P}(2nm):\sigma\wedge 2n\hat{1}_m=2n\hat{1}_m\}|$. It was shown in Theorem 5.3.4 of [BS98] that $\lim_{n\to\infty}(d_n^m)^{1/2n}=(m+1)$ (note that our use of m and n is the opposite of theirs). Therefore

$$\begin{aligned}\Bigg\|\sum_{\substack{(i_1,i_2,\ldots,i_k)\\ \in\{1,2,\ldots,N\}_\pi^k}} X_{i_1}X_{i_2}\cdots X_{i_k}\Bigg\| &= \lim_{n\to\infty}\Bigg[\varphi\Bigg(\Bigg(\Bigg(\sum_{\substack{(i_1,i_2,\ldots,i_k)\\ \in\{1,2,\ldots,N\}_\pi^k}} X_{i_1}X_{i_2}\cdots X_{i_k}\Bigg)\\ &\quad\times\Bigg(\sum_{\substack{(i_1,i_2,\ldots,i_k)\\ \in\{1,2,\ldots,N\}_\pi^k}} X_{i_1}X_{i_2}\cdots X_{i_k}\Bigg)^*\Bigg)^n\Bigg)\Bigg]^{1/2n}\\ &\leqslant 4^{2k}\,\|X\|^k\,(|\pi|+1)\,N^{-c(\pi)}.\end{aligned}$$

In particular,

$$\lim_{N\to\infty}\Bigg\|\sum_{\substack{(i_1,i_2,\ldots,i_k)\\ \in\{1,2,\ldots,N\}_\pi^k}} X_{i_1}X_{i_2}\cdots X_{i_k}\Bigg\|=0,$$

unless π is noncrossing. ∎

The following is the analog of Proposition 1 from [RW97] for the lattice of noncrossing partitions. Note that that Proposition for the lattice of all partitions follows directly from Definition 2 and so remains true as well.

Corollary 1. *The measures* Pr_π *and* St_π *are related as follows: for* $\pi\in NC(k)$

$$\mathrm{Pr}_\pi=\sum_{\substack{\sigma\in NC(k)\\ \sigma\geqslant\pi}}\mathrm{St}_\sigma,$$

$$\mathrm{St}_\pi=\sum_{\substack{\sigma\in NC(k)\\ \sigma\geqslant\pi}}\mu(\pi,\sigma)\,\mathrm{Pr}_\sigma,$$

where $\mu(\pi,\sigma)$ *is the Möbius function on the lattice of noncrossing partitions* [Kre72].

Proof. By Definition 2 and Eq. (1), $\mathrm{Pr}_\pi=\sum_{\sigma\in\mathscr{P}(k),\,\sigma\geqslant\pi}\mathrm{St}_\sigma$. By the Theorem, for $\sigma\notin NC(k)$, $\mathrm{St}_\sigma=0$. The second equality follows from the first one by the use of Möbius inversion on the lattice of noncrossing partitions. ∎

Hereafter, it is reasonable to consider only product and stochastic measures corresponding to noncrossing partitions, for the following reason. Following [RW97], we call a measure *multiplicative* if for any partition π (of the type to be determined below)

$$\varphi(\mathrm{St}_\pi) = \prod_{B \in \pi} \varphi(\Delta_{|B|}). \tag{4}$$

It is easy to see that if the process is not commutative, the measures are not in general multiplicative with respect to the lattice of all partitions. For example, let $\pi = \{(1, 3), (2, 4)\}$. The partition is crossing, so for the free Brownian motion the left-hand-side of Eq. (4) is 0, while the right-hand-side is $\varphi(\Delta_2(A))^2 = |A|^2 \neq 0$.

Corollary 2. *Stochastic processes with freely independent increments are multiplicative with respect to the lattice of noncrossing partitions. That is, for* $\pi \in NC(k)$, $\varphi(\mathrm{St}_\pi) = \prod_{B \in \pi} \varphi(\Delta_{|B|})$.

Proof. By Lemma 1,

$$\varphi(\mathrm{St}_\pi) = R_\pi(X) = \prod_{B \in \pi} r_{|B|}(X) = \prod_{B \in \pi} \varphi(\Delta_{|B|}).$$

Here the last equality follows from applying the first equality to $\pi = \hat{1}$. ∎

An immediate consequence of Corollary 1 is the analog of Theorem 1 from [RW97], which expresses the stochastic measure of degree k as a linear combination of various product measures. In fact more is true, as in this particular case not even all noncrossing partitions contribute to the sum. Call a *singleton* a class of a partition consisting of one element.

Proposition 1. *For a noncrossing partition* π *that contains an inner singleton*, $\mathrm{St}_\pi = 0$ *if the process is centered, that is*, $\varphi(X) = 0$.

Proof. If π contains a singleton, and the process is centered, then

$$\varphi\left(\sum_{\substack{(i_1, i_2, ..., i_k) \\ \in \{1, 2, ..., N\}^k_\pi}} X_{i_1} X_{i_2} \cdots X_{i_k} \right) = 0.$$

In fact each term is 0, for any N. On the other hand, if π contains an inner singleton, so does any partition $\sigma \geqslant n(\pi + \pi^{op})$, $\sigma \wedge (2n\hat{1}_k) = n(\pi + \pi^{op})$ with at most $(n-1)$ crossings, for any n. Thus in the sum in the equation (3), only the partitions σ with $c(\sigma) \geqslant n$ enter, and so by the argument following that equation we see that the limit defining St_π is 0. ∎

The following statement fits well with the original definition of free independence of Voiculescu [Voi85, VDN92].

COROLLARY 3. *If the process is centered,*

$$\psi_k = \mathrm{St}_{\hat{0}} = \lim_{N\to\infty} \sum_{\substack{i_1 \neq i_2 \neq \cdots \neq i_k \\ \textit{all distinct}}}^{N} X_{i_1} X_{i_2} \cdots X_{i_k}$$

$$= \lim_{N\to\infty} \sum_{\substack{i_1 \neq i_2 \neq \cdots \neq i_k \\ \textit{neighbors distinct}}}^{N} X_{i_1} X_{i_2} \cdots X_{i_k}.$$

Proof. The only noncrossing partition with no inner singletons for which no consecutive elements lie in the same class is $\hat{0}$. ∎

Finally, note that for any triangular array as above, we always have convergence in distribution for the diagonal measures (and hence for the stochastic measures by the results in Section 7):

THEOREM 2. *The free cumulants of the kth diagonal measure of the process are given by*

$$r_n(\Delta_k) = \lim_{N\to\infty} r_n\left(\sum_{i=1}^{N} X_i^k\right) = r_{nk}(X).$$

Proof. For $\mu(\pi) := \mu(\pi, \hat{1})$ the Möbius function on the lattice of noncrossing partitions,

$$\begin{aligned}
r_n\left(\sum_{i=1}^{N} X_i^k\right) &= \sum_{i=1}^{N} \sum_{\pi\in NC(n)} \mu(\pi)\, M_\pi(X_i^k) \\
&= \sum_{i=1}^{N} \sum_{\pi\in NC(n)} \mu(\pi) \prod_{B_j\in\pi} m_{|B_j|}(X_i^k) \\
&= \sum_{i=1}^{N} \sum_{\pi\in NC(n)} \mu(\pi) \prod_{B_j\in\pi} m_{k\,|B_j|}(X_i) \\
&= \sum_{i=1}^{N} \sum_{\pi\in NC(n)} \mu(\pi) \prod_{B_j\in\pi} \sum_{\sigma_j\in NC(k\,|B_j|)} \prod_{A\in\sigma_j} r_{|A|}(X_i) \\
&= \sum_{\pi\in NC(n)} \mu(\pi) \prod_{B_j\in\pi} \sum_{\sigma_j\in NC(k\,|B_j|)} N^{1-\sum_{j=1}^{|\pi|} |\sigma_j|} \prod_{A\in\sigma_j} r_{|A|}(X) \\
&= \mu(\hat{1})\, r_{nk}(X) + O(1/N) \\
&= r_{nk}(X) + O(1/N). \quad ∎
\end{aligned}$$

4. DIAGONAL MEASURES: EXAMPLES

4.1. *Free Brownian motion.* It follows from the results of [BS98] that for the free Brownian motion the quadratic variation is the scalar process $I \mapsto |I|$, and the higher diagonal measures are 0.

4.2. *Free Poisson process.* The free Poisson distribution is a distribution obtained by the Poisson-type limit process for freely independent variables [VDN92]. It is characterized by the property that all its free cumulants are equal to 1. For the free Poisson process, a remarkable representation was given in [NS96] (see also the Appendix to that paper). Let $I \mapsto p_I$ be a projection-valued process $[0,1] \to \mathcal{A}$. That is, $(I \cap J = \varnothing \Rightarrow p_I \perp p_J)$, $(I_1 \cap I_2 = \varnothing,\ I_1 \cup I_2 = J \Rightarrow p_{I_1} + p_{I_2} = p_J)$, $\varphi(p_I) = |I|$. Let $s \in \mathcal{A}$ be an element with a standard semicircular distribution, freely independent from the family $\{p_I\}$. Then $I \mapsto \{s p_I s\}$ is a free Poisson process for $I \subset [0,1]$. To get the full process, pick a countable collection of such families of projections, $p_I^{(n)}$, $I \subset [0,1]$, $n \in \mathbb{Z}$, which are freely independent from each other. Then $[t_1, t_2) \mapsto s(p^{([t_1])}_{[t_1 - [t_1], 1)} + \sum_{i=[t_1]+1}^{[t_2]-1} p^{(i)}_{[0,1)} + p^{([t_2])}_{[0, t_2 - [t_2])}) s$ is a free Poisson process. Here $[\cdot]$ again denotes the integer part of a real number.

By Theorem 2 we know the distributions of its diagonal measures. In fact in this case we can prove convergence in norm to a specific limit. First we have a technical theorem.

THEOREM 3. *Let $Z_1, Z_2, ..., Z_k$ be centered, and fix e, a self-adjoint element freely independent from the family $\{Z_i\}$. For $N \in \mathbb{N}$, let the family $e_1^{(N)}, e_2^{(N)}, ..., e_N^{(N)}$ (where henceforth we will again omit the dependence on N) be self-adjoint, freely independent from the family $\{Z_i\}$ and be an orthogonal family, that is, $e_i e_j = 0$ for $i \neq j$. In addition, let the e_i's be identically distributed and $\sum_{i=1}^N e_i = e$. Then for arbitrary indices $m_1, m_2, ..., m_{k+1} \in \mathbb{N}$*

$$\lim_{N \to \infty} \sum_{i=1}^N e_i^{m_1} Z_1 e_i^{m_2} Z_2 \cdots e_i^{m_k} Z_k e_i^{m_{k+1}} = 0.$$

Proof. First note that since the e_i's are orthogonal and identically distributed,

$$\varphi(e^n) = \varphi\left(\left(\sum_{i=1}^N e_i\right)^n\right) = \varphi\left(\sum_{i=1}^N e_i^n\right) = N\varphi(e_1^n).$$

$$\varphi\left(\left(\sum_{i=1}^N e_i^{m_1} Z_1 e_i^{m_2} Z_2 \cdots e_i^{m_k} Z_k e_i^{m_{k+1}}\right)^n\right)$$

$$= \varphi\left(\sum_{i=1}^N (e_i^{m_1} Z_1 e_i^{m_2} Z_2 \cdots e_i^{m_k} Z_k e_i^{m_{k+1}})^n\right)$$

$$=\varphi\left(\sum_{i=1}^{N}\left(e_i^{m_1+m_{k+1}}Z_1e_i^{m_2}Z_2\cdots e_i^{m_k}Z_k\right)^n\right)$$

$$=\sum_{i=1}^{N}\sum_{\pi\in NC(nk)}R_{K(\pi)}(Z_1,Z_2,...,Z_k,Z_1,...,Z_k)$$
$$\times M_\pi(e_i^{m_1+m_{k+1}},e_i^{m_2},...,e_i^{m_k},e_i^{m_1+m_{k+1}},...,e_i^{m_k}). \tag{5}$$

Z_i's are centered, so $r_1(Z_i)=0$. Thus only those partitions π contribute to the sum (5) for which $K(\pi)$ has no single-element classes. In particular $|K(\pi)|\leqslant(nk)/2$ and so $|\pi|\geqslant(nk)/2$ (since $|K(\pi)|+|\pi|=nk+1$). Then the sum (5) is

$$\sum_{i=1}^{N}\sum_{\substack{\pi\in NC(nk)\\|\pi|\geqslant(nk)/2}}R_{K(\pi)}(Z_1,Z_2,...,Z_k,Z_1,...,Z_k)\,N^{-|\pi|}$$
$$\times M_{(m_1+m_{k+1},m_2,...,m_k,m_1+m_{k+1},...,m_k)\,\pi}(e,e,...,e),$$

where $(u_1,u_2,...,u_{nk})\pi\in NC(\sum_{i=1}^{nk}u_i)$ is the partition obtained by expanding the ith point of the set on which π operates into u_i points. That is,

$$(i_1\overset{(u_1,u_2,...,u_{nk})\,\pi}{\sim}i_2)$$
$$\Leftrightarrow\left(\sum_{j=1}^{s_1}u_j+1\leqslant i_1\leqslant\sum_{j=1}^{s_1+1}u_j,\ \sum_{j=1}^{s_2}u_j+1\leqslant i_2\leqslant\sum_{j=1}^{s_2+1}u_j,\text{ and }s_1\overset{\pi}{\sim}s_2\right).$$

(Note that $(m,m,...,m)\,\pi=\pi^m$.) Therefore the sum (5) is less in absolute value than

$$4^{2nk}(\max\|Z_i\|)^{nk}\,\|e\|^{n\sum_{i=1}^{k+1}m_i}N^{1-nk/2}.$$

We can always choose Z_i's to be self-adjoint, while the e_i's are so by assumption. Hence

$$\left\|\sum_{i=1}^{N}e_i^{m_1}Z_1e_i^{m_2}Z_2\cdots e_i^{m_k}Z_ke_i^{m_{k+1}}\right\|\leqslant4^{2k}(\max\|Z_i\|)^k\,\|e\|^{\sum_{i=1}^{k+1}m_i}N^{-k/2}$$

and so converges to 0 as $N\to\infty$. ∎

Corollary 4. *For the free Poisson process, the kth diagonal measure is equal to the process itself.*

Proof. First consider the process on the interval $[0,1]$. In the notation from the beginning of the section, $X=sps$ and $X_i=sp_is$.

$$\begin{aligned}\sum_{i=1}^{N} (p_i s^2 p_i)^k &= \sum_{i=1}^{N} p_i s^2 p_i s^2 \cdots p_i s^2 p_i \\ &= \sum_{i=1}^{N} \sum_{\pi \in Int(k+1)} \varphi(s^2)^{k-|\pi|+1} p_i((s^2)^{\circ} p_i)^{|\pi|-1} \\ &= \sum_{i=1}^{N} \sum_{j=0}^{k} \binom{k}{j} p_i((s^2)^{\circ} p_i)^j\end{aligned}$$

since $\varphi(s^2)=1$. In the limit $N \to \infty$, by the Theorem the only term that survives is the one for $j=0$ (i.e. $\pi = \hat{1}$, $|\hat{1}|=1$), and that term is $\sum_{i=1}^{N} p_i = p$.

Therefore

$$\lim_{N \to \infty} \sum_{i=1}^{N} (s p_i s)^k = s \lim_{N \to \infty} \sum_{i=1}^{N} (p_i s^2 p_i)^{k-1} s = sps.$$

For the full Poisson process, the result follows from the free independence of the increments corresponding to disjoint intervals. ∎

In fact we can calculate explicitly some product measures for the free Poisson process; see Section 5.

4.3. *Free compound Poisson processes.* Let e_I be a process with identically distributed orthogonal increments on $[0, 1]$. It is a map from all finite half-open intervals $I \subset [0, 1]$ to the self-adjoint part of $(\mathcal{A}, \varphi)$ such that $(I \cap J = \varnothing \Rightarrow e_I e_J = 0)$, $(I_1 \cap I_2 = \varnothing,\ I_1 \cup I_2 = J \Rightarrow e_{I_1} + e_{I_2} = e_J)$, and the distribution of X_I depends only on $|I|$. Note that this implies that if μ is the distribution of $e_{[0,1)}$ then the distribution of e_I is $((1-|I|)\,\delta_0 + |I|\,\mu)$. Let s be a random variable with the standard semicircle distribution, freely independent from this process. Then by [NS96] the process $s e_I s$ is a process with free increments. Its distribution is

$$r_n(s^2 e) = \sum_{\pi \in NC(n)} R_\pi(e)\, R_{K(\pi)}(s^2) = \sum_{\pi \in NC(n)} R_\pi(e) = m_n(e).$$

That is, the process $s e_I s$ is a free compound Poisson process (these have been considered previously in [GSS92] and [Spe98]). In fact, that will be true even if the process does not have moments that are finite. Indeed, for general distributions we can use the "free Fourier transform", which has appeared for example in [NS97]. Namely, there is a relationship between Voiculescu's R- and S-transforms: denoting $\alpha(z) = zR(z)$, one has $S(w) = w^{-1}\alpha^{-1}(w)$.

LEMMA 2. *Let x be a random variable, and s a standard semicircular random variable freely independent from x. Let $y=sxs$. Then $S_y(w)=\frac{1}{w+1}S_x(w)$. Therefore $R_y(w)=w^{-1}\psi_x(w)$. Consequently the distribution of y is a free compound Poisson distribution with the Levy measure equal to the distribution of x.*

Proof. For a free Poisson element s^2, $R(z)=\frac{1}{1-z}$, so $S(w)=\frac{1}{1+w}$. Therefore $S_y(w)=\frac{1}{1+w}S_x(w)$ and so $\alpha_y(w)=\frac{w}{1+w}S_x(w)=\chi_x(w)$, which implies $R_y(z)=z^{-1}\psi_x(z)$. ∎

For a free compound Poisson process given in the standard form se_Is, we will call the process with orthogonal increments e_I the *generator* of the process (cf. [GSS92]).

To extend the process from $[0, 1]$ to the whole real line we can, once again, take a countable family of processes $e_I^{(n)}$ with orthogonal increments on $[0, 1]$ which are freely independent from each other and, thinking of $e_I^{(n)}$ as acting on $[n, n+1]$, take their sum.

Again we know the distributions of the diagonal measures from Theorem 2, but can in fact find them explicitly:

COROLLARY 5. *The diagonal measures of a free compound Poisson process with generator e are free compound Poisson processes with generators e^k.*

Proof.

$$\begin{aligned}\sum_{i=1}^{N}(se_is)^k &= s\sum_{i=1}^{N} e_is^2e_is^2\cdots e_is\\ &= s\sum_{i=1}^{N}\sum_{\pi\in Int(k)}\varphi(s^2)^{k-|\pi|}\,e_i^{|B_1|}(s^2)^{\circ}\,e_i^{|B_2|}(s^2)^{\circ}\cdots(s^2)^{\circ}\,e_i^{|B_{|\pi|}|}s,\end{aligned}$$

where $B_1, B_2, ..., B_{|\pi|}$ are the blocks of π. By Theorem 3, once again the only term that survives in the limit $N\to\infty$ is the one for $\pi=\hat{1}$, $|\hat{1}|=1$, and that term is $s\sum_{i=1}^{N}e_i^k s=s(\sum_{i=1}^{N}e_i)^k s=se^ks$. ∎

5. PRODUCT MEASURES

The key point of the paper [RW97] is that the stochastic measures St_π can be expressed combinatorially through the product measures Pr_σ, and the latter are indeed product vector measures. In the free case the situation is more complicated.

For a noncrossing partition π, let $i(\pi)$ and $o(\pi)$ be, respectively, the numbers of the inner and outer classes of π; in particular, $i(\pi)+o(\pi)=|\pi|$.

Here's another description of $i(\pi)$ and $o(\pi)$. There is a partial order on the classes of the partition π, by the height, i.e. for two classes $B \neq C$ of π, $B > C \Leftrightarrow \exists i, j \in B, k \in C : i < k < j$. Then $i(\pi)$ is the number of edges in the incidence graph of this partial order, and $o(\pi)$ is the number of maximal elements under this order.

PROPOSITION 2. *For X the free Brownian motion, for a partition π with no inner singletons*

$$\mathrm{Pr}_\pi(A) = X^{|\{B \in \pi : |B|=1\}|}\, |A|^{|\{B \in \pi : |B|=2\}|}\, 0^{|\{B \in \pi : |B|>2\}|} = \prod_{B \in \pi} \Delta_{|B|}(A).$$

That is, Pr_π *are indeed product measures.*

Proof. The proof will be by induction on the level in the above partial order on the classes of the partition π. For the product measure $\mathrm{Pr}_\pi = \lim_{N \to \infty} \sum_{(i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, N\}^k_{\geqslant \pi}} X_{i_1} X_{i_2} \cdots X_{i_k}$ one can sum over the indices corresponding to the minimal classes, which are precisely the classes that are intervals, without disturbing the rest of the partition. If any of these classes contains at least 3 elements, the result is 0, while each two-element class gives $|A|$, which is a scalar and can be factored out. In the partition resulting from factoring out the minimal two-element classes, all the classes which were at the distance of 1 from the minimum now become minimal. ∎

For a partition π *with* an inner singleton, $\mathrm{Pr}_\pi(A) = \sum_{\sigma \geqslant \pi} \mathrm{St}_\sigma(A)$. Each of such σ's will either contain an inner singleton, or a class of at least 3 elements. In either case $\mathrm{St}_\sigma = 0$. Thus in this case $\mathrm{Pr}_\pi = 0$, and so need not in general be a product measure.

PROPOSITION 3. *Let X be the free Poisson process. Suppose π is a partition with the following property: if U, V are inner classes of π covered by a class W and such that $\forall u \in U, v \in V : u < v$, then there exists $w \in W$ such that $\forall u \in U, v \in V : u < w < v$. Then*

$$\mathrm{Pr}_\pi(A) = X^{o(\pi)}(1 + |A|)^{i(\pi)}.$$

In particular, Pr_π *in this case are not product measures. Note, however, that if π is an interval partition, then $i(\pi) = 0$ and $o(\pi) = |\pi|$, and so one does again get a product measure decomposition.*

Proof. Once again, the proof is by induction on the level of the class of π in the partial order. Since the diagonal measures of the free Poisson process are equal to the process itself, each minimal (interval) class can be shrunk to a one-element class. Again, consider a class which used to be at a distance of 1 from the minimum, and after the above shrinking, covers only a number of singletons:

$$\sum_{i, j_1, j_2, ..., j_k, ... = 1}^{N} \cdots X_i X_{j_1} X_i X_{j_2} X_i \cdots X_{j_k} X_i \cdots$$
$$= \sum_{\substack{\sigma \in Int(k+1) \\ \sigma = (B_1, B_2, ..., B_{|\sigma|})}} \sum_{\substack{i, j_1, ..., j_{|\sigma|-1}, ... = 1 \\ i \neq j_1, ..., i \neq j_{|\sigma|-1}}}^{N} \cdots X_i^{|B_1|} X_{j_1} X_i^{|B_2|} \cdots X_{j_{|\sigma|-1}} X_i^{|B_{|\sigma|}|} \cdots .$$

Even though the free Poisson process is not centered, it is easy to see that each inner singleton contributes a factor of $\varphi(X) = |A|$. Thus all the singletons covered by the i class contribute $\sum_{\sigma \in Int(k+1)} |A|^{|\sigma|-1} = (|A|+1)^k$. Note that k is precisely the number of classes covered by the i class, i.e. the number of edges emanating down from it in the partial order incidence graph. The result easily follows by induction. ∎

Now we consider products of free stochastic measures.

Definition 3. For $\pi \in \mathscr{P}(k)$, define

$$\overrightarrow{\prod_{B \in \pi}} \psi_{|B|} = \lim_{N \to \infty} \sum X_{i_1} X_{i_2} \cdots X_{i_k},$$

where the sum is taken over all collections of indices in $\{1, 2, ..., N\}^k$ such that

$$m \overset{\pi}{\sim} l \Rightarrow i_m \neq i_l. \tag{6}$$

That is, on each of the classes of π we consider the measure $\mathrm{St}_{\hat{0}}$. In particular, if π is an interval partition, then the result is the ordered product of measures corresponding to all the classes.

The following is the analog of Theorem 4 of [RW97]. This is a combinatorial form of the general Itô product formula.

Proposition 4. *Let $\pi \in \mathscr{P}(k)$ Then*

$$\overrightarrow{\prod_{B \in \pi}} \psi_{|B|} = \sum_{\substack{\sigma \in \mathscr{P}(k) \\ \sigma \wedge \pi = \hat{0}}} \mathrm{St}_\sigma = \sum_{\substack{\sigma \in NC(k) \\ \sigma \wedge \pi = \hat{0}}} \mathrm{St}_\sigma . \tag{7}$$

Proof. Any k-tuple of indices determines a partition $\sigma \in \mathcal{P}(k)$ by $(m \overset{\sigma}{\sim} l) \Leftrightarrow (i_m = i_l)$. Condition (6) restricts σ precisely to those for which $\sigma \wedge \pi = \hat{0}$, and the union of all collections of indices corresponding to such σ's is precisely the set given by condition (6). This decomposition of the set of indices gives the representation (7).

The second equality follows from Theorem 1. ∎

COROLLARY 6. *For* $\pi \in \mathcal{P}(k)$,

$$\overrightarrow{\prod_{B \in \pi}} \psi_{|B|} = \sum_{\hat{0} \leqslant \sigma \leqslant \pi} \mu_{\mathcal{P}}(\hat{0}, \sigma) \operatorname{Pr}_\sigma = \sum_{\substack{\sigma \in NC(k) \\ \hat{0} \leqslant \sigma \leqslant \pi}} \mu_{NC}(\hat{0}, \sigma) \operatorname{Pr}_\sigma,$$

where $\mu_{\mathcal{P}}$ *and* μ_{NC} *are the Möbius functions on the lattices of all and of noncrossing partitions, respectively.*

Proof. The proof of Theorem 4 in [RW97] shows that

$$\sum_{\hat{0} \leqslant \sigma \leqslant \pi} \mu(\hat{0}, \sigma) \operatorname{Pr}_\sigma = \sum_{\substack{\sigma \in \mathcal{P}(k) \\ \sigma \wedge \pi = \hat{0}}} \operatorname{St}_\sigma.$$

That proof is purely combinatorial, and works for noncommutative stochastic measures as well as for the commutative ones. Therefore the first equality follows from Proposition 4 above. Moreover, the proof also works for the lattice of noncrossing partitions, provided we use the appropriate Möbius function. Therefore the second equality holds as well. ∎

6. ORTHOGONALITY RELATIONS AND THE EXISTENCE OF THE LIMITS

6.1. *Orthogonality relations.* The following Proposition is the analog of Proposition 9, Theorem 9, and Proposition 10 of [RW97].

PROPOSITION 5. 1. *Let* $\pi \in NC(n)$. *Then*

$$\varphi\left(\overrightarrow{\prod_{B \in \pi}} \psi_{|B|}\right) = \sum_{\substack{\sigma \in NC(n) \\ \sigma \wedge \pi = \hat{0}}} \prod_{B \in \sigma} \varphi(\Delta_{|B|}).$$

2. *Assume that the process is centered. Then*

$$\varphi(\psi_n \psi_m) = \begin{cases} \varphi(\Delta_2)^n = r_2(X)^n = |A|^n, & \text{if } m = n \\ 0, & \text{if } m \neq n. \end{cases}$$

Proof. The first part is obtained by taking expectations of both sides in Proposition 4 and applying the multiplicativity property of Corollary 2. The second part follows from the first part. ∎

6.2. *The existence of the limits.* By the results in the previous subsection, standard Itô isometry arguments show that the limit defining ψ_n exists in L^2. But in fact, we want to show that the limits exist, in norm, for all stochastic measures.

Without loss of generality, let $A = [0, 1)$. Choose two different partitions of the interval A: $X_i = X_{[i-1/N,\, i/N)}$, $i = 1, 2, ..., N$, $Y_j = X_{[j-1/M,\, j/M)}$, $j = 1, 2, ..., M$. Let $Z_m = X_{[m-1/MN,\, m/MN)}$, $m = 1, 2, ..., MN$ be their common refinement. Fix a partition π.

$$\left[\varphi\left(\left(\sum_{\substack{(i_1, i_2, ..., i_k)\\ \in \{1, 2, ..., N\}^k_\pi}} X_{i_1} X_{i_2} \cdots X_{i_k} - \sum_{\substack{(j_1, j_2, ..., j_k)\\ \in \{1, 2, ..., M\}^k_\pi}} Y_{j_1} Y_{j_2} \cdots Y_{j_k}\right)^n\right)\right]^{1/n}$$

$$= \left[\varphi\left(\left(\sum_{\substack{(i_1, i_2, ..., i_k)\\ \in \{1, 2, ..., N\}^k_\pi}} \left(\sum_{s_1 = (i_1 - 1) M + 1}^{i_1 M} Z_{s_1}\right)\right.\right.\right.$$

$$\times \left(\sum_{s_2 = (i_2 - 1) M + 1}^{i_2 M} Z_{s_2}\right) \cdots \left(\sum_{s_k = (i_k - 1) M + 1}^{i_k M} Z_{s_k}\right)$$

$$\left.\left.\left. - \sum_{\substack{(j_1, j_2, ..., j_k)\\ \in \{1, 2, ..., M\}^k_\pi}} \left(\sum_{t_1 = (j_1 - 1) N + 1}^{j_1 N} Z_{t_1}\right) \cdots \left(\sum_{t_k = (j_k - 1) N + 1}^{j_k N} Z_{t_k}\right)\right)^n\right)\right]^{1/n}. \tag{8}$$

This expression can be written as a sum over partitions $\sigma \in \mathcal{P}(nk)$ of free cumulants of $X = X_A$ with weights depending on N, M, π, σ, raised to the $1/n$th power. While I have not been able to estimate it directly, a trick similar to the one in Theorem 5.3.4 of [BS98] is applicable. Namely, from Proposition 3 and Corollary 1 we know that the expressions defining various stochastic measures do converge to a limit for the free Poisson process. For that process the expression (8) is the sum with all the free cumulants equal to 1. Since, as before, in general the cumulants grow no faster than an exponential, we have a uniform estimate on the expression (8) for $N, M > N_0$ for a fixed N_0. Therefore the sequence of approximations to the stochastic measure is a Cauchy sequence, and so converges to a limit.

7. THE FREE KAILATH–SEGALL FORMULA

The purpose of this section is to investigate the issues surrounding the free Kailath–Segall formula, the analog of Theorem 2 of [RW97].

The free Poisson distribution for the value of the parameter $t := \varphi(X_A) = |A| = 1$ is the image of the standard semicircle measure under the squaring map. Therefore if T_n are the orthogonal polynomials with respect to the semicircle measure, namely the Chebyshev polynomials of the second kind, then the orthogonal polynomials with respect to free Poisson (1) measure are $P_n(x) = T_{2n}(\sqrt{x})$. In particular from the usual Chebyshev recursion $xT_n(x) = T_{n+1}(x) + T_{n-1}(x)$ we get the recursion relations

$$xP_n(x) = P_{n+1}(x) + 2P_n(x) + P_{n-1}(x).$$

The orthogonal polynomials for the compensated (i.e. centered) Poisson (1) measure satisfy the same relations. We will see a generalization of these relations in Corollary 10.

Denote $\alpha_{n,m} = \Delta_n \psi_m$ and $\beta_{n,m} = \mathrm{St}_{\hat{1}_n + \hat{0}_m}$. Let $t := \varphi(X)$ be the expectation of the process.

LEMMA 3. *We have the following recursion relation,*

$$\alpha(n, m) = \beta(n, m) + \sum_{l=0}^{m-1} t^{m-1-l} \beta(n+1, l),$$

for $n, m \geqslant 1$, *with boundary conditions* $\alpha(0, n) = \beta(0, n) = \beta(1, n-1)$, $\alpha(n, 0) = \beta(n, 0)$.

Proof. The boundary conditions follow directly from the definitions.

$$\begin{aligned}
\Delta_n \psi_m &= \lim_{N \to \infty} \sum_{j=1}^{N} X_j^n \sum_{\substack{(i_1, i_2, \ldots, i_m) \\ \in \{1, 2, \ldots, N\}_{\hat{0}_m}}} X_{i_1} X_{i_2} \cdots X_{i_m} \\
&= \mathrm{St}_{\hat{1}_n + \hat{0}_m} + \lim_{N \to \infty} \sum_{j=1}^{m} \sum_{\substack{(i_1, i_2, \ldots, i_m) \\ \in \{1, 2, \ldots, N\}_{\hat{0}_m}^{m}}} X_{i_j}^n X_{i_1} X_{i_2} \cdots X_{i_m} \\
&= \mathrm{St}_{\hat{1}_n + \hat{0}_m} + \sum_{j=1}^{m} \mathrm{St}_{\pi_j},
\end{aligned}$$

where $\pi_j = \{(1, 2, \ldots, n, n+j), (n+1), \ldots, (n+j-1), (n+j+1), \ldots, (n+m)\}$. By an argument similar to the one in the proof of Proposition 3, each inner singleton contributes a factor of t, and so $\mathrm{St}_{\pi_j} = t^{j-1} \, \mathrm{St}_{\hat{1}_{n+1} + \hat{0}_{m-j}}$. ∎

THEOREM 4 (free Kailath–Segall formula). *We have the following expression for the nth stochastic measure*:

$$\psi_n = X\psi_{n-1} + \sum_{j=2}^{n} (-1)^{j-1} \sum_{q=0}^{n-j} \binom{n-q-2}{j-2} t^{n-j-q} \Delta_j \psi_q$$
$$= X\psi_{n-1} + \sum_{j=2}^{n} (-1)^{j-1} \sum_{m=0}^{n-j} \binom{m+j-2}{j-2} t^m \Delta_j \psi_{n-j-m}.$$

In particular, ψ_n is a polynomial in the diagonal measures $\Delta_1, \Delta_2, ..., \Delta_n$ and t.

Proof. By a repeated use of the Lemma,

$$\beta(0, n) = \beta(1, n-1) = \alpha(1, n-1) - \sum_{l_1=0}^{n-2} t^{n-2-l_1} \beta(2, l_1)$$
$$= \alpha(1, n-1) - \sum_{l_1=0}^{n-2} t^{n-2-l_1} \left[\alpha(2, l_1) - \sum_{l_2=0}^{l_1-1} t^{l_1-1-l_2} \beta(3, l_2) \right]$$
$$= \alpha(1, n-1) - \sum_{l_1=0}^{n-2} t^{n-2-l_1} \alpha(2, l_1) + \sum_{l_1=0}^{n-2} \sum_{l_2=0}^{l_1-1} t^{n-3-l_2} \beta(3, l_2) = \cdots$$
$$= \alpha(1, n-1) + \sum_{j=2}^{k} (-1)^{j-1} \sum_{l_1=0}^{n-2} \sum_{l_2=0}^{l_1-1} \cdots \sum_{l_{j-1}=0}^{l_{j-2}-1} t^{n-j-l_{j-1}} \alpha(j, l_{j-1})$$
$$+ (-1)^k \sum_{l_1=0}^{n-2} \cdots \sum_{l_k=0}^{l_{k-1}-1} t^{n-k-1-l_k} \beta(k+1, l_k). \tag{9}$$

For $k = n-1$, using the boundary conditions, the last term in (9) is

$$(-1)^k \sum_{l_1=0}^{n-2} \cdots \sum_{l_k=0}^{l_{k-1}-1} t^{n-(k+1)-l_k} \alpha(k+1, l_k).$$

Thus, continuing the expression (9),

$$= \alpha(1, n-1) + \sum_{j=2}^{n} (-1)^{j-1} \sum_{l_1=0}^{n-2} \sum_{l_2=0}^{l_1-1} \cdots \sum_{l_{j-1}=0}^{l_{j-2}-1} t^{n-j-l_{j-1}} \alpha(j, l_{j-1})$$
$$= \alpha(1, n-1) + \sum_{j=1}^{n} (-1)^{j-1}$$
$$\times \sum_q |\{l_1, ..., l_{j-1} \colon q = l_{j-1} < \cdots < l_1 < n-1\}| \, t^{n-j-q} \alpha(j, p)$$
$$= \alpha(1, n-1) + \sum_{j=1}^{n} (-1)^{j-1} \sum_{q=0}^{n-j} \binom{n-q-2}{j-2} t^{n-j-q} \alpha(j, p).$$

The result follows from the definitions of α and β. ∎

Corollary 7. *If the process is centered, i.e. $t=0$, then*

$$\psi_n = \sum_{j=1}^{n} (-1)^{j-1} \Delta_j \psi_{n-j}$$
$$= \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{j_1, j_2, ..., j_k \geqslant 1 \\ j_1+j_2+\cdots+j_k=n}} \Delta_{j_1} \Delta_{j_2} \cdots \Delta_{j_k}.$$

Corollary 8. *For the free Brownian motion X, for $n \geqslant 2$*

$$\psi_n = X\psi_{n-1} - |A|\, \psi_{n-2}$$
$$= \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} |A|^j X^{n-2j}.$$

These are the Chebyshev polynomials of the second kind.

Corollary 9 (Recursion relation for the free Poisson process). *For the free Poisson process X, for $n \geqslant 2$,*

$$X\psi_n = \psi_{n+1} + (1-t)\, \psi_n + tX\psi_{n-1}.$$

Proof.

$$\psi_n = X\psi_{n-1} + \sum_{j=2}^{n} (-1)^{j-1} \sum_{q=0}^{n-j} \binom{n-q-2}{j-2} t^{n-j-q} X\psi_q$$
$$= X\psi_{n-1} + \sum_{q=0}^{n-2} \sum_{j=2}^{n-q} (-1)^{(j-2)+1} \binom{n-q-2}{j-2} t^{(n-q-2)-(j-2)} X\psi_q$$
$$= X\psi_{n-1} - \sum_{q=0}^{n-2} (t-1)^{n-q-2} X\psi_q.$$

Therefore

$$(t-1)\, \psi_{n-1} = (t-1)\, X\psi_{n-2} - \sum_{q=0}^{n-3} (t-1)^{n-q-2} X\psi_q$$

and so

$$\psi_n = (t-1)\, \psi_{n-1} - X\psi_{n-2} - (t-1)\, X\psi_{n-2} + X\psi_{n-1}$$
$$= (X+t-1)\, \psi_{n-1} - tX\psi_{n-2}. \quad \blacksquare$$

Now we consider the compensated Poisson process. From Corollary 4 it follows that the diagonal measures of this process are linear functions in X. Therefore by Corollary 7, the stochastic measures ψ_n are nth degree polynomials in X. By Proposition 5, they are precisely the polynomials orthogonal with respect to the compensated free Poisson measure. It is natural to call them free Poisson–Charlier polynomials. Note that these polynomials have appeared in [HT98].

Corollary 10 (Recursion relation for the free Poisson–Charlier polynomials). *Let X be the free Poisson process, and ψ_i be the stochastic measures for the free compensated Poisson process $X-t$. Then*

$$X\psi_n = \psi_{n+1} + (1+t)\,\psi_n + t\psi_{n-1}.$$

Proof. Since the diagonal measures for the free compensated Poisson process are $\Delta_1 = X-t$, $\Delta_i = X$, $i \geqslant 2$, by Corollary 7

$$\psi_n = -t\psi_{n-1} + \sum_{j=0}^{n-1} (-1)^{n-j-1}\, X\psi_j.$$

Therefore

$$\psi_n + \psi_{n-1} = -t\psi_{n-1} - t\psi_{n-2} + X\psi_{n-1}. \quad \blacksquare$$

Corollary 11. *The free Poisson–Charlier polynomials are*

$$\psi_n = (X-t)^n + \sum_{i=0}^{n-2} (X-t)^i \sum_{k=1}^{[n-i/2]} \binom{n-i-k-1}{k-1} (-1)^{n-k-i}\, X^k.$$

Proof. By Corollary 7,

$$\begin{aligned}
\psi_n &= \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{j_1, j_2, \ldots, j_k \geqslant 1 \\ j_1 + j_2 + \cdots + j_k = n}} \Delta_{j_1} \Delta_{j_2} \cdots \Delta_{j_k} \\
&= \sum_{i=0}^{n} (X-t)^i \sum_k (-1)^{n-k-i}\, X^k \sum_{\substack{j_1, j_2, \ldots, j_k \geqslant 2 \\ j_1 + j_2 + \cdots + j_k = n-i}} 1 \\
&= (X-t)^n + \sum_{i=0}^{n-2} (X-t)^i \sum_{k=1}^{[n-i/2]} \binom{n-i-k-1}{k-1} (-1)^{n-k-i}\, X^k. \quad \blacksquare
\end{aligned}$$

COROLLARY 12. *For a free compound Poisson process X with generator e, $X = ses$,*

$$\psi_n = X\psi_{n-1} - \sum_{q=0}^{n-2} s(t-e)^{n-q-2}\, e^2 s\psi_q.$$

Proof. By Corollary 5, the diagonal measures of X are $\Delta_i = se^i s$. Therefore by the Theorem

$$\begin{aligned}
\psi_n &= X\psi_{n-1} + \sum_{j=2}^{n} \sum_{q=0}^{n-j} (-1)^{j-1} \binom{n-q-2}{j-2} t^{(n-j-q)} se^j s\psi_q \\
&= X\psi_{n-1} + \sum_{q=0}^{n-2} \sum_{j=2}^{n-q} (-1)^{j-1} \binom{n-q-2}{j-2} t^{(n-q-2)-(j-2)} se^{j-2} e^2 s\psi_q \\
&= X\psi_{n-1} - \sum_{q=0}^{n-2} s(t-e)^{n-q-2}\, e^2 s\psi_q. \quad \blacksquare
\end{aligned}$$

8. INTEGRATION OF FUNCTIONS

One of the important points in the proof of the free Itô formula of [BS98] is the observation that (in the language of this paper) for X the free Brownian motion, for Z centered $\lim_{N\to\infty} \sum_{i=1}^{N} X_i Z X_i = 0$. In fact this is true more generally.

THEOREM 5. *Let Z be centered and freely independent from the process $\{X_I\}$. Then*

$$\lim_{N\to\infty} \sum_{i=1}^{N} X_i Z X_i = 0.$$

Proof. First, let X be a free Poisson process, $X = sps$. Then $X_i Z X_i = s(p_i(sZs)\, p_i)\, s)$. The joint distribution of Z and sps is determined by the condition that Z is freely independent from sps. Thus we may assume that in fact Z is freely independent from p. In that case sZs is freely independent from p_i, and $\varphi(sZs) = \varphi(s^2)\, \varphi(Z) = 0$. Then the result follows from Theorem 3.

In general,

$$
\begin{aligned}
\varphi\left(\left(\sum_{i=1}^{N} X_i Z X_i\right)^n\right)
&= \sum_{i_1, i_2, ..., i_n = 1}^{N} \varphi(X_{i_1} Z X_{i_1} X_{i_2} Z X_{i_2} \cdots X_{i_n} Z X_{i_n}) \\
&= \sum_{\pi \in \mathcal{P}(n)} \sum_{\substack{(i_1, i_2, ..., i_n) \\ \in \{1, 2, ..., N\}^n_\pi}} \varphi(X_{i_1} Z X_{i_1} X_{i_2} Z X_{i_2} \cdots X_{i_n} Z X_{i_n}) \\
&= \sum_{\pi \in \mathcal{P}(n)} \sum_{\substack{(i_1, i_2, ..., i_n) \\ \in \{1, 2, ..., N\}^n_\pi}} \sum_{\sigma \in NC(n)} \sum_{\substack{\tau \in NC(2n) \\ \sigma \cup \tau \in NC(3n) \\ \tau \leqslant \pi^2}} R_\sigma(Z, Z, ... Z) \\
&\quad \times R_\tau(X_{i_1}, X_{i_1}, X_{i_2}, X_{i_2}, ..., X_{i_n}, X_{i_n}) \\
&= \sum_{\pi \in \mathcal{P}(n)} \sum_{\substack{(i_1, i_2, ..., i_n) \\ \in \{1, 2, ..., N\}^n_\pi}} \sum_{\sigma \in NC(n)} \sum_{\substack{\tau \in NC(2n) \\ \sigma \cup \tau \in NC(3n) \\ \tau \leqslant \pi^2}} R_\sigma(Z)\, N^{-|\tau|} R_\tau(X) \\
&= \sum_{\pi \in \mathcal{P}(n)} (N)_{|\pi|} \sum_{\sigma \in NC(n)} \sum_{\substack{\tau \in NC(2n) \\ \sigma \cup \tau \in NC(3n) \\ \tau \leqslant \pi^2}} R_\sigma(Z, ..., Z)\, N^{-|\tau|} R_\tau(X, X, ..., X).
\end{aligned} \tag{10}
$$

Here in the above equation (10), for partitions $\sigma \in NC(n)$, $\tau \in NC(2n)$, in the partition $\sigma \cup \tau$ we let σ act on $\{2, 5, ..., 3n-1\}$ while τ acts on $\{1, 3, 4, 5, ..., 3n-2, 3n\}$.

This sum can probably be estimated directly using the properties of Stirling numbers, but instead we will use the method of Section 6.2: the result follows from the fact that the cumulants grow no faster than an exponential, and the fact that the limit exists for the free Poisson process. ∎

COROLLARY 13. *In general, for Z freely independent from the process $\{X_I\}$*

$$\lim_{N \to \infty} \sum_{i=1}^{N} X_i Z X_i = \varphi(Z) \langle X, X \rangle,$$

where $\langle X, X \rangle = \Delta_2$ is the quadratic variation of the process.

PROPOSITION 6. *For $Z_1, Z_2, ..., Z_k$ centered and freely independent from the process $\{X_I\}$,*

$$\lim_{N \to \infty} \sum_{i=1}^{N} X_i^{m_1} Z_1 X_i^{m_2} Z_2 \cdots X_i^{m_k} Z_k X_i^{m_{k+1}} = 0.$$

Proof. As in Theorem 5,

$$\begin{aligned}
&\varphi\left(\left(\sum_{i=1}^{N} X_i^{m_1} Z_1 X_i^{m_2} \cdots X_i^{m_k} Z_k X_i^{m_{k+1}}\right)^n\right) \\
&\quad = \sum_{i_1, i_2, ..., i_n = 1}^{N} \varphi(X_{i_i}^{m_1} Z_1 X_{i_1}^{m_2} \cdots X_{i_1}^{m_k} Z_k X_{i_1}^{m_{k+1}} \cdots X_{i_n}^{m_k} Z_k X_{i_n}^{m_{k+1}}) \\
&\quad = \sum_{\pi \in \mathscr{P}(n)} (N)_{|\pi|} \sum_{\sigma \in NC(nk)} \sum_{\substack{\tau \in NC(n \sum_{i=1}^{k+1} m_i) \\ \sigma \cup \tau \in NC(n(k + \sum_{i=1}^{k+1} m_i)) \\ \tau \leqslant \pi^{\sum m_i}}} R_\sigma(Z_1, ..., Z_k, Z_1, ..., Z_k) \\
&\qquad \times N^{-|\tau|} R_\tau(X).
\end{aligned}$$

Here in $\sigma \cup \tau$, σ acts on the subset $\{m_1 + 1, m_1 + m_2 + 2, ..., \sum_{i=1}^{k} m_i + k, \sum_{i=1}^{k+1} m_i + k + m_1 + 1, ..., (n-1) \sum_{i=1}^{k+1} + (n-1) k + \sum_{i=1}^{k} m_i + k\}$ while τ acts on its complement.

The rest of the proof proceeds as in Theorem 5. ∎

COROLLARY 14. *In general, for $Z_1, Z_2, ..., Z_k$ freely independent from the process $\{X_I\}$*

$$\begin{aligned}
&\lim_{N \to \infty} \sum_{i=1}^{N} X_i^{m_1} Z_1 X_i^{m_2} Z_2 \cdots X_i^{m_k} Z_k X_i^{m_{k+1}} \\
&\quad = \varphi(Z_1)\, \varphi(Z_2) \cdots \varphi(Z_k) \lim_{N \to \infty} \sum_{i=1}^{N} X_i^{m_1 + m_2 + \cdots + m_{k+1}} \\
&\quad = \varphi(Z_1)\, \varphi(Z_2) \cdots \varphi(Z_k)\, \varDelta_{\sum_{j=1}^{k+1} m_j}.
\end{aligned}$$

These properties should allow us to consider integration with respect to free stochastic processes. We will return to this subject elsewhere.

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