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On tracial approximation

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Abstract

Let $\mathscr{C}$ be a class of unital C*-algebras. The class TA$\mathscr{C}$ of C*-algebras which can be tracially approximated (in the Egorov-like sense first considered by Lin) by the C*-algebras in $\mathscr{C}$ is studied (Lin considered the case that $\mathscr{C}$ consists of finite-dimensional C*-algebras or the tensor products of such with $C([0,1])$). In particular, the question is considered whether, for any simple separable $A \in \mathrm{TA}\mathscr{C}$, there is a C*-algebra B which is a simple inductive limit of certain basic homogeneous C*-algebras together with C*-algebras in $\mathscr{C}$, such that the Elliott invariant of A is isomorphic to the Elliott invariant of B. An interesting case of this question is answered. In the final part of the paper, the question is also considered which properties of C*-algebras are inherited by tracial approximation. (Results of this kind are obtained which are used in the proof of the main theorem of the paper, and also in the proof of the classification theorem of the second author given in [Z. Niu, A classification of tracially approximately splitting tree algebra, in preparation] and [Z. Niu, A classification of certain tracially approximately subhomogeneous C*-algebras, PhD thesis, University of Toronto, 2005]—which also uses the main result of the present paper.)

Keywords: Elliott invariant; Tracial approximation

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1. Introduction

The Elliott program for the classification of amenable C*-algebras might be said to have begun with the K-theoretical classification of AF-algebras [8]. (This was closely related to the Bratteli diagram classification given earlier in [3], but differed in introducing the K-functor.) Since then (but only after a fifteen-year hiatus), many classes of amenable C*-algebras have been found to be classified by their Elliott invariants. Among them, one important class is the class of simple unital inductive limits of homogeneous C*-algebras (AH-algebras for short). In this paper, by a unital homogeneous C*-algebra, we refer to a C*-algebra which is isomorphic to

$$p\mathrm{M}_n\big(\mathrm{C}(X)\big)p$$

for some compact Hausdorff space X, and some projection p in $\mathrm{M}_n(\mathrm{C}(X))$. (Noted that these C*-algebras are exactly the homogeneous C*-algebras with trivial Dixmier–Douady class. See, for example, p. 14 of [1]. In general, a homogeneous C*-algebra may not have this form.) In [16] and [14], Elliott, Gong, and Li showed that C*-algebras in this class can be classified by their Elliott invariant, provided that the dimensions of the base spaces of their building blocks are uniformly bounded. (Such AH-algebras are referred as AH-algebras without dimension growth.) Many naturally arising C*-algebras—for instance, the irrational rotation C*-algebras [12]—are known to be AH-algebras without dimension growth. (Note that AH-algebras not in this class were constructed by Villadsen in [33] and [34].)

A very important axiomatic version of the classification of the AH-algebras without dimension growth was given by Huaxin Lin. Instead of assuming inductive limit structure, he started with a certain abstract approximation property, and showed that C*-algebras with this abstract approximation property and certain additional properties are AH-algebras without dimension growth [18, 20,21,23]. More precisely, Lin introduced the class of tracially approximate interval algebras—TAI-algebras for short. Recall that an interval algebra is a C*-algebra isomorphic to $F \otimes \mathrm{C}([0,1])$ for a finite-dimensional C*-algebra F. Then TAI-algebras are defined by the following.

Definition. A unital C*-algebra A is a TAI-algebra if for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subseteq A$, and any non-zero $a \in A^+$, there exist a non-zero projection $p \in A$ and a sub-C*-algebra $I \subseteq A$ such that I is an interval algebra, $1_I = p$, and for all $x \in \mathcal{F}$,

(1) $\|xp - px\| < \varepsilon$,
(2) there exists $x' \in I$ such that $\|pxp - x'\| < \varepsilon$, and
(3) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

The classification theorem for TAI-algebras was given by Lin in [23]. (The second author also contributed to the final part of this work; see [26].)

Theorem. *(See [23].) Let A and B be two simple amenable unital TAI-algebras satisfying the UCT. Then A is isomorphic to B if the Elliott invariant of A is isomorphic to the Elliott invariant of B. Moreover, the isomorphism between the algebras can be chosen in such a way that it induces the given isomorphism between the invariants.*

Since AH-algebras without dimension growth are TAI-algebras by the work of Gong [16], and since AH-algebras exhaust the invariant for the class in question (see [32], combined with

Lemma 10.9 of [23]; see also Corollary 4.20 below), the TAI-algebras which are classified by the theorem above must be AH-algebras. Thus the theorem above also provides a method to verify whether a given C*-algebra is an AH-algebras. In this way, the higher-dimensional simple noncommutative tori were shown to be AT-algebras in [22] and [29], and the crossed-product C*-algebras arising from certain minimal homeomorphisms were shown to be AH-algebras in [24].

Motivated by Lin's work on AH-algebras and TAI-algebras, the second author succeeded on using the axiomatic approach to obtain a classification for certain simple inductive limits of subhomogeneous C*-algebras in his PhD thesis [27]. More precisely, C*-algebras which can be tracially approximated by splitting interval algebras—TASI-algebras for short—were introduced and studied in [27]. Under a certain assumption on the value of the invariant arising—to be shown now to be redundant!—these C*-algebras were classified. Recall that a splitting interval algebra is defined as follows.

Definition. Let k be a natural number and $(\bar{k}_i = \{k_{i1}, \dots, k_{ij_i}\})_{i=0,1}$ be two partitions of k. (All numbers non-zero.) The *splitting interval algebra* $S(\bar{k}_0, \bar{k}_1)$ is defined as follows:

$$S(\bar{k}_0, \bar{k}_1) := \big\{ f \in \mathrm{M}_k\big(\mathrm{C}\big([0,1]\big)\big);\ f(i) \in \mathrm{M}_{k_{i1}}(\mathbb{C}) \oplus \dots \oplus \mathrm{M}_{k_{ij_i}}(\mathbb{C}),\ i = 0, 1 \big\}.$$

These building blocks were introduced by Su in his work [31], in which he classified simple unital inductive limits of splitting interval algebras with real rank zero. General unital simple inductive limits of splitting interval algebras were classified by Jiang and Su in [17]. These authors also pointed out that there exist simple inductive limits of splitting interval algebras with K_0-groups failing to have the Riesz decomposition property, which is one of the properties held by simple AH-algebras without dimension growth. (Such examples were also constructed by the first author in [11].)

The classification theorem of [25] or [27] was as the following.

Theorem. *(See [25,27].) Let A be an amenable simple separable TASI-algebra which satisfies the UCT. If there exists a C*-algebra B which is a simple unital inductive limit of splitting interval algebras together with certain basic homogeneous C*-algebras* (*specified in* [27]; *see also below, where we refer to these algebras as the Gong standard homogeneous algebras*)*, such that*

$$\big(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]_0, \mathrm{K}_1(A), T(A)\big) \cong \big(\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]_0, \mathrm{K}_1(B), T(B)\big),$$

then $A \cong B$. Moreover, the $*$-isomorphism can be chosen to induce the given isomorphism between the invariants.

As one can see, the above classification theorem covers a priori only a subclass of TASI-algebras, namely, those with the same invariant as an inductive limit C*-algebra of certain kind. The main purpose of the present paper is to show that the assumption of the theorem is satisfied automatically.

Recall that the Gong standard homogeneous C*-algebras consist of

(1) matrix algebras over the C*-algebras of continuous functions on $T_{2,k}$, and
(2) matrix algebras over the C*-algebras of continuous functions on $S^1 \vee \dots \vee S^1 \vee T_{3,k_i} \vee \dots \vee T_{3,k_i}$,

where $T_{2,k}$ is the two-dimensional CW complex obtained by attaching a two-dimensional disk D to S^1 via a map S^1 $(\cong \partial D) \to S^1$ of degree k, and $T_{3,k}$ is the three-dimensional CW complex obtained by attaching a three-dimensional ball B to S^2 via a map S^2 $(\cong \partial B) \to S^2$ of degree k. (See [13,16], and [14].) We shall prove the following theorem in this paper:

Theorem A. *Let $\mathscr{S}$ be a class of splitting tree algebras containing the interval algebras, and let A be a simple separable C*-algebra in the class* TA$\mathscr{S}$. *There exists a simple inductive limit C*-algebra B of C*-algebras in the class $\mathscr{S}'$ consisting of $\mathscr{S}$ together with the Gong standard homogeneous C*-algebras such that the Elliott invariant of A is isomorphic to the Elliott invariant of B.*

Hence with $\mathscr{S}$ the class of splitting interval algebras, we have that for any TASI-algebra, there is a simple inductive limit C*-algebras of certain building blocks which shares the same invariant. As a consequence, the class of all separable simple amenable TASI-algebras satisfying the UCT is classified by the Elliott invariant.

The proof of this theorem is quite different from the proof for TAI-algebras. For a unital simple TAI-algebra, the K_0-group is weakly unperforated and has the Riesz decomposition property, and the pairing map preserves extreme points. Thus, by the Effros–Handelman–Shen type theorem for simple AH-algebras without dimension growth given by Villadsen in [32], the invariants of such AH-algebras and the invariants of TAI-algebras coincide. However, we do not have an Effros–Handelman–Shen theorem for simple inductive limits of our building blocks (whether splitting tree algebras, or just splitting interval algebras).

The main argument of the decomposition theorem (Theorem A) uses the following local criterion (Lemma 3.1) to determine whether an ordered group is an inductive limit of certain building blocks.

Lemma. *Let $\mathcal{G}$ be a set of ordered groups closed under direct sums and containing the group of integers with the usual order, and assume that the positive cone of every group in $\mathcal{G}$ is finite generated. Then a countable ordered group G is an inductive limit of ordered groups in $\mathcal{G}$ if G has the following lifting property*:

For any $G_1 \in \mathcal{G}$, any ordered group homomorphism $\theta : G_1 \to G$ and any $\alpha \in \ker(\theta)$, there exist $G_2 \in \mathcal{G}$ and ordered homomorphisms $\iota : G_1 \to G_2$ and $\theta' : G_2 \to G$, such that $\iota(\alpha) = 0$ and the following diagram commutes:

$$\begin{array}{ccc} G_2 & & \\ \uparrow{\scriptstyle \iota} & \searrow^{\theta'} & \\ G_1 & \xrightarrow[\theta]{} & G. \end{array}$$

Let G be the ordered group $\mathrm{K}_0(A)$, where A is a simple C*-algebra which can be tracially approximated by splitting tree algebras (a class of subhomogeneous C*-algebras which contains the class of splitting interval algebras). We are going to show that the ordered group G satisfies the criterion above for the class $\mathcal{G}$ consisting of the ordered K_0-groups of certain splitting tree algebras, and hence the ordered group G is an inductive limit of ordered groups in $\mathcal{G}$. Furthermore, we show that the groups G_1 and G_2 in the lemma above can be chosen in such a way that a large piece of them comes from a splitting tree algebra sitting inside A, and the restrictions of the lifting maps to these large pieces are induced by maps between the splitting tree algebras which come from the tracial approximation structure and the semiprojectivity of the splitting tree algebras. Therefore, we get an inductive system (not necessary unital) of certain splitting tree algebras sitting inside large pieces of A, such that the inductive system realizes most of the torsion free part of the K_0-group and the pairing map of A. Then, we shall put suitable Gong standard homogeneous C*-algebras into the inductive system to make it to be a unital inductive system, such that it realizes the K_1-group and the torsion part of K_0-group. Since these new building blocks only sit inside small pieces of the new inductive system, it will not change the pairing map. In this way, we construct an inductive limit C*-algebra using certain special building blocks, such that its Elliott invariant is isomorphic to that of A.

In the final section of this paper, we report some results on the tracial approximation by general C*-algebras, showing that certain properties of C*-algebras in a class $\mathscr{C}$ are possessed by simple C*-algebras which can be tracially approximated by C*-algebras in $\mathscr{C}$. More precisely, we investigate the following properties:

(1) being (stably) finite;
(2) having stable rank one;
(3) having at least one tracial state;
(4) the strict order on projections is determined by traces;
(5) any state on the K_0-group comes from a tracial state of the algebra;
(6) if the restriction of a tracial state to the order-unit K_0-group is the average of two distinct states on K_0, then it is the average of two distinct tracial states;
(7) the canonical map from the unitary group modulo the connected component containing the identity to the K_1-group being injective.

2. TAS-algebras

In this paper, $\mathscr{F}$ denotes the class of finite-dimensional C*-algebras, and $\mathscr{I}$ denotes the class of interval algebras (recall that an interval algebra is a C*-algebra isomorphic to $F \otimes \mathrm{C}([0,1])$ for a finite-dimensional C*-algebra F). If A is a unital C*-algebra, let us denote by $\mathrm{T}(A)$ the simplex of tracial states of A. If a, b are two elements of a C*-algebra A, $a =_\varepsilon b$ means that $\|a-b\| < \varepsilon$. If B is a subset of A, then for any $a \in A$ and $\varepsilon > 0$, the notation $a \in_\varepsilon B$ refers to the relation that there is an element $b \in B$ such that $\|b-a\| < \varepsilon$.

Recall that a splitting tree algebra is a certain subhomogeneous C*-algebra defined as follows.

Definition 2.1. (*See [31].*) Let T be a tree (as a topological space) with finitely many vertices $\{v_i\}_{i=1}^n$, k be a natural number, and $(\bar{k}_i = \{k_{i1}, \ldots, k_{ij_i}\})_{i=1}^n$ be n partitions of k (with all numbers non-zero). The *splitting tree algebra* $S(\bar{k}_1, \ldots, \bar{k}_n; T)$ is defined as follows:

$$S(\bar{k}_1, \ldots, \bar{k}_n; T) := \big\{ f \in \mathrm{M}_k\big(\mathrm{C}(T)\big);\ f(v_i) \in \mathrm{M}_{k_{i1}}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{k_{ij_i}}(\mathbb{C}) \text{ for all } i \big\}.$$

Let us call the vertices $\{v_i\}$ the singular points of S. In the case that T consists of only two vertices, let us call S a *splitting interval algebra*.

Following the notion of Lin on the tracial approximation by interval algebras, let us consider tracial approximation by more general C*-algebras. Let $\mathscr{C}$ be a class of unital C*-algebras. Then the class of C*-algebras which can be tracially approximated by C*-algebras in $\mathscr{C}$, denoted by TA$\mathscr{C}$, is defined as follows:

Definition 2.2. A unital C*-algebra A is said to belong to the class TA$\mathscr{C}$ if for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subseteq A$, and any non-zero $a \in A^+$, there exist a non-zero projection $p \in A$ and a sub-C*-algebra $C \subseteq A$ such that $C \in \mathscr{C}$, $1_C = p$, and for all $x \in \mathcal{F}$,

(1) $\|xp - px\| < \varepsilon$,
(2) $pxp \in_\varepsilon C$, and
(3) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

Lemma 2.3. *If the class $\mathscr{C}$ is closed under tensoring with matrix algebras, or closed under taking unital hereditary sub-C*-algebras, then* TA$\mathscr{C}$ *is also closed under passing to matrix algebras or unital hereditary sub-C*-algebras.*

Proof. Let us only verify the lemma for unital hereditary sub-C*-algebras. The matrix algebras can be verified similarly.

Let $A \in$ TA$\mathscr{C}$, and consider the unital hereditary sub-C*-algebra eAe where e is a projection in A. Let us show that $eAe \in$ TA$\mathscr{C}$.

For any $\varepsilon > 0$ (without loss of generality, let us assume that $\varepsilon < 1/32$), any $a \in (eAe)^+$, and any finite subset $\mathcal{F}$ in the unit ball of eAe, since $A \in$ TA$\mathscr{C}$, there is a sub-C*-algebra C with $p = 1_C$ and $C \in \mathscr{C}$ such that for any $x \in \mathcal{F} \cup \{e\}$,

(1) $\|xp - px\| < \varepsilon$,
(2) $pxp \in_\varepsilon C$, and
(3) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

Therefore, there is a projection $e_2 \in eAe$ such that $\|e_2 - pep\| < 4\varepsilon$. Since $pep \in_\varepsilon C$, there is a projection $e_2' \in C$ such that $\|e_2 - e_2'\| < 8\varepsilon$. Therefore, there is a unitary u with $\|u\| < 16\varepsilon$ such that $e_2 = ue_2'u^*$.

Consider the sub-C*-algebra $C' := uCu^*$. We assert that $e_2C'e_2$ satisfies the lemma. Indeed, since $e_2 \in eAe$, we have that $e_2C'e_2 \in eAe$. For any $x \in \mathcal{F} \subset eAe$,

$$e_2xe_2 =_{4\varepsilon} (pep)x(pep) = (pep)pxp(pep) =_{4\varepsilon} e_2pxpe_2 =_{16\varepsilon} e_2u(pxp)u^*e_2 \in_\varepsilon e_2C'e_2,$$

therefore $e_2xe_2 \in_{33\varepsilon} e_2C'e_2$.

Moreover, for any $x \in \mathcal{F}$, we have

$$xe_2 =_{4\varepsilon} x(pep) =_\varepsilon x(pe) =_\varepsilon pepx =_{4\varepsilon} e_2x,$$

therefore, $\|xe_2 - e_2x\| < 10\varepsilon$. Since $\|(e - e_2) - (1 - p)e(1 - p)\| < 8\varepsilon$, the projection $e - e_2$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$. Therefore, the C*-algebra eAe is in the class TA$\mathscr{C}$. □

The class TA$\mathscr{F}$ is the class of tracially AF C*-algebras [21], and TA$\mathscr{I}$ is the class of C*-algebras of tracial topological rank one (TAI-algebras) in sense of Lin [23]. If $\mathscr{S}$ denotes the class of finite direct sums of splitting tree algebras, for convenience, we shall refer to the C*-algebras in TA$\mathscr{S}$ as TAS-algebras.

The following lemma provides a criterion for the stable finiteness of simple separable C*-algebras in TA$\mathscr{C}$. (In fact, the lemma only uses the conditions (1) and (2) of Definition 2.2.)

Lemma 2.4. *Let $\mathscr{C}$ be a class of unital C*-algebras. Suppose that for any C*-algebra $C \in \mathscr{C}$, there is a unital $*$-homomorphism from C to a matrix algebra $\mathrm{M}_k(\mathbb{C})$ (with $k > 0$). Then any separable simple unital C*-algebra satisfying the conditions (1) and (2) of Definition 2.2 with respect to the class $\mathscr{C}$ can be unitally embedded into the asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_i \mathrm{M}_{k_n}(\mathbb{C})$ for some sequence of positive integers k_n. In particular, it has at least one tracial state and is stably finite.*

Proof. Let A be a separable simple unital C*-algebra which satisfies the conditions (1) and (2) of Definition 2.2. Choose a countable dense subset $\mathcal{F} = \{a_1, a_2, \ldots\}$ of the unit ball of A with $a_1 = 1$, and set $\varepsilon_n = 1/2^n$ for $n = 1, 2, \ldots$. Applying the conditions (1) and (2) of Definition 2.2 to ε_n and $\mathcal{F}_n = \{a_1, \ldots, a_n\}$, we obtain a sub-C*-algebra $C_n \in \mathscr{C}$ and a projection $p_n = 1_{C_n}$ such that $p_n a_i p_n \in_{\varepsilon_n} C_n$ and $\|p_n a_i - a_i p_n\| < \varepsilon_n$ for all a_i with $1 \leqslant i \leqslant n$. Pick a unital $*$-homomorphism $\phi_n : C_n \to \mathrm{M}_{k_n}(\mathbb{C})$ for some k_n. By Arveson's extension theorem, there exists an extension of ϕ_n to a positive linear contraction from $p_n A p_n$ to the same matrix algebra; denote this still by ϕ_n. Then Φ_n, defined by $\Phi_n(a) = \phi_n(pap)$, $a \in A$, is a positive linear contraction from A to the matrix algebra $\mathrm{M}_{k_n}(\mathbb{C})$ with $\|\Phi_n(a_i a_j) - \Phi_n(a_i)\Phi_n(a_j)\| < 2\varepsilon_n$ and $\|\Phi_n(a_i{}^*) - \Phi_n(a_i)^*\| < 2\varepsilon_n$ for any $1 \leqslant i, j \leqslant n$.

Applying this procedure for each n, we obtain a sequence (Φ_n) of unital positive linear contractions from A to various matrix algebras—the sequence $(\mathrm{M}_{k_n}(\mathbb{C}))$—with the approximation properties as above. The unital map Φ from A to the asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$, which is induced by the Cartesian product of $(\Phi_1, \Phi_2, \ldots)$, is then a unital $*$-homomorphism. By simplicity of A, the $*$-homomorphism Φ maps A injectively into the asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$.

Since the asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$ has tracial states, the C*-algebra A (being a unital sub-C*-algebra) has at least one tracial state. Since A is simple, it follows that A is stably finite. □

Corollary 2.5. *Any separable simple TAS-algebra can be embedded into the asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$ for some positive integers k_n. In particular, such a C*-algebra has at least one tracial state, and hence is stably finite.*

Proof. For any splitting tree algebra S and any point t in its spectrum, there is a unital $*$-homomorphism from S to a non-zero matrix algebra consisting of the evaluation map at the point t. Thus, the corollary follows from Lemma 2.4. □

The following lemma concerns the sizes of the matrix algebras in Lemma 2.4.

Lemma 2.6. *Let A be a C*-algebra of Lemma 2.4. If A is not of type I, then the sizes of the matrix algebras M_{k_n} are unbounded.*

Proof. Suppose that this were not true. Then A can be embedded into an asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$ with $\{k_n\}$ uniformly bounded. But $\prod_n \mathrm{M}_{k_n}(\mathbb{C})/\bigoplus_n \mathrm{M}_{k_n}(\mathbb{C})$ is a type I C*-algebra (it is a quotient of the type I C*-algebra $\prod_n \mathrm{M}_{k_n}$). Therefore, A must be of type I, which is in contradiction with the assumption. □

Recall that a C*-algebra has the property (SP) if every non-zero hereditary sub-C*-algebra of A contains a non-zero projection. Let $\mathscr{C}$ be a class of C*-algebras, and let $A \in \mathrm{TA}\mathscr{C}$. If A does not have the property (SP), then there is $a \in A^+$ such that the hereditary sub-C*-algebra $\overline{aAa}$ does not contain non-zero projection. Apply Definition 2.2 to any finite subset $\mathcal{F}$, any ε, and a, we conclude that A can be approximated by C*-algebras in $\mathscr{C}$. This observation will be used several times in Section 4 in which certain properties of C*-algebras in $\mathscr{C}$ are shown to be inherited by C*-algebras in the class $\mathrm{TA}\mathscr{C}$.

However, if only the conditions (1) and (2) of Definition 2.2 are satisfied with respect to splitting tree algebras, then the property (SP) holds automatically. More precisely, we have the following proposition.

Proposition 2.7. *If a separable simple unital C*-algebra A satisfies the conditions* (1) *and* (2) *of Definition* 2.2 *with respect to splitting tree algebras, then A has the property* (SP).

Proof. We must show that for any non-zero positive element a, there is a non-zero projection in $\overline{aAa}$. Given such an element a, without loss of generality, we may assume that $\|a\| = 1$.

For any $\varepsilon > 0$, there exists a sub-C*-algebra S with unit p (not necessarily equal to 1) such that there is an element $b \in S^+$ with $\|pap - b\| < \varepsilon$ by the conditions (1) and (2) of Definition 2.2. We assert that we may choose S and b such that for any irreducible representation π of S, the norm of $\pi(b)$ is greater than $1 - \varepsilon$.

Suppose that this were not true. Then there exists an $\varepsilon_0 > 0$ such that for any S_n and b_n obtained by the conditions (1) and (2) of Definition 2.2, there exists an irreducible representation π_n of S_n such that the $\|\pi_n(b_n)\| < 1 - \varepsilon_0$. By the proof of Lemma 2.4, these irreducible representations induce a unital $*$-homomorphism Φ from A to an asymptotic sequence algebra $\prod_n \mathrm{M}_{k_n} / \bigoplus_n \mathrm{M}_{k_n}$, sending a to the quotient of the Cartesian product $(\pi_1(b_1), \pi_2(b_2), \ldots)$. Since $\|\pi_n(b_n)\| < 1 - \varepsilon_0$ for any n, it follows that $\|\Phi(a)\| \leqslant 1 - \varepsilon_0$. But since A is simple, the unital map Φ must be injective, and thus is isometric. This is a contradiction to the assumption that $\|a\| = 1$.

Therefore, we may assume that for any $\varepsilon > 0$, there exist a sub-C*-algebra S of A with unit p, and $b \in S^+$ such that $\|pap - b\| < \varepsilon$, $\|pa - ap\| < \varepsilon$, and the norm of b is greater than $1 - \varepsilon$ pointwisely on the spectrum of S. By perturbation methods as for interval algebras [10], we may assume that b does not have multiple eigenvalues in each canonical quotient of S. Therefore there exists a projection (a spectral projection) $q \in S$ such that $\|qb - q\| < \varepsilon$, and hence $\|qpap - q\| < 2\varepsilon$. Since $\|pa - ap\| < \varepsilon$, we have that $\|q(pap) - qpa\| \leqslant 3\varepsilon$. Thus we have that $\|q - qpa\| \leqslant 4\varepsilon$, and $\|q - apqpa\| \leqslant 8\varepsilon$. Since q is a projection, the element $apqpa \in \overline{aAa}$ has disconnected spectrum when $\varepsilon < 1/16$. By functional calculus, there is a non-zero projection in $\overline{aAa}$. □

Let (G, G^+, u) be an order-unit group. Denote by $\mathrm{S}(G, G^+, u)$ the convex set of order-unit group homomorphisms from (G, G^+, u) to $(\mathbb{R}, \mathbb{R}^+, 1)$. It is compact with respect to the pointwise convergence topology. In the case that there is no confusion, we write $\mathrm{S}(G)$ for $\mathrm{S}(G, G^+, u)$ in short. Any element in $\mathrm{S}(G, G^+, u)$ is called a *state* of the order-unit group (G, G^+, u). The real-valued affine functions on the convex set $\mathrm{S}(G, G^+, u)$, denoted by $\mathrm{Aff}(\mathrm{S}(G, G^+, u))$, is an order-unit vector space with respect to positive functions as the positive cone and the constant function **1** as the order unit. Note that there is a canonical order-unit homomorphism ρ from (G, G^+, u) to $\mathrm{Aff}(\mathrm{S}(G, G^+, u))$, defined by

$$\rho(g)(s) = s(g)$$

for any $g \in G$ and $s \in \mathrm{S}(G)$.

Let A be a unital stably finite C*-algebra. Then $(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1]_0)$ is an ordered-unit group (see [2]). Moreover, any tracial state τ of A induces a state of $(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1]_0)$ by the restriction to projections of matrix algebras of A. This defines an affine map from $\mathrm{T}(A)$ to $\mathrm{S}(\mathrm{K}_0(A))$. We shall refer it as the pairing map (between the simplex of traces and the order-unit K_0-group), and denote it by r_A. Then the *Elliott invariant* of A is defined as the tuple

$$\big((\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1]_0), \mathrm{K}_1(A), \mathrm{T}(A), r_A\big).$$

This invariant has been shown to be the complete invariant for simple unital AH-algebras without dimension growth (the same class as the simple separable unital amenable TAI-algebras satisfying the UCT). (See [14,16], and [23].)

3. The Elliott invariants of TAS-algebras

Let $\mathscr{S}$ denote a class of splitting tree algebras which contains all interval algebras and is closed under taking finite direct sums. For example, $\mathscr{S}$ may denote the class of all splitting interval algebras. Any C*-algebra in the class $\mathscr{S}$ can be generated by a finite subset with respect to stable relations (see [7]). (Explicit generating sets and relations for certain splitting tree algebras are found in [28].)

As shown in Section 4, any simple separable C*-algebra A in $\mathrm{TA}\mathscr{S}$ has stable rank one (see Corollary 4.4), and thus has cancellation property for equivalent class of projections. Moreover, the strict order on projections of A is determined by traces, and thus A has weakly unperforated ordered K_0-group (see Corollary 4.14).

We shall investigate the Elliott invariant of C*-algebras in the class $\mathrm{TA}\mathscr{S}$, showing that for any simple separable C*-algebra $A \in \mathrm{TA}\mathscr{S}$, there exists a simple C*-algebra B, of an inductive limit of $\mathscr{S}'$ contaning $\mathscr{S}$ and the Gong standard homogeneous C*-algebras, such that the Elliott invariant of A is isomorphic to the Elliott invariant of B.

Fix a simple separable C*-algebra $A \in \mathrm{TA}\mathscr{S}$ be. Let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of A with dense union, and let (ε_n) be a decreasing sequence of positive numbers converging to zero. Since A belongs to the class $\mathrm{TA}\mathscr{S}$, there is a sub-C*-algebra S_1 of A in the class $\mathscr{S}$, such that if p_1 denotes 1_{S_1}, then for any $a \in \mathcal{F}_1$,

(1) $\|p_1a - ap_1\| < \varepsilon_1$,
(2) $p_1ap_1 \in_{\varepsilon_1} S_1$, and
(3) $\tau(1_A - p_1) < \varepsilon_1$ for any $\tau \in \mathrm{T}(A)$.

Let S_1' denotes the finite set consisting of b.

We may assume that $\mathcal{F}_2$ is sufficently large such that $\mathcal{F}_2$ contains a generating set of S_1. Since A belongs to the class $\mathrm{TA}\mathscr{S}$, for any $\varepsilon' > 0$, there is a C*-algebra S_2 of A which belongs to the class $\mathscr{S}$, such that if p_2 denotes 1_{S_2}, then for any $a \in \mathcal{F}_2$,

(1) $\|p_2 a - a p_2\| < \varepsilon'$,
(2) $p_2 a p_2 \in_{\varepsilon'} S_2$, and
(3) $\tau(1_A - p_2) < \varepsilon'$ for any $\tau \in \mathrm{T}(A)$.

We may assume that ε' is sufficiently small such that there is a $*$-homomorphism $\phi_1 : S_1 \to S_2$ satisfying

$$\left\|\phi_1(b) - p_2 b p_2\right\| \leqslant \varepsilon_2 \quad \text{for any } b \in S_1'.$$

Repeating this procedure, we obtain an inductive system (not necessarily unital) (S_n, ϕ_n). Denote by S the inductive limit C*-algebra of (S_n, ϕ_n). Since each S_n are unital, S is a stably finite C*-algebra with an approximate unit $\{e_n\}$ consisting of projections. Denote by $\mathrm{T}_{u'}(S)$ the set of traces τ on S satisfying $\sup_n \tau(e_n) = 1$. It is a Choquet simplex. Also denote by $\mathrm{S}_{u'}(\mathrm{K}_0(S))$ the set of the positive homomorphisms s from $\mathrm{K}_0(S)$ to $\mathbb{R}$ such that $\sup_n s([e_n]) = 1$. The same argument as that of Lemma 10.8 of [23] shows that there exist affine continuous isomorphisms $t : \mathrm{T}_{u'}(S) \to \mathrm{T}(A)$ and $s : \mathrm{S}_{u'}(\mathrm{K}_0(S)) \to \mathrm{S}(\mathrm{K}_0(A))$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{S}(\mathrm{K}_0(A)) & \xleftarrow{\;r_A\;} & \mathrm{T}(A) \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle t} \\ \mathrm{S}_{u'}(\mathrm{K}_0(S)) & \xleftarrow[r_S]{} & \mathrm{T}_{u'}(S). \end{array}$$

In the following part of this section, we shall construct a new inductive system $(C_n \oplus S_n, \phi_n')$ based on (S_n, ϕ_n) where C_n are the Gong standard homogeneous C*-algebras (see 3.3), such that $B := \varinjlim(C_n \oplus S_n)$ is simple and the K-theory of B is isomorphic to that of A. Moreover, the C*-algebras C_n and the maps ϕ_n' are chosen in such a way that $\mathrm{T}(B)$ is isomorphic to $\mathrm{T}_{u'}(S)$, $\mathrm{S}(\mathrm{K}_0(B))$ is isomorphic to $\mathrm{S}_{u'}(\mathrm{K}_0(S))$ and hence isomorphic to $\mathrm{S}(\mathrm{K}_0(A))$, and the map $r_B : \mathrm{T}(B) \to \mathrm{S}(\mathrm{K}_0(B))$ is compatible with the map r_s. Moreover, the isomorphism between $\mathrm{S}(\mathrm{K}_0(B))$ and $\mathrm{S}(\mathrm{K}_0(A))$ is also compatible with the isomorphism between the K_0-groups. Thus the pairing maps between the simplex of traces and the ordered K_0-group of A is isomorphic to that of B, and the Elliott invariant of A is isomorphic to the Elliott invariant of B.

3.1. Two ordered group lemmas

In analogy with Chao-Liang Shen's local criterion for dimension groups, we have a local criterion to determinate whether an ordered group is an inductive limit of certain basic building blocks. Let $\mathcal{G}$ be a set of ordered groups closed under direct sum and containing the group of integers with the usual order. Furthermore, let us assume that the positive cone of every group in $\mathcal{G}$ is finitely generated. Then we have

Lemma 3.1. *A countable ordered group G is an inductive limit of ordered groups in $\mathcal{G}$ if G has the following property*:

For any $G_1 \in \mathcal{G}$, any ordered group homomorphism $\theta : G_1 \to G$ and any $\alpha \in \ker(\theta)$, there exist $G_2 \in \mathcal{G}$ and ordered group homomorphisms $\iota : G_1 \to G_2$ and $\theta' : G_2 \to G$, such that $\iota(\alpha) = 0$ and the following diagram commutes:

$$\begin{array}{ccc} G_2 & & \\ {\scriptstyle\iota}\big\uparrow & \searrow^{\theta'} & \\ G_1 & \xrightarrow[\theta]{} & G. \end{array}$$

Proof. The proof is similar to the case considered by Shen. Since G is countable, we may write $G^+ = \{g_1, g_2, \ldots\}$. To prove G is an inductive limit of $\mathcal{G}$, it is enough to construct the following commutative diagram:

$$\begin{array}{ccc} G_1 & \xrightarrow{\theta_1} & G \\ {\scriptstyle\iota_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ G_2 & \xrightarrow{\theta_2} & G \\ {\scriptstyle\iota_2}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ G_3 & \xrightarrow{\theta_3} & G \\ {\scriptstyle\iota_3}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \vdots & & \vdots \end{array}$$

where θ_i and ι_i are positive homomorphisms such that $\ker(\theta_i) \subseteq \ker(\iota_i)$ and $\bigcup_i G_i^+ = G^+$.

Take $G_1 = \mathbb{Z}$ and define $\theta_1 : n \mapsto ng_1$. Now, assume the diagram is well defined for G_n with $\theta_n : G_n \to G$. Then, take $\iota' : G_n \to G_n \oplus \mathbb{Z}$ by $s \mapsto (s, 0)$, and $\kappa : G_n \oplus \mathbb{Z} \to G$ by $(s, m) \mapsto \theta_n(s) + mg_n$. Since $G_n \oplus \mathbb{Z} \in \mathcal{G}$, there are an $G_{n+1} \in \mathcal{G}$ and a positive homomorphisms $\theta_{n+1} : G_{n+1} \to G$, $\iota'' : G_n \oplus \mathbb{Z} \to G_{n+1}$ such that $\ker(\kappa) \subseteq \ker(\iota'')$ and we have the following commutative diagram:

$$\begin{array}{ccccc} G_n & & \xrightarrow{\quad\theta_n\quad} & & G \\ & \searrow^{\iota'} & & \nearrow^{\kappa} & \\ {\scriptstyle \iota_n = \iota'' \circ \iota'}\big\downarrow & & G_n \oplus \mathbb{Z} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ & \swarrow_{\iota''} & & & \\ G_{n+1} & & \xrightarrow[\quad\theta_{n+1}\quad]{} & & G. \end{array}$$

It is easy to verify that $\ker(\theta_n) \subseteq \ker(\iota_n)$. Therefore, the ordered group G is isomorphic to the inductive limit of $\{G_n, \iota_n\}$. $\square$

Recall that if $S(\bar{k}_1, \ldots, \bar{k}_n; T)$ is a splitting tree algebra defined by Definition 2.1, its K_0-group can be described as

$$\left\{ (m_1, \ldots, m_n) \in \bigoplus_{i=1}^{n} \mathbb{Z}^{|\bar{k}_i|};\ \sum_i m_1^{(i)} = \cdots = \sum_i m_n^{(i)} \right\}$$

with the usual order. (See [31].) A map $r : \mathrm{K}_0(S) \to \mathbb{Z}$ is called a *point evaluation map* if it is induced by a point evaluation map $S \to \mathrm{M}_n(\mathbb{C})$ for some n. By the *point evaluation map on a regular point*, we refer to the positive map $\mathrm{K}_0(S) \to \mathbb{Z}$ defined by

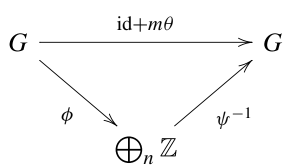
$$(m_1, \ldots, m_n) \mapsto \sum_{i=1}^{|\bar{k}_1|} m_1^{(i)}.$$

The K_0-groups of splitting tree algebras are not necessarily dimension groups (the Riesz decomposition may fail); however, certain positive homomorphisms factor through dimension groups, provided these maps have a large corner factoring though dimension groups. More precisely, we have the following lemma.

Lemma 3.2. *Let $G = \mathrm{K}_0(S)$ where S is a splitting tree algebra, and let $r : G \to \mathbb{Z}$ be the point evaluation map on a regular point. Then there exist $u \in G^+$ and a natural number m such that, if $\theta : G \to G$ is defined by $g \mapsto r(g)u$, the positive homomorphism $\mathrm{id} + m\theta : G \to G$ factors through $\bigoplus_1^n \mathbb{Z}$ for some n.*

Proof. The group G is torsion free and the positive cone of G is finite generated. Therefore, there is a basis of G (as an abelian group) consisting of the positive elements. Denote it by $\{u_1, \ldots, u_n\}$. Then, there is an isomorphism ψ from G to $\bigoplus_n \mathbb{Z}$ (as groups) sending $\{u_i\}$ to the canonical basis of $\bigoplus_n \mathbb{Z}$. This isomorphism may not be a positive homomorphism. But its inverse ψ^{-1} is positive.

Define a positive map $\theta_2 : \mathbb{Z} \to G$ by sending 1 to $u := u_1 + \cdots + u_n$, and define $\theta = \theta_2 \circ r$. Since $\psi(u) = (1, 1, \ldots, 1)$, we have that for any positive element g in G, there is a natural number m_g such that $\psi(g) + m_g \psi \circ \theta(g)$ is positive in $\bigoplus_n \mathbb{Z}$ (with the usual order). Since the positive cone of G is finite generated, there is a natural number m such that $\psi(g) + m\psi \circ \theta(g)$ is positive for any $g \in G^+$. Consider the positive map $\phi = \psi + m\psi \circ \theta$ from G to $\bigoplus_n \mathbb{Z}$. Then the diagram



commutes, as desired. □

3.2. The K_0*-groups of C*-algebras in* TA$\mathscr{S}$

Let A be a simple separable C*-algebra in the class TA$\mathscr{S}$. We shall prove that $\mathrm{K}_0(A)$ is an inductive limit of certain basic building blocks. Let $\mathcal{KS}$ denote the class of ordered K_0-groups of C*-algebras in $\mathscr{S}$, and let $\mathcal{ZT}$ denote the class of finite direct sums of ordered groups

$$\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}, \quad k = 1, 2, \ldots,$$

with respect to the order

$$(m, n) > 0 \quad \text{if and only if} \quad m > 0.$$

Denote by $\mathcal{K}$ the class of finite direct sums of ordered groups in $\mathcal{KS}$ and ordered groups in $\mathcal{ZT}$. We also refer to the ordered groups in $\mathcal{K}$ as basic building blocks (of ordered groups). We shall prove the following proposition in this subsection:

Proposition 3.3. *Let $H = \mathrm{K}_0(A)$ for some simple separable C*-algebra $A \in \mathrm{TA}\mathscr{S}$. Then, for any $G_0 \in \mathcal{K}$, any positive homomorphism $\theta : G_0 \to H$, and any $\alpha \in \ker(\theta)$, there exist $G_1 \in \mathcal{K}$, positive homomorphism $\iota : G_0 \to G_1$, and positive homomorphism $\theta' : G_1 \to H$, such that $\iota(\alpha) = 0$ and the following diagram commutes*:

$$\begin{array}{ccc} G_1 & & \\ {\scriptstyle\iota}\uparrow & \searrow{\scriptstyle\theta'} & \\ G_0 & \xrightarrow[\theta]{} & H. \end{array}$$

Moreover, if $G_0 = G_0' \oplus G_0''$, where G_0'' is the K_0-group of a sub-C-algebra S_n of A as described at the beginning of this section and the restriction of θ to G_0'' is induced by the inclusion map, and $u \in G_0$ is a positive element with $\theta(u) = [1_A]$, then, for any natural number N and $\varepsilon > 0$, there is a sub-C*-algebra S_k of A as described at the beginning of this section with k sufficiently large, such that G_1 can be chosen as $G_1' \oplus \mathrm{K}_0(S_k)$, where G_1' is an ordered group in $\mathcal{ZT}$. If denoted by u' the restriction of $\iota(u)$ to G_1' and u'' the restriction of $\iota(u)$ to $\mathrm{K}_0(S_k)$, we have that $u'' = [1_{S_k}]$ and $N\theta'(u') < \theta'(u'')$ in $\mathrm{K}_0(A)$. Furthermore, we may assume that $\tau(1_A - 1_{S_k}) < \varepsilon$ for all $\tau \in \mathrm{T}(A)$ and the restriction of ι to G_0'' and $\mathrm{K}_0(S_k)$ can be chosen to be induced by the $*$-homomorphism $\phi_{n,k}$ from S_n to S_k.*

As a straightforward consequence of the first part of this proposition, a certain inductive limit decomposition of the K_0-groups of a simple separable C*-algebra in $\mathrm{TA}\mathscr{S}$ is obtained.

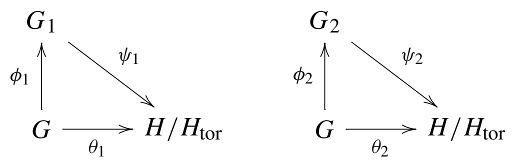
Corollary 3.4. *Let H be the K_0-group of a simple separable C*-algebra in $\mathrm{TA}\mathscr{S}$. Then H is an inductive limit of K_0-groups of splitting tree algebras in the class $\mathscr{S}$ and ordered groups $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, $n = 1, 2, \ldots$.*

Proof. It follows from Lemma 3.1 and the first part of Proposition 3.3. □

The proof of Proposition 3.3 needs several lemmas.

Let H be an ordered group with positive cone H^+. Denote by H_{tor} the subgroup of torsion elements of H. Then, one has that $H^+ \cap H_{\mathrm{tor}} = \{0\}$. It follows that the quotient group H/H_{tor}, with the image of H^+ as the positive cone, is an ordered group.

Lemma 3.5. *Let $G \in \mathcal{KS}$ and $H = \mathrm{K}_0(A)$ for a simple C*-algebra $A \in \mathrm{TA}\mathscr{S}$. Then, for any positive homomorphism $\theta : G \to H/H_{\mathrm{tor}}$ and an $\alpha \in \ker(\theta)$, there exist positive homomorphisms θ_1 and θ_2 from G to H/H_{tor} such that $\theta = \theta_1 + \theta_2$ and the following diagrams commute*:



where $G_1 = \bigoplus_k \mathbb{Z}$ for some natural number k, $G_2 \in \mathcal{KS}$ and $\phi_2(\alpha) = 0$. Moreover, the positive homomorphism ψ_2 can be lifted to a positive homomorphism to H.

Furthermore, if $G = G' \oplus G''$, where G'' is the K_0-group of a sub-C-algebra S_n of A as described at the beginning of this section and the restriction of θ to G'' is induced by the inclusion map, and $u \in G$ is a positive element with $\theta(u) = [1_A]$, then, for any natural number N and $\varepsilon > 0$, the group G_2 can be chosen to be the K_0-group of a sub-C*-algebra S_k of A as described at the beginning of this section with $\tau(1_A - 1_{S_k}) < \varepsilon$ for every $\tau \in \mathrm{T}(A)$, and if let $u_1 = \phi_1(u)$ and $u_2 = \phi_2(u)$, we have that $N\psi_1(u_1) < \psi_2(u_2)$ in the ordered group H/H_{tor}. Moreover, the restriction of the map ϕ_2 to G'' can be chosen to be induced by the $*$-homomorphism $\phi_{n,k}$ from S_n to S_k.*

Proof. We may assume that θ always send a non-zero positive element in G to a non-zero element in H/H_{tor}. Otherwise, the positive map θ factors through a quotient group of G which still belongs to $\mathcal{KS}$, and we then can investigate the map from this quotient group to H/H_{tor}. Therefore, we may assume that neither α nor $-\alpha$ is positive.

Step 1. We construct a point evaluation map θ' from G to H/H_{tor} which sends α to 0, and θ' is small enough such that $\theta - \theta'$ is still a positive homomorphism.

Let $r_1, \dots, r_k$ be the point evaluation maps on regular points of each simple summand of the group G. Denote by m the integer $r_1(\alpha) + \cdots + r_k(\alpha)$. If $m \geqslant 0$, one then chooses a singular point t_0 of α such that the value of α at t_0 is negative (we can do this since $\alpha \notin G^+$). Let s denote the point evaluation at t_0, and let $-n = s(\alpha)$. Since the positive cone of G is finite generated and A has the property (SP), there is $h \in H/H_{\mathrm{tor}}$ such that $(m+n)h$ is less than the images of the generators of G^+. Then, we define the positive homomorphism $\theta' : G \to H/H_{\mathrm{tor}}$ by

$$\theta' : g \mapsto s(g)mh + \sum_{i=1}^{k} r_i(g)nh.$$

It is easy to see $\theta - \theta'$ is a positive homomorphism and $\theta'(\alpha) = 0$. Since $\theta(u) = [1_A]$, we can also choose h sufficiently small such that $4\rho(\theta'(u)) \leqslant \min\{\varepsilon, 1/N\}$ for each state ρ of H/H_{tor}.

If $m < 0$, a construction similar to the above also gives us a desired positive homomorphism θ'.

Step 2. Denote by φ the map $\theta - \theta'$. Since $\theta'(\alpha) = 0$, we have that $\varphi(\alpha) = 0$. Apply Lemma 3.2 to each direct summand of G to obtain an integer m_i and a positive element u_i for each $i = 1, \dots, k$, such that the positive map $(\mathrm{id} + \sum_{i=1}^{k} m_i r_i u_i)$ factors through a finite direct sum of the group of integers.

Since the positive cone of G is finitely generated, one can denote the images of these generators in H/H_{tor} by $\{p_1, \dots, p_m\}$. Choose a pre-image q_i in $H = \mathrm{K}_0(A)$ for each p_i. Since A is a TAS-algebra, one has that

$$q_i = q_i' + q_i'', \quad i = 1, \dots, m,$$

where each q_i'' is in the K_0-group of a splitting tree algebra S_k as described at the beginning of this section. Since the positive cone of the group G is generated by finite relations and the K_0-groups of splitting tree algebras are torsion free, with a suitable choosing of the splitting tree algebra S_k (choose S_k far enough), the map sending each generator of G to the corresponding q'' induces a positive homomorphism from G to H and sends α to 0. Denote this positive homomorphism by ϕ_2, and denote by θ_2 the positive map from G to H/H_{tor} induced by ϕ_2. The map sending the generators of G to corresponding q_i' may not extend to a positive map from G to H, but it induces a positive map to H/H_{tor} by passing to the quotient. Denote this positive map by θ_1'. Therefore, we get a decomposition $\varphi = \theta_1' + \theta_2$, where θ_1' and θ_2 are positive homomorphisms from G to H/H_{tor} which satisfy $\theta_1'(\alpha) = 0$, $\theta_2(\alpha) = 0$, and θ_2 factors through G_2, the K_0-group of the splitting tree algebra S_k. Thus, one has the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\theta_2} & H/H_{\mathrm{tor}} \\ & \searrow^{\phi_2} \quad \nearrow_{\psi_2} & \\ & G_2 & \end{array}$$

with $\phi_2(\alpha) = 0$.

If $G = G' \oplus G''$ where G' is nonempty, G'' comes from a sub-C*-algebra S_n of A as described at the beginning of this section and the restriction of θ to G'' is induced by the inclusion map, one can choose the element h in Step 1 sufficiently small such that the map θ' is less that the restriction map of θ to the summand G'. Thus the restriction of the map φ to G'' is still induced by the inclusion map of S_n into A, and then we can choose the sub-C*-algebra S_k such that the restriction of the map ϕ_2 to G'' is induced by the $*$-homomorphism $\phi_{n,k}$ constructed at the beginning of this section.

Note that we may choose q_i' sufficiently small such that $m_i\theta_1'(u_i) < h$ and $4\rho(\theta_1'(u)) \leqslant \min\{\varepsilon, 1/N\}$ for all positive state ρ on the ordered group H/H_{tor}. Thus we have the decomposition

$$\theta = \theta' + \varphi = \big(\theta' + \theta_1'\big) + \theta_2,$$

and for any g in a direct summand of G, the following holds:

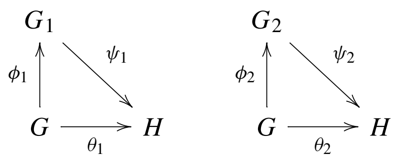
$$\begin{aligned} \theta'(g) + \theta_1'(g) &= s(g)mh + \sum_{i=1}^{k} r_i(g)nh + \theta_1'(g) \\ &= s(g)mh + \sum_{i=1}^{k} r_i(g)\big(nh - m_i\theta_1'(u_i)\big) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k} r_i(g)m_i\theta_1'(u_i)+\theta_1'(g)\\
&= s(g)mh+\sum_{i=1}^{k} r_i(g)\big(nh-m_i\theta_1'(u_i)\big)\\
&+\theta_1'\left(\sum_{i=1}^{k} m_ir_i(g)u_i+g\right).
\end{aligned}
$$

The first two terms of the decomposition are point evaluation maps, and thus factor through a finite direct sum of the group of integers. By Lemma 3.2, the positive homomorphism $(\sum_{i=1}^{k} m_ir_iu_i+\mathrm{id})$ also factors through a finite direct sum of the group of integers. Therefore, the positive homomorphism $\theta'+\theta_1'$ factors through the group $G_1=\bigoplus_k \mathbb{Z}$ for some k. Set $\theta_1=\theta'+\theta_1'$. Then the positive homomorphisms θ_1 and θ_2 have the desired factorization property. □

Using a slightly modification of the argument of the lemma above, we have the following decomposition for the positive homomorphisms from an ordered group in $\mathcal{K}$ to $H=\mathrm{K}_0(A)$.

Lemma 3.6. *Let $G\in\mathcal{K}$ and $H=\mathrm{K}_0(A)$ for a simple C*-algebra $A\in\mathrm{TA}\mathscr{S}$. Then, for any positive homomorphism $\theta: G\to H$ and $\alpha\in\ker(\theta)$, there exist positive homomorphisms θ_1 and θ_2 from G to H such that $\theta=\theta_1+\theta_2$ and the following diagrams commute*:



where $G_1\in\mathcal{ZT}$, $G_2\in\mathcal{KS}$, and $\phi_2(\alpha)=0$.

Furthermore, if $G=G'\oplus G''$, where G'' is the K_0-group of a sub-C-algebra S_n of A as described at the beginning of this section and the restriction of θ to G'' is induced by the inclusion map, and $u\in G$ is a positive element with $\theta(u)=[1_A]$, then, for any natural number N and $\varepsilon>0$, the group G_2 can be chosen to be the K_0-group of a sub-C*-algebra S_k of A as described at the beginning of this section with $\tau(1_A-1_{S_k})<\varepsilon$ for each $\tau\in\mathrm{T}(A)$, and if we let $u_1=\phi_1(u)$ and $u_2=\phi_2(u)$, we have that $N\psi_1(u_1)<\psi_2(u_2)$ in the ordered group H. Moreover, the restriction of the map ϕ_2 to G'' can be chosen to be induced by the $*$-homomorphism $\phi_{n,k}$ from S_n to S_k.*

Proof. Write $G=G_{\mathrm{ZT}}\oplus G_{\mathrm{KS}}$ where $G_{\mathrm{ZT}}\in\mathcal{ZT}$ and $G_{\mathrm{KS}}\in\mathcal{KS}$. We consider the restriction of θ to G_{ZT}. Since the positive cone of G_{ZT} is finite generated and A is a simple C*-algebra in the class $\mathrm{TA}\mathscr{S}$, the map $G_{\mathrm{ZT}}\to H$ can be decomposed into the sum of positive maps κ_1 and κ_2, where the κ_2 factors through the K_0-group of a sub-C*-algebra of A in the class $\mathscr{S}$. Since the K_0-groups of splitting tree algebras are torsion free, the map κ_2 factors through a finite direct sum of the group of integers. Moreover, given G, $u\in G^+$, natural number N, and $\varepsilon>0$ as the second part of the lemma, we may choose κ_1 and κ_2 such that κ_2 factors through the K_0-group of S_k and $2N\kappa_1(u) < \kappa_2(u)$.

Note that the map θ can be decomposed as the sum of the map κ_1, the map κ_2, and the restriction of θ to G_{KS}. Since the map κ_1 is chosen to be sufficiently small, it is enough to prove the lemma for the sum of the map κ_2 and the map $G_{\mathrm{KS}} \to H$. Since the map κ_2 factors through a finite direct sum of the group of integers, it is enough to prove the lemma for $G \in \mathcal{KS}$. Then, a repeating of the argument of Lemma 3.5 gives a proof of the lemma (even more straightforward, one does not need to find liftings of positive elements as that in Step 2 of the proof of Lemma 3.5). □

By Lemma 3.5 (or Lemma 3.6), the map θ can be lifted to a map $\phi_1 \oplus \phi_2$. However, we only have that $\phi_2(\alpha) = 0$. Thus, in order to determine whether the ordered group H (or H/H_{tor}) is an inductive limit of certain basic building blocks, we have to handle the map ψ_1. Note that the domain of the map ψ_1 is a finite direct sum of the group of integers. We can use certain arguments of the Effros–Handelman–Shen theorem in [6].

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \bigoplus_n \mathbb{Z}$ for some natural number n. Set

$$m = \max\{|\alpha_1|, \ldots, |\alpha_n|\},$$

and l to be the number of $|\alpha_i|$'s equal to m. Then we define the *degree* of α to be the pair (m, l). Denote it by $\deg(\alpha)$. The degrees are well ordered by $(m, l) < (m', l')$ if $m < m'$ or $m = m'$ and $l < l'$. Note that $\deg(\alpha) = 0$ implies $\alpha = 0$.

Lemma 3.7. *Let $G \cong \bigoplus_n \mathbb{Z}$ for some natural number n. Let $H = \mathrm{K}_0(A)$ for a simple C*-algebra $A \in \mathrm{TA}\mathscr{S}$. Then, for any positive homomorphism $\theta : G \to H/H_{\mathrm{tor}}$ and $\alpha \in \ker(\theta)$, there exist an ordered group $(\bigoplus_m \mathbb{Z}) \oplus G_1$ with $G_1 \in \mathcal{KS}$ and positive homomorphisms ϕ_1, ϕ_2 and ψ such that the following diagram commutes*:

$$\begin{array}{ccc} (\bigoplus_m \mathbb{Z}) \oplus G_1 & & \\ \uparrow{\scriptstyle \phi_1 \oplus \phi_2} & \searrow{\scriptstyle \psi} & \\ G & \xrightarrow[\theta]{} & H/H_{\mathrm{tor}}, \end{array}$$

such that $\phi_1(\alpha) = \phi_2(\alpha) = 0$. Moreover, G_1 comes from a sub-C-algebra of A which is a splitting tree algebra, and the restriction of ψ on G_1 is induced by the inclusion map.*

Furthermore, if the map θ is the quotient of a positive homomorphism $\theta' : G \to H$, the maps ϕ_1, ϕ_2 and the restrictions of ψ can be chosen to satisfy the following commutative diagrams:

$$\begin{array}{ccccc} \bigoplus_m \mathbb{Z} & & & & \\ \uparrow{\scriptstyle \phi_1} & & \searrow{\scriptstyle \psi_1} & & \\ G & \xrightarrow[\theta_1']{} & H & \xrightarrow[\pi]{} & H/H_{\mathrm{tor}} \end{array} \qquad \begin{array}{ccccc} G_1 & & & & \\ \uparrow{\scriptstyle \phi_2} & \searrow{\scriptstyle \psi_2} & & & \\ G & \xrightarrow[\theta_2']{} & H & \xrightarrow[\pi]{} & H/H_{\mathrm{tor}} \end{array}$$

where $\theta_1' + \theta_2' = \theta'$.

Proof. The proof follows the same line as the proof of the Effros–Handelman–Shen theorem in [6]. Let

$$\alpha=\sum_{i=1}^{r}m_i e_i-\sum_{j=1}^{s}n_j f_j,\quad m_i,n_j\in\mathbb{N},$$

where e_i $(1\leqslant i\leqslant r)$, f_j $(1\leqslant j\leqslant s)$ and g_k $(1\leqslant k\leqslant t)$ $(r+s+t=n)$ are the standard basis of $\bigoplus_n\mathbb{Z}$.

Let $\theta(e_i)=a_i$ and $\theta(f_j)=b_j$. Assume $\deg(\alpha)=(m,l)$ and $m_1=m$. Since $\theta(\alpha)=0$, we have

$$\sum_{i=1}^{r}m_i a_i=\sum_{j=1}^{s}n_j b_j.$$

Therefore,

$$m_1a_1\leqslant\sum_{i=1}^{r}m_i a_1=\sum_{j=1}^{s}n_j b_j\leqslant m_1\left(\sum_{j=1}^{s}b_j\right).$$

Since $H=\mathrm{K}_0(A)$, the ordered group H/H_{tor} is unperforated. Hence we have

$$a_1\leqslant\sum_{j=1}^{s}b_j.$$

Case 1. If $a_1=\sum_{j=1}^{s}b_j$, we have that $r=1$ and $n_j=m_1$ for each j. Then, we can define a positive homomorphism $\phi:\bigoplus_n\mathbb{Z}\to\bigoplus_{n-1}\mathbb{Z}$ by

$$\begin{aligned}e_1&\mapsto\sum_{j=1}^{s}f'_j,\\ f_j&\mapsto f'_j\quad(1\leqslant j\leqslant s),\\ g_k&\mapsto g'_k\quad(1\leqslant k\leqslant t),\end{aligned}$$

where f'_j $(1\leqslant j\leqslant s)$, g'_k $(1\leqslant k\leqslant t)$ are the standard basis of $\bigoplus_{n-1}\mathbb{Z}$. We also define the positive homomorphism $\psi:\bigoplus_{n-1}\mathbb{Z}\to H/H_{\mathrm{tor}}$ by

$$\begin{aligned}f'_j&\mapsto b_j\quad(1\leqslant j\leqslant s),\\ g'_k&\mapsto\theta(g_k)\quad(1\leqslant k\leqslant t).\end{aligned}$$

A direct calculation shows that $\theta=\psi\circ\phi$. Moreover, we have

$$\phi(\alpha)=m_1\sum_{j=1}^{s}f'_j-\sum_{j=1}^{s}n_j f'_j=m_1\left(\sum_{j=1}^{s}f'_j-\sum_{j=1}^{s}f'_j\right)=0.$$

Hence we get the desired lifting of θ (with $G_1=\{0\}$).

Case 2. If $a_1 < \sum_{j=1}^{s} b_j$, we have that $r > 1$ or $n_j < m$ for some j. Let a_i''s and b_j''s be preimages of a_i's and b_j's, respectively, in H. Let θ' be the lifting of θ according to a_i''s and b_j''s. We assert that there is a factorization of θ' as follows:

$$\begin{array}{ccc} G_1 \oplus (\bigoplus_k \mathbb{Z}) & & \\ {\scriptstyle \phi_1 \oplus \iota} \uparrow & \overset{\psi}{\searrow} & \\ G & \xrightarrow[\theta']{} & H, \end{array}$$

where $G_1 \in \mathcal{KS}$ comes from a sub-C*-algebra of A, $\phi_1(\alpha) = 0$. Moreover, $\deg(\iota(\alpha)) < \deg(\alpha)$.

To show this, we shall use the same technique of Lemma 3.5. Let $d_1 = \sum_{i=1}^{r} m_i$ and $d_2 = \sum_{j=1}^{s} n_j$. Define the positive homomorphism $r : G \cong \bigoplus_n \mathbb{Z} \to \mathbb{Z}$ by

$$(c_1, \ldots, c_n) \mapsto d_2\left(\sum_{i=1}^{r} c_i\right) + d_1\left(\sum_{i=r+1}^{r+s} c_i\right) + \sum_{i=r+s+1}^{n} c_i.$$

Since A has the property (SP), there is an element $h \in H$ sufficiently small such that $g \mapsto \theta'(g) - r(g)h$ is a positive homomorphism. To save notation, let us still use r to denote the map $g \mapsto r(g)h$. Note that $r(\alpha) = 0$ and r is faithful on the positive cone of G. We have

$$\sum_{i=1}^{r} m_i r(e_i) = \sum_{j=1}^{s} n_j r(f_j).$$

Therefore, we get

$$mr(e_1) < m\left(\sum_{j=1}^{s} r(f_j)\right).$$

In the other words, $r((\sum_{j=1}^{s} f_j) - e_1) > 0$.

Since A is a TAS-algebra and the positive homomorphism $\theta' - r$ is positive, there is a decomposition $\theta' - r = \phi_1' + \phi_1$ such that $\phi_1'(\alpha) = \phi_1(\alpha) = 0$ and ϕ_1 factors through G_1 which is the K_0-group of a splitting tree algebra inside A. Moreover, we may assume that $s\phi_1'(e_1) < r((\sum_{j=1}^{s} f_j) - e_1)$.

Therefore, we have that $\theta' = r + \phi_1' + \phi_1$. We wish to show that the map $r + \phi_1'$ factors through a dimension group and decreases the degree of α strictly. Since $r((\sum_{j=1}^{s} f_j) - e_1) > 0$, we have

$$r(e_1) < \sum_{j=1}^{s} r(f_j).$$

Moreover, the map r factors through $\mathbb{Z}$ by construction. Therefore, there exist $a_{11}', \ldots, a_{1s}'$ such that

$$r(e_1)=\sum_{j=1}^{s}a'_{1j}$$

and $a'_{1j}<r(f_j)$ for any $1\leqslant j\leqslant s$.

Among the $\{a_{ij}\}$, let us show there exists a_{1j_0} such that $\phi'_1(e_1)<r(f_{j_0})-a'_{1j_0}$. Suppose this were not true. Then for any $0\leqslant j\leqslant s$, $r(f_j)-a'_{1j}\leqslant\phi'_1(e_1)$. Therefore, one has

$$\sum_{j=1}^{s}r(f_j)-\sum_{j=1}^{s}a'_{1j}\leqslant s\phi'_1(e_1),$$

and $\sum_{j=1}^{s}r(f_j)-r(e_1)\leqslant s\phi'_1(e_1)$ which is a contradiction to the choice of ϕ'_1.

Set $a_{1j}=a'_{1j}$ for any $1\leqslant j\leqslant s$, $j\neq j_0$ and $a_{1j_0}=a'_{1j_0}+\phi'(e_1)$. Since $\phi'_1(e_1)<r(f_{j_0})-a'_{1j_0}$, we have

$$a_{1j}\leqslant r(f_j)\leqslant r(f_j)+\phi'_1(f_j)\quad\text{for any } 1\leqslant j\leqslant s,$$

and

$$r(e_1)+\phi'_1(e_1)=\left(\sum_{j=1}^{s}a'_{1j}\right)+\phi'_1(e_1)=\sum_{j=1}^{s}a_{1j}.$$

As in the proof of Effros–Handelman–Shen theorem in [6], let e'_{1j} $(0\leqslant j\leqslant s)$, e'_i $(2\leqslant i\leqslant r)$, f_j $(1\leqslant j\leqslant s)$ and g_k $(1\leqslant k\leqslant t)$ be a standard basis for $\mathbb{Z}^k$, where $k=r+2s+t-1$. Define the positive homomorphisms $\iota:G\to\mathbb{Z}^k$ by

$$\begin{aligned}
e_1&\mapsto\sum_j e'_{1j},\\
e_i&\mapsto e'_i\quad(2\leqslant i\leqslant r),\\
f_j&\mapsto f'_j+e'_{1j},\\
g_k&\mapsto g'_k,
\end{aligned}$$

and $\psi_1:\mathbb{Z}^k\to H$ by

$$\begin{aligned}
e'_{1j}&\mapsto a_{1j},\\
e'_i&\mapsto r(e_i)+\phi'_1(e_i)\quad(i\geqslant 2),\\
f'_j&\mapsto r(f_j)+\phi'_1(f_j)-a_{1j},\\
g'_k&\mapsto r(g_k)-\phi'_1(g_k).
\end{aligned}$$

A direct calculation shows that $\psi_1\circ\iota=r+\phi'_1$. Moreover,

$$\begin{aligned}\iota(\alpha) &= \iota\left(\sum_{i=1}^{r} m_i e_i - \sum_{j=1}^{s} n_j f_j\right)\\ &= \sum_j (m_1 - n_j)e'_{1j} + \sum_{i=2}^{r} m_i e'_i - \sum_j n_j f'_j,\end{aligned}$$

where $n_j > 0$ for any j. Thus, $0 \neq \deg(\iota(\alpha)) < \deg(\alpha)$ as desired.

Applying the factorization above to the restriction of ψ to $\bigoplus_k \mathbb{Z}$ iteratively, after finite steps, we get the following commutative diagram:

$$\begin{array}{ccc} G_1 \oplus (\bigoplus_{k'} \mathbb{Z}) & & \\ \uparrow{\scriptstyle \phi_1 \oplus \iota} & \searrow^{\psi} & \\ G & \xrightarrow[\theta']{} & H \end{array}$$

for some $G_1 \in \mathcal{KS}$ which comes from a sub-C*-algebra of A, such that $\phi_1(\alpha) = 0$ and the element $\iota(\alpha)$ has the form $m'e_1 - \sum m' f_j$ where e_1 and f_j from the standard basis of $\bigoplus_{k'} \mathbb{Z}$.

Therefore, after passing to the quotient H/H_{tor}, in order to prove the lemma, we shall find a suitable factorization of the restriction of ψ to $\bigoplus_{k'} \mathbb{Z}$. Since $\iota(\alpha)$ has the special form, it reduces the lemma to Case 1. Thus, the first part of the lemma holds.

If the positive map θ can be lifted to $\theta' : G \to H$, we can choose this positive homomorphism as the lifting of θ as what we used in the proof of Case 2. The commutative diagrams of the second part of the lemma follows directly from the constructions in Case 2. □

From Lemmas 3.5 and 3.7, we have the following corollary:

Corollary 3.8. *Let* $G \cong (\bigoplus_n \mathbb{Z}) \oplus G_0$ *with* $G_0 \in \mathcal{KS}$. *Let* $H = \mathrm{K}_0(A)$ *for a simple C*-algebra* $A \in \mathrm{TA}\mathscr{S}$. *Then, for any positive homomorphism* $\theta : G \to H/H_{\mathrm{tor}}$, *there are an ordered group* $(\bigoplus_m \mathbb{Z}) \oplus G_1$ *where* $G_1 \in \mathcal{KS}$ *and positive homomorphisms* ϕ_1, ϕ_2 *and* ψ *such that the following diagram commutes*:

$$\begin{array}{ccc} (\bigoplus_m \mathbb{Z}) \oplus G_1 & & \\ \uparrow{\scriptstyle \phi_1 \oplus \phi_2} & \searrow^{\psi} & \\ G & \xrightarrow[\theta]{} & H/H_{\mathrm{tor}}, \end{array}$$

such that $\ker(\phi_1) = \ker(\phi_2) = \ker(\theta)$. *Moreover, the group* G_1 *can be chosen to be the* K_0*-group of a sub-C*-algebra of* A *which is a splitting tree algebra, and the restriction of* ψ *on* G_1 *is induced by the inclusion map.*

Proof. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of generators of $\ker(\theta)$. By Lemma 3.5, there is a factorization

$$\begin{array}{ccc} (\bigoplus_{n_1'} \mathbb{Z}) \oplus G_1' & & \\ {\scriptstyle \phi_1' \oplus \phi_2'} \uparrow & \overset{\psi}{\searrow} & \\ G & \xrightarrow[\theta]{} & H/H_{\mathrm{tor}} \end{array}$$

with some $G_1' \in \mathcal{KS}$, such that $\phi_2'(\alpha_1) = 0$ and the positive homomorphisms in the commutative diagram satisfy the second part of the corollary. Applying Lemma 3.7 to the restriction of ψ' to $\bigoplus_{n_1'} \mathbb{Z}$, we obtain the factorization of θ

$$\begin{array}{ccc} (\bigoplus_{n_1} \mathbb{Z}) \oplus G_1 & & \\ {\scriptstyle \phi_1 \oplus \phi_2} \uparrow & \overset{\psi}{\searrow} & \\ G & \xrightarrow[\theta]{} & H/H_{\mathrm{tor}} \end{array}$$

with $\phi_1(\alpha_1) = \phi_2(\alpha_1) = 0$.

Then, one can consider the image of α_2 in $(\bigoplus_{n_1} \mathbb{Z}) \oplus G_1$ by $\phi_1 \oplus \phi_2$. If it is not zero, by the same argument above, one gets a factorization of the positive homomorphism ψ, and sends $\phi_1(\alpha_2) \oplus \phi_2(\alpha_2)$ to zero simultaneously. Repeat this procedure for each α_i, we obtain the desired factorization of θ. □

Corollary 3.9. *Let $H = \mathrm{K}_0(A)$ for a simple TAS-algebra A. Then H/H_{tor} is an inductive limit of the* K_0*-groups of splitting tree algebras.*

Proof. This follows directly from the first part of Corollary 3.8 and Lemma 3.1. □

Before proving Proposition 3.3, we need the following lemma about the K_0-groups of simple TAS-algebras.

Lemma 3.10. *Let $H = \mathrm{K}_0(A)$ for some simple separable TAS-algebra A. Then, for any $a \in H^+$ and $b \in H_{\mathrm{tor}}$, b is majorised by a.*

Proof. Since A is a simple separable TAS-algebra, A has the cancellation property for equivalence classes of projections. Since any matrix algebra over a TAS-algebra is a TAS-algebra again, we may assume

$$a = [e], \qquad b = [p] - [q]$$

for projections e, p and q in A with $e \perp q$, $e \perp p$. Since $b \in H_{\mathrm{tor}}$, we have

$$\tau(p) = \tau(q) \quad \text{for any } \tau \in T(A).$$

Hence

$$\tau(e+q) = \tau(e) + \tau(q) > \tau(p) \quad \text{for any } \tau \in T(A).$$

Since A is a simple TAS-algebra, by Theorem 4.12, the strict order on projections is determined by traces. Hence,

$$[p] \preccurlyeq [e+q],$$

which implies $b < a$. □

Proof of Proposition 3.3. Let α be an element in the kernel of the map $\theta : G_0 \to H$. Let us first show that it is sufficient to prove the proposition for positive maps from basic building blocks in $\mathcal{ZT}$—i.e., the basic building blocks $\mathbb{Z} \oplus$ (finitecyclic)—to the ordered group H. It follows from Lemma 3.6 that the map $G_0 \to H$ factors through $G_1 \oplus G_2$ with $G_1 \in \mathcal{ZT}$ and $G_2 \in \mathcal{KS}$ such that the map $G_0 \to G_2$ sends α to 0, and the groups G_1 and G_2 satisfy the second part of the proposition. Thus, to prove the proposition, it is sufficient to show that the map $G_1 \to H$ has a lifting in $\mathcal{K}$ which sends the image of α in G_1 to 0.

Therefore, let us assume that $G_0 \in \mathcal{ZT}$, and follows an argument similar to that of Theorem 3.2 of [9].

Step 1. Let us show that it is sufficient to verify Lemma 3.1 with the basic building blocks $\mathbb{Z} \oplus$ (finite cyclic) replaced by the ordered groups $\mathbb{Z} \oplus$ (finite) (with the order determined by the first coordinate). To see this, it is enough to show that a map $G_1' \to H$ with $G_1' = \mathbb{Z} \oplus$ (finite) can be factorised as $G_1' \to G_1 \to H$ with $G_1 \in \mathcal{ZT}$. With the finite part of G_1' be expressed as $F_n \oplus \cdots \oplus F_n$, denote the images of their generator by $t_1, \dots, t_n \in H_{\text{tor}}$, and denote the image in H of the positive generator of $\mathbb{Z} \subseteq G_1'$ by a. Since $H = \mathrm{K}_0(A)$ and A has property (SP), one has that $a = a_1 + \cdots + a_n$ with some $a_1, \dots, a_n \in H^+$. By Lemma 3.10, we have $t_i < a_i$ for each $1 \leqslant i \leqslant n$. With G_1 to be the direct sum of the basic building blocks $\mathbb{Z} \oplus F_1, \dots, \mathbb{Z} \oplus F_n$, let us define the map $G_1' \to G_1$ by mapping $F_1 \oplus \cdots \oplus F_n \subseteq G_1'$ identically onto $F_1 \oplus \cdots \oplus F_n \subseteq G_1$, and $1 \in \mathbb{Z} \subseteq G_1'$ into $(1, \dots, 1) \in \mathbb{Z}^n \subseteq G_1$. We also define the map $G_1 \to H$ by sending the fixed generators of F_i to $t_i \in H_{\text{tor}}$ and the positive generator of the ith summand of G_1 into $a_i \in H$. We then have the factorization $G_1' \to G_1 \to H$ as desired.

Step 2. In order to verify the local criterion for the generized building blocks of Step 1, it is enough to consider the case that the restriction of the map $G_0 \to H$ to the subgroup $(G_0)_{\text{free}}$ generated by the free direct summands of the generalized basic building blocks $\mathbb{Z} \oplus$ (finite) of G_0 factorises through the direct sum of $(G_1)_{\text{free}}$ a torsion free basic building blocks in $\mathcal{ZT}$ and $(G_1)_{\text{KS}}$ a basic building blocks in $\mathcal{KS}$, in such a way that the kernel of $(G_0)_{\text{free}} \to (G_1)_{\text{free}}$ is equal not only to the kernel of $(G_0)_{\text{free}} \to H$, but also to the kernel of $(G_0)_{\text{free}} \to H/H_{\text{tor}}$.

Let $G_0 \to H$ be a map of ordered groups with G_0 a finite direct sum of generalized basic building blocks, and consider the map $(G_0)_{\text{free}} \to H/H_{\text{tor}}$ obtained by restricting $G_0 \to H$ to the (noncanonical) subgroup $(G_0)_{\text{free}}$ and composing $(G_0)_{\text{free}} \to H$ with the canonical map $H \to H/H_{\text{tor}}$. By Lemma 3.7, there exists an ordered group $(G_1)_{\text{free}} \oplus (G_1)_{\text{KS}}$, with $(G_1)_{\text{free}}$ the finite direct sum of the ordered groups $\mathbb{Z}$, $(G_1)_{\text{KS}}$ the K_0-group of a splitting tree algebra, and a factorization $(G_0)_{\text{free}} \to (G_1)_{\text{free}} \oplus (G_1)_{\text{KS}} \to H/H_{\text{tor}}$ such that

$$\ker\big((G_0)_{\text{free}} \to H/H_{\text{tor}}\big) = \ker\big((G_0)_{\text{free}} \to \big((G_1)_{\text{free}} \oplus (G_1)_{\text{KS}}\big)\big).$$

Moreover, the restriction of the map $(G_1)_{\text{free}} \oplus (G_1)_{\text{KS}} \to H/H_{\text{tor}}$ to $(G_1)_{\text{KS}}$ has a lifting to H by the second part of Lemma 3.7. Choose a positive map $(G_1)_{\text{free}} \to H$ lifting the restriction of the map $(G_1)_{\text{free}} \oplus (G_1)_{\text{KS}} \to H/H_{\text{tor}}$ to $(G_1)_{\text{free}}$. Then, we get a map $(G_1)_{\text{free}} \oplus (G_1)_{\text{KS}} \to H$ lifting the map $(G_1)_{\text{free}} \oplus (G_1)_{\text{KS}} \to H/H_{\text{tor}}$.

The map $(G_0)_{\text{free}} \to H$ may not equal to the combined map $(G_0)_{\text{free}} \to (G_1)_{\text{free}} \oplus (G_1)_{\text{KS}} \to H$. By the second part of Lemma 3.7, we only need to fix the map which factors through $(G_1)_{\text{free}}$. This can be done by the same argument as that in the proof of Theorem 3.2 of [9].

Step 3. Let us show that the local criterion holds for generalized basic building blocks. Let $G_0 \to H$ be a map of ordered groups with G_0 a finite sum of generalized basic building blocks. By Step 2, we may assume there is a factorization $(G_0)_{\text{free}} \to (G_1)_{\text{free}} \oplus (G_1)_{\text{KS}} \to H$ of the map $(G_0)_{\text{free}} \to H$, where $(G_0)_{\text{free}}$ denotes the direct sum of the free parts of the generalized basic building blocks $\mathbb{Z} \oplus$ (finite) of G_0, and $(G_1)_{\text{free}}$ is some finite ordered group direct sum of copies of $\mathbb{Z}$, such that

$$\ker\big((G_0)_{\text{free}} \to H/H_{\text{tor}}\big) = \ker\big((G_0)_{\text{free}} \to (G_1)_{\text{free}} \oplus (G_1)_{\text{KS}}\big).$$

Let us construct G_1 a finite direct sum of generalized basic building blocks and ordered groups in $\mathcal{KS}$, with torsion free part equal to $(G_1)_{\text{free}} \oplus (G_1)_{\text{KS}}$ as given above, and a factorization of $G_0 \to G_1 \to H$ of $G_0 \to H$ such that

$$\ker(G_0 \to H) = \ker(G_0 \to G_1).$$

Denote by F the image of the finite part of G_0 in H. By Lemma 3.10, the subgroup F is majorised by any positive element in H. Pick a minimal direct summand $\mathbb{Z}$ of $(G_1)_{\text{free}}$ which has non-zero image in H, and append F to this summand. This direct sum can be ordered (in a unique way) so that it is a generalized basic building block. Thus, the group $G_1 = ((G_1)_{\text{free}} \oplus (G_1)_{\text{KS}}) \oplus F$ became an ordered group direct sum of basic building blocks and ordered group in $\mathcal{KS}$. The extension of $(G_0)_{\text{free}} \to (G_1)_{\text{free}} \oplus (G_1)_{\text{KS}} \subseteq G_1$ to G_0 by factoring $(G_0)_{\text{tor}}$ through the maps $(G_0)_{\text{tor}} \to F \subseteq H$ and $F \to (G_1)_{\text{tor}} \subseteq G_1$, and the obvious extension of $(G_1)_{\text{free}} \oplus (G_1)_{\text{KS}} \to H$ to $G_1 = ((G_1)_{\text{free}} \oplus (G_1)_{\text{KS}}) \oplus F$, mapping $F \subseteq G_1$ identically onto $F \subseteq H$, factorise $G_0 \to H$ obviously, and fulfill the condition

$$\ker(G_0 \to H) = \ker(G_0 \to G_1).$$

Positivity of $G_0 \to G_1$ follows directly from the construction and simplicity of the group H/H_{tor}: The finite part of a generalized basic building block of G_0 is majorised by the positive generators of the free part in G_0. By simplicity of the ordered group H/H_{tor}, for each positive element of direct summand of $(G_0)_{\text{free}}$ which does not vanish in H, we may assume that its image in each simple summand of $(G_1)_{\text{free}}$ under the map $(G_0)_{\text{free}} \to (G_1)_{\text{free}} \oplus (G_1)_{\text{KS}}$ is a positive element. Therefore, positive elements of G_0 are sent to positive element of G_1 by the map $G_0 \to G_1$.

Positivity of the map $G_1 \to H$ follows directly from Lemma 3.10: By Lemma 3.10, the image of F is majorized by any positive element of H. Thus, the map $G_1 \to H$ sends positive elements of G_1 to positive elements of H.

Step 4. The proposition follows from Steps 1 and 3. □

Remark 3.11. In fact, in order to prove Proposition 3.3, one only needs Lemmas 3.6, 3.7 and 3.10.

3.3. The K_1*-groups and the pairing maps of C*-algebras in* $\mathrm{TA}\mathcal{S}$

If A is a separable simple C*-algebra in the class $\mathrm{TA}\mathcal{S}$, we have shown that the K_0-group of A can be realized as an inductive limit of K_0-groups of the C*-algebras in $\mathcal{S}$ and certain ordered groups in the class $\mathcal{ZT}$. We shall go further to construct a C*-algebra B, a simple inductive limit of algebras in $\mathcal{S}$ together with certain homogeneous C*-algebras, such that A and B not only have the same K_0-group, but also have the same K_1-group and pairing.

The basic building blocks of C*-algebras we are going to use are

(1) splitting tree algebras in the class $\mathcal{S}$,
(2) matrix algebras over the C*-algebras of continuous functions on $T_{2,k}$, and
(3) matrix algebras over the C*-algebras of continuous functions on $S^1 \vee \cdots \vee S^1 \vee T_{3,k_i} \vee \cdots \vee T_{3,k_i}$,

where $T_{2,k}$ is the two-dimensional CW complex obtained by attaching a two-dimensional disk D to S^1 via a map S^1 $(\cong \partial D) \to S^1$ of degree k, and $T_{3,k}$ is the three-dimensional CW complex obtained by attaching a three-dimensional ball B to S^2 via a map S^2 $(\cong \partial B) \to S^2$ of degree k. Building blocks in (1) provide the torsion free part of the K_0-group, building blocks in (2) provide the torsion part of the K_0-group, and building blocks in (3) provide the K_1-group in the construction.

The homogeneous C*-algebras of (2) and (3) are called *the Gong standard homogeneous C*-algebras* (see [13,16], and [14]). Denote by $\mathcal{S}'$ the class of C*-algebras containing $\mathcal{S}$ and the Gong standard homogeneous C*-algebras.

Let A be a simple separable C*-algebra in the class $\mathrm{TA}\mathcal{S}$. Since $K_1 := \mathrm{K}_1(A)$ is a countable abelian group, we can write K_1 as an inductive limit (in the category of abelian groups) of finitely generated abelian groups with injective maps:

$$K_1^{(1)} \xhookrightarrow{\eta_{12}} K_1^{(2)} \xhookrightarrow{\eta_{23}} \cdots \hookrightarrow \varinjlim K_1^{(i)} = K_1.$$

We may assume

$$K_1^{(i)} \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m_i} \oplus \big(\mathbb{Z}/n_1^{(i)}\mathbb{Z}\big) \oplus \cdots \oplus \big(\mathbb{Z}/n_{k_i}^{(i)}\mathbb{Z}\big).$$

Then the compact topological space

$$E_i = \underbrace{S^1 \vee \cdots \vee S^1}_{m_i} \vee T_{3,n_1^{(i)}} \vee \cdots \vee T_{3,n_{k_i}^{(i)}}$$

has $K_1^{(i)}$ as its K_1-group and has $(\mathbb{Z}, \mathbb{Z}^+)$ as its ordered K_0-group, i.e., $\mathrm{K}_1(\mathrm{C}(E_i)) = K_1^{(i)}$ and $(\mathrm{K}_0(\mathrm{C}(E_i)), \mathrm{K}_0^+(\mathrm{C}(E_i))) = (\mathbb{Z}, \mathbb{Z}^+)$. Moreover, for any group homomorphism

$$\eta_{ij} : K_1^{(i)} \to K_1^{(j)},$$

there is a $*$-homomorphism

$$\phi_{ij} : \mathrm{C}(E_i) \to \mathrm{M}_{12}\big(\mathrm{C}(E_j)\big)$$

such that

$$[\phi_{ij}]_1 = \eta_{ij} \quad \text{and} \quad [\phi_{ij}]_0 = \mathrm{id},$$

where $[\phi_{ij}]_*$ denotes the homomorphism induced by ϕ_{ij} on K_*-groups.

We have shown that $\mathrm{K}_0(A)$ is an inductive limit (as ordered groups) of basic building blocks. Moreover, by Proposition 3.3 and the argument of Lemma 3.1, the inductive limit decomposition of $\mathrm{K}_0(A)$ can be chosen to have special forms: let $\{\varepsilon_n\}$ be a sequence of positive numbers which converges to 0. The ordered group $\mathrm{K}_0(A)$ can be realized as an inductive limit of $(G_i = G_i' \oplus G_i'', \iota_i)$ where G_i'' comes from a sub-C*-algebra S_i of A satisfying $\tau(1_A - 1_{S_i}) < \varepsilon_i$ as being described at the beginning of this section, and the restriction of ι_i to $G_i'' \to G_{i+1}''$ is induced by the homomorphism (not necessarily unital) $\phi_i : S_i \to S_{i+1}$.

Set the positive map θ_1 in the proof of Lemma 3.1 to be $n \mapsto n[1_A]$. Since A belongs to the class TA$\mathscr{S}$, the map θ_1 can be decomposed into $\theta_1' + \theta_1''$ such that θ_1' has the factorization $\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to \mathrm{K}_0(A)$ and θ_1'' has a factorization $\mathbb{Z} \to G_1'' \to \mathrm{K}_0(A)$, where G_1'' is the K_0-group of sub-C*-algebra S_1 of A with $\tau(1_A - 1_{S_1}) < \varepsilon_1$. Now suppose that the inductive system is constructed up to the nth level; that is, we have groups $G_i = G_i' \oplus G_i''$ and maps θ_i and ι_i such that the following diagram commutes:

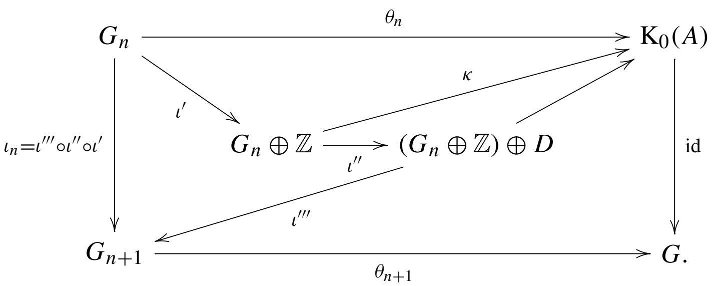
$$\begin{array}{ccc}
G_1 & \xrightarrow{\theta_1} & \mathrm{K}_0(A) \\
\downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \mathrm{id}} \\
G_2 & \xrightarrow{\theta_2} & \mathrm{K}_0(A) \\
\downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \mathrm{id}} \\
\vdots & & \vdots \\
\downarrow{\scriptstyle \iota_{n-1}} & & \downarrow{\scriptstyle \mathrm{id}} \\
G_n & \xrightarrow{\theta_n} & \mathrm{K}_0(A),
\end{array}$$

and moreover, the group G_i'' is the K_0-group of sub-C*-algebra S_i, the restriction of θ_i to G_i'' is induced by the inclusion map, and the restriction of ι_i to $G_i'' \to G_{i+1}''$ is induced by $\phi_{i,i+1}$.

We shall construct an ordered group $G_{n+1} = G_{n+1}' \oplus G_{n+1}''$ and certain positive maps. As in the proof of Lemma 3.1, define the positive maps $\iota' : G_n \to G_n \oplus \mathbb{Z}$ by $s \mapsto (s, 0)$, and $\kappa : G_n \oplus \mathbb{Z} \to \mathrm{K}_0(A)$ by $(s, m) \mapsto \theta_n(s) + mg_n$. Since A belongs to the class of TA$\mathscr{C}$, the map κ has a factorization $G_n \oplus \mathbb{Z} \to G_n \oplus \mathbb{Z} \oplus D \to \mathrm{K}_0(A)$, where D is the K_0-group of a sub-C*-algebra S_{n+1} of A as being described at the beginning of this section (we may pass to a subsequence of the original sequence (S_n), and assume that S_{n+1} is far enough from S_n). Moreover, since the restriction of κ to G_n'' is induced by the inclusion map, the factorization can be chosen such that the map $G''_n \to D$ is induced by the $*$-homomorphism $\phi_n : S_n \to S_{n+1}$. We then apply Proposition 3.3 to the map $(G_n \oplus \mathbb{Z}) \oplus D \to \mathrm{K}_0(A)$ to get a factorization

$$(G_n \oplus \mathbb{Z}) \oplus D \to \big(G_{n+1} = G'_{n+1} \oplus G''_{n+1}\big) \to \mathrm{K}_0(A),$$

such that $\ker((G_n \oplus \mathbb{Z}) \oplus D \to \mathrm{K}_0(A))$ and $\ker((G_n \oplus \mathbb{Z}) \oplus D \to G_{n+1})$ are the same. Moreover, the ordered group G''_{n+1} is the K_0-group of a sub-C*-algebra S_{n+2} of A as being described at the beginning of this section, and the map $D \to G''_{n+1}$ is induced by the $*$-homomorphism $\phi_{n+1} : S_{n+1} \to S_{n+2}$. Set the map $G_n \to G_{n+1}$ to be the composition of $G_n \to (G_n \oplus \mathbb{Z}) \oplus D$ and $(G_n \oplus \mathbb{Z}) \oplus D \to G_{n+1}$, and it is the desired map. We may pass to the subsequence of the inductive system (S_n, ϕ_n) described at the beginning of this section, and still denote the sub-C*-algebra of A associated with G_{n+1} by S_{n+1}. This procedure can be illustrated by the following diagram:



Thus, we have an inductive limit decomposition of the ordered group $\mathrm{K}_0(A)$

$$G'_1 \oplus G''_1 \to G'_2 \oplus G''_2 \to \cdots \to \mathrm{K}_0(A)$$

such that the ordered groups G''_n are sub-C*-algebras S_n of A, the maps $G''_n \to G''_{n+1}$ are induced by the $*$-homomorphisms $\phi_n : S_n \to S_{n+1}$, and the maps $G''_n \to \mathrm{K}_0(A)$ are induced by the inclusion maps.

Since each simple direct summand of the ordered group G'_n belongs to the class $\mathcal{K}$, we can choose a certain C*-algebra to have this direct summand as its ordered K_0-group: if the simple direct summand is $\mathbb{Z}$, we choose the algebra of complex number; if the simple direct summand is $\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$, we choose the algebra of the continuous function over $T_{2,k}$; if the simple direct summand is the K_0-group of a C*-algebra in $\mathscr{S}$, we choose the corresponding splitting tree algebra. Therefore, there is a C*-algebra C_n of the finite direct sum of basic building blocks such that $\mathrm{K}_0(C_n) = G'_n$. For each G'_n, choose $u'_n \in G'^{+}_n$ such that the image of (u'_n, u''_n) is $[1_A]$ in $\mathrm{K}_0(A)$. By taking matrix algebras or their cut-down over C_n, there is a C*-algebra of the finite direct sum of the basic building blocks such that its order-unit K_0-group is (G'_n, u'_n). Still denote this C*-algebra by C_n.

Set $B_n = C_n \oplus S_n$. It is clear that $(\mathrm{K}_0(B_n), [1_{B_n}]) = (G'_n \oplus G''_n, (u'_n, u''_n))$. For any positive homomorphism $\iota_n : G_n \to G_{n+1}$, since any order-unit K_0-map between basic building block C*-algebras can be lifted to a $*$-homomorphism (see [14] for homogeneous building blocks, and see [27] for splitting tree algebras), and the restriction of ι_n to $G''_n \to G''_{n+1}$ is induced by a $*$-homomorphism $\phi_n : S_n \to S_{n+1}$, there is a $*$-homomorphism $\psi_n : B_n \to B_{n+1}$ in the form

$$\begin{pmatrix} * & * \\ * & \phi_n \end{pmatrix}$$

such that $[\psi_n]_0 = \iota_n$. Moreover, we can choose the maps ψ_n in such a way that $\varinjlim B_n$ is simple. It is clear that the inductive limit of the system (B_n, ψ_n) has the same order-unit K_0-group as that of A. However, the C*-algebra B has trivial K_1-group.

To get the desired K_1-group, we shall replace one direct summand of each B_n by a certain basic building block C*-algebra with nontrivial K_1-groups, and modify the connection homomorphisms without changing the K_0-group. Since at least one of the simple direct summands of G'_n is $\mathbb{Z}$, there is one of the direct summand of C_n which is a matrix algebra, say $M_k(\mathbb{C})$, and it does not vanish in the inductive limit. We then replace this direct summand by $M_k(C(E_n))$, and still denote the new building block by B_n. We then see that $K_1(B_n) = K_1^{(n)}$, and the K_0-group of B_n remains same. Set the map between two such building blocks $C(E_n)$ and $M_k(C(E_{n+1}))$ by

$$f \mapsto \operatorname{diag}\{\phi_n, f(x_{k'-12}), \ldots, f(x_{k'})\},$$

where each x_i is a point in E_n, ϕ_n is a $*$-homomorphism from $C(E_n)$ to $M_{12}(C(E_{n+1}))$ which induces $\eta_{n,n+1}$ as K_1-map, and k' is the multiplicity of the K_0-map between the two matrix algebras being replaced. Moreover, since the K-group maps do not depend on the choices of the points $\{x_i\}$, we again can choose suitable points such that the inductive limit C*-algebra is simple. Thus, there is a $*$-homomorphism $\psi_n : B_n \to B_{n+1}$ in the form

$$\begin{pmatrix} * & * \\ * & \phi_n \end{pmatrix}$$

such that $[\psi_n]_0 = \iota_n$ and $[\psi_n]_1 = \eta_n$. Denote the inductive limit of (B_n, η_n) by B. We have

$$K_0(B) = \varinjlim(G_n, \iota_n) = K_0(A)$$

and

$$K_1(B) = \varinjlim\big(K_1^{(n)}, \eta_n\big) = K_1(A).$$

Thus A and B have the same K-groups.

We assert that A and B have the same pairing map between the simplex of traces and the ordered K_0-group. At the beginning of this section, we have that the map $r_A : T(A) \to S(K_0(A))$ is isomorphic to the map $r_S : T_{u'}(S) \to S_{u'}(K_0(S))$ where S is the inductive limit of (S_n, ϕ_n). We shall first show that $r_B : T(B) \to S(K_0(B))$ is also isomorphic to the above maps.

Since B is a simple inductive limit of splitting tree algebra and homogeneous C*-algebras with bounded dimensional spectra, the strict order on the projections in B is determined by traces. By the construction, there are homomorphisms $\lambda_n : S_n \to B$ by sending S_n to the corresponding direct summand of B_n. Note that we have commutative relations $\lambda_n = \lambda_{n+1} \circ \phi_n$. Since A is simple, the system (S_n, ϕ_n) is injective (by an asymptotic argument as that of Lemma 2.4). Therefore, the $*$-homomorphism $\lambda_n : S_n \to B$ is one-to-one, and we may consider the C*-algebra S and S_n as sub-C*-algebras of B.

Note that for any $\varepsilon > 0$, we have that $\rho([1_B]_0 - [\phi_{n,\infty}(1_{S_n})]) < \varepsilon$ for any $\rho \in \mathrm{S}(\mathrm{K}_0(B))$. Since B is a unital nuclear C*-algebra, any state on $\mathrm{K}_0(B)$ comes from a tracial state on B; we conclude that $\tau(1_B - 1_{S_n'}) < \varepsilon$ for any tracial state $\tau \in \mathrm{T}(B)$. Thus, let $\{\mathcal{F}_1, \dots, \mathcal{F}_n, \dots\}$ be an increasing sequence of finite subsets of B with dense union, and $\{\varepsilon_n\}$ be a decreasing sequence of positive number converging to zero. It is easy to see that the sub-C*-algebra S_n of B (we may pass to a subsequence of the system (B_n, ϕ_n)) satisfies the following: let $p_n = [1_{S_n}]_0$, then, for any $b \in \mathcal{F}_n$, we have

(1) $\|pb - bp\| < \varepsilon_n$,
(2) $pbp \in_{\varepsilon_n} S_n$, and
(3) $\tau(1_B - p_n) < \varepsilon_n$ for any $\tau \in \mathrm{T}(B)$.

Using the same argument as that of Lemma 10.8 of [23], we conclude that the map $r_B : \mathrm{T}(B) \to \mathrm{S}(\mathrm{K}_0(B))$ is isomorphic to $r_S : \mathrm{T}_{u'}(S) \to \mathrm{S}_{u'}(\mathrm{K}_0(S))$. In particular, it is isomorphic to the map $r_A : \mathrm{T}(A) \to \mathrm{S}(\mathrm{K}_0(A))$.

In order to prove that the pairing of A is isomorphic to the pairing of B, we must show that the isomorphism between the K_0-groups and the isomorphisms between $\mathrm{S}(\mathrm{K}_0(A))$ and $\mathrm{S}(\mathrm{K}_0(B))$ are compatible; that is, if ψ denotes the isomorphism $\mathrm{K}_0(B) \to \mathrm{K}_0(A)$ and ϱ denote the isomorphism $\mathrm{S}(\mathrm{K}_0(A)) \to \mathrm{S}(\mathrm{K}_0(A))$, one has that

$$s\big(\psi(p)\big) = \varrho(s)(p) \quad \text{for any } p \in \mathrm{K}_0(B),\ s \in \mathrm{S}\big(\mathrm{K}_0(A)\big).$$

Denote by ϱ_A the isomorphism $\mathrm{S}_{u'}(\mathrm{K}_0(S)) \to \mathrm{S}(\mathrm{K}_0(A))$ and denote by ϱ_B the isomorphism $\mathrm{S}_{u'}(\mathrm{K}_0(S)) \to \mathrm{S}(\mathrm{K}_0(B))$. To prove the compatibility, it is sufficient to show that for any $s \in \mathrm{S}_{u'}(\mathrm{K}_0(C))$ and any projection p in B, the equality

$$\varrho_B(s)\big([p]\big) = \varrho_A(s)\big(\psi\big([p]\big)\big)$$

holds. By (iv) of Lemma 10.8 of [23], if the projection q stands for the K_0-element $\psi([p])$, one has that

$$\varrho_A(s)\big(\psi\big([p]\big)\big) = \lim_{n\to\infty} \tau_s\big(\phi_{n,\infty}\big(L_n^{(A)}(q)\big)\big)$$

and

$$\varrho_B(s)\big([p]\big) = \lim_{n\to\infty} \tau_s\big(\phi_{n,\infty}\big(L_n^{(B)}(p)\big)\big),$$

where τ_s is a trace on S which induces s, and $\{L_n^{(A)}\}$ (or $\{L_n^{(B)}\}$) are certain completely positive linear maps from A (or B) to S_n. Let $\varepsilon > 0$. By the construction of B, one may assume that $p \in B_n$ and $[p] = [p'] \oplus [p''] \in G_n' \oplus G_n''$. By the construction of the isomorphism ψ (induced by the positive homomorphisms $\{\theta_n\}$), one has that $\psi([p]) = \theta_n([p])$. Thus, there is a projection $q \in A$ such that $[q] = \psi([p])$ and $q = q' + q''$ where $q'' \in S_{n+1}$. Moreover, since the restriction of θ_n to G_n'' is induced by the inclusion map of S_n, the projection q can be chosen such that

$$\big|\tau\big(\phi_n(p'') - q''\big)\big| \leqslant \varepsilon$$

for any $\tau \in \mathrm{T}(A)$. By (iii) of Lemma 10.8 of [23], this implies

$$\left|\tau'\left(\phi_n(p'') - q''\right)\right| \leqslant \varepsilon$$

for any $\tau' \in \mathrm{T}_{u'}(S)$. Since $L_n^{(A)}(q) = q''$ and $L_n^{(B)}(p) = p''$, one has

$$\left|\tau_s\left(L_n^{(A)}(q) - L_n^{(B)}(p)\right)\right| \leqslant \varepsilon$$

when n is sufficiently large. Thus, the equality

$$\varrho_B(s)\left([p]\right) = \varrho_A(s)\left(\psi\left([p]\right)\right)$$

holds, and hence the pairing of the simplex of traces and the K_0-groups of A is isomorphic to that of B. Thus, we proved Theorem A.

Theorem A. *Let A be a simple separable C*-algebra in the class* $\mathrm{TA}\mathscr{S}$*. There exists a simple inductive limit C*-algebra B of C*-algebras in the class $\mathscr{S}'$ such that the Elliott invariant of A is isomorphic to the Elliott invariant of B.*

Remark 3.12. Denote by $\mathscr{S}$ the class of splitting interval algebras. Theorem A concludes that for any simple separable C*-algebra in the class $\mathrm{TA}\mathscr{S}$, the model algebra in the sense of [27] exists. Therefore, by the classification theorem of [25] or [27], the class of simple separable amenable C*-algebras in $\mathrm{TA}\mathscr{S}$ which satisfy the UCT can be classified by the Elliott invariant.

4. Certain properties preserved by tracial approximation

The question of the behaviour of C*-algebra properties under passage from a class $\mathscr{C}$ to the class $\mathrm{TA}\mathscr{C}$ is interesting and sometimes important. In fact, the property of having tracial states, the property of being of stable rank one, and the property that the strict order on projections is determined by traces were used in the proof of Theorem A and also in the proof of the classification theorem of [27].

In this section, we shall show that the following properties of C*-algebras in a class $\mathscr{C}$ are inherited by simple C*-algebras in the class $\mathrm{TA}\mathscr{C}$:

(1) being (stably) finite;
(2) having stable rank one;
(3) having at least one tracial state;
(4) the strict order on projections is determined by traces (if this property holds for every matrix algebra over the given C*-algebra, then it in particular implies that the K_0-group of the given C*-algebra is weakly unperforated);
(5) any state of the order-unit K_0-group comes from a tracial state of the algebra;
(6) if the restriction of a tracial state to the order-unit K_0-group is the average of two distinct states on the K_0-group, then it is the average of two distinct tracial states (in particular, the restriction map preserves extreme points);
(7) the canonical map from the unitary group modulo the connected component containing the identity to the K_1-group being injective.

Theorem 4.1. *Let $\mathscr{C}$ be a class of finite unital C*-algebras. Then any simple C*-algebra in the class* $\mathrm{TA}\mathscr{C}$ *is finite. Moreover, if C*-algebras in $\mathscr{C}$ are stably finite, then simple C*-algebras in the class* $\mathrm{TA}\mathscr{C}$ *are also stably finite.*

Proof. Let $A \in \mathrm{TA}\mathscr{C}$. For any finite subset $\mathcal{F}$ of the unit ball of A, there is a sub-C*-algebra $C_{\mathcal{F}}$ of A with unit $p_{\mathcal{F}}$ such that for any $a \in \mathcal{F}$

(1) $\|p_{\mathcal{F}}a - ap_{\mathcal{F}}\| \leqslant \frac{1}{|\mathcal{F}|}$, and
(2) there is an element $a_{\mathcal{F}} \in C_{\mathcal{F}}$ such that $\|a_{\mathcal{F}} - p_{\mathcal{F}}ap_{\mathcal{F}}\| \leqslant \frac{1}{|\mathcal{F}|}$,

where $|\mathcal{F}|$ denotes the cardinality of $\mathcal{F}$. Let $\Phi_{\mathcal{F}}$ be a unital map (not necessarily a linear map, just a set theoretical map) from A to $C_{\mathcal{F}}$ such that $\Phi_{\mathcal{F}}$ sends a to $a_{\mathcal{F}}$.

Note that the collection of all the finite subsets of the unit ball of A forms an upward directed family. Define the map (which is not a linear map a priori)

$$\Phi : A \to \prod_{\mathcal{F}} C_{\mathcal{F}} \Big/ \bigoplus_{\mathcal{F}} C_{\mathcal{F}} \quad \text{by } a \mapsto \prod_{\mathcal{F}} \Phi_{\mathcal{F}},$$

where $\mathcal{F}$ runs over all the finite subsets of A.

We show that the map Φ is in fact a $*$-homomorphism. Fix a, b in the unit ball of A. Let $\mathcal{F}$ be an element of an upward chain of finite subsets containing $\{a, b, a^*, b^*, a+b, ab\}$. Then, we have that

$$\big\|\Phi_{\mathcal{F}}(a) - p_{\mathcal{F}}ap_{\mathcal{F}}\big\| = \|a_{\mathcal{F}} - p_{\mathcal{F}}ap_{\mathcal{F}}\| \leqslant \frac{1}{|\mathcal{F}|}.$$

The same argument shows that

$$\big\|\Phi_{\mathcal{F}}(b) - p_{\mathcal{F}}bp_{\mathcal{F}}\big\| \leqslant \frac{1}{|\mathcal{F}|}$$

and

$$\big\|\Phi_{\mathcal{F}}(a+b) - p_{\mathcal{F}}(a+b)p_{\mathcal{F}}\big\| \leqslant \frac{1}{|\mathcal{F}|}.$$

Therefore, one has that

$$\big\|\Phi_{\mathcal{F}}(a+b) - \big(\Phi_{\mathcal{F}}(a) + \Phi_{\mathcal{F}}(b)\big)\big\| \leqslant 3\frac{1}{|\mathcal{F}|}.$$

A similar argument also shows that

$$\big\|\Phi_{\mathcal{F}}(a^*) - \big(\Phi_{\mathcal{F}}(a)\big)^*\big\| \leqslant 2\frac{1}{|\mathcal{F}|}$$

and

$$\big\|\Phi_{\mathcal{F}}(ab) - \Phi_{\mathcal{F}}(a)\Phi_{\mathcal{F}}(b)\big\| \leqslant 5\frac{1}{|\mathcal{F}|}.$$

Therefore, the families

$$\big(\Phi_{\mathcal{F}}(a+b)-\big(\Phi_{\mathcal{F}}(a)+\Phi_{\mathcal{F}}(b)\big)\big)_{\mathcal{F}},$$
$$\big(\Phi_{\mathcal{F}}(a^*)-\big(\Phi_{\mathcal{F}}(a)\big)^*\big)_{\mathcal{F}}$$

and

$$\big(\Phi_{\mathcal{F}}(ab)-\Phi_{\mathcal{F}}(a)\Phi_{\mathcal{F}}(b)\big)_{\mathcal{F}}$$

are in the ideal $\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$. Note that Φ is also unital, hence the map Φ is a $*$-homomorphism.

Since the C*-algebra A is simple, the map Φ is injective, and the C*-algebra A can be considered as a unital sub-C*-algebra of $\prod_{\mathcal{F}} C_{\mathcal{F}}/\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$. We shall show that the C*-algebra $\prod_{\mathcal{F}} C_{\mathcal{F}}/\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$ is finite if the C*-algebras $C_{\mathcal{F}}$ are finite, and hence the C*-algebra A is finite.

Let v be a partial isometry in $\prod_{\mathcal{F}} C_{\mathcal{F}}/\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$ such that

$$vv^*=1 \quad \text{and} \quad v^*v=p.$$

Note that any partial isometry in $\prod_{\mathcal{F}} C_{\mathcal{F}}/\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$ can be lifted to a partial isometry in $\prod_{\mathcal{F}} C_{\mathcal{F}}$. Pick a lifting and denote it by $(v_{\mathcal{F}})_{\mathcal{F}}$ where $v_{\mathcal{F}}$ is a partial isometry in $C_{\mathcal{F}}$. Therefore, we have that

$$\lim_{\mathcal{F}\to\infty}\big\|v_{\mathcal{F}}v_{\mathcal{F}}^*-1_{C_{\mathcal{F}}}\big\|=0.$$

In particular, $v_{\mathcal{F}}v_{\mathcal{F}}^*=1_{C_{\mathcal{F}}}$ if $\mathcal{F}$ is sufficiently large. Since $C_{\mathcal{F}}$ is finite, we conclude that $v_{\mathcal{F}}^*v_{\mathcal{F}}=1_{C_{\mathcal{F}}}$ if $\mathcal{F}$ is sufficiently large, and hence $p=v^*v=1$. Thus the C*-algebra $\prod_{\mathcal{F}} C_{\mathcal{F}}/\bigoplus_{\mathcal{F}} C_{\mathcal{F}}$ is finite, as desired.

If C*-algebras in $\mathscr{C}$ are stably finite, the argument above applies to matrix algebras of A, and shows that A is stably finite. □

Remark 4.2. In the proof of the theorem above, we only use conditions (1) and (2) of Definition 2.2. In other words, only a piece of A is required to be approximated by C*-algebras in $\mathscr{C}$ without assuming this piece to be large.

Recall that a unital C*-algebra is said to have *stable rank one* if the invertible elements are dense.

Theorem 4.3. *Let $\mathscr{C}$ be a class of unital C*-algebras with stable rank one. Then any simple C*-algebra in the class* TA$\mathscr{C}$ *has stable rank one.*

Proof. Let A be a C*-algebra in TA$\mathscr{C}$. Let us show that it is enough to assume that A has the property (SP). If A does not have the property (SP), then there is a positive element $a\in A$ such that the only projection in $\overline{aAa}$ is the zero projection. Since $A\in \mathrm{TA}\mathscr{C}$, by applying Definition 2.2 to a given finite subset $\mathcal{F}\subseteq A$, $\varepsilon>0$, and a, we conclude that $\mathcal{F}$ can be approximated by a C*-algebra in the class $\mathscr{C}$ (which has stable rank one) to within ε. In particular, this implies that the C*-algebra A has stable rank one. Hence it remains to consider the case that A has the property (SP).

We must show that the invertible elements are dense in A. Let a be an element of A which is not invertible. Let ε be any positive number. Note that any C*-algebra with stable rank one is stably finite. By Theorem 4.1, A is stably finite, and thus a is not one-sided invertible. By [30], there exists a zero-divisor a' such that $\|a - a'\| < \varepsilon/2$. Thus, in order to prove the proposition, it is enough to prove that a' can be approximated by invertible elements.

Since A has the property (SP), there is a projection e which is orthogonal to a'. Recall that stable rank one is preserved by unital hereditary sub-C*-algebras. Hence, we may assume that the class $\mathscr{C}$ is closed under passing to unital hereditary sub-C*-algebras, and therefore we may assume that the C*-algebra pAp belongs to the class $\mathrm{TA}\mathscr{C}$ for any projection $p \in A$. Since A has the property (SP), eAe also has the property (SP). Since A is simple, we have that $e = e_1 + e_2$ with $e_2 \preccurlyeq e_1$ by Lemma 4.9 below. Since a' is orthogonal to e_1, we have that $a' \in (1 - e_1)A(1 - e_1)$. Furthermore, since $(1 - e_1)A(1 - e_1)$ is also a C*-algebra in $\mathrm{TA}\mathscr{C}$, there is a projection $p \in (1 - e_1)A(1 - e_1)$, and a sub-C*-algebra $C \in \mathscr{C}$ with $1_C = p$ such that a' is close to $pa'p - (1 - e_1 - p)a'(1 - e_1 - p)$ up to $\varepsilon/4$, $pa'p$ is almost inside C up to $\varepsilon/4$, and $1 - e_1 - p \preccurlyeq e_2$. Since C has stable rank one, there exists an invertible element $b \in C$ which is close to $pa'p$ up to $\varepsilon/2$. In the unital hereditary sub-C*-algebra $(1 - p)A(1 - p)$, if we denote by μ the partial isometry with $\mu\mu^* = 1 - e_1 - p$ and $\mu^*\mu \leqslant e_1$, the element $(1 - e_1 - p)a(1 - e_1 - p) + (\varepsilon/2)\mu + (\varepsilon/2)\mu^* + (\varepsilon/2)(e_1 - \mu^*\mu)$, with the matrix form

$$\begin{pmatrix} (\varepsilon/2)(e_1 - \mu\mu^*) & 0 & 0 \\ 0 & 0 & (\varepsilon/2)\mu^* \\ 0 & (\varepsilon/2)\mu & (1 - e_1 - p)a(1 - e_1 - p) \end{pmatrix},$$

is invertible, and is close to $(1 - e_1 - p)a'(1 - e_1 - p)$ up to $\varepsilon/2$. Thus, the element a' can be approximated by the invertible elements of A. This shows that A has stable rank one. □

Corollary 4.4. *Any simple TAS-algebra has stable rank one.*

Proof. This follows from Theorem 4.3 and Corollary 2.5. □

Remark 4.5. The authors thank the referee for informing us that the same statement as that of Theorem 4.3 has appeared in Qingzhai Fan's PhD thesis, and a weaker version appeared in [15].

Let $\mathscr{T}$ be a class of unital C*-algebras which have tracial states.

Theorem 4.6. *Any C*-algebra in the class* $\mathrm{TA}\mathscr{T}$ *has at least one tracial state.*

Proof. Let A be a C*-algebra in $\mathrm{TA}\mathscr{T}$ and $\mathcal{F}$ be a finite subset of A. Then there is a sub-C*-algebra $B_{\mathcal{F}}$ (with unit p) of A such that for any $x \in \mathcal{F}$,

$$\bigl\|x - \bigl(pxp + (1 - p)x(1 - p)\bigr)\bigr\| < \frac{1}{|\mathcal{F}|},$$

and there is $b \in B_{\mathcal{F}}$

$$\|pxp - b\| < \frac{1}{|\mathcal{F}|},$$

where $|\mathcal{F}|$ is the cardinality of $\mathcal{F}$. Choose a tracial state τ_B of $B_{\mathcal{F}}$ and extend it to a state of pAp; still denoted by τ_B. Define a state $\tau_{\mathcal{F}}$ on A by

$$\tau_{\mathcal{F}}: a \mapsto \tau_B(pap).$$

The finite subsets of A form an upward directed collection with respect to inclusion. Therefore, all the states $\{\tau_{\mathcal{F}}\}$ on A form an upward directed family. Since the state space is compact, there is a state τ on A such that τ is an accumulation point of $\{\tau_{\mathcal{F}}\}$.

As expected, τ is a trace. To verify this, fix $a, b \in A$, and let us show that $\tau(ab) = \tau(ba)$. For any $\varepsilon > 0$, by definition, there exists a finite subset $\mathcal{F}$ of A containing $\{a, b, ab, ba\}$ and $|\mathcal{F}| > 1/\varepsilon$, such that

$$\left|\tau(ab) - \tau_{\mathcal{F}}(ab)\right| < \varepsilon \quad \text{and} \quad \left|\tau(ba) - \tau_{\mathcal{F}}(ba)\right| < \varepsilon.$$

(One can take $\mathcal{F}_1 = \{a, b, ab, ba\}$; then take an increasing sequence of finite subsets containing $\mathcal{F}_1$.) Then we have

$$\begin{aligned} \tau(ab) &=_\varepsilon \tau_{\mathcal{F}}(ab) = \tau_B(pabp) =_\varepsilon \tau_B(pappbp) \\ &=_\varepsilon \tau_B(a'b') \quad \text{where } a', b' \in B_{\mathcal{F}} \\ &= \tau_B(b'a') \\ &=_{2\varepsilon} \tau_{\mathcal{F}}(ba) \\ &=_\varepsilon \tau(ba), \end{aligned}$$

where $a =_\varepsilon b$ refers to $|a - b| < \varepsilon$. Thus we have

$$\left\|\tau(ab) - \tau(ba)\right\| < 6\varepsilon.$$

Since ε is arbitrary, $\tau(ab) = \tau(ba)$. □

An immediately consequence of the theorem stated above is

Corollary 4.7. *Any simple C*-algebra in the class* TA$\mathscr{T}$ *is stably finite.*

Remark 4.8. In the proof of Theorem 4.6 and corollary above, we only need the conditions (1) and (2) of Definition 2.2. In other words, one does not need the unknown piece in tracial approximation to be small.

Note that with A a C*-algebra in the class TA$\mathscr{C}$ for some class $\mathscr{C}$, any element $a \in A$ can be approximated by

$$pap + (1 - p)a(1 - p),$$

where pap is approximately inside a sub-C*-algebra of A which is in the class $\mathscr{C}$. We shall see in the following that if A is an infinite-dimensional simple C*-algebra with the property (SP), we can make the piece $(1 - p)a(1 - p)$ to be uniformly small with respect to tracial states. First, we have the following well-known lemma (for example, see Lemma 3.5.6(b) of [19]):

Lemma 4.9. *Let A be a simple C*-algebra with the property* (SP). *Then, for any finite set of projections $\{p_1, \dots, p_n\}$, there is a non-zero subprojection e of p_1 such that e is Murray–von Neumann equivalent to a subprojection of p_i for all $1 \leqslant i \leqslant n$.*

Combining the above lemma with the fact that any infinite-dimensional unital simple C*-algebra contains a positive element with infinite points in its spectrum, we have the following observation: for any unital simple infinite-dimensional C*-algebra A with the property (SP), and for any $n > 0$, there exist n mutually orthogonal projections $\{q_1, \dots, q_n\}$ in A which are Murray–von Neumann equivalent to each other. Applying this observation to any unital hereditary sub-C*-algebra of A, we have the following lemma.

Lemma 4.10. *Let A be a infinite-dimensional simple unital C*-algebra with the property* (SP). *Then, for any natural number N and any non-zero projection p, there is a non-zero projection $q \in A$ such that $N[q]_0 \leqslant [p]$. Hence, the projection q is less that $1/N$ with respect to any tracial state of A* (*if tracial states exist*).

In particular, this lemma holds for $p = 1$. Therefore, if $A \in \mathrm{TA}\mathscr{C}$ for some class $\mathscr{C}$, by applying tracial approximation to any finite subset $\mathcal{F} \subseteq A$, any $\varepsilon > 0$, and q, we get the following lemma.

Lemma 4.11. *Let $A \in \mathrm{TA}\mathscr{C}$ for some class $\mathscr{C}$ of C*-algebras. If A is simple, infinite-dimensional, and has the property* (SP)*, then, for any natural number N and any finite subset $\mathcal{F} \subseteq A$, any $\varepsilon > 0$ and any $a \in A^+$, there is $C \in \mathscr{C}$ with $I_C = p$ satisfying the approximation with respect to $\mathcal{F}$, ε and a, such that*

$$N[1 - p]_0 \leqslant [1]_0.$$

Recall that a unital C*-algebra A has *the Blackadar comparison property* if $\mathrm{T}(A)$ is non-empty, and for any projections $p, q \in A$, if $\tau(p) < \tau(q)$ holds for any $\tau \in \mathrm{T}(A)$, then p is Murray–von Neumann equivalent to a subprojection of q.

Theorem 4.12. *Let $\mathscr{C}$ be a class of unital C*-algebras with the Blackadar comparison property. Then any simple C*-algebra $A \in \mathrm{TA}\mathscr{C}$ has the Blackadar comparison property.*

Proof. By Theorem 4.6, the C*-algebra A has tracial state. Let p and q be two projections in A with $\tau(p) < \tau(q)$ for any $\tau \in \mathrm{T}(A)$. We shall show that $p \preccurlyeq q$.

Let us assume that A has the (SP) property. Since the simplex of tracial states is compact under the pointwise convergence topology, there exists $\delta > 0$ such that $\tau(p) < \tau(q) - \delta$ for any $\tau \in \mathrm{T}(A)$. Since A is simple and has the (SP) property, by Lemma 4.10, there is a subprojection q' of q such that $\tau(q') < \delta/2$ for any $\tau \in \mathrm{T}(A)$.

Applying tracial approximation (Definition 2.2) to $\mathcal{F} = \{p, q - q'\}$, $\varepsilon = 1/16$ and $a = q'$, we obtain a sub-C*-algebra $C \in \mathscr{C}$ with unit e such that for $f = p, q - q'$,

(1) $\|fe - ef\| < 1/16$,
(2) $efe \in_{1/16} C$, and
(3) $1 - e$ is Murray–von Neumann equivalent to a subprojection of q'.

Therefore, we get

$$p \sim p_1 + p_2 \quad \text{and} \quad q - q' \sim q_1 + q_2,$$

where p_1, q_1 are subprojections of $1 - e$ and p_2, q_2 are projections in S.

We assert that we can choose such a sub-C*-algebra C such that $\tau(p_2) < \tau(q_2)$ for any tracial state τ of C. Indeed, if this were not true, then for any sub-C*-algebra C stated above, there would exist a trace τ on C such that $\tau(p_2) \geqslant \tau(q_2)$. This trace could be extended to a positive linear functional τ with norm 1 on eAe. Let us still denote it by τ. Then the map $\phi: a \mapsto \tau(eae)$ would be a positive linear functional on A with norm 1. Note that $\phi(p) \geqslant \phi(q)$. As the argument of Theorem 4.6, we could apply this construction with any finite subset of A, to obtain an upward directed family of states (ϕ_λ) inside the unit ball of the dual space of A. Choose an accumulation point τ_0, and it is easy to verify that τ_0 is a trace of A. But then we have that

$$\tau_0(p) = \lim \phi_\lambda\big(p_2^{(\lambda)}\big) \geqslant \lim \phi_\lambda\big(q_2^{(\lambda)}\big) = \tau_0(q),$$

which is in contradiction with the assumption on p and q.

Therefore, we may assume that $\tau(p_2) < \tau(q_2)$ for every trace τ of S. Then p_2 is Murray–von Neumann equivalent to a subprojection of q_2. Note that p_1 is a subprojection of $1 - e$ and $1 - e$ is Murray–von Neumann equivalent to a subprojection of q', so that p_1 is Murray–von Neumann equivalent to a subprojection of q'. Therefore we have

$$p \sim p_1 + p_2 \precsim q' + q_2 \leqslant q' + q - q' = q.$$

If A does not have the (SP) property, there is a positive element $a \in A$ such that the only projection in $\overline{aAa}$ is 0. Apply Definition 2.2 to any finite subset $\mathcal{F}$, any ε, and a, we conclude that there is a unital sub-C*-algebra C of A such that $\mathcal{F} \subseteq_\varepsilon C$. With an argument same as that used above, we conclude that the unital sub-C*-algebra C can be chosen such that there are projections p' and q' in C with $\|p - p'\| < 1/2$, $\|q - q'\| < 1/2$, and $\tau(p') < \tau(q')$ for any $\tau \in \mathrm{T}(C)$. Since C has the Blackadar comparison property, one has that

$$p \sim p' \precsim q' \sim q,$$

as desired. □

Remark 4.13. The proof above is similar to the proof of Theorem 3.7.2 of [19], which states that any simple TAI-algebra has the Blackadar comparison property.

Corollary 4.14. *Let A be a simple TAS-algebra. The ordered group $\mathrm{K}_0(A)$ is weakly unperforated*; *that is, if $g \in \mathrm{K}_0(A)$ and $ng \in \mathrm{K}_0(A)^+ \setminus \{0\}$ for any $n \in \mathbb{Z}^+$, then $g \in \mathrm{K}_0(A)^+$.*

Proof. We must show that for any two projections p and q in a matrix algebra $\mathrm{M}_k(A)$, if $\mathrm{diag}\{\underbrace{p, \dots, p}_{n}\}$ is Murray–von Neumann equivalent to a proper subprojection of $\mathrm{diag}\{\underbrace{q, \dots, q}_{n}\}$ in $\mathrm{M}_{nk}(A)$ for some $n \in \mathbb{Z}^+$, then p is Murray–von Neumann equivalent to a subprojection of q.

Since the projection $\operatorname{diag}\{\underbrace{p, \ldots, p}_{n}\}$ is Murray–von Neumann equivalent to a proper subprojection of $\operatorname{diag}\{\underbrace{q, \ldots, q}_{n}\}$, one has that $n\tau(p) < n\tau(q)$ for any $\tau \in \mathrm{T}(A)$, and hence $\tau(p) < \tau(q)$ for any τ of $\mathrm{T}(A)$. Since $\mathrm{M}_k(A)$ is also a simple TAS-algebra, by Theorem 4.12, the projection p is Murray–von Neumann equivalent to a subprojection of q, and hence the ordered group $\mathrm{K}_0(A)$ is weakly unperforated. □

Let $\mathscr{E}$ be a class of unital stably finite unital C*-algebras, such that any state on the order-unit K_0-group comes from a tracial state. It is well known that any stably finite unital exact C*-algebra has this property. We shall show that this property still holds for C*-algebras in $\mathrm{TA}\mathscr{E}$.

Theorem 4.15. *The map $r : T(A) \to S(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1])$ is always a surjective map for any simple C*-algebra $A \in \mathrm{TA}\mathscr{E}$, where r is induced by the canonical restriction of tracial states to projections in matrix algebras of A.*

Proof. Let ρ be a positive state over $(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1])$, and let $\mathcal{F}$ be a finite subset of A with $|\mathcal{F}| > 2$ where $|\mathcal{F}|$ is the cardinality of $\mathcal{F}$.

Let us assume that A has the property (SP). Since A is a simple C*-algebra in $\mathrm{TA}\mathscr{E}$, there is a sub-C*-algebra E of A with $p = 1_E$ such that $E \in \mathscr{E}$, and for any $a \in \mathcal{F}$:

(1) $\|pa - ap\| \leqslant 1/|\mathcal{F}|$,
(2) $pap \in_{1/|\mathcal{F}|} E$, and
(3) $M[1-p] \preccurlyeq [1]$ for a natural number $M \geqslant |\mathcal{F}|$.

Then the map $\phi : a \mapsto pap$ is $\mathcal{F} - \frac{2}{|\mathcal{F}|}$ multiplicative, and $\rho([1-p]) \leqslant \frac{1}{M} \leqslant \frac{1}{|\mathcal{F}|}$.

Define the map $\rho_1 : \mathrm{K}_0(S) \to \mathbb{R}$ to be $\rho_1([q]) = \frac{1}{\rho([p])}\rho([q])$ for any projection q in a matrix algebra of E (since E is a sub-C*-algebra of A). It is clear that ρ_1 is a positive state of $\mathrm{K}_0(E)$. Since E is in the class $\mathscr{E}$, ρ_1 arises from a tracial state of S, denote it by τ_1'. One can extend τ_1' to a state on pAp; we still denote it by τ_1'. Then define a positive linear contraction τ_1 on A by $\tau_1 = \tau_1' \circ \phi$. It is clear that

$$\left|\tau_1(ab) - \tau_1(ba)\right| \leqslant 4/|\mathcal{F}| \quad \text{for any } a, b \in \mathcal{F}.$$

Moreover, for any projection $q \in \mathcal{F}$, the projection q will be unitarily equivalent to a sum of two orthogonal projections q' and q'' where $q' \leqslant 1-p$, $q'' \in E$, and $\|q'' - \phi(q)\| \leqslant 1/|\mathcal{F}|$. Then for any projection $q \in \mathcal{F}$, we have that

$$\begin{aligned}
\left|\rho\big([q]\big) - \tau_1(q)\right| &= \left|\rho\big([q'] + [q'']\big) - \tau_1(q)\right| \\
&\leqslant \frac{1}{|\mathcal{F}|} + \left|\rho\big([q'']\big) - \tau_1' \circ \phi(q)\right| \\
&\leqslant \frac{2}{|\mathcal{F}|} + \left|\rho\big([q'']\big) - \tau_1'(q'')\right|
\end{aligned}$$

$$\leqslant \frac{2}{|\mathcal{F}|} + \left|\rho\big([q'']\big) - \rho_1\big([q'']\big)\right|$$
$$\leqslant \frac{4}{|\mathcal{F}|}.$$

Repeating this construction for all finite subsets $(\mathcal{F}_\lambda)$, we obtain an upward directed family of positive linear contractions (τ_λ). Pick an accumulation point τ of (τ_λ) in the unit ball of the dual space of A (with respect to the weak topology). It is a state of A. Moreover, for any a and b in the unit ball of A, we have that

$$\tau(ab) - \tau(ba) = \lim_{\lambda\to\infty}\big(\tau_\lambda(ab) - \tau_\lambda(ba)\big) = 0.$$

Therefore, τ is a tracial state of A.

For any projection $q \in A$, there is a finite subset $\mathcal{F}$ containing q with sufficiently large cardinality. Therefore, we have

$$\tau(q) = \lim_{\lambda\to\infty} \tau_\lambda(q) = \rho\big([q]_0\big).$$

Set T_1 to be the set of all the traces on A which have the property $\tau(f) = \rho([f]_0)$ for any projection f in A. T_1 is a compact set with respect to the weak topology. By the argument above, T_1 is not empty. Since the n by n matrix algebra over a C*-algebra in $\mathrm{TA}\mathscr{E}$ still belongs to the class $\mathrm{TA}\mathscr{E}$, we can apply the construction above to $\mathrm{M}_n(A)$, and get a nonempty compact set of traces on A such that $\tau(f) = \rho([f]_0)$ for any projection f in $\mathrm{M}_n(A)$. Denote by T_n all such tracial states of $\mathrm{M}_n(A)$. Since $T_n \supset T_{n+1}$ and all T_n are compact, there exists a tracial state τ in all of T_n. Therefore the state on $(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1])$ induced by τ is exactly ρ. And the map $r: \mathrm{T}(A) \to \mathrm{S}(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1])$ is surjective.

If A does not have the property (SP)—say, there is no nontrivial projection in the hereditary sub-C*-algebra generated by $a \in A$—one can apply the tracial approximation property (Definition 2.2) to any finite subset, any ε, and a. One then concludes that A can be locally approximated by C*-algebras in the class $\mathscr{E}$. The same arguments as above (even more directly) show that the map $r: \mathrm{T}(A) \to \mathrm{S}(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1])$ is surjective. □

Unital stably finite exact C*-algebras are in the class $\mathscr{E}$. However, not all simple C*-algebras in the class $\mathrm{TA}\mathscr{E}$ are exact. There even exist non-exact separable simple C*-algebras in the class $\mathrm{TA}\mathscr{F}$. Here is an example constructed by M. Dadarlat in [5]:

Example 4.16. Let B be a non-exact unital separable C*-algebra which has a separating sequence of finite-dimensional representations $\{\pi_n\}$. (For instance, we can take B to be the full C*-algebra of the group $\mathbb{F}_2$, the free group with two generators. See [4,35].) Denote by d_n the dimension of the representation π_n.

We are going to construct A as an inductive limit. Set $A_1 = B \oplus \mathrm{M}_{d_1}(\mathbb{C})$. Assume that we already have $A_n = \mathrm{M}_{i_n}(B) \oplus \mathrm{M}_{j_n}(\mathbb{C})$, we set $i_{n+1} = i_n + i_n j_n$ and $j_{n+1} = d_n i_n + j_n$. We then set

$$A_{n+1} = \mathrm{M}_{i_{n+1}}(B) \oplus \mathrm{M}_{j_{n+1}}(\mathbb{C}),$$

and set a map $\varphi_n : A_n \to A_{n+1}$ by

$$A_n \ni (b, m) \mapsto \big(\mathrm{diag}\{b, \underbrace{m, \ldots, m}_{i_n}\}, \mathrm{diag}\big\{\pi_n(b), m\big\}\big) \in A_{n+1}.$$

Denote by A the inductive limit C*-algebra of the system (A_n, φ_n). We claim that A is a simple C*-algebra in the class $\mathrm{TA}\mathscr{F}$. But A contains B, a non-exact C*-algebra, as a sub-C*-algebra; thus A cannot be an exact C*-algebra.

The simplicity of A follows directly from the separability of $\{\pi_n\}$ and the construction of the maps $\{\varphi_n\}$. We wish to show that A satisfies Definition 2.2 for the class of C*-algebras $\mathscr{F}$. For any finite subset $\mathcal{F} \subseteq A$ and $\varepsilon > 0$, we may assume $\mathcal{F} \subseteq A_n$ for some n. Thus, A satisfies conditions (1) and (2) of Definition 2.2; by Proposition 2.7, the C*-algebra A has the property (SP). Therefore, in order to verify the condition (3) of Definition 2.2, it is enough to verify it for any unital hereditary sub-C*-algebra of A. In other words, to see A belongs to the class $\mathrm{TA}\mathscr{F}$, it is enough to verify Definition 2.2 for any finite subset $\mathcal{F} \subseteq A$, any $\varepsilon > 0$, and any projection q. We then may assume $\mathcal{F}$, and q are in a building block of A, say A_n. Since $A_n = \mathrm{M}_{i_n}(B) \oplus \mathrm{M}_{j_n}(\mathbb{C})$, we even may assume that $q \in \mathrm{M}_{j_n}(\mathbb{C})$.

Denote by p' the unit of $\mathrm{M}_{i_n}(B)$ in A_n. Passing to $A_{n+1} = \mathrm{M}_{i_{n+1}}(B) \oplus \mathrm{M}_{j_{n+1}}(\mathbb{C})$, the image of p' in $\mathrm{M}_{i_{n+1}}(B)$ by the map φ_n has the form of $(1, 0, \ldots, 0)$. By the construction of φ_n, restricting to the sub-C*-algebra $\mathrm{M}_{i_{n+1}}(B)$, the projection $(1, 0, \ldots, 0)$ is Murray–von Neumann equivalent to a subprojection of any image of minimal projections of $\mathrm{M}_{j_n}(\mathbb{C})$ by the map φ_n; hence it is Murray–von Neumann equivalent to a subprojection of q. Therefore, we may take the sub-C*-algebra F to be the direct sum of the sub-C*-algebra $\mathrm{M}_{j_{n+1}}(\mathbb{C})$ and the image of $\mathrm{M}_{j_n}(\mathbb{C})$ in $\mathrm{M}_{i_{n+1}}(B)$. It is easy to see that this sub-C*-algebra satisfies Definition 2.2 for the finite subset $\mathcal{F}$ and projection q. Thus, A is a simple separable C*-algebra in the class $\mathrm{TA}\mathscr{F}$ which is not an exact C*-algebra.

Let A be unital stably finite C*-algebra. Denote by r the canonical affine map

$$\mathrm{T}(A) \to \mathrm{S}\big(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]_0\big)$$

induced by restricting traces to projections in matrix algebras.

Definition 4.17. A unital stably finite C*-algebra A is said to have the property (M) if for any tracial state τ of A satisfying

$$r(\tau) = \lambda_1\rho_1 + \lambda_2\rho_2$$

for two states ρ_1, ρ_2 on $\mathrm{K}_0(A)$ where $\lambda_1, \lambda_2 \geqslant 0$ with $\lambda_1 + \lambda_2 = 1$, there exist two tracial states τ_1 and τ_2 of A such that $r(\tau_1) = \rho_1$ and $r(\tau_2) = \rho_2$ respectively, and

$$\tau = \lambda_1\tau_1 + \lambda_2\tau_2.$$

Remark 4.18. Let A be a C*-algebra with the property (M). Then the map r is surjective (indeed, one can set one of the λ_i's to be zero). If a tracial state τ is sent to a midpoint of $\mathrm{S}(\mathrm{K}_0(A))$ by the map r, then τ must be a midpoint of $\mathrm{T}(A)$. In particular, the map r preserves extreme points.

Theorem 4.19. *Let $\mathscr{M}$ be a class of unital stably finite C*-algebras with the property* (M). *Then any simple C*-algebras in the class* TA$\mathscr{M}$ *has the property* (M).

Proof. Let A be a C*-algebra in TA$\mathscr{M}$. Let τ be a tracial state of A satisfying

$$r(\tau) = \lambda_1\rho_1 + \lambda_2\rho_2$$

for some states ρ_1, ρ_2 on $\mathrm{K}_0(A)$ where $\lambda_1, \lambda_2 \geqslant 0$ with $\lambda_1 + \lambda_2 = 1$.

Suppose that A has the property (SP). For any subset $\mathcal{F}$ of the unit ball of A, since $A \in \mathrm{TA}\mathscr{M}$, there is a sub-C*-algebra M with unit p such that p commutes with the elements of $\mathcal{F}$ to within $1/|\mathcal{F}|$, and pap belongs to M up to $1/|\mathcal{F}|$ for any $a \in \mathcal{F}$, where $|\mathcal{F}|$ is the cardinality of $\mathcal{F}$. Therefore, if we define the map $\phi : A \to pAp$ by $a \mapsto pap$, ϕ is unital and $\mathcal{F} - \frac{1}{|\mathcal{F}|}$ multiplicative, and the map ϕ sends $\mathcal{F}$ to M to within $1/|\mathcal{F}|$. Moreover, we may assume that $\tau'(1-p) < 1/|\mathcal{F}|$ for any tracial state τ' of A.

Consider the restriction of τ to M and the restrictions of ρ_1 and ρ_2 to $\mathrm{K}_0(M)$, one then has

$$r_M(\tau|_M) = \lambda_1\rho_1|_{\mathrm{K}_0(M)} + \lambda_2\rho_2|_{\mathrm{K}_0(M)}.$$

Since M has the property (M), there exist tracial states τ_1' and τ_2' of M such that $r_M(\tau_1') = \rho_1|_{\mathrm{K}_0(M)}$, $r_M(\tau_2') = \rho_2|_{\mathrm{K}_0(M)}$ and

$$\tau|_M(b) = \lambda_1\tau_1'(b) + \lambda_2\tau_2'(b) \quad \text{for any } b \in M.$$

We then extend these two tracial states to states on pAp; still denote them by τ_1' and τ_2' respectively.

For any $a \in \mathcal{F}$, we have

$$\begin{aligned} &\bigl|\tau(a) - \bigl(\lambda_1\tau_1'\bigl(\phi(a)\bigr) + \lambda_2\tau_2'\bigl(\phi(a)\bigr)\bigr)\bigr| \\ &\quad\leqslant \bigl|\tau\bigl((1-p)a(1-p) + pap\bigr) - \bigl(\lambda_1\tau_1'\bigl(\phi(a)\bigr) + \lambda_2\tau_2'\bigl(\phi(a)\bigr)\bigr)\bigr| + 1/|\mathcal{F}| \\ &\quad\leqslant 2/|\mathcal{F}| + \bigl|\tau\bigl(\phi(a)\bigr) - \bigl(\lambda_1\tau_1'\bigl(\phi(a)\bigr) + \lambda_2\tau_2'\bigl(\phi(a)\bigr)\bigr)\bigr| \\ &\quad\leqslant 5/|\mathcal{F}|. \end{aligned}$$

Denote by $\tau_i^{\mathcal{F}}$ the state $\tau_i' \circ \phi$ of A for $i = 1, 2$.

Thus, for any finite subset of the unit ball of A, on can construct two states $\tau_1^{\mathcal{F}}$ and $\tau_2^{\mathcal{F}}$ of A such that for any $a \in \mathcal{F}$,

$$\bigl|\tau(a) - \bigl(\lambda_1\tau_1^{\mathcal{F}}(a) - \lambda_2\tau_2^{\mathcal{F}}(a)\bigr)\bigr| \leqslant 1/|\mathcal{F}|,$$

and for any projection $q \in \mathcal{F}$,

$$\bigl|\tau_1(q) - \rho_1\bigl([q]\bigr)\bigr| \leqslant 1/|\mathcal{F}| \quad \text{and} \quad \bigl|\tau_2(q) - \rho_2\bigl([q]\bigr)\bigr| \leqslant 1/|\mathcal{F}|.$$

Moreover, for any $a, b \in \mathcal{F}$, one has that $|\tau_i^{\mathcal{F}}(ab) - \tau_i^{\mathcal{F}}(ba)| < 4/|\mathcal{F}|$ for $i = 1, 2$.

Since the finite subsets of the unital ball of A form an upward directed collection, there are accumulate points τ_1^{∞} and τ_2^{∞} for the upward directed families $\{\tau_1^{\mathcal{F}}\}$ and $\{\tau_2^{\mathcal{F}}\}$ in the unital ball of the dual space of A. A simple argument shows that τ_1^∞ and τ_2^∞ are tracial states of A, and moreover,

$$\tau = \lambda_1 \tau_1^\infty + \lambda_2 \tau_2^\infty,$$

and

$$\tau_1^\infty(q) = \rho_1([q]) \quad \text{and} \quad \tau_2^\infty(q) = \rho_2([q])$$

for any projection q in A. Using the same trick as that of the argument of Theorem 4.15, we can find two traces τ_1 and τ_2 such that $\tau_1(q) = \rho_1([q])$ and $\tau_2(q) = \rho_2([q])$ for any projection q in matrix algebras of A, and $\tau = \lambda_1\tau_1 + \lambda_2\tau_2$. Therefore, we have

$$r(\tau_1) = \rho_1, \qquad r(\tau_2) = \rho_2$$

and

$$\tau = \lambda_1 \tau_1 + \lambda_2 \tau_2,$$

as desired.

If A does not have the property (SP), one has that A can be locally approximated by C*-algebras in the class $\mathscr{M}$ by the argument of Theorem 4.15. Using an argument similar with above (even more directly), one can show that A has the property (M). □

Note that all unital homogeneous C*-algebras have the property (M). Therefore, we have the following corollary.

Corollary 4.20. *Let A be a separable simple C*-algebras in the class* TA$\mathscr{H}$*, where $\mathscr{H}$ is the class of unital homogeneous C*-algebras. Then the C*-algebra A has the property* (M). *In particular, the map r preserves extreme points.*

Remark 4.21. A direct consequence of the above corollary is that any separable simple TAI-algebra has this property. This is important for TAI classification. (See [23,26]. Note that this fact is included in [23], as Lemma 10.9, but was erroneously just assumed in [26].) (The present proof is different from that of Lemma 10.9 of [23].)

Let A be a unital C*-algebra. Denote by $\mathrm{U}(A)$ and $\mathrm{U}_0(A)$ the unitary group of A and the path connected component of 1_A, respectively. Then the group $\mathrm{U}_0(A)$ is a normal subgroup of $\mathrm{U}(A)$, and there is a canonical map π from $\mathrm{U}(A)/\mathrm{U}_0(A)$ to $\mathrm{K}_1(A)$.

Let $\mathscr{C}$ be a class of unital C*-algebras such that the map π is injective for any member of this class. We shall show that for any simple C*-algebra $A \in \mathrm{TA}\mathscr{C}$ with the cancellation property for projections, the map π is still injective.

It is well known that if two unitaries u, v satisfy $\|u - v\| < 2$, then u is path connected to v. There is a little improvement of this fact.

Lemma 4.22. *Let u, v be two unitaries in a C*-algebra A. If $\|u - v\| < 2$ and there is a projection $p \in A$ with $[p, u] = [p, v] = 0$, then there exists a path $w(t)$ in* $\mathrm{U}(A)$ *such that $w(0) = v$,*

$w(1) = u$ *and* $[p, w(t)] = 0$, $\forall t \in [0,1]$. *Moreover, if* pup *and* pvp *are in a unital sub-C*-algebra* B *of* pAp, *then the path* $w(t)$ *can be chosen such that* $pw(t)p$ *is a path of unitaries in the C*-algebra* B.

Proof. Since $\|u - v\| < 2$, the point -1 is not in the spectrum of v^*u. Therefore, there is a real-valued continuous function ψ on $\mathrm{sp}(v^*u)$ such that $z = \exp(i\psi(z))$. Set $h = \psi(v^*u) \in A^{s.a}$, and one has $v^*u = \exp(ih)$. Set $w'(t) = \exp(ith)$. It is a path in the unitary group of A such that $w'(0) = 1_A$ and $w'(1) = v^*u$. Moreover, since $[p, u] = 0$ and $[p, v] = 0$, one has $[p, h] = 0$. Hence, $[p, w'(t)] = 0$. Set $w(t) = vw'(t)$. It is clear that $w(0) = v$, $w(1) = u$ and $[p, w(t)] = 0$ for all $t \in [0,1]$.

If pup and pvp are in a unital sub-C*-algebra B of pAp, then one has that

$$php = \psi\big((pv^*p)(pup)\big) \in B,$$

and hence for any $t \in [0,1]$,

$$pw(t)p = pvw'(t)p = (pvp)\big(\exp\big(it(php)\big)\big) \in B,$$

as desired. □

Theorem 4.23. *Let* A *be a simple C*-algebra in* $\mathrm{TA}\mathscr{C}$ *with the cancellation property for projections. Let* $\mathrm{U}(A)$ *and* $\mathrm{U}_0(A)$ *denote the unitary group of* A *and the path connected component containing the identity* 1, *respectively. Then the canonical map* $\mathrm{U}(A)/\mathrm{U}_0(A) \to \mathrm{K}_1(A)$ *is injective.*

Proof. To show the injectivity of the map $\mathrm{U}(A)/\mathrm{U}_0(A) \to \mathrm{K}_1(A)$, it is enough to show for any unitary $u \in A$, if

$$\mathrm{diag}\{\underbrace{u, 1, \ldots, 1}_{n+1}\}$$

is path connected to

$$\mathrm{diag}\{\underbrace{1, 1, \ldots, 1}_{n+1}\}$$

in the unitary group of $\mathrm{M}_{n+1}(A)$ for some n, then u is path connected to 1 in $\mathrm{U}(A)$.

Suppose that A has the property (SP). Let $\mathrm{diag}\{u, 1, \ldots, 1\}$ be a unitary in $\mathrm{M}_{n+1}(A)$ which is path connected to $\mathrm{diag}\{1, 1, \ldots, 1\}$ by a path $W(t)$. Then there is a partition $0 = t_1 < t_2 < \cdots < t_s = 1$ such that

$$\big\|W(t_k) - W(t_{k+1})\big\| < 2, \quad 0 \leqslant k \leqslant s-1.$$

Since A is simple and has the property (SP), there are $n+1$ mutually orthogonal projections $\{q_1, \ldots, q_{n+1}\}$ in A which is Murray–von Neumann equivalent to each other. Since $A \in \mathrm{TA}\mathscr{C}$, one may assume that there is a sub-C*-algebra $C \in \mathrm{TA}\mathscr{C}$ with $0 \neq p \in 1_C$ such that

$$W(t_k)_{i,j} = W'(t_k)_{i,j} + W''(t_k)_{i,j}$$

with $W'(t_k)_{i,j} \in (1-p)A(1-p)$, $W''(t_k)_{i,j} \in C$ for each $0 \leqslant k \leqslant s$ and $1 \leqslant i, j \leqslant n+1$, and $(1-p) \preccurlyeq q_1$. Moreover, one can assume $(W'(t_k)_{i,j})$ is a unitary in $\mathrm{M}_{n+1}((1-p)A(1-p))$ and $(W''(t_k)_{i,j})$ is a unitary in $\mathrm{M}_{n+1}(C)$. By Lemma 4.22, there is a path of unitaries $w_k(t)$ between $W(t_k)$ and $W(t_{k+1})$ for each $0 \leqslant k \leqslant s-1$ and w_k commutes with $Q = \mathrm{diag}\{1-p, 1-p, \dots, 1-p\}$. Therefore, $Qw_k(t)Q$ is a path of unitaries in $\mathrm{M}_{n+1}((1-p)A(1-p))$ which connects $(W'(t_k)_{i,j})$ and $(W'(t_{k+1})_{i,j})$. Moreover, $(1-Q)w_k(t)(1-Q)$ is a path of unitaries in $\mathrm{M}_{n+1}(C)$ which connects $(W''(t_k)_{i,j})$ and $(W''(t_{k+1})_{i,j})$. Then, there is a path of unitaries $Qw(t)Q$ in $\mathrm{M}_{n+1}((1-p)A(1-p))$ which connects $(W'(t_0)_{i,j})$ and $(W'(t_s)_{i,j}) = Q$, and there is a path of unitaries $(1-Q)w(t)(1-Q)$ in $\mathrm{M}_{n+1}(C)$ which connects the unitary

$$\left(W''(t_0)_{i,j}\right) = \mathrm{diag}\left\{W''(t_0)_{1,1}, p, \dots, p\right\}$$

and the unitary

$$\left(W''(t_s)_{i,j}\right) = 1 - Q = \mathrm{diag}\{p, \dots, p\}.$$

Therefore, the unitary $W''(t_0)_{1,1}$ has trivial K_1 class. Since the canonical map

$$\pi : \mathrm{U}(C)/\mathrm{U}_0(C) \to \mathrm{K}_1(C)$$

is injective, one has that $W''(t_0)_{1,1}$ is path connected to $p = 1_C$ in the unitary group of C.

Note that $(1-p) \preccurlyeq q_1$ and the mutually orthogonal projections $q_1, \dots, q_{n+1}$ are Murray–von Neumann equivalent to each other. Since A has the cancellation property for projections, there are partial isometries $\{v_1 = 1-p, v_2, \dots, v_{n+1}\}$ such that the source projections are $1-p$ and the range projections are mutually orthogonal. Set

$$V = \{v_1, v_2, \dots, v_{n+1}\} \in \mathrm{M}_{1,n+1}(A).$$

It is easy to verify that $V^*V = Q$. Then

$$c(t) = VQw(t)QV^* + (1 - VV^*)$$

is a path of unitaries in A. One has

$$\begin{aligned} c(0) &= V\,\mathrm{diag}\left\{(1-p)u(1-p), 1-p, \dots, 1-p\right\}V^* + (1 - VV^*) \\ &= (1-p)u(1-p) + p \\ &= W'(t_0)_{1,1} + p \end{aligned}$$

and

$$\begin{aligned} c(1) &= V\,\mathrm{diag}\{1-p, 1-p, \dots, 1-p\}V^* + (1 - VV^*) \\ &= (1-p) + p \\ &= 1. \end{aligned}$$

Therefore $W'(t_0)_{1,1} + p$ is path connected to 1 in the unitary group of A.

Note that

$$u = W(t_0)_{1,1} = W'(t_0)_{1,1} + W''(t_0)_{1,1}.$$

Since $W''(t_0)_{1,1}$ is path connected to $p = 1_C$ in the unitary group of C, one has that u is path connected to $c(0) = W'(t_0)_{1,1} + p$ in the unitary group of A, and hence u is path connected to $c(1) = 1$ as desired.

If A does not have the property (SP), A can be locally approximated by C*-algebras in the class $\mathscr{C}$. Therefore, using an argument same as above, we may assume that the unitary

$$\mathrm{diag}\{\underbrace{u, 1, \ldots, 1}_{n+1}\}$$

is path connected to

$$\mathrm{diag}\{\underbrace{1, 1, \ldots, 1}_{n+1}\}$$

in a unitary group of the matrix algebra $\mathrm{M}_{n+1}(C)$ for a unital sub-C*-algebra C of A which is in the class $\mathscr{C}$. Hence, the unitary u is path connected to the identity in the unitary group of C, and in particular, u is path connected to the identity in the unitary group of A. $\square$

Remark 4.24. If A is a simple TAS-algebra, then A has stable rank one by Corollary 4.4, and therefore, the natural map $\mathrm{U}(A)/\mathrm{U}_0(A) \to \mathrm{K}_1(A)$ is an isomorphism.

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