Available online at www.sciencedirect.com

ScienceDirect

Journal of Pure and Applied Algebra 212 (2008) 401–410

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Separated Lie models and the homotopy Lie algebra

Peter Bubenik*

Department of Mathematics, Cleveland State University, 2121 Euclid Ave. RT 1515, Cleveland OH, 44115–2214, USA

Received 21 June 2004; received in revised form 23 March 2007; accepted 15 May 2007
Available online 27 June 2007

Communicated by F. Morel

Abstract

A simply connected topological space X has homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$. Following Quillen, there is a connected differential graded free Lie algebra (dgL) called a Lie model, which determines the rational homotopy type of X, and whose homology is isomorphic to the homotopy Lie algebra. We show that such a Lie model can be replaced with one that has a special property that we call being *separated*. The homology of a separated dgL has a particular form which lends itself to calculations.

MSC: Primary: 55P62; secondary: 17B55

1. Introduction

All of our topological spaces are assumed to be simply connected, and have finite rational homology in each dimension. For such a space X, there exists a differential graded Lie algebra $(\mathbb{L}V, d)$, where $\mathbb{L}V$ denotes the free graded Lie algebra on the rational vector space V, called a *Lie model*, which determines the rational homotopy type of X, and whose homology is isomorphic to $\pi_*(\Omega X) \otimes \mathbb{Q}$, the *(rational) homotopy Lie algebra* of X. While this description is pleasing in principle, it is less satisfying when explicit calculations are desired. Here, we want to put a certain structure on a Lie model that will prove conducive to calculation.

Assume that our Lie algebras are finite in each dimension and concentrated in positive dimension. Our assumption on our spaces implies that their Lie models satisfy this assumption. We are particularly interested in dgLs where V can be bigraded as follows: $V_* = \oplus_{i=1}^{N} V_{i,*}$, such that

$$dV_{i,*} \subset \mathbb{L}\left(\bigoplus_{j=1}^{i-1} V_{j,*}\right). \tag{1.1}$$

We will call the first gradation *degree* and the second, usual gradation *dimension*.

* Tel.: +1 216 687 4684; fax: +1 216 523 7340.
E-mail address: p.bubenik@csuohio.edu.
URL: http://academic.csuohio.edu/bubenik_p/.

doi:10.1016/j.jpaa.2007.05.018

Our *separated* condition will say that the attaching map from degree $i+1$ has no effect on the homology coming from degree $\leq i-1$. We need to introduce some notation to make this precise.

Given a Lie algebra $L = (\mathbb{L}V, d)$ as above, let

$$L_i = (\mathbb{L}V_{\leq i,*}, d). \tag{1.2}$$

Let ZL_i and BL_i denote the cycles and boundaries in L_i. There is an induced map

$$\tilde{d}_i : V_{i,*} \xrightarrow{d} Z(L_{i-1}) \twoheadrightarrow HL_{i-1}. \tag{1.3}$$

The inclusion $L_i \hookrightarrow L_{i+1}$ induces inclusions $BL_i \hookrightarrow BL_{i+1}$ and $ZL_i \hookrightarrow ZL_{i+1}$. Thus there is an induced map $HL_i \to HL_{i+1}$.

We introduce some notation for two Lie subalgebras of HL_i to which we will often refer. Let HL_i^- be the Lie subalgebra of HL_i given by $\mathrm{im}(HL_{i-1} \to HL_i)$, and let HL_i^+ be the Lie ideal generated by $\mathrm{im}(\tilde{d}_{i+1} : V_{i+1,*} \to HL_i)$.

Definition 1.1. Let L be a dgL satisfying (1.1). Say that L is *separated* if for all i,

$$HL_i^+ \cap HL_i^- = 0.$$

Let $(\mathbb{L}W, d)$ and $(\mathbb{L}W', d')$ be two dgLs satisfying (1.1). Say that $(\mathbb{L}W', d')$ is a bigraded extension of $(\mathbb{L}W, d)$ if $W' = W \oplus \bar{W}$ as bigraded R-modules and $d'|_W = d$.

Theorem 1.2. *Let L be a dgL satisfying (1.1). Then L has a bigraded extension L' such that L' is separated and the inclusion $L \hookrightarrow L'$ induces an isomorphism on homology.*

Corollary 1.3. *Let X be a simply connected space with finite rational homology in each dimension and finite (rational) LS category. Then X has a separated Lie model.*

Define

$$\underline{\mathbf{L}}_i = (HL_{i-1} \amalg \mathbb{L}V_{i,*}, \tilde{d}_i), \tag{1.4}$$

where $L \amalg L'$ denotes the free product of Lie algebras (i.e., their coproduct), $\tilde{d}_i|_{HL_{i-1}} = 0$ and $\tilde{d}_i|_{V_{i,*}}$ is given in (1.3).

Theorem 1.4. *Let $L = (\mathbb{L}V, d)$ be a separated dgL. Let*

$$\hat{L}_i = \mathbb{L}((H\underline{\mathbf{L}}_i)_{1,*})/[\tilde{d}_{i+1}V_{i+1,*}].$$

Then $HL \cong \bigoplus_i \hat{L}_i$ as $\mathbb{Q}$-modules. In particular, if L is a Lie model for a space X, then its homotopy Lie algebra has the form $\pi_*(\Omega X) \otimes \mathbb{Q} \cong \oplus_i \hat{L}_i$ as rational vector spaces.

To state a more precise version of this theorem, which elucidates some of the Lie algebra structure, we need to define some more notation.

Given an increasing filtration $\cdots \subset F_{i-1}M \subset F_iM \subset F_{i+1}M \subset \cdots$ of an R-module M, there is an associated graded R-module $\mathrm{gr}(M) = \bigoplus_i \mathrm{gr}_i M$ where $\mathrm{gr}_i M = F_iM/F_{i-1}M$. If M has the structure of an algebra or Lie algebra, then there is an induced algebra or Lie algebra structure on $\mathrm{gr}(M)$. If M has a separate grading, then $\mathrm{gr}(M)$ is bigraded. If M has a differential d and $dF_iM \subset F_{i-1}M$ then the filtration is called a differential filtration and there is an induced filtration on $H(M, d)$.

Recall the definition of L_i and $\tilde{d}_{i+1}$ from (1.2) and (1.3). There is an increasing differential filtration $\{F_kL_i\}$ on $L_i = (\mathbb{L}(V_{\leq i}), d) \cong (\mathbb{L}V_{<i} \amalg \mathbb{L}V_i, d)$ given by $F_{-1}L_i = 0$, $F_0L_i = \mathbb{L}V_{<i}$, and for $k \geq 0$, $F_{k+1}L_i = F_kL_i + [F_kL_i, V_i]$. This induces a filtration on HL_i and $HL_i/[\tilde{d}V_{i+1}]$, where we write $L/[V]$ to denote the quotient of L by the Lie ideal generated by $V \subset L$. When we write $\mathrm{gr}(HL_i)$ and $\mathrm{gr}(HL_i/[\tilde{d}V_{i+1}])$ we will always be using this filtration.

There is a short exact sequence $0 \to I \xrightarrow{i} L \xrightarrow{p} A \to 0$ of Lie algebras that splits (that is, there is a Lie algebra map $j : A \to L$ such that $pj = \mathrm{id}_A$) if and only if L is the semi-direct product of I and A, written $L \cong A \ltimes I$. The semi-direct product is isomorphic as modules to $A \oplus I$ and the product of elements in A and I is given by the action of A on I. We will show the following.

Theorem 1.5. *Let $L = (\mathbb{L}V, d)$ be a free dgL over $\mathbb{Q}$ which is separated. Let L_i, $\tilde{d}_i$, $\underline{\mathbf{L}}_i$, HL_i^- and HL_i^+ be as in (1.2)–(1.4) and just before Definition 1.1. Then for all i, there are Lie algebra isomorphisms*

$$\mathrm{gr}(HL_i) \cong (H\underline{\mathbf{L}}_i)_0 \rtimes \mathbb{L}((H\underline{\mathbf{L}}_i)_1), \qquad \mathrm{gr}(HL_{i+1}^-) \cong HL_i^- \rtimes \hat{L}_i,$$
$$(H\underline{\mathbf{L}}_i)_0 \cong HL_{i-1}/[\tilde{d}_i V_i] = HL_{i-1}/HL_{i-1}^+ \cong HL_i^-,$$

where $\hat{L}_i = \mathbb{L}((H\underline{\mathbf{L}}_i)_1)/[\tilde{d}_{i+1}V_{i+1}] = \mathbb{L}((H\underline{\mathbf{L}}_i)_1)/HL_i^+$, and HL_i^- is a Lie subalgebra of HL. Furthermore, if we can choose a preimage $W_i \subset HL_i$ of $(H\underline{\mathbf{L}}_i)_1$ such that $\mathbb{L}W_i \subset HL_i$ is a Lie ideal, then $HL_i \cong HL_i^- \rtimes \mathbb{L}W_i$.

Let $* = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$ be a spherical cone decomposition (see Section 2) corresponding to a separated Lie model of X. In particular $X = X_{n-1} \cup_f (\bigvee_j D^{n_j+1})$ for some map f from a wedge of spheres. The separated condition implies that f is *free* [3]. Let L_{X_i} denote the homotopy Lie algebra of X_i. Let i denote the inclusion $X_{n-1} \hookrightarrow X$. Then there is an induced map $i_\# : L_{X_{n-1}} \to L_X$. In general, this map is neither injective nor surjective [8]. Let $R(L_X)$ denote the *radical* of L_X. That is, the sum of the solvable ideals of L_X [5].

Corollary 1.6. *Either,*

(a) $i_\# : L_{X_{n-1}} \to L_X$ *is surjective,*
(b) $\dim(H\underline{\mathbf{L}}_n)_1 = 1$, *or*
(c) $R(L_X) \subset \mathrm{im}(i_\#)$ *and L_X contains a free Lie algebra on two generators.*

Remark 1.7. (a) An equivalent condition to (a) is the condition that f is *inert* [8].
(b) The only example of (b) seems to be $\mathbb{CP}^n$ (see Example 1.8).
(c) It is known that $\dim R(L_X)_{\mathrm{even}} \leq n$ [5]. The Avramov–Félix conjecture [2,6] states that either L_X is finite dimensional, or L_X contains a free Lie algebra on two generators.

Proof. Applying Theorem 1.5, $\mathrm{gr}(L_X) \cong \mathrm{im}(i_\#) \rtimes \mathbb{L}((H\underline{\mathbf{L}}_n)_1)$. The result follows. $\square$

Example 1.8. We show how the well-known homotopy Lie algebras of $\mathbb{CP}^n$ and $\mathbb{CP}^\infty$ can be easily calculated using separated Lie models.

$\mathbb{CP}^\infty$ has the Lie model $L = (\mathbb{L}\langle v_1, v_2, v_3, \ldots\rangle, d)$ where v_k has dimension $2k-1$ and $dv_k = \frac{1}{2}\sum_{i+j=k}[v_i, v_j]$. The Lie subalgebra $L_i = (\mathbb{L}\langle v_1, \ldots, v_n\rangle, d)$ is a Lie model for $\mathbb{CP}^n$. Then $L_1 = (\mathbb{L}\langle v_1\rangle, 0) = \underline{\mathbf{L}}_1$ and $L_2 = (\mathbb{L}\langle v_1, v_2\rangle, d) = \underline{\mathbf{L}}_2$ where $dv_2 = \frac{1}{2}[v_1, v_1]$. We remark that for any dgL, L_2 is always separated, since $HL_1^- = 0$ and with respect to L_2 (not L), $HL_2^+ = 0$. Now $(H\underline{\mathbf{L}}_2)_{0,*} = \mathbb{L}\langle v_1\rangle/[v_1, v_1] \cong \hat{L}_1$ and $(H\underline{\mathbf{L}}_2)_{1,*} \cong \mathbb{Q}\{u_2\}$ where $u_2 = [v_1, v_2]$. By Theorem 1.5, the Lie algebra structure of HL_2 is determined by the action of $(H\underline{\mathbf{L}}_2)_0$ on $\mathbb{L}((H\underline{\mathbf{L}}_2)_{1,*})$ in $H\underline{\mathbf{L}}_2$. Since $d[v_2, v_2] = [v_1, [v_1, v_2]]$ in L_2, $[v_1, u_2] = 0$ in $H\underline{\mathbf{L}}_2$. So as Lie algebras, $HL_2 \cong \mathbb{L}_{ab}\langle v_1, u_2\rangle$, where $\mathbb{L}_{ab}$ denotes the free abelian Lie algebra.

Next we will show by induction that L_n is separated, as Lie algebras, $HL_n \cong \mathbb{L}_{ab}\langle v_1, u_n\rangle$, where u_n has dimension $2n$, and $HL_n^- = \mathbb{L}_{ab}\langle v_1\rangle$. By definition $\underline{\mathbf{L}}_n \cong (\mathbb{L}_{ab}\langle v_1, u_{n-1}\rangle \amalg \mathbb{L}\langle v_n\rangle, \tilde{d}_n)$ where $\tilde{d}_n v_1 = \tilde{d}_n u_{n-1} = 0$ and $\tilde{d}_n v_n = u_{n-1}$. So $HL_{n-1}^- \cap HL_{n-1}^+ = \mathbb{L}_{ab}\langle v_1\rangle \cap \mathbb{L}\langle u_{n-1}\rangle = 0$. Since $(H\underline{\mathbf{L}}_n)_0 \cong \mathbb{L}_{ab}\langle v_1\rangle$, $(H\underline{\mathbf{L}})_1 \cong \mathbb{Q}\{[v_1, v_n]\}$ and $[v_1, [v_1, v_n]] = 0$ by the Jacobi identity, the claim follows.

Finally, this implies that $HL \cong \mathbb{L}_{ab}\langle v_1\rangle$.

Example 1.9. In the next example we show that the Lie model obtained from the minimal spherical cone decomposition of a product of spheres is separated, and use this to calculate the homotopy Lie algebra for the wedges of spheres of various "thickness".

Let $X = \prod_{i=1}^r S^{n_i}$, where $n_i \geq 2$. Let $N = \dim X = \sum_{i=1}^r n_i$. Let X_k denote the subcomplex of X consisting of those points in X such that at least $r-k$ of the coordinates are the basepoint. In particular, X_1 is the wedge, and X_{r-1} is the fat wedge. Also, X_{k+1} can be obtained from X_k by attaching a wedge of spheres. Then,

$$* = X_0 \subset X_1 \subset \cdots \subset X_r = X \tag{1.5}$$

is a spherical cone decomposition for X. (It is minimal, since the cone length of X is bounded below by the rational LS category of X, which is r.) In addition, X_k has a Lie model $L_k = \left(\mathbb{L}\left\langle \oplus_{i=1}^{k} V_i \right\rangle, d\right)$, where $V_i = \mathbb{Q}\{\alpha_{i,j}\}$ with the $\alpha_{i,j}$ in one-to-one correspondence with the i-fold products of the spheres S^{n_ℓ} and the dimension of $\alpha_{i,j}$, $|\alpha_{i,j}|$, is one less than the dimension of the corresponding product, and $dV_i \subset \mathbb{L}\left\langle \oplus_{j=1}^{i-1} V_j \right\rangle$. Let L_{X_k} denote the homotopy Lie algebra of X_k. Then $HL_k \cong L_{X_k}$ and $HUL_k \cong UHL_k \cong UL_{X_k}$ where U denotes the universal enveloping algebra functor.

For a non-negatively graded $\mathbb{Q}$-vector space M, let $M(z)$ be the formal power series $\sum_{i \geq 0} (\dim M_i) z^i$. $M(z)$ is called the Hilbert series for M, and $H_*(\Omega X; \mathbb{Q})(z) = (UL_X)(z)$ is called the Poincaré series for ΩX. Let $M(z)^{-1}$ denote the power series $\frac{1}{M(z)}$. Define $A_i(z)$ by

$$\prod_{i=1}^{r} (1 - z^{n_i - 1} x) = \sum_{i=0}^{r} A_i(z) x^i .$$

Let $A(z) = \prod_{i=1}^{r} (1 - z^{n_i - 1}) = \sum_{i=0}^{r} A_i(z)$, and finally define $B_k(z) = (-z)^{k-1} \sum_{i=k+1}^{r} A_i(z)$.

Theorem 1.10. *As Lie algebras, $L_{X_1} \cong \mathbb{L}\langle x_1, \ldots, x_r \rangle$, where $|x_i| = n_i - 1$, $L_X \cong \oplus_{i=1}^{r} \mathbb{L}\langle x_i \rangle$ and for $r \geq 3$, $L_{X_{r-1}} \cong L_X \amalg \mathbb{L}\langle u \rangle$ where $|u| = N - 2$. Furthermore, for $k \geq 2$, $UL_{X_k}(z)^{-1} = A(z) - B_k(z)$.*

Proof. The first two Lie algebra isomorphisms in the statement of the theorem follow directly from the well-known formulas for the homotopy Lie algebra of a wedge and a product.

The remainder of the statement of the theorem is obtained inductively. We assume that $k \geq 2$. Notice that the inclusion $X_1 \hookrightarrow X$ induces a surjection $L_{X_1} \twoheadrightarrow L_X$. So, $L_X \cong HL_2^- \hookrightarrow HL_k^-$. Assume that L_k is separated, which is trivial for $k = 2$. Then using Theorem 1.5 one can show that in fact, $HL_k^- \cong L_X$. Thus $HL_k^- \hookrightarrow L_X$ and hence $HL_k^- \cap HL_k^+ = 0$. Therefore L_{k+1} is separated. Thus, the minimal cone decomposition (1.5) yields a separated Lie model for X.

For $k \geq 2$, the Poincaré series for ΩX_k is obtained as follows. By Theorem 4.2 and [3, Lemma 3.8 and Theorem 3.5] we can apply Anick's formula [1, Theorem 3.7],

$$\begin{aligned} (UL_{X_k})(z)^{-1} &= (U(H\underline{\mathbf{L}}_k)_0)(z)^{-1} - [V_{k+1}(z) + z[(UL_{X_{k-1}})(z)^{-1} - (U(H\underline{\mathbf{L}}_k)_0)(z)^{-1}]] \\ &= A(z) + (-z)^{k-1} A_k(z) - z[(UL_{X_{k-1}})(z)^{-1} - A(z)], \end{aligned}$$

where the second equality is by Theorem 1.5 and since $(UL_X)(z)^{-1} = A(z)$. By induction, this is equal to $A(z) - B_k(z)$.

The Lie algebra isomorphism for $L_{X_{r-1}}$ follows since the cell attachment from X_{r-2} to X_{r-1} is semi-inert [3]. $\square$

We remark that the fact that the top-cell attachment of X is inert [8] is witnessed by $\underline{\mathbf{L}}_r = (L_X \amalg \mathbb{L}\langle u, v \rangle, d)$, where $d|_{L_X} = 0$ and $dv = u$.

From Theorem 1.10 it is easy to check that a fat wedge of odd-dimensional spheres has the maximum possible gap in the rational homotopy groups [7, Theorem 33.3] if and only if all the spheres have the same dimension.

Example 1.11. For the final example, we calculate the homotopy Lie algebra of a connected sum of products of spheres. For example, a Lie group M is rationally equivalent to a product of odd spheres (and so L_M is a free abelian Lie algebra).

For $s \geq 2$ and $1 \leq i \leq s$, let M_i be simply connected, of dimension N and rationally equivalent to a product of at least three spheres (e.g. $SU(n)$, $n \geq 4$). Let $X = \#_{i=1}^{s} M_i$. Applying the previous example gives:

Theorem 1.12. *As Lie algebras,*

$$L_X \cong \coprod_{i=1}^{s} L_{M_i} \amalg \mathbb{L}\langle u_1, \ldots u_s \rangle / (u_1 + \cdots + u_s),$$

where $|u_i| = N - 2$.

Proof. By the previous example M_i has a separated Lie model of the form $(L_{(i)} \amalg \mathbb{L}\langle v_i \rangle, d)$ such that $\underline{\mathbf{L}}_{r_i} = (L_{M_i}, 0) \amalg (\mathbb{L}\langle u_i, v_i \rangle, \tilde{d})$ with $\tilde{d}v_i = u_i$. Thus X has a separated Lie model $(\coprod_{i=1}^s L_{(i)} \amalg \mathbb{L}\langle v \rangle, d)$ and $\underline{\mathbf{L}}_r = (\coprod_{i=1}^s L_{M_i}, 0) \amalg (\mathbb{L}\langle u_1, \dots u_s, v \rangle, \tilde{d}v = u_1 + \cdots + u_s)$, where $r = \max_i r_i$. $\square$

In particular, L_X contains a free Lie algebra on $2s - 1$ generators. So X satisfies the Avramov–Félix conjecture.

In the Appendix we state and prove a generalization of the Schreier property of free Lie algebras: that any Lie subalgebra is also free, which may be of independent interest. This generalization is used in the proof of Theorem 1.5.

2. Background

In their landmark papers [12,14], Quillen and Sullivan construct algebraic models for rational homotopy theory. Sullivan constructs a contravariant functor A_{PL} which serves as a fundamental bridge between topology and algebra. For a space X, $A_{PL}(X)$ is a commutative cochain algebra which has the property that $H(A_{PL}(X)) \cong H^*(X; \mathbb{Q})$ as algebras. Quillen gives a construction for a differential graded Lie algebra (dgL) $(\mathbb{L}V, d)$, where $\mathbb{L}V$ denotes the free Lie algebra on a rational vector space V, one of whose properties is that $H(\mathbb{L}V, d) \cong \pi_*(\Omega X) \otimes \mathbb{Q}$. We will follow convention and call this a free dgL even though it is almost always not a free object in the category of *differential* graded Lie algebras. For an excellent reference on these models and their applications, the reader is referred to [7, Parts II and IV].

A *quasi-isomorphism* is a morphism which induces an isomorphism in homology. A *Lie model* for a space X is a differential graded Lie algebra (L, d) equipped with a quasi-isomorphism $m : C^*(L, d) \xrightarrow{\simeq} A_{PL}(X)$. Here, $C^*(L, d) = \mathrm{Hom}(C_*(L, d), \mathbb{Q})$, is the contravariant functor induced by Quillen's functor C_*. $C_*(L, d)$ is called the Cartan–Eilenberg–Chevalley construction on (L, d). Quillen's free dgL above is a Lie model. Given a Lie model (L, d) for a space X and a quasi-isomorphism $(L, d) \xrightarrow{\simeq} (L', d')$, their is an induced quasi-isomorphism $C^*(L', d') \xrightarrow{\simeq} C^*(L, d) \xrightarrow{\simeq} A_{PL}(X)$. In particular, a quasi-isomorphic bigraded extension of a Lie model for X is also a Lie model for X.

It is the Samelson product on $\pi_*(\Omega X)$, which corresponds the Whitehead product under the canonical isomorphism $\pi_*(\Omega X) \cong \pi_{*+1}(X)$, which gives it the structure of a graded Lie algebra, which we call the *(rational) homotopy Lie algebra*.

A *rational homotopy equivalence* is a continuous map $f : X \to Y$ such that $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism. Two spaces X and Y are said to have the same *rational homotopy type*, written $X \simeq_{\mathbb{Q}} Y$, if they are connected by a sequence of rational homotopy equivalences (in either direction). The *Lusternik–Schnirelmann (LS) category* of a space X, denoted $\mathrm{cat}(X)$, is the smallest integer n such that X is the union of $n + 1$ open subsets, each contractible in X. The *rational LS category* of a space X, denoted $\mathrm{cat}_{\mathbb{Q}}(X)$, is the smallest integer n such that there exists a space Y with $\mathrm{cat}(Y) = n$ and $X \simeq_{\mathbb{Q}} Y$. So $\mathrm{cat}_{\mathbb{Q}}(X) \le \mathrm{cat}(X)$. A space X_n is said to be a *spherical n-cone* if there exists a sequence of spaces

$$* = X_0 \subset X_1 \subset \cdots \subset X_n \tag{2.1}$$

such that for $k = 0 \dots n - 1$, $X_{k+1} = X_k \cup_{f_{k+1}} \left(\bigvee_j D_{j,k}^{n_j+1}\right)$, where D^n denotes the n-dimensional disk and $f_{k+1} : \bigvee_j S_{j,k}^{n_j} \to X_k$ is an attaching map. A space X is said to have *rational cone length* n, written $\mathrm{cl}_{\mathbb{Q}}(X) = n$, if n is the smallest number such that $X \simeq_{\mathbb{Q}} X_n$ for some spherical n-cone X_n. (This is equivalent to the more usual definition of rational cone length, see [7, Proposition 28.3], for example.) It is a theorem of Cornea [4] that if $\mathrm{cat}_{\mathbb{Q}}(X) = n$ then $\mathrm{cl}_{\mathbb{Q}}(X) = n$ or $n + 1$. A CW complex is said to have finite type if it has finitely many cells in each dimension.

A free dgL $(\mathbb{L}V, d)$ is said to have *length N* if we can decompose $V_* = \oplus_{i=1}^N V_{i,*}$ such that $\mathrm{d}V_{i,*} \subset \mathbb{L}(\oplus_{j=1}^{i-1} V_{j,*})$, where $dV_{1,*} = 0$. Note that V is now bigraded, though $(\mathbb{L}V, d)$ is typically not a bigraded dgL. We will call the first gradation *degree* and the second, usual gradation *dimension*. For any free dgL $(\mathbb{L}V, d)$ of length N there is a spherical N-cone X such that $(\mathbb{L}V, d)$ is a Quillen model for X. We call $(\mathbb{L}V, d)$ the cellular Lie model of X. Furthermore, any spherical N-cone has a cellular Lie model of length N. We outline the construction as follows. For each $n \ge 1$ let the $(n + 1)$-cells D_α^{n+1} of X correspond to a basis $\{v_\alpha\}$ of V_n. Let X_n denote the n-skeleton of X. By induction, $(\mathbb{L}V_{<n}, d)$ is a cellular Lie model of X_n. The attaching maps $f_\alpha : S^n \to X_n$ are given by representatives of $[dv_\alpha] \in H_{n-1}(\mathbb{L}V_{<n}, d) \cong \pi_{n-1}(\Omega X_n) \cong \pi_n(X_n)$.

A dgL is said to be of *finite type* if it is finite in each dimension. It is said to be *connected* if it is concentrated in strictly positive dimensions. Given two (Lie) algebras L and L', let $L \amalg L'$ denote their free product (i.e., coproduct). We remark that $\mathbb{L}V \amalg \mathbb{L}V' \cong \mathbb{L}(V \oplus V')$.

3. Replacing dgLs with separated dgLs

To prove Theorem 1.2, we will need the following lemmas. Let $\mathbf{k}$ be a field of characteristic 0.

Lemma 3.1. *Let W be a free $\mathbf{k}$-module and let $\alpha, \beta \in (\mathbb{L}W, d)$ with $d\beta = \alpha$. Let $W' = W \oplus \mathbf{k}\{a, b\}$ where $|a| = |\beta|$ and $|b| = |\beta| + 1$. Define $(\mathbb{L}W', d')$ by letting d' be an extension of d determined by taking $d'a = \alpha$ and $d'b = a - \beta$. Then the inclusion $(\mathbb{L}W, d) \hookrightarrow (\mathbb{L}W', d')$ is a quasi-isomorphism.*

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
(\mathbb{L}W, d) \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & (\mathbb{L}(W \oplus \mathbf{k}\{a, b\}), d') \\
& (\mathbb{L}(W \oplus \mathbf{k}\{\hat{a}, \hat{b}\}), \hat{d}) \arrow[ur, "\theta"'] &
\end{tikzcd}$$

where $\hat{d}\hat{b} = \hat{a}$, ϕ and ψ are injections and θ is defined on generators as follows: $\theta|_W = \mathrm{id}_W$, $\theta\hat{a} = a - \beta$, and $\theta\hat{b} = b$. It is easy to check that θ is a chain map and a dgL isomorphism. Furthermore,

$$(\mathbb{L}(W \oplus \mathbf{k}\{\hat{a}, \hat{b}\}), \hat{d}) \cong (\mathbb{L}W, d) \amalg (\mathbb{L} < \hat{a}, \hat{b} >, \hat{d}) \simeq (\mathbb{L}W, d).$$

Thus ψ is a quasi-isomorphism. It follows that ϕ is one as well. $\square$

Corollary 3.2. *Let $L = (\mathbb{L}W, d)$ with $\{\alpha_j, \beta_j\}_{j \in J} \subset \mathbb{L}W$ where $d\alpha_j = 0$ and $d\beta_j = \alpha_j$. Taking $\bar{W} = \mathbf{k}\{a_j, b_j\}_{j \in J}$ with $|a_j| = |\beta_j|$ and $|b_j| = |\beta_j| + 1$, let $L' = \mathbb{L}(W \oplus \bar{W}, d')$ where $d'|_W = d$, $d'a_j = \alpha_j$, and $d'b_j = a_j - \beta_j$. Then $L' \simeq L$.*

Given an $\mathbf{k}$-module M, let ZM, BM denote the $\mathbf{k}$-submodules of cycles and boundaries.

Lemma 3.3. *Let $L = (\mathbb{L}W, d)$ and let $V = \{v_j\}_{j \in J} \subset HL_n$ with $v_j \neq 0$. For each v_j choose a representative cycle $\hat{v}_j \in ZL$. Let $L' = (\mathbb{L}W', d')$ where $W' = W \oplus \mathbf{k}\{a_j, b_j\}_{j \in J}$, $d'|_W = d$, $da_j = \hat{v}_j$ and b_j is in dimension $n + 2$. Then $H_{\leq n}L' \cong H_{\leq n}L/V$.*

Proof. Since $Z_{\leq n}L' = Z_{\leq n}L$ and $B_{\leq n}L' \cong B_{\leq n}L \oplus \mathbf{k}\{da_j\}_{j \in J}$, $H_{\leq n}L' \cong H_{\leq n}L/V$. $\square$

Lemma 3.4. *Given $\mathbf{k}$-modules $C \subset A \subset B$ and $D \subset B$, let $p : B \to B/C$ denote the quotient map. If $D \cap A \subseteq C$ then $pD \cap pA = 0$.*

Proof. Let $x \in pD \cap pA$. Then there are $y \in D$, and $y' \in A$ such that $py = py' = x$. Let $z = y - y'$. Since $pz = 0$, $z \in C \subset A$. Thus $y = y' + z \in A \cap D \subseteq C$. Therefore $x = py = 0$. $\square$

The proof of Theorem 1.2 will rely on a two-step inductive procedure given in Proposition 3.8. To help with the book-keeping, we introduce the following definitions. We will use the notation HL_i^+ and HL_i^- defined just before Definition 1.1.

Definition 3.5. Let $(\mathbb{L}W, d)$ be a dgL satisfying (1.1). Say that $(\mathbb{L}W, d)$ is *k-separated* if for all $i < k$, $HL_i^+ \cap HL_i^- = 0$. Say $(\mathbb{L}W, d)$ is *(k, n)-separated* if it is k-separated and $H_{<n}L_k^+ \cap H_{<n}L_k^- = 0$.

Let $\delta_{i,j} = 1$ if $i = j$ and let it be 0 otherwise. Let s denote the suspension homomorphism. That is, $(sM)_k = M_{k-1}$.

Lemma 3.6. *Let $L = (\mathbb{L}W, d)$ be a dgL of length N that is (k, n)-separated. Let V be the component of $HL_k^+ \cap HL_k^-$ in dimension n.*

Then there exists a bigraded extension $\hat{L} = (\mathbb{L}\hat{W}, \hat{d}) \supset L$ *of length* $N + \delta_{k+2,N+1}$ *with* $\hat{W}_i = W_i$ *for* $i \neq k, k+2$, $\hat{W}_k = W_k \oplus \bar{W}$, $\hat{W}_{k+2} = W_{k+2} \oplus s\bar{W}$, *where* $\bar{W}$ *is in dimension* $n+1$, *such that* $L' \simeq L$. *Furthermore* $H_{\leq n}\hat{L}_k \cong H_{\leq n}L_k/V$, *and*

$$H_{\leq n}\hat{L}_k^+ \cap H_{\leq n}\hat{L}_k^- = 0. \tag{3.1}$$

Proof. Let $\{\alpha_j\}_{j\in J}$ be a basis (as an $\mathbf{k}$-module) for V. Since $\alpha_j \in HL_k^-$, α_j has a preimage $\alpha'_j \in HL_{k-1}$. Let α''_j be a representative cycle in L_{k-1} for α'_j. Since $\alpha_j \in HL_k^+$, there is a $\beta_j \in L_{k+1}$ such that $d\beta_j = \alpha''_j$. Let $\{a_j, b_j\}_{j\in J}$ be pairs of elements of bidegree $(k, n+1)$ and $(k+2, n+2)$ respectively. Define $\hat{L} = (\mathbb{L}\hat{W}, \hat{d}) \supset L$ by letting $\hat{W}_i = W_i$ for $i \neq k, k+2$, $\hat{W}_k = W_k \oplus \mathbf{k}\{a_j\}$ and $\hat{W}_{k+2} = W_{k+2} \oplus \mathbf{k}\{b_j\}$. Extend d to $\hat{d}$ by letting $da_j = \alpha''_j$ and $db_j = a_j - \beta_j$. Note that $\hat{L}$ has length $N + \delta_{k+2,N+1}$. By Corollary 3.2, $\hat{L} \simeq L$.

By Lemma 3.3,

$$(H_{\leq n}\hat{L}_k) \cong (H_{\leq n}L_k)/V. \tag{3.2}$$

In other words, in dimensions $\leq n$ the map from HL_k to $H\hat{L}_k$ is just the quotient by V. Thus, $H_{\leq n}\hat{L}_k^+ \cong H_{\leq n}L_k^+/V$, and $H_{\leq n}\hat{L}_k^- \cong H_{\leq n}L_k^-/V$. Applying Lemma 3.4 to $V \subset H_{\leq n}L_k^- \subset H_{\leq n}L_k$ and $H_{\leq n}L_k^+ \subset H_{\leq n}L_k$, one gets that $H_{\leq n}L_k^+/V \cap H_{\leq n}L_k^-/V = 0$. □

Definition 3.7. Say that a bigraded $\mathbf{k}$-module M is in the (k,n)-*region* if $W_{*,\leq n} = W_{\geq k+3,*} = W_{k+1,n+1} = W_{k+2,n+1} = 0$. Let $L = (\mathbb{L}W, d)$ be a free dgL. Say that $L' = (\mathbb{L}W', d')$ is a (k,n)-*extension* of L if $W' = W \oplus \bar{W}$ as bigraded modules, where $\bar{W}$ is in the (k,n)-region, and $d'|_W = d$.

Proposition 3.8. *Let* $L = (\mathbb{L}W, d)$ *of length* N *that is* (k,n)*-separated. Let* V *be the component of* $HL_k^+ \cap HL_k^-$ *in dimension* n.

(a) *Then there is an* (k,n)*-extension of* L, $L' = (\mathbb{L}W', d')$ *of length* $N + \delta_{k+2,N+1}$ *which is* $(k, n+1)$*-separated, such that* $L' \simeq L$. *Furthermore* $H_{\leq n}L'_k \cong H_{\leq n}L_k/V$.

(b) *In addition there an* (k,n)*-extension of* L, $L'' = (\mathbb{L}W'', d'')$ *of length* $N + \delta_{k+2,N+1}$ *which is* $(k+1)$*-separated, such that* $L'' \simeq L$. *Furthermore* $H_{\leq n}L''_k \cong H_{\leq n}L_k/V$, *and if* $H_{\leq n}L_{k+1}^+ \cap H_{\leq n}L_{k+1}^- = 0$ *then* $H_{\leq n}L''^+_{k+1} \cap H_{\leq n}L''^-_{k+1} = 0$.

Remark 3.9. Recall that $d(W_{k+1}) \subset \mathbb{L}(W_{\leq k})$ and that HL_k^+ is the ideal generated by the images of W_{k+1} by the induced map $W_{k+1} \to HL_k$. All of the elements in this ideal have preimages in $\mathbb{L}(W_{\leq k})$. V consists of those dimension n elements in this ideal which have preimages in lower filtration.

Proof. We prove the proposition by induction on k. The statement of the proposition is trivial if $k = 0$. We will assume the statement of the proposition is true for $k-1$.

Let L, V be as in the statement of the proposition. By Lemma 3.6, there exists $\hat{L} = (\mathbb{L}\hat{W}, \hat{d}) \supset L$ of length $N + \delta_{k+2,N+1}$ with $\hat{W}_i = W_i$ for $i \neq k, k+2$, $\hat{W}_k = W_k \oplus \bar{W}$, $\hat{W}_{k+2} = W_{k+2} \oplus s\bar{W}$, where $\bar{W}$ is in dimension $n+1$, such that $L' \simeq L$. Furthermore $H_{\leq n}\hat{L}_k \cong H_{\leq n}L_k/V$, and

$$H_{\leq n}\hat{L}_k^+ \cap H_{\leq n}\hat{L}_k^- = 0. \tag{3.3}$$

Since $\hat{L}_{k-1} = L_{k-1}$ and L is k-separated, $\hat{L}$ is (at least) $(k-1)$-separated. Also $H\hat{L}_{k-2} = HL_{k-2}$ and $H\hat{L}_{k-1} = HL_{k-1}$.

Let $\hat{\mathcal{L}} = H\hat{L}_{k-1}^+ \cap H\hat{L}_{k-1}^-$. Since L is k-separated, this equals $[\hat{d}\bar{W}] \cap HL_{k-1}^-$. Since $\bar{W}$ is n-connected, $\hat{\mathcal{L}}$ is $(n-1)$-connected, but it is not necessarily n-connected. In the non-trivial case, there are elements of V which not only have preimages in filtration $k-1$ but also have preimages in filtration $k-2$. Let $\hat{V} = \hat{\mathcal{L}}_n$.

By induction there exists a $(k-1,n)$-extension $L' = (\mathbb{L}V', d') \supset \hat{L}$ of length $N + \delta_{k+2,N+1}$ that is k-separated such that $L' \simeq \hat{L}$, and since $H_{\leq n}\hat{L}_k^+ \cap H_{\leq n}\hat{L}_k^- = 0$, $H_{\leq n}L'^+_k \cap H_{\leq n}L'^-_k = 0$. So L' is a (k,n)-extension of L such that $L' \simeq L$. Furthermore $H_{\leq n}L'_{k+1} \cong H_{\leq n}\hat{L}_{k+1}/\hat{V} = H_{\leq n}L_{k-1}/\hat{V}$. This proves part (a) of the statement.

To prove part (b) of the statement we simply iterate part (a). By iterating (a) we get a sequence of dgLs

$$L = L^{(0)} \subset L^{(1)} \subset \cdots \subset L^{(i)} \subset L^{(i+1)} \subset \cdots$$

where $L^{(i)}$ is $(k, n+i)$-separated, $L^{(i+1)}$ is a $(k, n+i)$-extension of $L^{(i)}$, and $L^{(i+1)} \simeq L^{(i)}$. Furthermore, $H_{\le n} L_k^{(i)} \cong H_{\le n} L_k / V$, and if $H_{\le n} L_{k+1}^{+} \cap H_{\le n} L_{k+1}^{-} = 0$ then $H_{\le n} {L^{(i)}}_{k+1}^{+} \cap H_{\le n} {L^{(i)}}_{k+1}^{-} = 0$. Let $L'' = \cup_i L^{(i)}$. Then L'' is a $(k+1)$-separated (k, n)-extension of L and $L'' \simeq L$. Furthermore, $H_{\le n} L_k^{(i)} \cong H_{\le n} L_k / V$, and if $H_{\le n} L_{k+1}^{+} \cap H_{\le n} L_{k+1}^{-} = 0$ then $H_{\le n} {L''}_{k+1}^{+} \cap H_{\le n} {L''}_{k+1}^{-} = 0$. □

Proof of Theorem 1.2. Since any dgL is 1-separated, Theorem 1.2 follows by applying Proposition 3.8, N times. □

4. Properties of separated dgLs

In this section we will use Theorems 1.2 and A.1 to prove Theorem 1.5. We will defer to the Appendix the proof of Theorem A.1, which is a generalization the well-known result that a Lie subalgebra of a free Lie algebra is also a free Lie algebra. We will use (1.2)–(1.4), and the notation HL_i^+ and HL_i^- defined just before Definition 1.1.

Definition 4.1. Let $(\mathbb{L}W, d)$ be a free dgL over a field. Say that $(\mathbb{L}W, d)$ is *strongly free* if for all i, $HL_i^+ \subset L_{M_i}$ is a free Lie subalgebra.

Theorem 4.2. *Let L be a free dgL over $\mathbb{Q}$ which is separated. Then L is strongly free and for all i, there are Lie algebra isomorphisms*

$$\mathrm{gr}(HL_i) \cong (H\underline{\mathbf{L}}_i)_0 \rtimes \mathbb{L}((H\underline{\mathbf{L}}_i)_1), \quad \textit{and} \quad (H\underline{\mathbf{L}}_i)_0 \cong HL_{i-1}/[\tilde{d}W_i] \cong HL_i^-.$$

We will prove Theorem 4.2 by induction. A main part of the induction will use the following theorem, which is a special case of the main algebraic result from [3]. Note that U denotes the universal enveloping algebra functor.

Theorem 4.3 ([3, Theorem 3.12]). *Over the field $\mathbb{Q}$, let $L' = (L \amalg \mathbb{L}V_1, d)$, where $dL \subset L$, $dV_1 \subset L$ and $HUL \cong UL_0$, such that there is an induced map $d' : V_1 \to L_0$. Let $\underline{\mathbf{L}} = (L_0 \amalg \mathbb{L}V_1, d')$. If $[d'V_1] \subset L_0$ is a free Lie algebra, then as algebras*

$$\mathrm{gr}(HUL') \cong UH\underline{\mathbf{L}}, \quad \textit{and} \quad H\underline{\mathbf{L}} \cong (H\underline{\mathbf{L}})_0 \rtimes \mathbb{L}(H\underline{\mathbf{L}})_1 \quad \textit{and} \quad (H\underline{\mathbf{L}})_0 \cong L_0/[d'V_1]$$

as Lie algebras.

The associated graded structure on HUL' is that induced from the filtration on L' given by $F_{-1}L' = 0$, $F_0 L' = L$ and $F_{i+1}L' = F_i L' + [V_1, F_i L']$.

Proof of Theorem 4.2. Let $L = (\mathbb{L}W, d)$ be a free dgL over $\mathbb{Q}$ which is separated. Recall that $\tilde{d}_i : W_i \to HL_{i-1}$ and $\underline{\mathbf{L}}_i = (HL_{i-1} \amalg \mathbb{L}W_i, \tilde{d}_i)$.

Assume that L_i is strongly free, $\mathrm{gr}(HL_i) \cong (H\underline{\mathbf{L}}_i)_0 \rtimes \mathbb{L}((H\underline{\mathbf{L}}_i)_1)$, and $(H\underline{\mathbf{L}}_i)_0 \cong HL_{i-1}/[\tilde{d}W_i] \cong HL_i^-$. This is trivial for $i \le 1$.

Note that $L_{i+1} = (L_i \amalg \mathbb{L}W_{i+1}, d)$. Both L_{i+1} and UL_{i+1} can be filtered by the 'length in W_{i+1}' filtration. That is, let $F_{-1}(L_{i+1}) = 0$, $F_0 L_{i+1} = L_i$, and for $i \ge 0$, $F_{i+1}(L_{i+1}) = F_i(L_{i+1}) + [F_i(L_{i+1}), W_{i+1}]$. This induces a similar filtration on UL_{i+1}.

By assumption, $\mathrm{gr}(HL_i) \cong HL_i^- \rtimes \mathbb{L}((H\underline{\mathbf{L}}_i)_1)$. Since L_{i+1} is separated, $HL_i^+ \cap HL_i^- = 0$. Thus by Theorem A.1, HL_i^+ is a free Lie algebra. Hence L_{i+1} is strongly free.

Then by Theorem 4.3, as algebras

$$\mathrm{gr}(HUL_{i+1}) \cong U\left((H\underline{\mathbf{L}}_{i+1})_0 \rtimes \mathbb{L}((H\underline{\mathbf{L}}_{i+1})_1)\right), \quad \text{and} \quad (H\underline{\mathbf{L}}_{i+1})_0 \cong HL_i/[\tilde{d}_{i+1}W_{i+1}].$$

It is a result of Quillen's that $HUL_{i+1} \cong UHL_{i+1}$. Thus $\mathrm{gr}(HUL_{i+1}) \cong \mathrm{gr}(UHL_{i+1}) \cong U\mathrm{gr}(HL_{i+1})$. So $U\mathrm{gr}(HL_{i+1}) \cong U\left((H\underline{\mathbf{L}}_{i+1})_0 \rtimes \mathbb{L}((H\underline{\mathbf{L}}_{i+1})_1)\right)$.

For any Lie algebra L, UL has a canonical cocommutative Hopf algebra structure. Let P denote the primitive functor. Over $\mathbb{Q}$, the composition PU is the identity functor [11]. Therefore $\mathrm{gr}(HL_{i+1}) \cong (H\underline{\mathbf{L}}_{i+1})_0 \rtimes \mathbb{L}((H\underline{\mathbf{L}}_{i+1})_1)$.

It follows that $HL_{i+1}^- \cong (H\underline{\mathbf{L}}_{i+1})_0$. This finishes the inductive step. □

We will now use Theorem 4.2 to prove Theorem 1.5.

Proof of Theorem 1.5. Let $L = (\mathbb{L}V, d)$ that is separated. Recall that $L_i = \mathbb{L}(V_{\leq i,*})$, $\tilde{d} : V_i \xrightarrow{d} ZL_{i-1} \twoheadrightarrow HL_{i-1}$, and $\underline{\mathbf{L}}_i = (HL_i \amalg \mathbb{L}V_i, \tilde{d})$.

Let $N_i = (H\underline{\mathbf{L}}_i)_1$ and recall that HL_i is filtered by the length in V_i filtration. By Theorem 4.2, $(\mathbb{L}V, d)$ is strongly free and for all i, it satisfies $\operatorname{gr}(HL_i) \cong HL_i^- \rtimes \mathbb{L}N_i$ where $HL_i^- \cong HL_{i-1}/[\tilde{d}_i V_i]$.

Also by Theorem 4.2, $HL_{i+1}^- \cong HL_i/[\tilde{d}_{i+1}V_{i+1}]$, which has a filtration induced by the filtration on HL_i. Thus we have the following short exact sequence of filtered, graded Lie algebras:

$$0 \to [\tilde{d}_{i+1}V_{i+1}] \to HL_i \to HL_{i+1}^- \to 0,$$

which induces a short exact sequence of bigraded Lie algebras:

$$0 \to \operatorname{gr}([\tilde{d}_{i+1}V_{i+1}]) \to \operatorname{gr}(HL_i) \to \operatorname{gr}(HL_{i+1}^-) \to 0.$$

Recall that $\operatorname{gr}(HL_i) \cong HL_i^- \rtimes \mathbb{L}N_i$.

Since $(\mathbb{L}V, d)$ is separated, $[\tilde{d}_{i+1}V_{i+1}] \cap HL_i^- = 0$, and thus $\operatorname{gr}([\tilde{d}_{i+1}V_{i+1}]) \subset \mathbb{L}N_i$ as Lie algebras. Since any Lie subalgebra of a free Lie algebra is automatically free, $\operatorname{gr}([\tilde{d}_{i+1}V_{i+1}]) \cong \mathbb{L}K_{i+1}$ for some $\mathbb{Q}$-module $K_{i+1} \subset \mathbb{L}N_i$. Therefore $\operatorname{gr}(HL_{i+1}^-) \cong HL_i^- \rtimes (\mathbb{L}N_i/\mathbb{L}K_{i+1})$ as Lie algebras. Let $\hat{L}_i = \mathbb{L}N_i/\mathbb{L}K_{i+1}$.

By Theorem 4.2 there is a split short exact sequence of Lie algebras

$$0 \to \mathbb{L}((H\underline{\mathbf{L}}_i)_1) \to \operatorname{gr}(HL_i) \to HL_i^- \to 0.$$

Let W_i be a preimage of $(H\underline{\mathbf{L}}_i)_1$ in HL_i. We have a short exact sequence of modules

$$0 \to \mathbb{L}W_i \to HL_i \to HL_i^- \to 0$$

but this is not necessarily a short exact sequence of Lie algebra since $\mathbb{L}W_i$ may not be a Lie ideal. Equivalently, the projection $HL_i \to HL_i^-$ may not be a Lie algebra morphism. However, if $\mathbb{L}W_i$ is a Lie ideal in HL_i then $HL_i \cong HL_i^- \rtimes \mathbb{L}W_i$. $\square$

Acknowledgments

I would like to thank Kathryn Hess for many helpful discussions and Greg Lupton and John Oprea for help in rewriting the introduction and suggesting Examples 1.9 and 1.11.

Appendix. A generalized Schreier property

In this chapter we give a simple criterion which we prove guarantees that certain Lie subalgebras are free.

It is a well-known fact that any (graded) Lie subalgebra of a (graded) Lie algebra is a free Lie algebra [9,13,10] This is often referred to as the *Schreier property*. In this chapter we generalize this result to the following.

Theorem A.1. *Over a field $\mathbb{F}$, let L be a finite-type graded Lie algebra with filtration $\{F_k L\}$ such that $\operatorname{gr}(L) \cong L_0 \rtimes \mathbb{L}V_1$ as Lie algebras, where $L_0 = F_0 L$ and $V_1 = F_1 L/F_0 L$. Let $J \subset L$ be a Lie subalgebra such that $J \cap F_0 L = 0$. Then J is a free Lie algebra.*

Before proving this theorem, we prove the following lemma.

Lemma A.2. *Let J be a finite-type filtered Lie algebra such that $\operatorname{gr}(J)$ is a free Lie algebra. Then J is a free Lie algebra.*

Proof. By assumption there is an $\mathbb{F}$-module $\bar{W}$ such that $\operatorname{gr}(J) \cong \mathbb{L}\bar{W}$.

Let $\{\bar{w}_i\}_{i \in I} \subset \operatorname{gr}(J)$ be an $\mathbb{F}$-module basis for $\bar{W}$. Let $m_i = \deg(\bar{w}_i)$. That is, $\bar{w}_i \in F_{m_i}J/F_{m_i-1}L$. For each $\bar{w}_i$ choose a representative $w_i \in F_{m_i}J$. Let $W = \mathbb{F}\{w_i\}_{i \in I} \subset J$.

Then there is a canonical map $\phi : \mathbb{L}W \to J$. Grade $\mathbb{L}W$ by letting $w_i \in W$ be in degree m_i. Then ϕ is a map of filtered objects and there is an induced map $\theta : \mathbb{L}W \to \operatorname{gr}(J)$. However the composite map

$$\mathbb{L}W \xrightarrow{\theta} \operatorname{gr}(J) \xrightarrow{\cong} \mathbb{L}\bar{W}$$

is just the canonical isomorphism $\mathbb{L}W \xrightarrow{\cong} \mathbb{L}\bar{W}$. So θ is an isomorphism. Therefore ϕ is an isomorphism and J is a free Lie algebra. $\square$

Proof of Theorem A.1. The filtration on L filters J by letting

$$F_k J = J \cap F_k L.$$

From this definition it follows that the inclusion $J \hookrightarrow L$ induces an inclusion $\mathrm{gr}(J) \hookrightarrow \mathrm{gr}(L)$. So $\mathrm{gr}(J) \hookrightarrow \mathrm{gr}(L) \cong L_0 \rtimes \mathbb{L}V_1$. Since $J \cap F_0 L = 0$ it follows that $(\mathrm{gr} J)_0 = 0$ and $\mathrm{gr}(J) \hookrightarrow (\mathrm{gr} J)_{\geq 1} \cong \mathbb{L}V_1$. By the Schreier property $\mathrm{gr}(J)$ is a free Lie algebra. Thus by Lemma A.2, J is a free Lie algebra. $\square$

The following corollary is a special case of this theorem.

Corollary A.3. *Over a field $\mathbb{F}$, if $J \subset L_0 \rtimes \mathbb{L}(L_1)$ is a Lie subalgebra such that $J \cap L_0 = 0$ then J is a free Lie algebra.*

Note that since J is not necessarily homogeneous with respect to degree $J \cap L_0 = 0$ does not imply that $J \subset \mathbb{L}L_1$.

References

[1] David J. Anick, A counterexample to a conjecture of Serre, Ann. of Math. (2) 115 (1) (1982) 1–33.
[2] Luchezar L. Avramov, Free Lie subalgebras of the cohomology of local rings, Trans. Amer. Math. Soc. 270 (2) (1982) 589–608.
[3] Peter Bubenik, Free and semi-inert cell attachments, Trans. Amer. Math. Soc. 357 (11) (2005) 4533–4553.
[4] Octavian Cornea, Cone-length and Lusternik–Schnirelmann category, Topology 33 (1) (1994) 95–111.
[5] Yves Félix, Stephen Halperin, Carl Jacobsson, Clas Löfwall, Jean-Claude Thomas, The radical of the homotopy Lie algebra, Amer. J. Math. 110 (2) (1988) 301–322.
[6] Yves Félix, Stephen Halperin, Jean-Claude Thomas, Sur l'homotopie des espaces de catégorie 2, Math. Scand. 55 (2) (1984) 216–228.
[7] Yves Félix, Stephen Halperin, Jean-Claude Thomas, Rational homotopy theory, in: Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001.
[8] Stephen Halperin, Jean-Michel Lemaire, Suites inertes dans les algèbres de Lie graduées (Autopsie d'un meurtre. II), Math. Scand. 61 (1) (1987) 39–67.
[9] Alexander A. Mikhalëv, Subalgebras of free colored Lie superalgebras, Mat. Zametki 37 (5) (1985) 653–661, 779.
[10] Alexander A. Mikhalev, Andrej A. Zolotykh, Combinatorial Aspects of Lie Superalgebras, CRC Press, Boca Raton, FL, 1995, With 1 IBM-PC floppy disk (3.5 inch; HD).
[11] John W. Milnor, John C. Moore, On the structure of Hopf algebras, Ann. of Math. (2) 81 (1965) 211–264.
[12] Daniel Quillen, Rational homotopy theory, Ann. of Math. (2) 90 (1969) 205–295.
[13] A.I. Širšov, Subalgebras of free Lie algebras, Mat. Sb. N.S. 33 (75) (1953) 441–452.
[14] Dennis Sullivan, Infinitesimal computations in topology, Inst. Hautes Études Sci. Publ. Math. 1977 (47) (1978) 269–331.