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Appl. Comput. Harmon. Anal. 22 (2007) 78–104

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Pseudo-splines, wavelets and framelets ☆

Bin Dong, Zuowei Shen *

Department of Mathematics, Science Drive 2, National University of Singapore, Singapore, 117543, Singapore

Received 16 February 2005; revised 17 April 2006; accepted 17 April 2006

Available online 23 June 2006

Communicated by Charles K. Chui

Abstract

The first type of pseudo-splines were introduced in [I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14 (1) (2003) 1–46; I. Selesnick, Smooth wavelet tight frames with zero moments, Appl. Comput. Harmon. Anal. 10 (2) (2001) 163–181] to construct tight framelets with desired approximation orders via the *unitary extension principle* of [A. Ron, Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$: The analysis of the analysis operator, J. Funct. Anal. 148 (2) (1997) 408–447]. In the spirit of the first type of pseudo-splines, we introduce here a new type (the second type) of pseudo-splines to construct symmetric or antisymmetric tight framelets with desired approximation orders. Pseudo-splines provide a rich family of refinable functions. B-splines are one of the special classes of pseudo-splines; orthogonal refinable functions (whose shifts form an orthonormal system given in [I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math. 41 (1988) 909–996]) are another class of pseudo-splines; and so are the interpolatory refinable functions (which are the Lagrange interpolatory functions at $\mathbb{Z}$ and were first discussed in [S. Dubuc, Interpolation through an iterative scheme, J. Math. Anal. Appl. 114 (1986) 185–204]). The other pseudo-splines with various orders fill in the gaps between the B-splines and orthogonal refinable functions for the first type and between B-splines and interpolatory refinable functions for the second type. This gives a wide range of choices of refinable functions that meets various demands for balancing the approximation power, the length of the support, and the regularity in applications. This paper will give a regularity analysis of pseudo-splines of the both types and provide various constructions of wavelets and framelets. It is easy to see that the regularity of the first type of pseudo-splines is between B-spline and orthogonal refinable function of the same order. However, there is no precise regularity estimate for pseudo-splines in general. In this paper, an optimal estimate of the decay of the Fourier transform of the pseudo-splines is given. The regularity of pseudo-splines can then be deduced and hence, the regularity of the corresponding wavelets and framelets. The asymptotical regularity analysis, as the order of the pseudo-splines goes to infinity, is also provided. Furthermore, we show that in all tight frame systems constructed from pseudo-splines by methods provided both in [I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14 (1) (2003) 1–46] and this paper, there is one tight framelet from the generating set of the tight frame system whose dilations and shifts already form a Riesz basis for $L_2(\mathbb{R})$.

Keywords: B-splines; Framelets; Pseudo-splines; Riesz wavelets; Tight frame

☆ The research is supported by several grants at Department of Mathematics, National University of Singapore.
* Corresponding author.
E-mail addresses: bdong@nus.edu.sg (B. Dong), matzuows@nus.edu.sg (Z. Shen).

1063-5203/$ – see front matter
doi:10.1016/j.acha.2006.04.008

1. Introductions

Pseudo-splines were first introduced in [12,24] in order to construct tight framelets with required approximation order of the truncated frame series. Pseudo-splines are refinable and compactly supported. They give a wide variety of choices of refinable functions and provide large flexibilities in wavelet and framelet constructions and filter designs. Functions such as B-splines, interpolatory, or orthogonal refinable functions are special cases of them. An optimal regularity analysis of pseudo-splines does not come easily, as it has already been illustrated in a regularity estimate of the orthonormal refinable functions, which is a special case of pseudo-splines (see [4] and [10]). This paper gives a systematic regularity analysis of both types of pseudo-splines. Furthermore, the technique used to estimate the regularity of pseudo-splines can be applied to discover that the tight frame systems derived from the methods given in both [12] and this paper have one framelet whose dilations and shifts already form a Riesz basis for $L_2(\mathbb{R})$. This leads to a new understanding of the structure of the pseudo-spline tight frame systems.

A function $\phi \in L_2(\mathbb{R})$ is *refinable* if it satisfies the refinement equation

$$\phi = 2\sum_{k\in\mathbb{Z}} a(k)\phi(2\cdot - k) \tag{1.1}$$

for some sequence $a \in \ell_2(\mathbb{Z})$, called *refinement mask* of ϕ.

By $L_p(\mathbb{R})$, for $1 \leqslant p < \infty$, we denote all the functions $f(x)$ satisfying

$$\|f(x)\|_{L_p(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty;$$

and $\ell_p(\mathbb{Z})$ the set of all sequences c defined on $\mathbb{Z}$ such that

$$\|c\|_{\ell_p(\mathbb{Z})} := \left(\sum_{j\in\mathbb{Z}} |c(j)|^p\right)^{\frac{1}{p}} < \infty.$$

The *Fourier transform* of a function $f \in L_1(\mathbb{R})$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(t)e^{-i\xi t}\,\mathrm{d}t, \quad \xi \in \mathbb{R},$$

which can be extended to more general function spaces (e.g., $L_2(\mathbb{R})$) naturally. Similarly, the *Fourier series* $\hat{c}$ of a sequence $c \in \ell_2(\mathbb{Z})$ is defined by

$$\hat{c}(\xi) := \sum_{j\in\mathbb{Z}} c(j)e^{-ij\xi}, \quad \xi \in \mathbb{R}.$$

With these, the refinement equation (1.1) can be written in terms of its Fourier transform as

$$\hat{\phi}(\xi) = \hat{a}(\xi/2)\hat{\phi}(\xi/2), \quad \xi \in \mathbb{R}.$$

We also call $\hat{a}$ a refinement mask for convenience.

Pseudo-splines are defined in terms of their refinement masks. It starts with the simple identity, for given nonnegative integers l and m with $l \leqslant m-1$,

$$1 = \left(\cos^2(\xi/2) + \sin^2(\xi/2)\right)^{m+l}. \tag{1.2}$$

The refinement masks of pseudo-splines are defined by the summation of the first $l+1$ terms of the binomial expansion of (1.2). In particular, the refinement mask of a *pseudo-spline of Type I with order* (m,l) is given by, for $\xi \in [-\pi,\pi]$,

$$\left|{}_1\hat{a}(\xi)\right|^2 := \left|{}_1\hat{a}_{(m,l)}(\xi)\right|^2 := \cos^{2m}(\xi/2)\sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2) \tag{1.3}$$

and the refinement mask of a *pseudo-spline of Type II with order* (m,l) is given by, for $\xi \in [-\pi,\pi]$,

$$ {}_2\hat{a}(\xi) := {}_2\hat{a}_{(m,l)}(\xi) := \cos^{2m}(\xi/2)\sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2). \tag{1.4}$$

Except for some special circumstances, we always drop the subscript "(m,l)" in ${}_1\hat{a}(\xi)_{(m,l)}$ and ${}_2\hat{a}(\xi)_{(m,l)}$ for simplicity. We note that the mask of Type I is obtained by taking the square root of the mask of Type II using the Fejér–Riesz lemma (see, e.g., [10] and [20]), i.e., ${}_2\hat{a}(\xi) = |{}_1\hat{a}(\xi)|^2$. Type I was introduced and used in [12] in their constructions of tight framelets.

The corresponding pseudo-splines can be defined in terms of their Fourier transforms, i.e.,

$$ {}_k\hat{\phi}(\xi) := \prod_{j=1}^{\infty} {}_k\hat{a}\big(2^{-j}\xi\big), \quad k=1,2. \tag{1.5}$$

The pseudo-splines with order $(m,0)$ for both types are B-splines. Recall that a B-spline with order m and its refinement mask are defined by

$$ \hat{B}_m(\xi) = e^{-ij\frac{\xi}{2}}\left(\frac{\sin(\xi/2)}{\xi/2}\right)^m \quad \text{and} \quad \hat{a}(\xi) = e^{-ij\frac{\xi}{2}}\cos^m(\xi/2), $$

where $j=0$ when m is even, $j=1$ when m is odd (for detailed discussions about B-splines, one may refer to [2]). The pseudo-splines of Type I with order $(m,m-1)$ are the refinable functions with orthonormal shifts (called orthogonal refinable functions) given in [11]. The pseudo-splines of Type II with order $(m,m-1)$ are the interpolatory refinable functions (which were first introduced in [14] and a systematic construction was given in [11]). Recall that a continuous function $\phi \in L_2(\mathbb{R})$ is interpolatory if $\phi(j) = \delta(j)$, $j \in \mathbb{Z}$, i.e., $\phi(0)=1$, and $\phi(j)=0$, for $j \neq 0$ (see, e.g., [14]). The other pseudo-splines fill in the gap between the B-splines and orthogonal or interpolatory refinable functions.

For fixed m, since the value of the mask $|{}_k\hat{a}(\xi)|$, for $k=1,2$ and $\xi \in \mathbb{R}$, increases with l (by (1) of Lemma 2.2 in Section 2), and the length of the mask ${}_k a$ also increases with l, we conclude that the decay rate of the Fourier transform of a pseudo-spline decreases with l and the support of the corresponding pseudo-spline increases with l. In particular, for fixed m, the pseudo-spline with order $(m,0)$ has the highest order of smoothness with the shortest support, the pseudo-spline with order $(m,m-1)$ has the lowest order of smoothness with the largest support in the family. When we move from B-splines to orthogonal or interpolatory refinable functions, we sacrifice the smoothness and short support of the B-splines to gain some other desirable properties, such as orthogonality or interpolatory property. What do we get for the pseudo-splines of the other orders? When we move from B-splines to pseudo-splines, we gain the approximation power of the truncated tight frame systems derived from them, as we will discuss below.

For a given $\phi \in L_2(\mathbb{R})$, a *shift* (integer translation) *invariant space* generated by $\phi \in L_2(\mathbb{R})$ is defined by

$$ V_0(\phi) := \overline{\operatorname{Span}\big\{\phi(\cdot - k),\ k \in \mathbb{Z}\big\}}. \tag{1.6}$$

Let

$$ V_n(\phi) := \big\{f\big(2^n\cdot\big)\colon\ f \in V_0(\phi),\ n \in \mathbb{Z}\big\}. \tag{1.7}$$

The function ϕ is the generator of V_0, hence the generator of $V_n(\phi)$, $n \in \mathbb{Z}$. It is easy to see that for fixed m and type, pseudo-splines of all orders (m,l), $0 \leqslant l \leqslant m-1$, satisfy the same order of the Strang–Fix (SF) condition. (Type I pseudo-splines are of order m and Type II are of order $2m$.) Recall that a function ϕ satisfies the SF condition of order m if

$$ \hat{\phi}(0) \neq 0, \qquad \hat{\phi}^{(j)}(2\pi k) = 0, \quad j = 0,1,2,\ldots,m-1,\ k \in \mathbb{Z}\backslash\{0\}. $$

Assume that ϕ satisfies the SF condition of order m_0. Then, the order of the best approximation of a sufficiently smooth function f from $(V_n)_{n\in\mathbb{Z}}$ is m_0. Recall that $(V_n(\phi))_{n\in\mathbb{Z}}$ *provides approximation order* m_0 (or we can say that *the refinable function* ϕ *provides approximation order* m_0), if for all the f in the Sobolev space $W_2^{m_0}(\mathbb{R})$,

$$ \operatorname{dist}(f,V_n) := \min\big\{\|f-g\|_{L_2(\mathbb{R})}\colon\ g \in V_n\big\} = O\big(2^{-nm_0}\big). $$

Therefore, even though the $(V_n)_{n\in\mathbb{Z}}$ may be generated by a different pseudo-spline for the fixed type with order (m,l), $0 \leqslant l \leqslant m-1$, the corresponding spaces $(V_n)_{n\in\mathbb{Z}}$ provide the same approximation order. However, in many applications of wavelets and framelets, we normally use

$$\mathcal{P}_n : f \mapsto \sum_{k\in\mathbb{Z}} \langle f, \phi_{n,k}\rangle \phi_{n,k} \tag{1.8}$$

to approximate f, where $\phi_{n,k} := 2^{n/2}\phi(2^n\cdot - k)$. The operation $\mathcal{P}_n f$ may not provide the best approximation of f from V_n. We say that the *operator* $\mathcal{P}_n$ *provides approximation order* m_1, if for all f in the Sobolev space $W_2^{m_1}(\mathbb{R})$

$$\|f - \mathcal{P}_n f\|_{L_2(\mathbb{R})} = O\big(2^{-nm_1}\big).$$

As shown in [12], the approximation order of $\mathcal{P}_n f$ depends on the order of the zero of

$$1 - \big|\hat{a}(\xi)\big|^2$$

at the origin. In fact, if $1 - |\hat{a}|^2 = O(|\cdot|^{m_2})$ at the origin, then $m_1 = \min\{m_0, m_2\}$. For B-splines, m_2 never exceeds 2. This indicates that the approximation order of $\mathcal{P}_n$ can never exceed 2 even if a high order B-spline is used. On the other hand, for the pseudo-spline of either type with order (m,l), $0 \leqslant l \leqslant m-1$, the corresponding $m_2 = 2l+2$ (see Theorem 3.10). Therefore, the approximation order of $\mathcal{P}_n$, with a pseudo-spline with order (m,l), $0 \leqslant l \leqslant m-1$, as the underlying refinable function, is $\min\{m, 2l+2\}$ for Type I and $2l+2$ for Type II. More importantly, the approximation order of $\mathcal{P}_n$ determines the approximation order of the truncated series of a tight frame system. For given $\Psi := \{\psi_1, \psi_2, \ldots, \psi_r\}$, the system

$$X(\Psi) := \big\{\psi_{n,k} = 2^{n/2}\psi\big(2^n\cdot - k\big),\ \psi \in \Psi,\ n,k \in \mathbb{Z}\big\}$$

is a *tight frame* for $L_2(\mathbb{R})$ if

$$\sum_{g\in X(\Psi)} \big|\langle f, g\rangle\big|^2 = \|f\|^2_{L_2(\mathbb{R})}, \quad \forall f \in L_2(\mathbb{R}).$$

For $X(\Psi)$, define the *truncated operator* as

$$\mathcal{Q}_n : f \mapsto \sum_{\psi\in\Psi,\ k\in\mathbb{Z},\ j<n} \langle f, \psi_{j,k}\rangle\ \psi_{j,k}. \tag{1.9}$$

When the tight framelets Ψ are obtained via the unitary extension principle (see, e.g., Section 4.2) from the multiresolution analysis generated by the same ϕ, then [12, Lemma 2.4] shows that $\mathcal{P}_n f = \mathcal{Q}_n f$ for all $f \in L_2(\mathbb{R})$. Recall that for a compactly supported refinable function $\phi \in L_2(\mathbb{R})$, we define V_0 and V_n as in (1.6) and (1.7). Then, the sequence of spaces $(V_n)_{n\in\mathbb{Z}}$ forms a *multiresolution analysis* (MRA) generated by ϕ, i.e., (i) $V_n \subset V_{n+1}, \forall n \in \mathbb{Z}$; (ii) $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L_2(\mathbb{R})$, $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$ (see, e.g., [3] and [18]). The wavelet system $X(\Psi)$ is said to be MRA-based if there exists an MRA $(V_n)_{n\in\mathbb{Z}}$, such that $\Psi \in V_1$. If, in addition, the system $X(\Psi)$ is a (*tight*) *frame system*, we refer to the elements of Ψ as (*tight*) *framelets*.

Therefore, the tight frame system derived from a pseudo-spline normally gives better approximation order when the truncated series is used to approximate the underlying functions than that derived from B-splines. For fixed m, the choice of l depends entirely on applications. One needs to balance among the approximation order, the length of support of the wavelet, and regularity according to the practical problems in hand.

The rest of the paper is organized as follows: Section 2 gives some technical lemmata used in other sections. Section 3 focuses on the analysis of regularity and approximation order. In particular, the exact decay of the Fourier transforms of the pseudo-splines for all orders of both types are given. The asymptotical analysis is also provided. In Section 4, we show that in all tight frame systems constructed from pseudo-splines by methods provided both in [12] and this paper, there is one tight framelet from the generating set of the tight frame system whose dilations and shifts already form a Riesz basis for $L_2(\mathbb{R})$. Furthermore, (anti)symmetric tight framelets, which have a Riesz wavelet as one of the framelets, are designed.

2. Two lemmata

This section gives two key technical lemmata that will be used to prove several key results of this paper. We start with the following lemma on binomial coefficients, where (1) is well known (see, e.g., [8]) and the proof of (3) is rather technical but needed in Section 4.

Lemma 2.1. *For given nonnegative integers m, j, l, we have*:

(1) $\binom{m+1}{j} = \binom{m}{j} + \binom{m}{j-1}$ *for* $j \geqslant 1$ *and* $(j+1)\binom{m+j}{j+1} = (m+j)\binom{m-1+j}{j}$.
(2) $2(m+1)\sum_{j=0}^{l-1}\binom{m+l}{j} - l\sum_{j=0}^{l}\binom{m+l}{j} \geqslant 0$ *for* $m \geqslant 1$ *and* $1 \leqslant l \leqslant m-1$.
(3) $\frac{2^l\binom{m+l}{l}^{\frac{1}{2}}}{\sum_{j=0}^{l}\binom{m+l}{j}} \leqslant 1$ *for all* $m \geqslant 1$ *and* $0 \leqslant l \leqslant m-1$.

Proof. The identities in (1) are well known and can be proven directly by the definition of the binomial coefficients.

For (2), since $m > l$, we only need to check if

$$(m+1)\sum_{j=0}^{l-1}\binom{m+l}{j} - l\binom{m+l}{l} \geqslant 0$$

holds, which follows from the identity $(m+1)\binom{m+l}{l-1} = l\binom{m+l}{l}$.

Finally, we prove (3) by induction with respect to m. Since (3) is obviously true for $l = 0$, we now focus on $1 \leqslant l \leqslant m-1$. When $m = 1$, the inequality trivially holds. Assume (3) holds when $m = m_0$, i.e.,

$$2^{2l}\binom{m_0+l}{l} \leqslant \left(\sum_{j=0}^{l}\binom{m_0+l}{j}\right)^2$$

for all $1 \leqslant l \leqslant m_0 - 1$. Consider the case $m = m_0 + 1$. We first show that (3) holds for all l, where $1 \leqslant l \leqslant m_0 - 1$. For $1 \leqslant l \leqslant m_0 - 1$, we have

$$\begin{aligned}
2^{2l}\binom{m_0+l+1}{l} &\leqslant \frac{m_0+l+1}{m_0+1}\left(\sum_{j=0}^{l}\binom{m_0+l}{j}\right)^2 \quad \text{(by induction hypothesis)} \\
&= \left(\sum_{j=0}^{l}\binom{m_0+l}{j} + \left(\sqrt{\frac{m_0+l+1}{m_0+1}} - 1\right)\sum_{j=0}^{l}\binom{m_0+l}{j}\right)^2 \\
&< \left(\sum_{j=0}^{l}\binom{m_0+l}{j} + \frac{l}{2m_0+2}\sum_{j=0}^{l}\binom{m_0+l}{j}\right)^2 \\
&\leqslant \left(\sum_{j=0}^{l}\binom{m_0+l}{j} + \sum_{j=0}^{l-1}\binom{m_0+l}{j}\right)^2 \quad \text{(from (2))} \\
&= \left(\sum_{j=0}^{l}\binom{m_0+l+1}{j}\right)^2 \quad \text{(from (1))}.
\end{aligned}$$

This shows that (3) holds for all $1 \leqslant l \leqslant m_0 - 1$, it remains to show (3) holds for $l = m_0$, i.e., to show

$$2^{2m_0}\binom{2m_0+1}{m_0} \leqslant \left(\sum_{j=0}^{m_0}\binom{2m_0+1}{j}\right)^2. \tag{2.1}$$

Observe that

$$\sum_{j=0}^{m_0}\binom{2m_0+1}{j}=\frac{1}{2}\sum_{j=0}^{2m_0+1}\binom{2m_0+1}{j}=2^{2m_0}.$$

Then (2.1) is equivalent to

$$\binom{2m_0+1}{m_0}\leqslant\sum_{j=0}^{m_0}\binom{2m_0+1}{j},$$

which is obviously true. This concludes the proof of (3). □

Define

$$P_{m,l}(y):=\sum_{j=0}^{l}\binom{m+l}{j}y^j(1-y)^{l-j} \tag{2.2}$$

and

$$R_{m,l}(y):=(1-y)^m P_{m,l}(y), \tag{2.3}$$

where $y=\sin^2(\xi/2)$ and m,l are nonnegative integers with $l\leqslant m-1$. Then, it is obvious that

$$R_{m,l}\big(\sin^2(\xi/2)\big)={}_2\hat{a}(\xi).$$

Next, we give several basic properties of the polynomials $P_{m,l}(y)$ and $R_{m,l}(y)$. Parts (2)–(4) of the following lemma are mainly used in Sections 3.3 and 4.

Lemma 2.2. *For nonnegative integers m and l with $l\leqslant m-1$, let $P_{m,l}(y)$ and $R_{m,l}(y)$ be the polynomials defined in* (2.2) *and* (2.3). *Then*

(1) $P_{m,l}(y)=\sum_{j=0}^{l}\binom{m-1+j}{j}y^j$.
(2) $R'_{m,l}(y)=-(m+l)\binom{m+l-1}{l}y^l(1-y)^{m-1}$.
(3) *Define* $Q(y):=R_{m,l}(y)+R_{m,l}(1-y)$. *Then,*

$$\min_{y\in[0,1]}Q(y)=Q\left(\frac{1}{2}\right)=2^{1-m-l}\sum_{j=0}^{l}\binom{m+l}{j}.$$

(4) *Define* $S(y):=R^2_{m,l}(y)+R^2_{m,l}(1-y)$. *Then,*

$$\min_{y\in[0,1]}S(y)=S\left(\frac{1}{2}\right)=2^{1-2m-2l}\left(\sum_{j=0}^{l}\binom{m+l}{j}\right)^2.$$

Proof. For fixed m, we prove (1) by induction with respect to l. It is obviously true for $l=0$. Now suppose (1) holds for l_0. Consider $l=l_0+1$,

$$\begin{aligned}P_{m,l}(y)&=\sum_{j=0}^{l_0+1}\binom{m+l_0+1}{j}y^j(1-y)^{l_0-j+1}\\&=(1-y)^{l_0+1}+\sum_{j=1}^{l_0+1}\binom{m+l_0+1}{j}y^j(1-y)^{l_0-j+1}.\end{aligned}$$

Applying the first identity in (1) of Lemma 2.1, we have,

$$\begin{aligned}
P_{m,l}(y) &= (1-y)^{l_0+1} + \sum_{j=1}^{l_0+1} \binom{m+l_0}{j} y^j (1-y)^{l_0-j+1} + \sum_{j=0}^{l_0} \binom{m+l_0}{j} y^{j+1} (1-y)^{l_0-j} \\
&= (1-y) P_{m,l_0}(y) + y P_{m,l_0}(y) + \binom{m+l_0}{l_0+1} y^{l_0+1} \\
&= \sum_{j=0}^{l_0} \binom{m-1+j}{j} y^j + \binom{m+l_0}{l_0+1} y^{l_0+1} \quad \text{(by induction hypothesis)} \\
&= \sum_{j=0}^{l_0+1} \binom{m-1+j}{j} y^j.
\end{aligned}$$

We prove (2) by induction with respect to l for given m. It is obviously true when $l = 0$. Suppose (2) holds for l_0, i.e.,

$$R'_{m,l_0}(y) = -(m+l_0)\binom{m+l_0-1}{l_0} y^{l_0}(1-y)^{m-1}$$

and consider the case $l = l_0 + 1 \leqslant m - 1$. Using (1) and the definition of $R_{m,l}(y)$ in (2.3), we have

$$R_{m,l_0+1}(y) = (1-y)^m P_{m,l_0+1}(y) = (1-y)^m \left(P_{m,l_0}(y) + \binom{m+l_0}{l_0+1} y^{l_0+1} \right).$$

Since $R_{m,l_0}(y) = (1-y)^m P_{m,l_0}(y)$, we have

$$R_{m,l_0+1}(y) = \binom{m+l_0}{l_0+1} y^{l_0+1}(1-y)^m + R_{m,l_0}(y).$$

Then,

$$\begin{aligned}
R'_{m,l_0+1}(y) &= (l_0+1)\binom{m+l_0}{l_0+1} y^{l_0}(1-y)^m - m\binom{m+l_0}{l_0+1} y^{l_0+1}(1-y)^{m-1} \\
&\quad - (m+l_0)\binom{m+l_0-1}{l_0} y^{l_0}(1-y)^{m-1}.
\end{aligned}$$

Pulling the common factor $y^{l_0}(1-y)^{m-1}$ out, one obtains

$$\begin{aligned}
R'_{m,l_0+1}(y) &= y^{l_0}(1-y)^{m-1}\left((l_0+1)\binom{m+l_0}{l_0+1} - (l_0+1)\binom{m+l_0}{l_0+1} y \right. \\
&\quad \left. - m\binom{m+l_0}{l_0+1} y - (m+l_0)\binom{m+l_0-1}{l_0} \right).
\end{aligned}$$

By using the second identity in (1) of Lemma 2.1, one obtains

$$R'_{m,l_0+1}(y) = -(m+l_0+1)\binom{m+l_0}{l_0+1} y^{l_0+1}(1-y)^{m-1}.$$

This concludes the proof of (2).

For (3), we compute $Q'(y)$, i.e.,

$$Q'(y) = R'_{m,l}(y) + \big(R_{m,l}(1-y)\big)' = R'_{m,l}(y) - R'_{m,l}(1-y).$$

Applying (2), one obtains

$$Q'(y) = (m+l)\binom{m+l-1}{l}\big(y^{m-1}(1-y)^l - (1-y)^{m-1}y^l\big).$$

Now we show that $Q'(y) \leqslant 0$ on $[0, \frac{1}{2}]$, $Q'(y) \geqslant 0$ on $[\frac{1}{2}, 1]$. Note that

$$y^{m-l-1} \leqslant (1-y)^{m-l-1} \quad \text{for all } y \in \left[0, \frac{1}{2}\right].$$

Multiplying both sides by $y^l(1-y)^l$,

$$y^{m-1}(1-y)^l \leqslant (1-y)^{m-1}y^l \quad \text{for all } y \in \left[0, \frac{1}{2}\right].$$

Similarly we have

$$y^{m-1}(1-y)^l \geqslant (1-y)^{m-1}y^l \quad \text{for all } y \in \left[\frac{1}{2}, 1\right].$$

We conclude that

$$Q'(y) \begin{cases} \leqslant 0, & y \in [0, \frac{1}{2}], \\ \geqslant 0, & y \in [\frac{1}{2}, 1]. \end{cases}$$

This means that $Q(y)$ reaches its minimum value at the point $y = \frac{1}{2}$. Now we compute $Q(\frac{1}{2})$. Note that $Q(\frac{1}{2}) = 2R_{m,l}(\frac{1}{2}) = 2^{1-m}P_{m,l}(\frac{1}{2})$. Recall that $P_{m,l}(y)$ is defined in (2.2), i.e., $P_{m,l}(y) = \sum_{j=0}^{l}\binom{m+l}{j}y^j(1-y)^{l-j}$. Then

$$\min_{y\in[0,1]} Q(y) = Q\left(\frac{1}{2}\right) = 2^{1-m}2^{-l}\sum_{j=0}^{l}\binom{m+l}{j} = 2^{1-m-l}\sum_{j=0}^{l}\binom{m+l}{j}.$$

With (3), the proof of (4) is simpler. Since

$$S'(y) = 2R_{m,l}(y)R'_{m,l}(y) + 2R_{m,l}(1-y)\big(R_{m,l}(1-y)\big)',$$

using the identities

$$R_{m,l}(y) = (1-y)^m P_{m,l}(y),$$
$$R'_{m,l}(y) = -(m+l)\binom{m+l-1}{l}y^l(1-y)^{m-1}$$

and

$$\big(R_{m,l}(1-y)\big)' = (m+l)\binom{m+l-1}{l}y^{m-1}(1-y)^l,$$

we obtain

$$\frac{S'(y)}{2(m+l)\binom{m+l-1}{l}} = \sum_{j=0}^{l}\binom{m-1+j}{j}\big((1-y)^{l+j}y^{2m-1} - y^{l+j}(1-y)^{2m-1}\big).$$

For each $0 \leqslant j \leqslant l$ and $y \in [0, \frac{1}{2}]$, we have $y^{2m-l-j-1} \leqslant (1-y)^{2m-l-j-1}$; and for $y \in [\frac{1}{2}, 1]$, we have $y^{2m-l-j-1} \geqslant (1-y)^{2m-l-j-1}$. Then by similar arguments as in (3) we conclude,

$$S'(y) \begin{cases} \leqslant 0, & y \in [0, \frac{1}{2}], \\ \geqslant 0, & y \in [\frac{1}{2}, 1]. \end{cases}$$

Thus $\min_{y\in[0,1]} S(y) = S(\frac{1}{2})$. Since $R_{m,l}(\frac{1}{2}) = 2^{-m-l}\sum_{j=0}^{l}\binom{m+l}{j}$, we have

$$\min_{y\in[0,1]} S(y) = S\left(\frac{1}{2}\right) = 2R^2_{m,l}\left(\frac{1}{2}\right) = 2^{1-2m-2l}\left(\sum_{j=0}^{l}\binom{m+l}{j}\right)^2. \qquad \square$$

Remark 2.3. From (1) of Lemma 2.2 we know that the refinement mask of the pseudo-spline of Type I in (1.3) can be written as

$$\big|{}_1\hat{a}(\xi)\big|^2 = \cos^{2m}(\xi/2)\sum_{j=0}^{l}\binom{m-1+j}{j}\sin^{2j}(\xi/2).$$

Hence, the pseudo-spline of Type I with order $(m, m-1)$ is indeed the refinable function whose shifts form an orthonormal system constructed in [11] and the pseudo-spline of Type II with order $(m, m-1)$ is indeed the autocorrelation of the orthogonal refinable function, which is interpolatory.

3. Basics of pseudo-splines

This section is devoted to a systematic analysis of the regularity of pseudo-splines and approximation order of quasi-interpolatory operator $\mathcal{P}_n$ (see (1.8)) defined by pseudo-splines. These two are basic and essential properties of pseudo-splines. Indeed, the regularity of pseudo-splines determines the regularity of the corresponding wavelets and framelets; and the approximation order of $\mathcal{P}_n$ determines that of the truncated wavelet and framelet series. These two properties, together with the length of support, are the key criteria in selecting wavelets or framelets in various applications.

3.1. Regularity

In this section the regularity of the pseudo-splines is analyzed. For $\alpha = n + \beta$, $n \in \mathbb{N}$, $0 \leqslant \beta < 1$, the Hölder space C^α (see, e.g., [10]) is defined to be the set of functions which are n times continuously differentiable and such that the nth derivative $f^{(n)}$ satisfies the condition,

$$\left|f^{(n)}(x+h) - f^{(n)}(x)\right| \leqslant C|h|^\beta, \quad \forall x, h.$$

It is well known (see [10]) that if

$$\int_{\mathbb{R}} \left|\hat{f}(\xi)\right|\left(1+|\xi|\right)^\alpha < \infty,$$

then $f \in C^\alpha$. In particular, if $|\hat{f}(\xi)| \leqslant C(1+|\xi|)^{-1-\alpha-\varepsilon}$, then $f \in C^\alpha$.

The main idea here is to estimate the decay of the Fourier transform of pseudo-splines with order (m, l) in order to get the lower bound of the regularity of the pseudo-splines. It turns out that this lower bound coincides with the upper bound when m goes to infinity, as shown in Section 3.2. It is well known that the exact Sobolev regularity of a given refinable function can be obtained via its mask by applying the transfer operator and it is a very well studied area (see, e.g., [10,23] and references in therein). Although the exact Sobolev exponent of a give refinable function can be computed exactly by computing the spectrum of the transfer operator derived from the corresponding refinement mask, it is hard to analyze the Sobolev exponents of a class of refinable functions, such as pseudo-splines discussed here, systematically from the transfer operator approach. This is simply because a different refinable function will lead to a different transfer operator. This is the main reason why we give here a systematic estimate of the decay of the Fourier transform of pseudo-splines instead.

Since for any compactly supported refinable function ϕ in $L_2(\mathbb{R})$ with $\hat{\phi}(0) = 1$, the refinement mask a must satisfy $\hat{a}(0) = 1$ and $\hat{a}(\pi) = 0$ (see, e.g., [10,19]), then $\hat{a}(\xi)$ can be factorized as

$$\hat{a}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^n \mathcal{L}(\xi),$$

where n is the maximum multiplicity of zeros of $\hat{a}$ at π and $\mathcal{L}(\xi)$ is a trigonometric polynomial with $\mathcal{L}(0) = 1$. Hence, we have

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} \hat{a}\left(2^{-j}\xi\right) = \prod_{j=1}^{\infty}\left(\frac{1+e^{-i(2^{-j}\xi)}}{2}\right)^n \prod_{j=1}^{\infty} \mathcal{L}\left(2^{-j}\xi\right) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^n \prod_{j=1}^{\infty} \mathcal{L}\left(2^{-j}\xi\right).$$

This shows that any compactly supported refinable function in $L_2(\mathbb{R})$ is the convolution of a B-spline of some order, say n, with a distribution (see [21]). Indeed, a B-spline of order n can also be defined via its Fourier transform by

$$\hat{B}_n := \left(\frac{1-e^{-i\xi}}{i\xi}\right)^n.$$

The B-spline of order n is a piecewise polynomial of degree $n-1$ in $C^{n-1-\varepsilon}(\mathbb{R})$, supported on $[0, n]$, and has refinement mask

$$\left(\frac{1+e^{-i\xi}}{2}\right)^n.$$

Since $\mathcal{L}(\xi)$ is bounded, $\mathcal{L}(\xi)$ is actually the refinement mask of a refinable distribution. Therefore, ϕ is the convolution of the B-spline B_n with the distribution. The regularity of ϕ comes from the B-spline factor while the distribution factor takes away the regularity. But the distribution component also provides some desirable properties for ϕ, such as interpolatory properties, orthogonality of its shifts and approximation order of certain quasi-interpolants.

The decay of $|\hat{\phi}|$ can be characterized by $|\hat{a}|$ as stated in the following theorem. The proof of this theorem can be found in [10]. Note that in the following theorem, we write $|\hat{a}|$ in the form of

$$\left|\hat{a}(\xi)\right| = \left|\left(\frac{1+e^{-i\xi}}{2}\right)^n \mathcal{L}(\xi)\right| = \cos^n(\xi/2)\left|\mathcal{L}(\xi)\right|, \quad \xi \in [-\pi, \pi].$$

Theorem 3.1. *Let a be the refinement mask of the refinable function ϕ of the form*

$$\left|\hat{a}(\xi)\right| = \cos^n(\xi/2)\left|\mathcal{L}(\xi)\right|, \quad \xi \in [-\pi, \pi].$$

Suppose that

$$\begin{aligned} &\left|\mathcal{L}(\xi)\right| \leqslant \left|\mathcal{L}\left(\frac{2\pi}{3}\right)\right| \quad \text{for } |\xi| \leqslant \frac{2\pi}{3}, \\ &\left|\mathcal{L}(\xi)\mathcal{L}(2\xi)\right| \leqslant \left|\mathcal{L}\left(\frac{2\pi}{3}\right)\right|^2 \quad \text{for } \frac{2\pi}{3} \leqslant |\xi| \leqslant \pi. \end{aligned} \tag{3.1}$$

Then $|\hat{\phi}(\xi)| \leqslant C(1+|\xi|)^{-n+\kappa}$, with $\kappa = \log\left(\left|\mathcal{L}\left(\frac{2\pi}{3}\right)\right|\right)/\log 2$, and this decay is optimal.

This theorem allows us to estimate the decay of the Fourier transform of a refinable function via its refinement mask. Since $|{}_1\hat{\phi}|^2 = |{}_2\hat{\phi}|$, the decay rate of $|{}_1\hat{\phi}|$ is half of that of $|{}_2\hat{\phi}|$. Thus we can focus on the analysis of the decay of the Fourier transforms of pseudo-splines of Type II. Based on (1) of Lemma 2.2, we will show that $P_{m,l}(y)$, defined in (2.2), satisfies (3.1). This will lead directly to the estimate of the regularity of pseudo-splines. Note that the corresponding result for $l = m-1$ was proven in [4] which led to the optimal estimates for the decay of the Fourier transforms of the orthogonal and interpolatory refinable functions. Here, the more general result for pseudo-splines is obtained by a simpler proof than the original one of [4] and [10].

Proposition 3.2. *Let $P_{m,l}(y)$ be defined as in* (2.2)*, where l, m are nonnegative integers with $l \leqslant m-1$. Then*

$$P_{m,l}(y) \leqslant P_{m,l}\left(\frac{3}{4}\right) \quad \text{for } y \in \left[0, \frac{3}{4}\right], \tag{3.2}$$

$$P_{m,l}(y)P_{m,l}\big(4y(1-y)\big) \leqslant \left(P_{m,l}\left(\frac{3}{4}\right)\right)^2 \quad \text{for } y \in \left[\frac{3}{4}, 1\right]. \tag{3.3}$$

Proof. It is clear that (3.2) is true. Indeed, by using (1) of Lemma 2.2 we have that $P_{m,l}(y)$ is monotonically increasing on $[0, \frac{3}{4}]$ (in fact, it is monotonically increasing on $(0, \infty)$). Hence, we focus on the proof of (3.3).

Throughout this proof, we let m be fixed. Let

$$W_{m,l}(y) := P_{m,l}(y)P_{m,l}\big(4y(1-y)\big) - \left(P_{m,l}\left(\frac{3}{4}\right)\right)^2.$$

Then, the inequality (3.3) is equivalent to

$$W_{m,l}(y) \leqslant 0 \quad \text{for all } y \in \left[\frac{3}{4}, 1\right]. \tag{3.4}$$

In order to show (3.4), we show, instead,

$$W_{m,l+1}(y) - W_{m,l}(y) \leqslant 0 \quad \text{for all } y \in \left[\frac{3}{4}, 1\right],\ l = 0, 1, \ldots, m-2. \tag{3.5}$$

Note that since for $l = 0$, $P_{m,0}(y) = 1$ for all $y \in [0, 1]$, (3.4) is obviously true for $l = 0$. Hence, (3.4) follows from (3.5) and (3.3) follows from (3.4).

We now compute $W_{m,l+1}(y) - W_{m,l}(y)$. By (1) of Lemma 2.2, one obtains

$$\begin{aligned} W_{m,l+1}(y) - W_{m,l}(y) &= \left(\sum_{j=0}^{l+1}\binom{m-1+j}{j}y^j\right)\left(\sum_{j=0}^{l+1}\binom{m-1+j}{j}\big(4y(1-y)\big)^j\right) \\ &\quad - \left(\sum_{j=0}^{l}\binom{m-1+j}{j}y^j\right)\left(\sum_{j=0}^{l}\binom{m-1+j}{j}\big(4y(1-y)\big)^j\right) \\ &\quad + P_{m,l}^2\left(\frac{3}{4}\right) - P_{m,l+1}^2\left(\frac{3}{4}\right). \end{aligned}$$

Splitting the sum $\sum_{j=0}^{l+1}\binom{m-1+j}{j}y^j$, one obtains

$$\begin{aligned} W_{m,l+1}(y) - W_{m,l}(y) &= \left(\sum_{j=0}^{l}\binom{m-1+j}{j}y^j\right)\left(\sum_{j=0}^{l+1}\binom{m-1+j}{j}\big(4y(1-y)\big)^j\right) \\ &\quad + \binom{m+l}{l+1}y^{l+1}\sum_{j=0}^{l+1}\binom{m-1+j}{j}\big(4y(1-y)\big)^j \\ &\quad - \left(\sum_{j=0}^{l}\binom{m-1+j}{j}y^j\right)\left(\sum_{j=0}^{l}\binom{m-1+j}{j}\big(4y(1-y)\big)^j\right) \\ &\quad + P_{m,l}^2\left(\frac{3}{4}\right) - P_{m,l+1}^2\left(\frac{3}{4}\right). \end{aligned}$$

Combining the first and the third terms, one obtains

$$\begin{aligned} W_{m,l+1}(y) - W_{m,l}(y) &= \binom{m+l}{l+1}\left(\big(4y(1-y)\big)^{l+1}\sum_{j=0}^{l}\binom{m-1+j}{j}y^j\right. \\ &\quad \left. + y^{l+1}\sum_{j=0}^{l+1}\binom{m-1+j}{j}\big(4y(1-y)\big)^j\right) + P_{m,l}^2\left(\frac{3}{4}\right) - P_{m,l+1}^2\left(\frac{3}{4}\right). \end{aligned} \tag{3.6}$$

Since $W_{m,l+1}(\frac{3}{4}) - W_{m,l}(\frac{3}{4}) = 0 - 0 = 0$, it suffices to show that $W_{m,l+1}(y) - W_{m,l}(y)$ monotonically decreases on $[\frac{3}{4}, 1]$, which is equivalent to showing that

$$G(y) := \big(4y(1-y)\big)^{l+1}\sum_{j=0}^{l}\binom{m-1+j}{j}y^j + y^{l+1}\sum_{j=0}^{l+1}\binom{m-1+j}{j}\big(4y(1-y)\big)^j$$

monotonically decreases on $[\frac{3}{4}, 1]$. For this purpose, we obtain G' as follows:

$$\begin{aligned} G'(y) &= (l+1)(4-8y)\big(4y(1-y)\big)^l\sum_{j=0}^{l}\binom{m-1+j}{j}y^j + \big(4y(1-y)\big)^{l+1}\sum_{j=0}^{l-1}\binom{m+j}{j+1}(j+1)y^j \\ &\quad + (l+1)y^l\sum_{j=0}^{l+1}\binom{m-1+j}{j}\big(4y(1-y)\big)^j + y^{l+1}(4-8y)\sum_{j=0}^{l}\binom{m+j}{j+1}(j+1)\big(4y(1-y)\big)^j. \end{aligned}$$

Applying (1) of Lemma 2.1 to the second and the fourth term above, one obtains

$$\begin{aligned}G'(y) &= (l+1)(4-8y)\big(4y(1-y)\big)^l \sum_{j=0}^{l}\binom{m-1+j}{j} y^j + \big(4y(1-y)\big)^{l+1}\sum_{j=0}^{l}\binom{m-1+j}{j}(m+j)y^j \\ &\quad - (m+l)\binom{m-1+l}{l} y^l\big(4y(1-y)\big)^{l+1} + (l+1)y^l\sum_{j=0}^{l}\binom{m-1+j}{j}\big(4y(1-y)\big)^j \\ &\quad + (l+1)\binom{m+l}{l+1} y^l\big(4y(1-y)\big)^{l+1} + y^{l+1}(4-8y)\sum_{j=0}^{l}\binom{m-1+j}{j}(m+j)\big(4y(1-y)\big)^j.\end{aligned}$$

Since $(l+1)\binom{m+l}{l+1} = (m+l)\binom{m-1+l}{l}$ by (1) of Lemma 2.1, we have

$$(l+1)\binom{m+l}{l+1} y^l\big(4y(1-y)\big)^{l+1} - (m+l)\binom{m-1+l}{l} y^l\big(4y(1-y)\big)^{l+1} = 0.$$

Hence,

$$\begin{aligned}G'(y) &= \sum_{j=0}^{l}\binom{m-1+j}{j}\Big((l+1)(4-8y)\big(4y(1-y)\big)^l y^j + (m+j)\big(4y(1-y)\big)^{l+1} y^j \\ &\quad + (l+1)y^l\big(4y(1-y)\big)^j + (m+j)(4-8y)y^{l+1}\big(4y(1-y)\big)^j\Big).\end{aligned}$$

Pulling the common factor $y^j(4y(1-y))^j$ out from each term of the above summation, one obtains

$$\begin{aligned}G'(y) &= \sum_{j=0}^{l}\binom{m-1+j}{j} y^j\big(4y(1-y)\big)^j\Big((l+1)(4-8y)\big(4y(1-y)\big)^{l-j} \\ &\quad + (m+j)\big(4y(1-y)\big)^{l+1-j} + (l+1)y^{l-j} + (m+j)(4-8y)y^{l+1-j}\Big).\end{aligned}$$

For $0 \leqslant j \leqslant l \leqslant m-2$, consider

$$\begin{aligned}g_{l,j}(y) &:= (l+1)(4-8y)\big(4y(1-y)\big)^{l-j} + (m+j)\big(4y(1-y)\big)^{l+1-j} \\ &\quad + (l+1)y^{l-j} + (m+j)(4-8y)y^{l+1-j} \\ &= (l+1)\big(4y(1-y)\big)^{l-j}\big(4y(1-y)-(8y-4)\big) + (l+1)y^{l-j}\big(1-(8y-4)\big) \\ &\quad + (m+j-l-1)\Big(\big(4y(1-y)\big)^{l+1-j} - (8y-4)y^{l+1-j}\Big).\end{aligned}$$

The inequality $4y(1-y) \leqslant y$ and $8y-4 \geqslant 2$ for $y \in [\frac{3}{4}, 1]$ show that $g_{l,j}(y) \leqslant 0$ and $G'(y) \leqslant 0$ on this interval. □

Remark 3.3. It is clear that $W_{m,0}(y) = 0$, $y \in [\frac{3}{4}, 1]$, because $P_{m,0} = 1$. It was also proven by [4] that $W_{m,m-1}(y) \leqslant 0$, $y \in [\frac{3}{4}, 1]$, which is equivalent to (3.3). The decreasing of $W_{m,l}(y)$, for $y \in [\frac{3}{4}, 1]$, as l increases shown above indicates some difficulties to prove (3.3) directly for an arbitrary l, $0 < l < m-1$, since it has a smaller margin than the case when $l = m-1$. In fact, to some extent, the proof of (3.3) for the case when $l = m-1$ relies on a numerical check for $m \leqslant 12$ (see [10]). Inequality (3.3) for the case $l = m-1$ as proven in [4] (also see [10]) is one of the cornerstones of the wavelet theory, because it immediately leads to the optimal estimate of the decay of the Fourier transforms (hence, an estimate of the regularity) of both interpolatory and orthogonal refinable functions. We take a different approach here by proving that $W_{m,l}(y)$, $y \in [\frac{3}{4}, 1]$, decreases as l increases. As a result, we obtain (3.3) for all $0 \leqslant l \leqslant m-1$ by the fact that $W_{m,0}(y) = 0$, $y \in [\frac{3}{4}, 1]$. This shows that introducing the concepts of the pseudo-splines gives a better understanding and a more complete picture of the proof of (3.3) and also, we hope, enriches the theory of wavelets. Note that the proof of (3.3) for all $0 \leqslant l \leqslant m-1$ given here does not rely on any numerical computation and is simpler than the original proof of [4] and [10].

With this proposition, one obtains the regularity of pseudo-splines by applying Theorem 3.1.

Theorem 3.4. *Let ${}_2\phi$ be the pseudo-spline of Type II with order (m,l). Then*

$$\left|{}_2\hat{\phi}(\xi)\right| \leqslant C\left(1+|\xi|\right)^{-2m+\kappa},$$

where $\kappa = \log\left(P_{m,l}\left(\frac{3}{4}\right)\right)/\log 2$. Consequently, ${}_2\phi \in C^{\alpha_2-\varepsilon}$ with $\alpha_2 = 2m-\kappa-1$. Furthermore, let ${}_1\phi$ be the pseudo-spline of Type I with order (m,l). Then

$$\left|{}_1\hat{\phi}(\xi)\right| \leqslant C\left(1+|\xi|\right)^{-m+\frac{\kappa}{2}}.$$

Consequently, ${}_1\phi \in C^{\alpha_1-\varepsilon}$ with $\alpha_1 = m-\frac{\kappa}{2}-1$.

Proof. Since

$$P_{m,l}\left(\sin^2(\xi)\right) = \sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2),$$

the refinement mask of the pseudo-spline of Type II with order (m,l) is

$${}_2\hat{a}(\xi) = \cos^{2m}(\xi/2)\sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2) = \left(\cos(\xi/2)\right)^{2m}P_{m,l}\left(\sin^2(\xi/2)\right).$$

Hence, $|\mathcal{L}(\xi)|$ in Theorem 3.1 is exactly $P_{m,l}(\sin^2(\xi/2))$ here. Let $y = \sin^2(\xi/2)$. Applying (3.2) of Proposition 3.2

$$P_{m,l}(y) \leqslant P_{m,l}\left(\frac{3}{4}\right), \quad y \in \left[0,\frac{3}{4}\right],$$

we have

$$\left|\mathcal{L}(\xi)\right| = P_{m,l}\left(\sin^2(\xi/2)\right) = P_{m,l}(y) \leqslant P_{m,l}\left(\frac{3}{4}\right) = P_{m,l}\left(\sin^2\left(\frac{\pi}{3}\right)\right) \quad \text{for } |\xi| \leqslant \frac{2\pi}{3}.$$

Note that

$$\left|\mathcal{L}(2\xi)\right| = P_{m,l}\left(\sin^2(\xi)\right) = P_{m,l}\left(4\sin^2(\xi/2)\left(1-\sin^2(\xi/2)\right)\right) = P_{m,l}\left(4y(1-y)\right).$$

Applying (3.3) of Proposition 3.2

$$P_{m,l}(y)P_{m,l}\left(4y(1-y)\right) \leqslant \left(P_{m,l}\left(\frac{3}{4}\right)\right)^2, \quad y \in \left[\frac{3}{4},1\right],$$

we have

$$\begin{aligned}\left|\mathcal{L}(\xi)\mathcal{L}(2\xi)\right| &= P_{m,l}\left(\sin^2(\xi/2)\right)P_{m,l}\left(4\sin^2(\xi/2)\left(1-\sin^2(\xi/2)\right)\right)\\ &= P_{m,l}(y)P_{m,l}\left(4y(1-y)\right) \leqslant \left(P_{m,l}\left(\frac{3}{4}\right)\right)^2 = \left(P_{m,l}\left(\sin^2\left(\frac{\pi}{3}\right)\right)\right)^2 \quad \text{for } \frac{2\pi}{3} \leqslant |\xi| \leqslant \pi.\end{aligned}$$

Hence, by Theorem 3.1, ${}_2\hat{\phi}$ satisfies

$$\left|{}_2\hat{\phi}(\xi)\right| \leqslant C\left(1+|\xi|\right)^{-2m+\kappa},$$

where $\kappa = \log\left(P_{m,l}\left(\frac{3}{4}\right)\right)/\log 2$. This leads to ${}_2\phi \in C^{\alpha_2-\varepsilon}$, where $\alpha_2 = 2m-\kappa-1$.

Since the decay of $|{}_1\hat{\phi}|$ is exactly half of $|{}_2\hat{\phi}|$, we have

$$\left|{}_1\hat{\phi}(\xi)\right| \leqslant C\left(1+|\xi|\right)^{-m+\frac{\kappa}{2}},$$

consequently, ${}_1\phi \in C^{\alpha_1-\varepsilon}$, where $\alpha_2 = m-\frac{\kappa}{2}-1$. □

Table 1
Decay rates $\beta_{m,l} = 2m - \kappa$ of pseudo-splines of Type II with order (m, l) for $2 \leqslant m \leqslant 8$ and $1 \leqslant l \leqslant m - 1$

(m,l)	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=7$
$m=2$	2.67807						
$m=3$	4.29956	3.27208					
$m=4$	6.00000	4.73321	3.82507				
$m=5$	7.75207	6.27890	5.19506	4.35316			
$m=6$	9.54057	7.88626	6.64465	5.66363	4.86449		
$m=7$	11.35614	9.54057	8.15608	7.04717	6.13261	5.36349	
$m=8$	13.19265	11.23182	9.71691	8.48992	7.46770	6.59988	5.85310

Table 1 gives the decay rates $\beta_{m,l}$ of the Fourier transform of pseudo-splines of Type II with order (m, l) for $2 \leqslant m \leqslant 8$ and $1 \leqslant l \leqslant m - 1$. The regularity exponent of the corresponding pseudo-spline is, at least, $\alpha_2 = \beta_{m,l} - 1 - \varepsilon$. The decay rate of the Fourier transform of the pseudo-spline of Type I with the same order is $\frac{\beta_{m,l}}{2}$ and its regularity exponent α_1 is $\frac{\alpha_2 - 1}{2}$. Therefore, the table shows that for either type of the pseudo-splines and fixed order m, the decay rate of their Fourier transform decreases as l increases, while for fixed l, it increases as m increases. This is true indeed as shown in the following proposition.

Proposition 3.5. *Let $\beta_{m,l} = 2m - \kappa$ with $\kappa = \log P_{m,l}\left(\frac{3}{4}\right)/\log 2$ as given in Theorem* 3.4 *and* $0 \leqslant l \leqslant m - 1$. *Then*:

(1) *For fixed m, $\beta_{m,l}$ decreases as l increases.*
(2) *For fixed l, $\beta_{m,l}$ increases as m increases.*
(3) *When $l = m - 1$, $\beta_{m,l}$ increases as m increases.*

Consequently, the decay rate $\beta_{2,1} = 2.67807$ is the smallest among all $\beta_{m,l}$, with $m \geqslant 2$ and $0 \leqslant l \leqslant m - 1$.

Proof. Part (1) follows directly from (1) of Lemma 2.2, which shows that $P_{m,l}\left(\frac{3}{4}\right)$ increases as l increases for fixed m.

For part (2), note that

$$\beta_{m,l} = 2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2}.$$

Consider

$$2^{\beta_{m,l}} = 2^{2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2}} = \frac{4^m}{P_{m,l}\left(\frac{3}{4}\right)} = \frac{1}{4^{-m} P_{m,l}\left(\frac{3}{4}\right)}.$$

Hence, part (2) is equivalent to the fact that

$$I_m := 4^{-m} P_{m,l}\left(\frac{3}{4}\right)$$

decreases as m increases for fixed l, which is equivalent to showing that for fixed $0 \leqslant l \leqslant m - 1$,

$$I_{m+1} - I_m < 0. \tag{3.7}$$

Note that

$$I_{m+1} - I_m = 4^{-m-1} P_{m+1,l}\left(\frac{3}{4}\right) - 4^{-m} P_{m,l}\left(\frac{3}{4}\right) = 4^{-m-1} \sum_{j=0}^{l} \left(\binom{m+j}{j} - 4\binom{m-1+j}{j} \right) \left(\frac{3}{4}\right)^j.$$

Inequality (3.7) follows from the fact that for $0 \leqslant j \leqslant m - 1$,

$$\binom{m+j}{j} = \frac{m+j}{m}\binom{m-1+j}{j} = \left(1 + \frac{j}{m}\right)\binom{m-1+j}{j} < 4\binom{m-1+j}{j}. \tag{3.8}$$

This concludes the proof of part (2).

For part (3), using a similar argument as in the proof of part (2), one can derive that it is equivalent to showing that

$$J_m := 4^{-m} P_{m,m-1}\left(\frac{3}{4}\right)$$

decreases as m increases, which, in turn, is equivalent to showing that

$$J_{m+1} - J_m < 0 \quad \text{for } m \geqslant 1. \tag{3.9}$$

Note that, similar to the proof of part (2), we have

$$J_{m+1} - J_m = 4^{-m-1}\left(\sum_{j=0}^{m}\binom{m+j}{j}\left(\frac{3}{4}\right)^j - 4\sum_{j=0}^{m-1}\binom{m-1+j}{j}\left(\frac{3}{4}\right)^j\right).$$

Let

$$M := \sum_{j=0}^{m}\binom{m+j}{j}\left(\frac{3}{4}\right)^j - 4\sum_{j=0}^{m-1}\binom{m-1+j}{j}\left(\frac{3}{4}\right)^j.$$

Then, (3.9) is equivalent to $M < 0$ for $m \geqslant 1$. It is easy to check that $M < 0$, when $m = 1$. We consider now the case when $m \geqslant 2$. First, we note that

$$\begin{aligned} M &= \sum_{j=0}^{m-1}\binom{m+j}{j}\left(\frac{3}{4}\right)^j - 4\sum_{j=0}^{m-1}\binom{m-1+j}{j}\left(\frac{3}{4}\right)^j + \binom{2m}{m}\left(\frac{3}{4}\right)^m \\ &= \sum_{j=1}^{m-1}\binom{m-1+j}{j-1}\left(\frac{3}{4}\right)^j - 3\sum_{j=0}^{m-1}\binom{m-1+j}{j}\left(\frac{3}{4}\right)^j + \binom{2m}{m}\left(\frac{3}{4}\right)^m, \end{aligned}$$

where the last identity follows from (1) of Lemma 2.1. Substituting j for $j-1$ in the first term, one obtains that

$$M = \frac{3}{4}\sum_{j=0}^{m-2}\binom{m+j}{j}\left(\frac{3}{4}\right)^j - 3\sum_{j=0}^{m-1}\binom{m-1+j}{j}\left(\frac{3}{4}\right)^j + \binom{2m}{m}\left(\frac{3}{4}\right)^m. \tag{3.10}$$

Splitting the second term in (3.10), one obtains

$$M = \frac{3}{4}\sum_{j=0}^{m-2}\binom{m+j}{j}\left(\frac{3}{4}\right)^j - 3\sum_{j=0}^{m-2}\binom{m-1+j}{j}\left(\frac{3}{4}\right)^j + \binom{2m}{m}\left(\frac{3}{4}\right)^m - 3\binom{2m-2}{m-1}\left(\frac{3}{4}\right)^{m-1}. \tag{3.11}$$

For the last two terms of (3.11), we have

$$\begin{aligned} \binom{2m}{m}\left(\frac{3}{4}\right)^m - 3\binom{2m-2}{m-1}\left(\frac{3}{4}\right)^{m-1} &= \left(\frac{3}{4}\right)^m\left(\binom{2m}{m} - 4\binom{2m-2}{m-1}\right) \\ &= \left(\frac{3}{4}\right)^m\left(\left(4 - \frac{2}{m}\right)\binom{2m-2}{m-1} - 4\binom{2m-2}{m-1}\right) \\ &< 0. \end{aligned}$$

Therefore,

$$\begin{aligned} M &< \frac{3}{4}\sum_{j=0}^{m-2}\binom{m+j}{j}\left(\frac{3}{4}\right)^j - 3\sum_{j=0}^{m-2}\binom{m-1+j}{j}\left(\frac{3}{4}\right)^j \\ &< \sum_{j=0}^{m-2}\binom{m+j}{j}\left(\frac{3}{4}\right)^j - 3\sum_{j=0}^{m-2}\binom{m-1+j}{j}\left(\frac{3}{4}\right)^j \\ &= \sum_{j=0}^{m-2}\left(\binom{m+j}{j} - 3\binom{m-1+j}{j}\right)\left(\frac{3}{4}\right)^j. \end{aligned}$$

Applying (3.8), one obtains, for $0 \leqslant j \leqslant m-2$,

$$\binom{m+j}{j} = \left(1+\frac{j}{m}\right)\binom{m-1+j}{j} < 3\binom{m-1+j}{j}.$$

Therefore, we conclude that $M < 0$ and part (3) follows.

Finally, note that the decay rate of the Fourier transform of the pseudo-spline of Type I with order $(2,1)$ is $\frac{\beta_{m,l}}{2} \approx 1.33903$. Hence, it follows from parts (1)–(3) that the decay rate of an arbitrary pseudo-spline of either type with order (m,l), $m > 2$, $0 \leqslant l \leqslant m-1$, is higher than 1.33903. □

3.2. Asymptotical analysis

Proposition 3.5 reveals that the decay rates of the Fourier transforms of pseudo-splines of either type increase as m increases for fixed l and decrease as l increases for fixed m. In this section, we give an asymptotical analysis of the decay rate which, in turn, gives an asymptotical analysis of the regularity of ${}_1\phi$ and ${}_2\phi$ as the order $(m,l) \to \infty$.

Theorem 3.6. *Let ${}_1\phi$ and ${}_2\phi$ be the pseudo-splines of Types I and II respectively with order (m,l). Fix $l = \lfloor \lambda m \rfloor$, $0 \leqslant \lambda \leqslant 1$, where $\lfloor \lambda m \rfloor$ denotes the largest integer which is smaller than or equal to λm. Then, we have*

$$\left|{}_1\hat{\phi}(\xi)\right| \leqslant C\big(1+|\xi|\big)^{-\frac{\mu}{2}m} \quad \textit{and} \quad \left|{}_2\hat{\phi}(\xi)\right| \leqslant C\big(1+|\xi|\big)^{-\mu m},$$

where $\mu = \frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^{\lambda}}{\log 2}$, asymptotically for large m. This means that the asymptotic rate of the pseudo-spline of Type I and Type II are $\frac{\mu}{2}$ and μ, respectively.

Proof. As the estimate of Type I follows immediately from that of Type II, we only give the estimate for pseudo-splines of Type II. We first prove the following fact:

$$x^{-l}P_{m,l}(x) \geqslant y^{-l}P_{m,l}(y) \quad \text{for } 0 < x \leqslant y \leqslant 1. \tag{3.12}$$

Indeed, assertion (1) of Lemma 2.2 gives for $0 < x \leqslant y \leqslant 1$,

$$x^{-l}P_{m,l}(x) = \sum_{j=0}^{l}\binom{m-1+j}{j}x^{j-l} \geqslant \sum_{j=0}^{l}\binom{m-1+j}{j}y^{j-l} = y^{-l}P_{m,l}(y).$$

The key step to compute the asymptotic rate is to estimate the upper and lower bound of $P_{m,l}\left(\frac{3}{4}\right)$ in terms of m and l. For this, let $x = \frac{3}{4}$ and $y = 1$ in (3.12). Then we obtain

$$P_{m,l}\left(\frac{3}{4}\right) \geqslant \left(\frac{3}{4}\right)^{l}P_{m,l}(1) = \left(\frac{3}{4}\right)^{l}\binom{m+l}{l}. \tag{3.13}$$

Next, let $x = \frac{1}{2}$ and $y = \frac{3}{4}$ in (3.12), we obtain

$$P_{m,l}\left(\frac{3}{4}\right) \leqslant \left(\frac{3}{2}\right)^{l}P_{m,l}\left(\frac{1}{2}\right).$$

Since

$$P_{m,l}\left(\frac{1}{2}\right) = \sum_{j=0}^{l}\binom{m+l}{j}2^{-j}2^{j-l} = 2^{-l}\sum_{j=0}^{l}\binom{m+l}{j},$$

one obtains

$$P_{m,l}\left(\frac{3}{4}\right) \leqslant \left(\frac{3}{4}\right)^{l}\sum_{j=0}^{l}\binom{m+l}{j}. \tag{3.14}$$

Putting (3.13) and (3.14) together, we obtain the following estimates of $P_{m,l}\left(\frac{3}{4}\right)$,

$$\left(\frac{3}{4}\right)^l \binom{m+l}{l} \leqslant P_{m,l}\left(\frac{3}{4}\right) \leqslant \left(\frac{3}{4}\right)^l \sum_{j=0}^{l} \binom{m+l}{j}.$$

For $l \leqslant m-1$, we have

$$\sum_{j=0}^{l} \binom{m+l}{j} \leqslant m \binom{m+l}{l}.$$

Hence,

$$\left(\frac{3}{4}\right)^l \binom{m+l}{l} \leqslant P_{m,l}\left(\frac{3}{4}\right) \leqslant m \left(\frac{3}{4}\right)^l \binom{m+l}{l}. \tag{3.15}$$

Next, we will use this estimate to analyze the decay of ${}_2\hat{\phi}$ with order (m,l) as m goes to infinity. The upper bound of $P_{m,l}\left(\frac{3}{4}\right)$ in (3.15) gives

$$2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2} \geqslant 2m - \frac{\log\left(m\left(\frac{3}{4}\right)^l \binom{m+l}{l}\right)}{\log 2}.$$

We estimate the right-hand side of the above inequality asymptotically for large (m,l) to obtain the asymptotical lower bound of $2m - \frac{\log P_{m,l}(\frac{3}{4})}{\log 2}$. For this, we first recall the Stirling approximation, i.e., $m! \sim \sqrt{2\pi}\, e^{(m+\frac{1}{2})\log m - m}$ (see, e.g., [15]), where $a_m \sim b_m$ means that $\frac{a_m}{b_m} \to 1,\ m \to \infty$. By Stirling approximation, we have

$$\log m! \sim \log \sqrt{2\pi}\, e^{(m+\frac{1}{2})\log m - m} \sim m \log m - m. \tag{3.16}$$

Applying (3.16), one obtains

$$\begin{aligned} \log \binom{m+l}{l} &= \log(m+l)! - \log m! - \log l! \\ &\sim (m+l)\log(m+l) - (m+l) - (m\log m - m) - (l\log l - l) \\ &\sim (m+l)\log(m+l) - m\log m - l\log l. \end{aligned}$$

Thus,

$$\begin{aligned} 2m - \frac{\log\left(m\left(\frac{3}{4}\right)^l \binom{m+l}{l}\right)}{\log 2} &= 2m - \frac{\log m + l\log\frac{3}{4} + \log\binom{m+l}{l}}{\log 2} \\ &\sim m\left(2 - \frac{\frac{l}{m}\log\frac{3}{4} + (1+\frac{l}{m})\log(m+l) - \log m - \frac{l}{m}\log l}{\log 2}\right). \end{aligned}$$

By the assumption, $l = \lfloor \lambda m \rfloor,\ 0 \leqslant \lambda \leqslant 1$. Hence, when m is sufficiently large, $\frac{l}{m} \sim \lambda$ and therefore,

$$2m - \frac{\log\left(m\left(\frac{3}{4}\right)^l \binom{m+l}{l}\right)}{\log 2} \sim m\left(2 - \frac{\log(1+\lambda)\left(\frac{3+3\lambda}{4\lambda}\right)^\lambda}{\log 2}\right) = m\left(\frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^\lambda}{\log 2}\right).$$

Now we obtain the asymptotical lower bound of $2m - \frac{\log P_{m,l}(\frac{3}{4})}{\log 2}$, i.e., asymptotically, for large m with $l = \lfloor \lambda m \rfloor$,

$$2m - \frac{\log\left|P_{m,l}\left(\frac{3}{4}\right)\right|}{\log 2} \geqslant m\left(\frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^\lambda}{\log 2}\right). \tag{3.17}$$

Next, we use the left-hand side of (3.15) to obtain the asymptotical upper bound of $2m - \frac{\log P_{m,l}(\frac{3}{4})}{\log 2}$. First note that (3.15) gives

$$2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2} \leqslant 2m - \frac{l\log\frac{3}{4} + \log\binom{m+l}{l}}{\log 2}.$$

Applying arguments similar to the estimate of the lower bound by using (3.16), we will obtain the following:

$$2m - \frac{l\log\frac{3}{4} + \log\binom{m+l}{l}}{\log 2} \sim m\left(2 - \frac{\frac{l}{m}\log\frac{3}{4} + (1+\frac{l}{m})\log(m+l) - \log m - \frac{l}{m}\log l}{\log 2}\right)$$
$$\sim m\left(\frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^{\lambda}}{\log 2}\right).$$

This leads to the asymptotical lower bound of $2m - \frac{\log P_{m,l}(\frac{3}{4})}{\log 2}$, i.e., asymptotically, for large m with $l = \lfloor \lambda m \rfloor$,

$$2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2} \leqslant m\left(\frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^{\lambda}}{\log 2}\right). \tag{3.18}$$

Combining (3.17) and (3.18), we conclude that for large m, the asymptotical upper and lower bounds coincide and equal to

$$2m - \frac{\log P_{m,l}\left(\frac{3}{4}\right)}{\log 2} \sim m\left(\frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^{\lambda}}{\log 2}\right) = \mu m. \tag{3.19}$$

Therefore Eq. (3.19) gives that, fixing $l = \lfloor \lambda m \rfloor$ and asymptotically, for large m, we have

$$\left|{}_2\hat{\phi}(\xi)\right| \leqslant C\left(1+|\xi|\right)^{-\mu m} \quad \text{and} \quad \left|{}_1\hat{\phi}(\xi)\right| \leqslant C\left(1+|\xi|\right)^{-\frac{\mu}{2}m},$$

where $\mu = \frac{\log\left(\frac{4}{1+\lambda}\right)^{\lambda+1}\left(\frac{\lambda}{3}\right)^{\lambda}}{\log 2}$. □

Remark 3.7. The above theorem shows that, asymptotically for large m, the smoothness of the pseudo-splines of Types I and II increases at a rate $\mu/2$ and μ, respectively. The proof of Theorem 3.6 also leads to the following two observations:

(1) Consider pseudo-splines of Type II with order $(m, m-p)$, where p is a fixed positive integer independent of m. The asymptotic rate is $2 - \frac{\log 3}{\log 2} \approx 0.4150$. Indeed, when $l = m - p$, $\lambda \sim \frac{l}{m} = \frac{m-p}{m} \sim 1$ for sufficiently large m. Similarly, for pseudo-splines of Type I with order $(m, m-p)$, the corresponding asymptotic rate is $1 - \frac{\log 3}{2\log 2} \approx 0.2075$.
(2) Assume that l is fixed for all m. The asymptotic rates of pseudo-splines of Types I and II with order (m, l) are 1 and 2, respectively. This is simply because, for the fixed integer l, $\lambda \sim \frac{l}{m} \sim 0$ for sufficiently large m.

Example 3.8. In Table 2, we give μ, the asymptotical rate of pseudo-splines of Type II with order $(m, \lfloor \lambda m \rfloor)$, as m goes to infinity and the parameter $\lambda = \frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, 1$. The asymptotic rate μ_0 for pseudo-splines of Type I with the same order is just $\mu_0 = \frac{\mu}{2}$.

3.3. Approximation order

We follow [12] to give a brief discussion of the approximation order of $\mathcal{P}_n$ through $\mathcal{Q}_n$, where $\mathcal{P}_n$ is given by (1.8) with the underlying refinable function ϕ and $\mathcal{Q}_n$ is given by (1.9) with the underlying tight framelets Ψ. Characterizations of approximation order of $\mathcal{Q}_n$ were given in [12, Theorem 2.8]. Furthermore, [12, Lemma 2.4] says that $\mathcal{P}_n = \mathcal{Q}_n$ on $L_2(\mathbb{R})$ when the tight framelets Ψ are obtained via the unitary extension principle (see Section 4.2 for the UEP) from the MRA generated by the same refinable function ϕ. The following theorem is a special case of [12, Theorem 2.8] with the understanding $\mathcal{P}_n = \mathcal{Q}_n$.

Table 2
Asymptotically for large m, the smoothness of ${}_2\phi$ increases at rate μ, which is given in the following table with some choices of l

$m \to \infty$	$l = 0$	$l = \frac{m}{10}$	$l = \frac{m}{8}$	$l = \frac{m}{6}$	$l = \frac{m}{4}$	$l = \frac{m}{2}$	$l = m - 1$
$\mu \approx$	2.0000	1.5581	1.4857	1.3789	1.2013	0.8301	0.4150

Theorem 3.9. *Let ϕ be a pseudo-spline of order (m,l) with refinement mask a. Let $\mathcal{P}_n$ be the operator as defined in* (1.8) *with ϕ as the underlying refinable function. Then the approximation order of the operator $\mathcal{P}_n$ is* $\min\{m, m_1\}$, *with m_1 the order of the zero of $1-|\hat{a}|^2$ at the origin.*

With this, we have the following:

Theorem 3.10. *Let m and l be nonnegative integers satisfying $l \leqslant m-1$.*

(1) *Let ${}_1\phi$ be the pseudo-spline of Type I with order (m,l) and ${}_1\hat{a}$ be its refinement mask. Then the corresponding operator $\mathcal{P}_n$ provides approximation order* $\min\{m, 2l+2\}$.
(2) *Let ${}_2\phi$ be the pseudo-spline of Type II with order (m,l) and ${}_2\hat{a}$ be its refinement mask. Then the corresponding operator $\mathcal{P}_n$ provides approximation order $2l+2$.*

Proof. It was shown in [12] that $1-|{}_1\hat{a}| = O(|\cdot|^{2l+2})$. Therefore, Theorem 3.9 gives the rest of the proof of (1). For (2), we compute the order of zeros of $1-|{}_2\hat{a}|^2$ at the origin. We rewrite $1-|{}_2\hat{a}|^2$ as

$$1-|{}_2\hat{a}|^2 = 1 - R_{m,l}^2\big(\sin^2(\xi/2)\big),$$

where $R_{m,l}(y)$ was defined in (2.3). It is obvious that for $\xi=0$, $1-R_{m,l}^2(\sin^2(\xi/2))=0$. Recall that the derivative of $R_{m,l}(y)$ was given by (2) of Lemma 2.2, i.e.,

$$R'_{m,l}(y) = -(m+l)\binom{m+l-1}{l} y^l (1-y)^{m-1}. \tag{3.20}$$

Applying (3.20) to take the first derivative of $1-R_{m,l}^2(\sin^2(\xi/2))$ with respect to ξ, one obtains

$$\begin{aligned}
\big(1-R_{m,l}^2\big(\sin^2(\xi/2)\big)\big)' &= -2R_{m,l}\big(\sin^2(\xi/2)\big)R'_{m,l}\big(\sin^2(\xi/2)\big)\big(\sin^2(\xi/2)\big)' \\
&= 2R_{m,l}\big(\sin^2(\xi/2)\big)\left((m+l)\binom{m+l-1}{l}\sin^{2l}(\xi/2)\cos^{2m-2}(\xi/2)\right)\big(\sin^2(\xi/2)\big)' \\
&= 2(m+l)\binom{m+l-1}{l}R_{m,l}\big(\sin^2(\xi/2)\big)\sin^{2l+1}(\xi/2)\cos^{2m-1}(\xi/2).
\end{aligned}$$

Since $R_{m,l}(\sin^2(\xi/2))$ and $\cos^{2m-1}(\xi/2)$ are equal to 1 when $\xi=0$ and since $\sin^{2l+1}(\xi/2)$ has zero of order $2l+1$ at $\xi=0$, we conclude that

$$1-\big|{}_2\hat{a}(\xi)\big|^2 = 1-R_{m,l}^2\big(\sin^2(\xi/2)\big) = O\big(|\xi|^{2l+2}\big).$$

Then Theorem 3.9 shows that the approximation order of $\mathcal{P}_n$ with the pseudo-spline of Type II as the underlying refinable function is $\min\{2m, 2l+2\} = 2l+2$ for $0 \leqslant l \leqslant m-1$.

Remark 3.11. The above result says that when $l \leqslant \frac{m}{2}-1$, the approximation order of a pseudo-spline of Type I with order (m,l) and one of Type II with the same order are the same, although the support of the Type I is half of that of Type II. When $l > \frac{m}{2}-1$, the approximation order of Type I is m and Type II is $2l+2 > m$. The regularity of Type II is about two times that of Type I with the same order. Furthermore, one can obtain symmetric short Riesz wavelets and tight framelets from pseudo-splines of Type II, as we will see in the last section.

4. Riesz wavelets in framelets

In this section, we focus on the structure of the tight frame systems constructed from pseudo-splines by applying the unitary extension principle [22]. We show that in almost all pseudo-spline tight frame systems constructed both in [12] and the symmetric tight frame systems constructed by a pseudo-spline of Type II in this section, there is one framelet whose dilations and shifts already form a Riesz basis for $L_2(\mathbb{R})$.

4.1. Riesz wavelets

For a given ψ, define the *wavelet system*

$$X(\psi) := \{\psi_{n,k} = 2^{n/2}\psi(2^n \cdot - k)\colon\ n, k \in \mathbb{Z}\}.$$

We call $X(\psi)$ a *Bessel system* if for some $C_1 > 0$ and for every $f \in L_2(\mathbb{R})$,

$$\sum_{g \in X(\psi)} \left|\langle f, g\rangle\right|^2 \leqslant C_1 \|f\|^2_{L_2(\mathbb{R})}.$$

A Bessel system $X(\psi)$ is a *Riesz basis* if there exists $C_2 > 0$ such that,

$$C_2 \|\{c_{n,k}\}\|_{\ell_2(\mathbb{Z}^2)} \leqslant \left\| \sum_{(n,k)\in\mathbb{Z}^2} c_{n,k}\psi_{n,k} \right\|_{L_2(\mathbb{R})} \quad \text{for all } \{c_{n,k}\} \in \ell_2(\mathbb{Z}^2)$$

and the span of $\{\psi_{n,k}\colon\ n, k \in \mathbb{Z}\}$ is dense in $L_2(\mathbb{R})$. The function ψ is called *Riesz wavelet* if $X(\psi)$ forms a Riesz basis for $L_2(\mathbb{R})$ and $X(\psi)$ is also called the *Riesz wavelet system*.

As all pseudo-splines are compactly supported, refinable and in $L_2(\mathbb{R})$, the sequence of spaces $(V_n)_{n\in\mathbb{Z}}$ defined via (1.7) forms an MRA. Since the objective here is to construct Riesz wavelets, one needs to start with stable refinable functions. Indeed, it was shown in [13, Proposition 1.1 and Lemma 2.2] that all pseudo-splines are stable (in fact, we proved in [13] that the shifts of them are linearly independent, which is stronger than stable).

For a given stable refinable function $\phi \in L_2(\mathbb{R})$, the key step in the construction of the Riesz wavelet ψ is to select some desirable sequence b, called a *wavelet mask*. The *wavelet* ψ is then defined by b and the corresponding refinable function ϕ as

$$\psi := 2\sum_{k\in\mathbb{Z}} b(k)\phi(2\cdot - k).$$

It can be written equivalently in the Fourier domain as

$$\hat{\psi}(\xi) = \hat{b}(\xi/2)\hat{\phi}(\xi/2).$$

When $\{\phi(\cdot - k)\colon\ k \in \mathbb{Z}\}$ forms an orthonormal basis for $V_0(\phi)$, e.g., ϕ is a pseudo-spline of Type I with order $(m, m-1)$, define

$$\psi := 2\sum_{k\in\mathbb{Z}} b(k)\phi(2\cdot - k) \quad \text{with } b(k) = (-1)^{k-1}\overline{a(1-k)},\ k \in \mathbb{Z}, \tag{4.1}$$

or equivalently,

$$\hat{b}(\xi) = e^{-i\xi}\overline{\hat{a}(\xi+\pi)}.$$

Then the corresponding wavelet system $X(\psi)$ with the pseudo-spline of Type I with order $(m, m-1)$ being the underlying refinable function forms an orthonormal basis for $L_2(\mathbb{R})$. We are interested to know whether the function ψ defined in (4.1) is a Riesz wavelet, when the refinable function ϕ is chosen to be the pseudo-splines with other orders. In fact, it was shown in [17] that it is true, when ϕ is a B-spline, i.e., a pseudo-spline with order $(m, 0)$ or when ϕ is a pseudo-spline of Type II with order $(m, m-1)$. In the rest of this section we will show that for all pseudo-splines, the wavelet defined by (4.1) is a Riesz wavelet. To prove this, we use the following theorem which is the special case of [17, Theorem 2.1]. When both refinement masks are finitely supported, a similar result was already obtained before in [5,6,9].

Theorem 4.1. *Let a be a finitely supported refinement mask of a refinable function $\phi \in L_2(\mathbb{R})$ with $\hat{a}(0) = 1$ and $\hat{a}(\pi) = 0$, such that $\hat{a}$ can be factorized into the form*

$$\left|\hat{a}(\xi)\right| = \left|\left(\frac{1+e^{-i\xi}}{2}\right)^n \mathcal{L}(\xi)\right| = \cos^n(\xi/2)\left|\mathcal{L}(\xi)\right|, \quad \xi \in [-\pi, \pi], \tag{4.2}$$

where $\mathcal{L}$ is the Fourier series of a finitely supported sequence with $\mathcal{L}(\pi) \neq 0$. Suppose that

$$\left|\hat{a}(\xi)\right|^2 + \left|\hat{a}(\xi+\pi)\right|^2 \neq 0, \quad \xi \in [-\pi, \pi].$$

Define

$$\hat{\psi}(2\xi) := e^{-i\xi}\overline{\hat{a}(\xi+\pi)}\hat{\phi}(\xi)$$

and

$$\tilde{\mathcal{L}}(\xi) := \frac{\mathcal{L}(\xi)}{|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2}. \tag{4.3}$$

Assume that

$$\rho_{\mathcal{L}} := \left\|\mathcal{L}(\xi)\right\|_{L_\infty(\mathbb{R})} < 2^{n-\frac{1}{2}} \quad \textit{and} \quad \rho_{\tilde{\mathcal{L}}} := \left\|\tilde{\mathcal{L}}(\xi)\right\|_{L_\infty(\mathbb{R})} < 2^{n-\frac{1}{2}}. \tag{4.4}$$

Then $X(\psi)$ is a Riesz basis for $L_2(\mathbb{R})$.

As we will show, the key step in the application of the above theorem is to estimate the upper bound of $|\mathcal{L}(\xi)|$ and $|\tilde{\mathcal{L}}(\xi)|$. Recall that the refinement masks of pseudo-splines of Types I and II are, for $\xi \in [-\pi, \pi]$,

$$\left|{}_1\hat{a}(\xi)\right| := \cos^m(\xi/2)\left(\sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2)\right)^{\frac{1}{2}} \tag{4.5}$$

and

$${}_2\hat{a}(\xi) := \cos^{2m}(\xi/2)\sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2). \tag{4.6}$$

Hence, the corresponding $\mathcal{L}$ function in (4.2) for pseudo-splines of Type I is

$$\left|{}_1\mathcal{L}(\xi)\right| = \left(\sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2)\right)^{\frac{1}{2}}$$

and for pseudo-splines of Type II is

$$\left|{}_2\mathcal{L}(\xi)\right| = \sum_{j=0}^{l}\binom{m+l}{j}\sin^{2j}(\xi/2)\cos^{2(l-j)}(\xi/2).$$

Denoting $y = \sin^2(\xi/2)$, we have

$$|{}_1\hat{a}| = \left((1-y)^m P_{m,l}(y)\right)^{\frac{1}{2}}, \qquad {}_2\hat{a} = (1-y)^m P_{m,l}(y), \tag{4.7}$$

and

$$|{}_1\mathcal{L}| = \left(P_{m,l}(y)\right)^{\frac{1}{2}}, \qquad |{}_2\mathcal{L}| = P_{m,l}(y). \tag{4.8}$$

Furthermore, we have

$$\left|{}_1\hat{a}(\xi)\right|^2 + \left|{}_1\hat{a}(\xi+\pi)\right|^2 = R_{m,l}(y) + R_{m,l}(1-y)$$

and

$$\left|{}_2\hat{a}(\xi)\right|^2 + \left|{}_2\hat{a}(\xi+\pi)\right|^2 = R^2_{m,l}(y) + R^2_{m,l}(1-y),$$

with $y = \sin^2(\xi/2)$. Hence,

$$|{}_1\tilde{\mathcal{L}}| = \frac{(P_{m,l}(y))^{\frac{1}{2}}}{R_{m,l}(y) + R_{m,l}(1-y)} \quad \text{and} \quad |{}_2\tilde{\mathcal{L}}| = \frac{P_{m,l}(y)}{R^2_{m,l}(y) + R^2_{m,l}(1-y)}. \tag{4.9}$$

The estimation of $\|{}_1\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})}$ and $\|{}_2\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})}$ are based on the following result:

Proposition 4.2. *Let m and l be given nonnegative integers with $l \leqslant m-1$ and $|_1\tilde{\mathcal{L}}|$ and $|_2\tilde{\mathcal{L}}|$ be defined in* (4.9). *Then,*

(1) $\|_1\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} = \sup_{y\in[0,1]} \frac{(P_{m,l}(y))^{\frac{1}{2}}}{R_{m,l}(y)+R_{m,l}(1-y)} < 2^{m-\frac{1}{2}}$.

(2) $\|_2\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} = \sup_{y\in[0,1]} \frac{P_{m,l}(y)}{R^2_{m,l}(y)+R^2_{m,l}(1-y)} < 2^{2m-\frac{1}{2}}$.

Proof. Note that from (1) of Lemma 2.2,

$$P_{m,l}(y) = \sum_{j=0}^{l} \binom{m+l}{j} y^j (1-y)^{l-j} = \sum_{j=0}^{l} \binom{m-1+j}{j} y^j, \quad y \in [0,1], \tag{4.10}$$

hence both $(P_{m,l}(y))^{\frac{1}{2}}$ and $P_{m,l}(y)$ attain their maximum on $[0,1]$ at the point 1 and the maximum values are:

$$\big(P_{m,l}(1)\big)^{\frac{1}{2}} = \binom{m+l}{l}^{\frac{1}{2}} \quad \text{and} \quad P_{m,l}(1) = \binom{m+l}{l}.$$

By (3) of Lemma 2.2, one obtains

$$\|_1\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} = \sup_{y\in[0,1]} \frac{(P_{m,l}(y))^{\frac{1}{2}}}{R_{m,l}(y)+R_{m,l}(1-y)} \leqslant \binom{m+l}{l}^{\frac{1}{2}} \max_{y\in[0,1]} \frac{1}{R_{m,l}(y)+R_{m,l}(1-y)} \leqslant \frac{2^{m+l-1}\binom{m+l}{l}^{\frac{1}{2}}}{\sum_{j=0}^{l}\binom{m+l}{j}}.$$

Applying (3) of Lemma 2.1, i.e.,

$$\frac{2^l\binom{m+l}{l}^{\frac{1}{2}}}{\sum_{j=0}^{l}\binom{m+l}{j}} \leqslant 1, \tag{4.11}$$

one obtains

$$\|_1\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} \leqslant 2^{m-1} < 2^{m-\frac{1}{2}}.$$

The proof of (2) is similar to that of (1). Indeed, by (4) of Lemma 2.2

$$\|_2\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} = \sup_{y\in[0,1]} \frac{P_{m,l}(y)}{R^2_{m,l}(y)+R^2_{m,l}(1-y)} \leqslant \binom{m+l}{l} \max_{y\in[0,1]} \frac{1}{R^2_{m,l}(y)+R^2_{m,l}(1-y)} = \frac{2^{2m+2l-1}\binom{m+l}{l}}{\big(\sum_{j=0}^{l}\binom{m+l}{j}\big)^2}.$$

Applying (4.11) again, we have

$$\|_2\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} \leqslant 2^{2m-1} < 2^{2m-\frac{1}{2}}. \qquad \square$$

Theorem 4.3. *Let ${}_k\phi$, $k=1,2$, be the pseudo-spline of Types I and II with order (m,l). The refinement masks ${}_ka$, $k=1,2$, are given in* (1.3) *and* (1.4). *Define*

$${}_k\hat{\psi}(2\xi) := e^{-i\xi}\,\overline{{}_k\hat{a}(\xi+\pi)}\,{}_k\hat{\phi}(\xi), \quad k=1,2, \tag{4.12}$$

then $X({}_k\psi)$ forms a Riesz basis for $L_2(\mathbb{R})$.

Proof. To apply Theorem 4.1, we first note that

$$\big|{}_1\hat{a}(\xi)\big|^2 + \big|{}_1\hat{a}(\xi+\pi)\big|^2 = R_{m,l}\big(\sin^2(\xi/2)\big) + R_{m,l}\big(\cos^2(\xi/2)\big) \neq 0$$

and

$$\big|{}_2\hat{a}(\xi)\big|^2 + \big|{}_2\hat{a}(\xi+\pi)\big|^2 = R^2_{m,l}\big(\sin^2(\xi/2)\big) + R^2_{m,l}\big(\cos^2(\xi/2)\big) \neq 0$$

for all $\xi \in [-\pi,\pi]$, where $R_{m,l}$ is defined in (2.3) (by (3) and (4) of Lemma 2.2).

Next, one needs to check whether

$$\rho_{1\mathcal{L}} = \|_1\mathcal{L}\|_{L_\infty(\mathbb{R})} < 2^{m-\frac{1}{2}}, \qquad \rho_{2\mathcal{L}} = \|_2\mathcal{L}\|_{L_\infty(\mathbb{R})} < 2^{2m-\frac{1}{2}}, \tag{4.13}$$

$$\rho_{1\tilde{\mathcal{L}}} = \|_1\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} < 2^{m-\frac{1}{2}} \quad \text{and} \quad \rho_{2\tilde{\mathcal{L}}} = \|_2\tilde{\mathcal{L}}\|_{L_\infty(\mathbb{R})} < 2^{2m-\frac{1}{2}} \tag{4.14}$$

hold. Inequalities in (4.14) follows from Proposition 4.2.

For (4.13), we note that for both $k = 1$ and $k = 2$, we have

$$\left|{}_k\hat{a}(\xi)\right|^2 + \left|{}_k\hat{a}(\xi+\pi)\right|^2 \leqslant 1 \quad \text{for all } \xi \in \mathbb{R}.$$

Hence,

$$\left|{}_k\mathcal{L}(\xi)\right| \leqslant \left|{}_k\tilde{\mathcal{L}}(\xi)\right| \quad \text{for all } \xi \in \mathbb{R}.$$

This concludes the proof. □

In [12], three constructions of tight framelets were given for pseudo-splines of Type I. The number of framelets is either two or three. Interested readers may consult [12] Section 3.1 for details. We observe that in all the three constructions, one of the framelets ψ_1 is defined by

$$\hat{\psi}_1 := \hat{b}_1(\xi/2)\hat{\phi}(\xi/2),$$

where

$$\hat{b}_1 := e^{-i\xi}\overline{\hat{a}(\cdot+\pi)}$$

and $\hat{a}$ is the refinement mask of a pseudo-spline. It was shown in Theorem 4.3 of this paper that $X(\psi_1)$ forms a Riesz basis for $L_2(\mathbb{R})$. This implies that all pseudo-spline tight frame systems constructed in [12] already have one of the subsystems form a Riesz basis for $L_2(\mathbb{R})$. We further remark that it was observed in [17] that the same phenomenon occurs for the tight spline frame systems constructed in [12]. This, together with our new finding here, gives insight into the redundant structure of tight frame systems given in [12].

4.2. Symmetric framelets

In this section, we give a construction of symmetric tight framelets from pseudo-splines of Type II by using the unitary extension principle of [22]. The constructions of symmetric tight framelets from a symmetric refinable functions by using the unitary extension principle have been discussed in [1,7,16].

We note that the constructions of tight framelets given in [12] can also be applied to pseudo-splines of Type II. However, the constructions there cannot guarantee that all tight framelets are symmetric, even though pseudo-splines of Type II are symmetric. Therefore, in this section, we make use of the symmetry of the pseudo-splines of Type II to obtain symmetric tight framelets. Furthermore, the results of the previous subsection reveal that Construction 4.4 below also has one framelet ψ such that $X(\psi)$ itself already forms a Riesz basis for $L_2(\mathbb{R})$.

The construction here is based on the *unitary extension principle* (UEP) of [22]. We give a brief discussion here while the more general version and comprehensive discussions of the UEP can be found in [12] and [22].

Let $\hat{a}$ be the refinement mask of $\phi \in L_2(\mathbb{R})$ with $\hat{a}(0) = 1$ and let $\hat{b}_j$, $j = 1, 2, \ldots, r$, be wavelet masks. If $\hat{a}$ and $\hat{b}_j$ are trigonometric polynomials that satisfy

$$\hat{a}(\xi)\overline{\hat{a}(\xi+\nu)} + \sum_{j=1}^{r} \hat{b}_j(\xi)\overline{\hat{b}_j(\xi+\nu)} = \begin{cases} 1, & \nu = 0, \\ 0, & \nu = \pi \end{cases} \tag{4.15}$$

for all $\xi \in [-\pi, \pi]$ and $\Psi := \{\psi_1, \psi_2, \ldots, \psi_r\} \subset L_2(\mathbb{R})$ are given by

$$\hat{\psi}_j(2\xi) := \hat{b}_j(\xi)\hat{\phi}(\xi), \quad j = 1, 2, \ldots, r,$$

then the UEP asserts that $X(\Psi)$ is a tight frame for $L_2(\mathbb{R})$.

When applying constructions in [12] on pseudo-splines of Type II to obtain a set of three tight framelets, only the first framelet is symmetric. To overcome this, one can apply [16, Construction 3.4] to convert these framelets to

a set of five symmetric or antisymmetric tight framelets. It was further shown in [16] that Construction 3.4 leads to new tight framelets from the same MRA as the old tight framelets whenever the old ones are derived from the MRA generated by a symmetric refinable function. We forgo the idea of giving the details of this construction and leave it to readers by consulting [16], because next we will give a different approach that leads to a symmetric tight frame system with only three generators. The ideas of this construction are based on those of [7] and one of the constructions of [12]. Note that the construction here is generic and can be applied to any symmetric refinable function whose mask is a trigonometric polynomial and satisfies

$$|\hat{a}|^2 + \left|\hat{a}(\cdot+\pi)\right|^2 \leqslant 1. \tag{4.16}$$

Construction 4.4. Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function with its trigonometric polynomial refinement mask $\hat{a}$ satisfying $\hat{a}(0) = 1$ and (4.16). Moreover, we assume that ϕ, hence its refinement mask $\hat{a}$, is symmetric about the origin. Let

$$T = 1 - |\hat{a}|^2 - \left|\hat{a}(\cdot+\pi)\right|^2 \quad \text{and} \quad \mathcal{A} := \frac{\sqrt{T}}{2},$$

where $\sqrt{T}$ is obtained via the Fejér–Riesz lemma. Define

$$\hat{b}_1(\xi) := e^{-i\xi}\overline{\hat{a}(\xi+\pi)}, \qquad \hat{b}_2(\xi) := \mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi) \quad \text{and} \quad \hat{b}_3(\xi) := e^{-i\xi}\overline{\hat{b}_2(\xi+\pi)}.$$

Let $\Psi := \{\psi_1,\ \psi_2,\ \psi_3\}$, where

$$\hat{\psi}_j(\xi) := \hat{b}_j(\xi/2)\hat{\phi}(\xi/2), \quad j = 1, 2, 3. \tag{4.17}$$

Then $X(\Psi)$ is a tight frame for $L_2(\mathbb{R})$. Moreover, ψ_1 is symmetric about $\frac{1}{2}$, ψ_2 is symmetric about $\frac{1}{4}$ and ψ_3 is antisymmetric about $\frac{1}{4}$. We also note that since ψ_1 is defined exactly the same as (4.12) in Theorem 4.3, $X(\psi_1)$ forms a Riesz basis for $L_2(\mathbb{R})$ when ϕ is a pseudo-spline. Furthermore, since $\hat{b}_2$ and $\hat{b}_3$ have zeros at both 0 and π, one can check easily that neither the shifts of ψ_2 nor those of ψ_3 can form a Riesz system. Hence, $X(\psi_2)$ and $X(\psi_3)$ cannot form a Riesz basis for $L_2(\mathbb{R})$.

Proof. In order to verify that $X(\Psi)$ is a tight frame for $L_2(\mathbb{R})$, one needs to show that the masks $\{\hat{a}, \hat{b}_1, \hat{b}_2, \hat{b}_3\}$ satisfy (4.15). Note that

$$\hat{b}_1 = e^{-i\xi}\overline{\hat{a}(\cdot+\pi)} \quad \text{and} \quad \hat{b}_3 = e^{-i\xi}\overline{\hat{b}_2(\cdot+\pi)}.$$

Hence,

$$\hat{a}\overline{\hat{a}(\cdot+\pi)} + \sum_{j=1}^{3}\hat{b}_j\overline{\hat{b}_j(\cdot+\pi)} = \hat{a}\overline{\hat{a}(\cdot+\pi)} - \hat{a}\overline{\hat{a}(\cdot+\pi)} + \hat{b}_2\overline{\hat{b}_2(\cdot+\pi)} - \hat{b}_2\overline{\hat{b}_2(\cdot+\pi)} = 0.$$

Next, we show that

$$|\hat{a}|^2 + \sum_{j=1}^{3}|\hat{b}_j|^2 = 1. \tag{4.18}$$

Since

$$|\hat{a}|^2 + |\hat{b}_1|^2 = |\hat{a}|^2 + \left|\hat{a}(\cdot+\pi)\right|^2,$$

it remains to show that

$$|\hat{b}_2|^2 + |\hat{b}_3|^2 = 1 - |\hat{a}|^2 - \left|\hat{a}(\cdot+\pi)\right|^2 = T.$$

Since

$$\left|\mathcal{A}(\xi)\right|^2 = \frac{1}{4}T(\xi) = \frac{1}{4}\left(1 - \left|\hat{a}(\xi)\right|^2 - \left|\hat{a}(\xi+\pi)\right|^2\right)$$

and since T is π-periodic, the spectral factorization (which is based on the Fejér–Riesz lemma) leads to the function $\mathcal{A}(\xi)$ also to be π-periodic. Furthermore, the Fourier coefficients of $\mathcal{A}(\xi)$ are real. Hence, we have

$$\mathcal{A}(\xi) = \mathcal{A}(\xi+\pi) \quad \text{and} \quad \left|\mathcal{A}(\xi)\right|^2 = \left|\mathcal{A}(-\xi)\right|^2 \quad \text{for all } \xi \in \mathbb{R}. \tag{4.19}$$

Since

$$b_2(\xi) = \mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi) \quad \text{and} \quad \hat{b}_3(\xi) = e^{-i\xi}\overline{\hat{b}_2(\cdot+\pi)} = e^{-i\xi}\mathcal{A}(-\xi) - \mathcal{A}(\xi),$$

applying (4.19), one obtains

$$\begin{aligned}\left|\hat{b}_2(\xi)\right|^2 &= \left(\mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi)\right)\left(\overline{\mathcal{A}(\xi)} + e^{i\xi}\overline{\mathcal{A}(-\xi)}\right)\\ &= \left|\mathcal{A}(\xi)\right|^2 + \left|\mathcal{A}(-\xi)\right|^2 + e^{i\xi}\mathcal{A}(\xi)\overline{\mathcal{A}(-\xi)} + e^{-i\xi}\mathcal{A}(-\xi)\overline{\mathcal{A}(\xi)}\\ &= 2\left|\mathcal{A}(\xi)\right|^2 + e^{i\xi}\mathcal{A}(\xi)\overline{\mathcal{A}(-\xi)} + e^{-i\xi}\mathcal{A}(-\xi)\overline{\mathcal{A}(\xi)}\end{aligned}$$

and

$$\begin{aligned}\left|\hat{b}_3(\xi)\right|^2 &= \left(e^{-i\xi}\mathcal{A}(-\xi) - \mathcal{A}(\xi)\right)\left(e^{i\xi}\overline{\mathcal{A}(-\xi)} - \overline{\mathcal{A}(\xi)}\right)\\ &= \left|\mathcal{A}(\xi)\right|^2 + \left|\mathcal{A}(-\xi)\right|^2 - e^{i\xi}\mathcal{A}(\xi)\overline{\mathcal{A}(-\xi)} - e^{-i\xi}\mathcal{A}(-\xi)\overline{\mathcal{A}(\xi)}\\ &= 2\left|\mathcal{A}(\xi)\right|^2 - e^{i\xi}\mathcal{A}(\xi)\overline{\mathcal{A}(-\xi)} - e^{-i\xi}\mathcal{A}(-\xi)\overline{\mathcal{A}(\xi)}.\end{aligned}$$

Hence,

$$\left|\hat{b}_2(\xi)\right|^2 + \left|\hat{b}_3(\xi)\right|^2 = 4\left|\mathcal{A}(\xi)\right|^2 = T(\xi),$$

which gives (4.18) and thus concludes that the masks $\{\hat{a}, \hat{b}_1, \hat{b}_2, \hat{b}_3\}$ satisfy (4.15). Therefore, $X(\Psi)$ is indeed a tight frame for $L_2(\mathbb{R})$ by the unitary extension principle.

Now we show that ψ_1 is symmetric about $\frac{1}{2}$ while ψ_2 is symmetric about $\frac{1}{4}$ and ψ_3 is antisymmetric about $\frac{1}{4}$. It is well known that a function $f \in L_2(\mathbb{R})$, is symmetric about the point $\gamma_1 \in \mathbb{R}$ if and only if

$$f(x) = f(2\gamma_1 - x) \quad \text{a.e.},$$

which is equivalent to

$$\hat{f}(\xi) = e^{-i2\gamma_1\xi}\hat{f}(-\xi) \quad \text{a.e.} \tag{4.20}$$

Similarly, a function $f \in L_2(\mathbb{R})$ is antisymmetric about the point $\gamma_2 \in \mathbb{R}$ if and only if

$$f(x) = -f(2\gamma_2 - x) \quad \text{a.e.},$$

which is equivalent to

$$\hat{f}(\xi) = -e^{-i2\gamma_2\xi}\hat{f}(-\xi) \quad \text{a.e.} \tag{4.21}$$

By the definition of $\hat{b}_1$ and the fact that $\hat{a}$ is symmetric about the origin and 2π-periodic, one obtains

$$\hat{b}_1(\xi) = e^{-i\xi}\overline{\hat{a}(\xi+\pi)} = e^{-2i\xi}\left(e^{i\xi}\overline{\hat{a}(-\xi+\pi)}\right) = e^{-2i\xi}\hat{b}_1(-\xi).$$

Since ϕ is symmetric about the origin, then by (4.20) one obtains

$$\hat{\phi}(\xi) = \hat{\phi}(-\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{4.22}$$

Therefore,

$$\hat{\psi}_1(\xi) = \hat{b}_1(\xi/2)\hat{\phi}(\xi/2) = e^{-i\xi}\hat{b}_1(-\xi/2)\hat{\phi}(-\xi/2) = e^{-i\xi}\hat{\psi}_1(-\xi),$$

which, by (4.20), means that ψ_1 is symmetric about $\frac{1}{2}$. Similarly by the definition of $\hat{b}_2$, one obtains

$$\hat{b}_2(\xi) = \mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi) = e^{-i\xi}\left(\mathcal{A}(-\xi) + e^{i\xi}\mathcal{A}(\xi)\right) = e^{-i\xi}\hat{b}_2(-\xi).$$

Applying (4.22) and the definition of $\hat{\psi}_2$, one obtains,

$$\hat{\psi}_2(\xi) = \hat{b}_2(\xi/2)\hat{\phi}(\xi/2) = e^{-i\frac{\xi}{2}}\hat{b}_2(-\xi/2)\hat{\phi}(-\xi/2) = e^{-i\frac{\xi}{2}}\hat{\psi}_2(-\xi),$$

which, by (4.20), means that ψ_2 is symmetric about $\frac{1}{4}$. Similarly, we can show that ψ_3 is antisymmetric about $\frac{1}{4}$. □

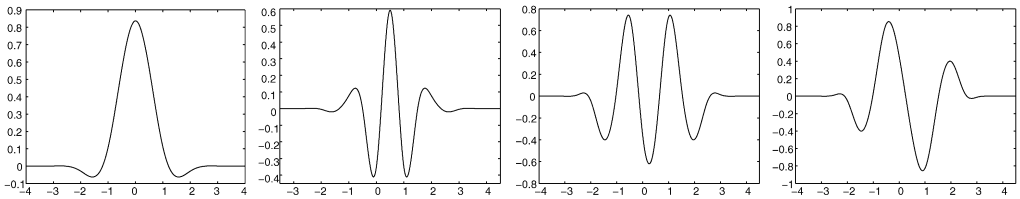

Fig. 1. (a) Pseudo-spline of Type II with order (3, 1) and (b)–(d) are the corresponding (anti)symmetric tight framelets.

The approximation order provided by a tight frame $X(\Psi)$ can be characterized by the approximation order of the corresponding operator $\mathcal{Q}_n$ (see [12]), which is defined in (1.9). We have shown in Section 3.3 that, for the operator $\mathcal{P}_n$ defined in (1.8), we have $\mathcal{Q}_n f = \mathcal{P}_n f$, for $f \in L_2(\mathbb{R})$, provided that Ψ is derived from the UEP and the underlying MRA is generated by the same ϕ as that defines $\mathcal{P}_n$. Therefore, by Theorem 3.10, if we start from the pseudo-spline of Type II with order (m, l) in Construction 4.4, the tight frame system $X(\Psi)$ provides approximation order $2l + 2$.

In the end, we give one example of (anti)symmetric tight framelets constructed from Construction 4.4 using pseudo-splines of Type II with order (3, 1).

Example 4.5. Let $\hat{a}$ to be the mask of the pseudo-spline of Type II with order (3, 1), i.e.,

$$\hat{a}(\xi) = \cos^6(\xi/2)\big(1 + 3\sin^2(\xi/2)\big).$$

We define

$$\hat{b}_1(\xi) := e^{-i\xi}\overline{\hat{a}(\xi+\pi)} = e^{-i\xi}\sin^6(\xi/2)\big(1 + 3\cos^2(\xi/2)\big),$$
$$\hat{b}_2(\xi) := \mathcal{A}(\xi) + e^{-i\xi}\mathcal{A}(-\xi) \quad \text{and} \quad \hat{b}_3(\xi) := e^{-i\xi}\mathcal{A}(-\xi) - \mathcal{A}(\xi),$$

where

$$\mathcal{A} = \frac{1}{2}\big(0.00123930398199e^{-4i\xi} + 0.00139868605052e^{-2i\xi} - 0.22813823298962 + 0.44712319189971e^{2i\xi} - 0.22162294894260e^{4i\xi}\big).$$

The graphs of Ψ are given by (b)–(d) in Fig. 1. The tight frame system has approximation order 4.

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