LP Inequalities for Polynomials

Q. I. Rahman

Département de mathématiques et de statistique,
Université de Montréal, Montréal, Québec H3C 3J7, Canada

AND

G. Schmeisser

Mathematisches Institut, Universität Erlangen-Nürnberg,
D-8520 Erlangen, West Germany

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1. Introduction

1.1. Let \( \mathcal{F}_n \) be the class of all trigonometric polynomials

\[ t_n(\theta) = \sum_{\nu=-n}^{n} c_{\nu} e^{i\nu\theta} \]

of degree \( n \). It was found by Zygmund [15] that if \( t_n \in \mathcal{F}_n \), then, for \( 1 \leq p < +\infty \),

\[ \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(\theta)|^p d\theta \right)^{1/p} \leq n \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(\theta)| d\theta \right)^{1/p} =: \|t_n\|_{[-\pi, \pi], p}. \tag{1} \]

Since (1) was deduced from M. Riesz's interpolation formula [12] by means of Minkowski's inequality, it was not clear whether the restriction on \( p \) was indeed essential. This question was open for a long time. Finally, Arestov [2] proved that (1) remains true for \( 0 < p < 1 \) and indeed for \( p = 0 \) as well, where

\[ \|t_n\|_{[-\pi, \pi], 0} := \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |t_n(\theta)| d\theta \right). \]

The difficulty which was associated with Zygmund's inequality (1) is characteristic of several other \( L^p \) inequalities involving polynomials,
trigonometric polynomials, etc. The purpose of this paper is to mention a few which, like (1), can be extended to $p \in [0, 1)$.

1.2. Let $\mathcal{P}_n$ be the set of all polynomials

$$P(z) = \sum_{v=0}^{n} a_v z^v$$

of degree at most $n$. For $P \in \mathcal{P}_n$ define

$$\|P\|_p := \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^p \, d\theta \right)^{1/p} \quad (0 < p < +\infty),$$

$$\|P\|_{\infty} := \max_{|z|=1} |P(z)|$$

and

$$\|P\|_0 := \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |P(e^{i\theta})| \, d\theta \right).$$

The above result of Zygmund and Arestov says that, in particular, for $P \in \mathcal{P}_n$ and $0 \leq p \leq +\infty$,

$$\|P'\|_p \leq n \|P\|_p,$$  \tag{2}

where equality holds if and only if $P(z) = cz^n$. If $P(z) \neq 0$ for $|z| < 1$ then, for $1 \leq p \leq +\infty$, inequality (2) can be replaced by [4,7]

$$\|P'\|_p \leq n \|P\|_p / \|1 + z^n\|_p.$$  \tag{3}

We prove

**Theorem 1.** Let $P \in \mathcal{P}_n$ and $P(z) \neq 0$ for $|z| < 1$. Then (3) holds for all $p \in [0, +\infty]$.

1.3. It is a simple consequence of a classical result of Hardy [6] that if $P \in \mathcal{P}_n$, then [11, Theorem 5.5], for $R \geq 1$ and $p \geq 0$,

$$\|P(Rz)\|_p \leq R^n \|P\|_p,$$  \tag{4}

where equality is attained if and only if $P(z) = cz^n$. If $P(z) \neq 0$ for $|z| < 1$, then for $1 \leq p \leq +\infty$ inequality (4) can be replaced by [1,3]

$$\|P(Rz)\|_p \leq \frac{\|1 + R^n z^n\|_p}{\|1 + z^n\|_p} \cdot \|P\|_p.$$  \tag{5}
We prove

**Theorem 2.** Let \( P \in \mathcal{P}_n \) and \( P(z) \neq 0 \) for \( |z| < 1 \). Then (5) holds for all \( p \in [0, +\infty] \).

1.4. If

\[
P(z) = \sum_{v=0}^{n} a_v z^v \in \mathcal{P}_n
\]

and \( a_u, a_v \) (\( u < v \)) are two coefficients such that for no other coefficient \( a_w \neq 0 \) we have \( w \equiv u \mod(v - u) \), then [14, 5, 9] for every \( p \in [1, +\infty] \),

\[
|a_u| + |a_v| \leq 2 \|P\|_p / \|1 + z^n\|_p.
\]

This result is best possible. We prove

**Theorem 3.** In the case \( u = 0, v = n \), inequality (6) holds for all \( p \in [0, +\infty] \).

Remark. It may be mentioned that inequality (6) in its full generality does not extend to \( p \in [0, 1) \). To see this let

\[
P(z) = (1 + z)^4 = 1 + 4z + 6z^2 + 4z^3 + z^4.
\]

Then the pair of indices \( (u, v) = (1, 3) \) is clearly admissible. But

\[
|a_1| + |a_3| = 8, \text{ whereas for } p = \frac{1}{2},
\]

\[
2 \|P\|_p / \|1 + z^n\|_p = \frac{8}{\|1 + z\|_{1/2}} < \frac{8}{\|1 + z\|_0} = 8.
\]

As an application of Theorem 3 we mention.

**Corollary.** Consider a polynomial \( \prod_{\nu=1}^{n} (z - \zeta_\nu) \). Then, for \( 1 \leq k \leq n \) and all \( p \in [0, +\infty] \),

\[
|\zeta_1 \zeta_2 \cdots \zeta_{k-1}| + |\zeta_k \zeta_{k+1} \cdots \zeta_n| \leq 2 \|P\|_p / \|1 + z^n\|_p.
\]

This result extends Theorem 2 in [10] and lends itself to the kind of applications mentioned therein.

2. A Lemma

For \( \gamma = (\gamma_0, ..., \gamma_n) \in \mathbb{C}^{n+1} \) and

\[
P(z) = \sum_{v=0}^{n} a_v z^v.
\]
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we define

\[ A, P(z) = \sum_{r = 0}^{n} \gamma_r a_r z^r. \]

The operator \( A, \) is said to be admissible if it preserves one of the following properties:

(i) \( P(z) \) has all its zeros in \( \{ z \in \mathbb{C} : |z| < 1 \} \).

(ii) \( P(z) \) has all its zeros in \( \{ z \in \mathbb{C} : |z| > 1 \} \).

**Lemma [2, Theorem 4].** Let \( \phi(x) = \psi(\log x) \), where \( \psi \) is a convex non-decreasing function on \( \mathbb{R} \). Then for all \( P \in \mathcal{P} \) and each admissible operator \( A, \)

\[ \max(\gamma, n) = \max(|\gamma_0|, |\gamma_n|). \]

In particular, the lemma applies with \( \phi : x \mapsto x^p \) for every \( p \in (0, +\infty) \) and with \( \phi : x \mapsto \log x \) as well. Therefore we have

\[ \| A, P \|_p \leq c(\gamma, n) \| P \|_p \quad (0 \leq p < +\infty). \]

3. PROOFS OF THE THEOREMS

**Proof of Theorem 1.** According to a theorem of Laguerre as stated in [4], if \( P \in \mathcal{P} \) does not vanish in \( K := \{ z \in \mathbb{C} : |z| < 1 \} \), then

\[ nP(z) - (z - \zeta) P'(z) \neq 0 \quad \text{for } z \in K \text{ and } \zeta \in K. \]

Setting \( \zeta = -e^{-i\alpha}z \), we readily see that the operator \( A \) defined by

\[ AP(z) := (e^{i\alpha} + 1)z P'(z) - ne^{i\alpha} P(z) \]

is admissible and so by (9)

\[ \int_{0}^{2\pi} \left| (e^{i\alpha} + 1) \frac{d}{d\theta} P(e^{i\theta}) - ine^{i\alpha} P(e^{i\theta}) \right|^p d\theta \leq n^p \int_{0}^{2\pi} |P(e^{i\theta})|^p d\theta \quad (10) \]

for \( p > 0 \). Rearranging the left-hand side of (10) and integrating the inequality with respect to \( \alpha \) on \([0, 2\pi]\), we obtain

\[ \int_{0}^{2\pi} \int_{0}^{2\pi} \left| \frac{d}{d\theta} P(e^{i\theta}) \right|^p \cdot |1 + e^{i\alpha} R(\theta)|^p d\theta d\alpha \leq 2\pi n^p \| P \|_p^p, \quad (11) \]
where

\[ R(\theta) := \left( \frac{d}{d\theta} P(e^{i\theta}) - inP(e^{i\theta}) \right) \left/ \frac{d}{d\theta} P(e^{i\theta}) \right. \]

It is known [4, Theorem 2] that if \( P(z) \neq 0 \) for \( |z| < 1 \), then \(|R(\theta)| \geq 1\), and therefore by a theorem of Hardy

\[ \int_0^{2\pi} |1 + e^{ix} R(\theta)|^p d\alpha \geq \int_0^{2\pi} |1 + e^{ix}|^p d\alpha \]

for all \( \theta \in [0, 2\pi] \). Using this in (11), the desired result follows immediately for \( p > 0 \). The extension to \( p = 0 \) is obtained by continuity.

**Proof of Theorem 2.** For \( R \geq 1 \) and \( \gamma \in \mathbb{R} \), the polynomial

\[ \sum_{\nu = 0}^{n} \binom{n}{\nu} \left( R^\nu + e^{i\gamma} R^{n-\nu} \right) z^\nu \]

has all its zeros on the unit circle (see [8, Problem 26, p. 108]). Hence, if

\[ P(z) = \sum_{\nu = 0}^{n} a_{\nu} z^\nu \in \mathbb{D}_n \]

does not vanish for \( |z| < 1 \), then by Szegő's convolution theorem [13] the same is true for

\[ \Lambda P(z) := (1 + e^{i\gamma} R^n) a_0 + (R + e^{i\gamma} R^{n-1}) a_1 z + \cdots + (R^n + e^{i\gamma}) a_n z^n = P(Rz) + e^{i\gamma} R^n P(z/R). \]

Therefore \( \Lambda \) is an admissible operator. Applying (9) we obtain

\[ \int_0^{2\pi} \left| P(Re^{i\theta}) + e^{i\gamma} R^n P(e^{i\theta}/R) \right|^p d\theta \leq |R^n e^{i\gamma} + 1| \cdot \| P \|_p^p \]  

(12)

for \( p > 0 \). Since

\[ f(z) := z^n P(1/z)/P(z) \]

is holomorphic for \( |z| \leq 1 \) with \( |f(z)| = 1 \) on the unit circle, it follows from the maximum principle that \( |f((1/R) e^{i\theta})| \leq 1 \) for \( 1/R < 1 \) and so

\[ |R^n P(e^{i\theta}/R)/P(Re^{i\theta})| \geq 1 \quad (R \geq 1). \]  

(13)

Now, integrating (12) with respect to \( \gamma \) on \([0, 2\pi]\) and using (13), the desired result is obtained in the same way as Theorem 1.
\textit{Proof of Theorem 3.} The operator $A$ defined by

$$A \left( \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \right) := a_{0} + a_{n} z^{n}$$

is obviously admissible. Hence by (9)

$$\int_{0}^{2\pi} |a_{0} + a_{n} e^{i\theta}|^{p} d\theta \leq \left( \int_{0}^{2\pi} \left| \sum_{\nu=0}^{n} a_{\nu} e^{i\nu \theta} \right|^{p} d\theta \right)^{1/p} \quad (14)$$

for all $p > 0$. From the inequality

$$\left| \frac{1 + re^{i\theta}}{1 + e^{i\theta}} \right| \geq \frac{1 + r}{2}$$

we deduce that

$$|a_{0}| + |a_{n}| \leq 2 \left( \int_{0}^{2\pi} |a_{0} + a_{n} e^{i\theta}|^{p} d\theta \right)^{1/p} \int_{0}^{2\pi} \left| 1 + e^{i\theta} \right|^{p} d\theta$$

Using this in conjunction with (14), the desired result follows.

The corollary is obtained by applying Theorem 3 to the polynomial

$$Q(z) := P(z) \prod_{j=1}^{n} \left( \frac{\xi_j z - 1}{z - \xi_j} \right)$$

\textbf{REFERENCES}

9. Q. I. RAHMAN, Inequalities concerning polynomials and trigonometric polynomials, 
10. Q. I. RAHMAN AND G. SCHMEISSER, Location of the zeros of polynomials with a prescribed 
13. G. SZEGÖ, Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer 
    392–400.