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Alexandria Engineering Journal

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ORIGINAL ARTICLE

Large amplitude free vibration of axially loaded beams resting on variable elastic foundation



Alborz Mirzabeigy^{a,b,*}, Reza Madoliat^a

^a School of Mechanical Engineering, Iran University of Science and Technology, Narmak, Tehran 16846, Iran

^b Young Researchers and Elite Club, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran

Received 27 October 2015; revised 14 March 2016; accepted 20 March 2016

Available online 1 April 2016

KEYWORDS

Free vibration;
Large amplitude;
Axial load;
Variable foundation;
Homotopy perturbation;
Cubic nonlinear term

Abstract In the present study, large amplitude free vibration of beams resting on variable elastic foundation is investigated. The Euler–Bernoulli hypothesis and the Winkler model have been applied for beam and elastic foundation, respectively. The beam is axially loaded and is restrained by immovable boundary conditions, which yields stretching during vibrations. The energy method and Hamilton's principle are used to derive equation of motion, where after decomposition an ordinary differential equation with cubic nonlinear term is obtained. The second order homotopy perturbation method is applied to solve nonlinear equation of motion. An explicit amplitude-frequency relation is achieved from solution with relative error less than 0.07% for all amplitudes. This solution is applied to study effects of variable elastic foundation, amplitude of vibration and axial load on nonlinear frequency of beams with simply supported and fully clamped boundary conditions. Proposed formulation is capable to dealing with any arbitrary distribution of elastic foundation.

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1. Introduction

Beams are fundamental component in engineering and have wide applications in design and fabrication of structures and machines such as tall building, huge cranes, bridges, turbine and compressor blades. They are also used as simple and accurate model for analysis of complex engineering structures. Natural frequencies and dynamic response of beam-like structure in small amplitude vibration studied by many researchers

through analytical and numerical methods and different aspects have been considered. Mathematical model of small amplitude vibration is in the form of linear differential equation which is relatively simple for handling. Nowadays, the demand for light-weight structures and machines is continuously increasing. These light-weight systems are more flexible due to their high aspect ratio and external excitation such as wind load causing large amplitude vibration in them. Large amplitude vibration induced nonlinear terms in differential equation of motion. In the case of beam with immovable ends, axial stretching of the beam during vibration with large amplitude is the source of nonlinearity. The nonlinear vibration of beams due to the large amplitude of vibration has received considerable attention by many researchers. Bhashyam and Prathap [1] presented a Galerkin finite element method for

* Corresponding author at: School of Mechanical Engineering, Iran University of Science and Technology, Narmak, Tehran 16846, Iran. E-mail address: mirzabeigy@mecheng.iust.ac.ir (A. Mirzabeigy).

Peer review under responsibility of Faculty of Engineering, Alexandria University.

<http://dx.doi.org/10.1016/j.aej.2016.03.021>

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studying nonlinear vibration of beams describable in terms of moderately large bending theory. Ozkaya et al. [2] study the nonlinear response of a beam-mass system with immovable ends by applying the method of multiple scales directly to partial differential equation of motion. Gayesh and Balar [3] study nonlinear parametric vibration and stability of axially moving viscoelastic Rayleigh beams, and they derived the partial-differential equation of motion for large amplitude vibration through geometrical, constitutive, and dynamical relations. Abdel-Jaber et al. [4] study nonlinear frequencies of an elastically restrained tapered beam. They used the nonlinear curvature and the axial shortening due to transverse deflection in the energy formulation of system. Merrimi et al. [5] investigate the geometrically nonlinear steady state periodic forced response of a clamped-clamped beam containing an open crack. The crack has been modeled as a linear spring in which, for a given depth, the spring constant remains the same for both directions. Sedighi et al. [6] derive an analytic solution of transversal oscillation of quintic nonlinear beam with homotopy analysis method. Baghani et al. [7] study large amplitude free vibrations and post-buckling of unsymmetrically laminated composite beams on nonlinear elastic foundation. Lai et al. [8] derive the analytical solutions for large amplitude vibration of thin functionally graded beams. Sedighi and Shirazi [9] study effect of deadzone nonlinear boundary condition on large amplitude vibration of cantilever beam. There are many other researches exist on buckling, linear and nonlinear vibrations of beams and plates [10–18].

It is obvious that the accurate analysis of structures required an understanding of soil-structure interaction. The surrounding soil increases resistance of buried structures and significantly changes modal parameters of them. Many practical cases in engineering related to soil-structure interaction can be modeled by means of a beam on elastic foundation. The well-known model for elastic foundations is Winkler. The Winkler model of elastic foundation is the most preliminary in which the vertical displacement is assumed to be proportional to the contact pressure at an arbitrary point [19], in another words, the foundation modeled as a series of closely spaced and mutually independent linear elastic springs. Different problems of beams resting on elastic foundation were studied and reported in the literature [20–25]. Often, researcher assumed that the foundation has constant value through the length of the beam and only limited studies exist for dynamic analysis of beams on variables foundations. Eisenberger and Clastornik [26] study free vibration and buckling of the Euler-Bernoulli beams on variable Winkler foundation. Zhou [27] presents a general solution to vibration of the Euler-Bernoulli beams on variable elastic foundation. He assumed the reaction force of the foundation on the beam as the external force acting on the beam. Pradhan and Murmu [28] study thermo-mechanical vibration of sandwich beams resting on variable Winkler foundation using differential quadrature method. Kacar et al. [29] apply differential transform method to investigate free vibration of the Euler-Bernoulli beams on variable Winkler foundation.

According to literature survey, large amplitude free vibration of beam resting on variable elastic foundation has not been studied and for the first time has been studied in this paper. Equation of motion is obtained from energy method by invoking Hamilton principle, and then homotopy perturbation method [30–33] is applied to solve governing nonlinear

differential equation. Comparisons are made with studies in the open literature in which special cases of the present problem have been studied and very good agreement observed. Finally, some new and more useful results are extracted from the present formulation.

2. Mathematical formulation

Consider a straight beam under axial load which resting on variable Winkler foundation as shown in Fig. 1. The beam has length L , rectangular cross section with the area of S , the cross-sectional moment of inertia of I and thickness of h . The beam was made from isotropic material with E as modulus of elasticity and ρ as mass per unit volume. Stiffness of Winkler foundation changes along the beam length and is the function of spatial coordinate along the beam length (i.e. \bar{x}). Using Euler-Bernoulli beam hypothesis, the strain energy induced by large displacement amplitude is given by the following:

$$\begin{aligned} \Pi = & \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 \bar{y}(\bar{x}, \bar{t})}{\partial \bar{x}^2} \right)^2 d\bar{x} \\ & + \frac{1}{2} \int_0^L ES \left(\frac{\partial \bar{u}(\bar{x}, \bar{t})}{\partial \bar{x}} + \frac{1}{2} \left(\frac{\partial \bar{y}(\bar{x}, \bar{t})}{\partial \bar{x}} \right)^2 \right)^2 d\bar{x} \\ & + \frac{1}{2} \int_0^L k(\bar{x}) \bar{y}(\bar{x}, \bar{t})^2 d\bar{x} \end{aligned} \quad (1)$$

where \bar{u} and \bar{y} are axial and transverse displacements, respectively. $k(\bar{x})$ is the mathematical expression for variable Winkler foundation. The kinetic energy is given by the following:

$$T = \frac{1}{2} \int_0^L \rho S \left(\frac{\partial \bar{y}(\bar{x}, \bar{t})}{\partial \bar{t}} \right)^2 d\bar{x} \quad (2)$$

The external work done by axial load can be written as follows:

$$W = \frac{f}{2} \int_0^L \left(\frac{\partial \bar{y}(\bar{x}, \bar{t})}{\partial \bar{x}} \right)^2 d\bar{x} \quad (3)$$

Using the Lagrangian of the system and invoking Hamilton's principle, we have the following:

$$\delta \int_{t_1}^{t_2} (T - \Pi + W) d\bar{t} = 0 \quad (4)$$

Substituting Eqs. (1)–(3) into Eq. (4), performing the necessary algebra and eliminating axial displacement, the following partial differential equation is obtained for motion:

$$\begin{aligned} EI \frac{\partial^4 \bar{y}(\bar{x}, \bar{t})}{\partial \bar{x}^4} + \rho A \frac{\partial^2 \bar{y}(\bar{x}, \bar{t})}{\partial \bar{t}^2} + f \frac{\partial^2 \bar{y}(\bar{x}, \bar{t})}{\partial \bar{x}^2} + k(\bar{x}) \bar{y}(\bar{x}, \bar{t}) \\ - \frac{EA}{2L} \frac{\partial^2 \bar{y}(\bar{x}, \bar{t})}{\partial \bar{x}^2} \int_0^L \left(\frac{\partial \bar{y}(\bar{x}, \bar{t})}{\partial \bar{x}} \right)^2 d\bar{x} = 0 \end{aligned} \quad (5)$$

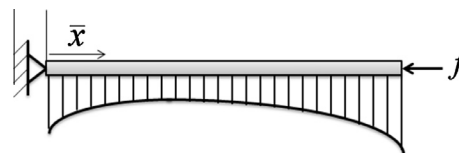


Figure 1 Schematic of the beam under axial load and resting on variable elastic foundation.

Variable Winkler foundation can be considered as follows:

$$k(\bar{x}) = kg(\bar{x}) \quad (6)$$

where k is a constant and $g(\bar{x})$ is function of the spatial coordinate along the beam length. Some dimensionless parameters are defined to better handling of equation and better representation of the numerical results, as follows:

$$x = \frac{\bar{x}}{L}, \quad y = \frac{\bar{y}}{R}, \quad t = \bar{t} \sqrt{\frac{EI}{\rho SL^4}}, \quad F = \frac{fL^2}{EI}, \quad K = \frac{kL^4}{EI} \quad (7)$$

where $R = \sqrt{\frac{I}{S}}$ is the radius of gyration of the cross section. Substituting dimensionless parameters into Eq. (5), yields the following:

$$\frac{\partial^4 y(x, t)}{\partial x^4} + \frac{\partial^2 y(x, t)}{\partial t^2} + F \frac{\partial^2 y(x, t)}{\partial x^2} + Kg(x)y(x, t) - \frac{1}{2} \frac{\partial^2 y(x, t)}{\partial x^2} \int_0^1 \left(\frac{\partial y(x, t)}{\partial x} \right)^2 dx = 0 \quad (8)$$

in order to solve Eq. (8), separation of variables is applied by assuming $y(x, t) = \phi(x)T(t)$ where $\phi(x)$ is the first eigenmode of the beam that depends on boundary conditions of beam and for simply support and clamped boundary conditions presented in Table 1. $T(t)$ is an unknown time-dependent function. Applying the weighted residual Bubnov–Galerkin method yields the following:

$$\int_0^1 \left(\frac{\partial^4 y(x, t)}{\partial x^4} + \frac{\partial^2 y(x, t)}{\partial t^2} + F \frac{\partial^2 y(x, t)}{\partial x^2} + Kg(x)y(x, t) - \frac{1}{2} \frac{\partial^2 y(x, t)}{\partial x^2} \int_0^1 \left(\frac{\partial y(x, t)}{\partial x} \right)^2 dx \right) \phi(x) dx = 0 \quad (9)$$

Finally, the nonlinear equation is obtained in terms of the time-dependent variable as follows:

$$\ddot{T} + \alpha T + \beta T^3 = 0 \quad (10)$$

where α and β are as follows:

$$\begin{aligned} \alpha &= \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 &= \frac{\int_0^1 \left(\frac{d^4 \phi(x)}{dx^4} \right) \phi(x) dx}{\int_0^1 \phi(x)^2 dx}, \quad \alpha_2 = F \frac{\int_0^1 \left(\frac{d^2 \phi(x)}{dx^2} \right) \phi(x) dx}{\int_0^1 \phi(x)^2 dx}, \\ \alpha_3 &= K \frac{\int_0^1 g(x) \phi(x)^2 dx}{\int_0^1 \phi(x)^2 dx}, \quad \beta = -\frac{1}{2} \frac{\int_0^1 \left(\frac{d^2 \phi(x)}{dx^2} \right) \left(\int_0^1 \left(\frac{d\phi(x)}{dx} \right)^2 dx \right) \phi(x) dx}{\int_0^1 \phi(x)^2 dx} \end{aligned} \quad (11)$$

Table 1 The first eigenmode of beam with various boundary conditions.[49].

Boundary conditions	$\phi(x)$
Simply supported beam	$\sin(\pi x)$
Fully clamped beam	$(\cos(\Omega x) - \cosh(\Omega x)) - \frac{\cos(\Omega) - \cosh(\Omega)}{\sin(\Omega) - \sinh(\Omega)} (\sin(\Omega x) - \sinh(\Omega x)),$ $\Omega = 4.7300$

It is assumed midpoint of beam subjected to an initial displacement and zero initial velocity, and accordingly, the nonlinear equation with initial conditions for large amplitude vibration of Euler–Bernoulli beam becomes the following:

$$\ddot{T} + \alpha T + \beta T^3 = 0, \quad T(0) = A, \quad \dot{T}(0) = 0 \quad (12)$$

Note that A is dimensionless initial displacement or dimensionless maximum amplitude of oscillation. Differential equation is obtained in Eq. (12) known as Duffing nonlinear equation.

3. Analytical solution

There are several methods developed by researchers to derive analytical solution for nonlinear differential equation such as homotopy perturbation method [30–33], energy balance method [34–36], max–min approach [37–40], Hamiltonian approach [41–43] and so on [44–47]. Possibility of parametric studies is most important advantage of these analytical methods rather than numerical methods. The homotopy perturbation method is utilized to derive analytical amplitude–frequency relationship for equation of motion in Eq. (12). This method does not depend upon the assumption of small parameter and is capable to solve ordinary and partial differential equations.

At first, we establish the following homotopy:

$$\ddot{T} + 0 \cdot T = p[-\alpha T - \beta T^3], \quad p \in [0, 1] \quad (13)$$

where p is homotopy parameter. When $p = 0$, Eq. (13) becomes a linear ordinary differential equation and when $p = 1$, it becomes the original nonlinear equation. We consider T and 0 as series of p in the following form:

$$T = T_0 + pT_1 + p^2T_2 \dots, \quad (14)$$

$$0 = \omega^2 - pv_1 - p^2v_2 \dots, \quad (15)$$

where v_1, v_2, \dots are to be determined.

Substituting Eqs. (14) and (15) into Eq. (13) and collecting terms with identical powers of p , we can find three first linear equations with initial conditions as follows:

$$p^0: \ddot{T}_0 + \omega^2 T_0 = 0, \quad T_0(0) = A, \quad \dot{T}_0(0) = 0 \quad (16)$$

$$p^1: \ddot{T}_1 + \omega^2 T_1 = (v_1 - \alpha)T_0 - \beta T_0^3, \quad T_1(0) = \dot{T}_1(0) = 0 \quad (17)$$

$$p^2: \ddot{T}_2 + \omega^2 T_2 = (v_1 - \alpha)T_1 + v_2 T_0 - 3\beta T_0^2 T_1, \quad T_2(0) = \dot{T}_2(0) = 0 \quad (18)$$

Analytical solution of Eq. (16) is obtained as follows:

$$T_0(t) = A \cos \omega t, \quad (19)$$

Substituting Eq. (19) into right-hand side of Eq. (17) gives the following:

$$\ddot{T}_1 + \omega^2 T_1 = (v_1 - \alpha)(A \cos \omega t) - \beta(A \cos \omega t)^3, \quad (20)$$

Avoiding secular terms in T_1 requires eliminating contributions proportional to $\cos \omega t$ on the right-hand side of Eq. (20), and therefore we have

$$\int_0^{2\pi} \left[(v_1 - \alpha)(A \cos \omega t) - \beta(A \cos \omega t)^3 \right] \cos \omega t dt = 0, \quad (21)$$

From Eq. (21) we obtain the following:

$$v_1 = \alpha + \frac{3}{4} \beta A^2 \quad (22)$$

First-order approximate frequency–amplitude relationship can be obtained by substituting Eq. (22) into Eq. (15) and setting $p = 1$ as follows:

$$\omega_1 = \sqrt{v_1} = \sqrt{\alpha + \frac{3}{4}\beta A^2} \tag{23}$$

The solution of Eq. (17) can be obtained from Eq. (23) as follows:

$$T_1(t) = \frac{\beta A^3}{32\omega_1^2}(\cos 3\omega t - \cos \omega t) \tag{24}$$

Avoiding secular terms in T_2 requires the following:

$$\int_0^{2\pi/\omega} [(v_1 - \alpha)T_1 + v_2T_0 - 3\beta T_0^2T_1] \cos \omega t dt = 0, \tag{25}$$

By substituting T_0, T_1 from Eqs. (19) and (24) into Eq. (25) and integrating, we obtain the following:

$$v_2 = -\frac{3}{32} \frac{\beta^2 A^4}{4\alpha + 3\beta A^2} \tag{26}$$

Second-order approximate frequency-amplitude relationship can be obtained by substituting Eqs. (22) and (26) into Eq. (15) and setting $p = 1$ as follows:

$$\omega_2 = \sqrt{v_1 + v_2} = \frac{1}{4} \sqrt{\frac{128\alpha^2 + 192\alpha\beta A^2 + 69\beta^2 A^4}{8\alpha + 6\beta A^2}} \tag{27}$$

4. Numerical results

In order to analyze nonlinear frequency of beam-foundation system, it is necessary to investigate accuracy of obtained solution for nonlinear Duffing oscillator because nonlinear frequency of system is determined based on this solution. Therefore, comparison has been made between solutions obtained from first-order homotopy perturbation method (FHPM) in Eq. (23), second-order homotopy perturbation method (SHPM) in Eq. (27), exact integration method and second-order energy balance method based on Galerkin method [48] and the results are presented in Table 2. Also, the comparison between FHPM and SHPM in conjugation with fourth-order Runge–Kutta numerical solution is presented in Fig. 2. It is obvious the amplitude–frequency relation obtained from SHPM yields high accuracy and is suitable for analysis of practical problems.

Derived formulation for nonlinear frequency yields fundamental natural or linear frequency of system by setting the initial displacement to zero. To validate accuracy of formulation, the fundamental natural frequency of beams resting on variable elastic foundation with linear and parabolic distribution studied by Kacar et al. [29] is re-examined and the results are tabulated in Table 3. The value of nonlinear to linear frequency of beam without axial load and elastic foundation is obtained from SHPM compared with other well-known studies and the results are shown in Table 4. The results of the present study show very good agreement with other studies for determination of linear and nonlinear frequencies.

We assume foundation stiffness continuously decreases or continuously increases along beam length. One time this continuous variation has linear pattern with mathematical model as $k(x) = 100(1 + \lambda x)$ and another time this has parabolic pattern with mathematical model as $k(x) = 100(1 + \lambda x^2)$. It is clear, in both cases, foundation stiffness continuously decreases from left support to right support when $-1 \leq \lambda < 0$ and continuously increases when $0 < \lambda \leq 1$. Fig. 3 is given to study

Table 2 Comparison between different solutions obtained for frequency of nonlinear equation in Eq. (12) ($\alpha = 1$).

βA^2	Exact	FHPM Error%	SHPM Error%	[48] Error%
1	1.3178	1.3228 (0.379%)	1.3178 (≈0%)	1.3164 (0.110%)
10	2.8666	2.9155 (1.706%)	2.8678 (0.042%)	2.8536 (0.455%)
100	8.5336	8.7178 (2.158%)	8.5391 (0.064%)	8.4843 (0.579%)
1000	26.8107	27.4044 (2.214%)	26.8289 (0.068%)	26.6519 (0.592%)
5000	59.9157	61.2454 (2.219%)	59.9566 (0.068%)	59.5599 (0.594%)

influence of λ on nonlinear frequency as observed, when foundation stiffness continuously decreases from left support to right support, parabolic pattern yields higher frequency while if foundation stiffness continuously increases from left support to right support, linear pattern yields higher frequency.

As mentioned in introduction, the foundation is modeled as a series of closely spaced and mutually independent linear elastic springs in the Winkler foundation theory. On this basis, we introduce total stiffness of variable Winkler foundation as follows:

$$K_{Total} = \int_0^1 k(x)dx = K \int_0^1 g(x)dx \tag{28}$$

We consider three different distributions of elastic foundation along length of beam as follows:

- Case 1 : $g(x) = 1 + 2x$
- Case 2 : $g(x) = 1 + 3x^2$
- Case 3 : $g(x) = \pi \sin(\pi x)$

where Case 1 shows linear variation, Case 2 shows parabolic variation and Case 3 shows sinusoidal variation of elastic foundation along length of beam, respectively. Total stiffness of variable foundation is equal for these cases, when value of K be the same. Fig. 4 is given to study effect of K on nonlinear frequency. Three different distributions of Eq. (29) are considered for variable Winkler foundation. It is clear, by increasing

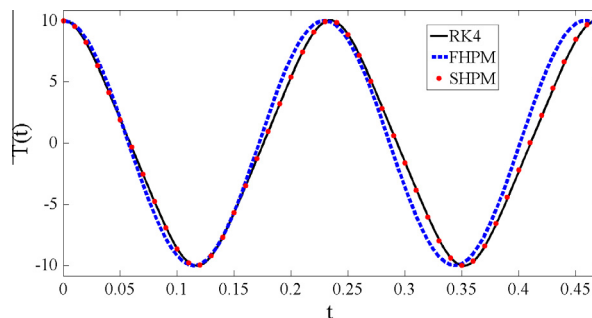


Figure 2 A comparison between the FHPM and SHPM in conjunction with the fourth-order Runge–Kutta method for Eq. (12) ($\alpha = 1, A = \beta = 10$).

Table 3 Fundamental linear frequency of the Euler–Bernoulli beam resting on variable Winkler foundation.

B.C.		$g(x) = 1 - 0.2x$		$g(x) = 1 - 0.2x^2$	
		$K = 10$	$K = 10^2$	$K = 10$	$K = 10^2$
S–S	Present	10.315	13.690	10.336	13.848
	[29]	10.315	13.690	10.336	13.848
C–C	Present	22.573	24.301	22.583	24.395
	[29]	22.573	24.301	22.583	24.395

Table 4 Nonlinear to linear frequency ratio obtained from different methods for beam without axial load and elastic foundation.

A	Simply supported beam			Fully clamped beam		
	[10]	[14]	Present	[10]	[14]	Present
1	1.0897	1.0892	1.0892	1.0572	1.0569	1.0550
2	1.3228	1.3178	1.3178	1.2125	1.2098	1.2031
3	1.6393	1.6257	1.6258	1.4344	1.4263	1.4138
4	1.9999	1.9761	1.9764	1.6171	1.6816	1.6629

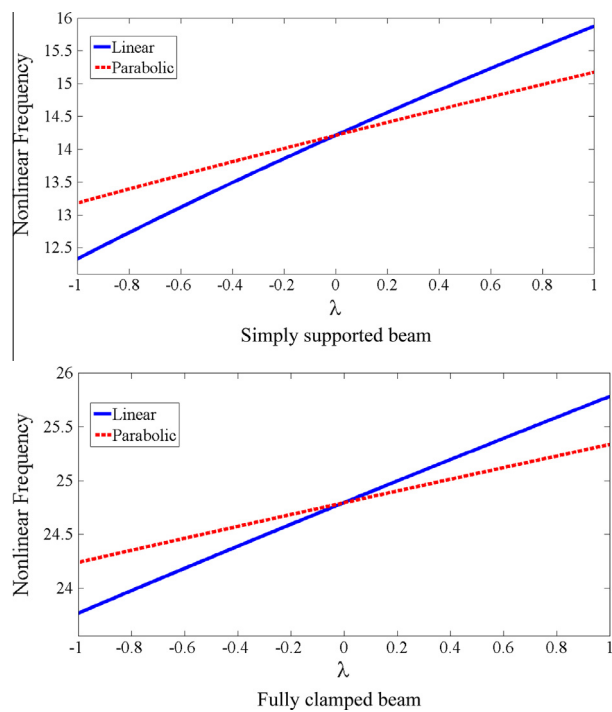


Figure 3 Effect of linear and parabolic distribution of foundation on nonlinear frequency ($K = 100$, $A = 0.5$).

K nonlinear frequency increased in all cases. The interesting point is that, for any value of K , sinusoidal variation yields higher frequency rather than linear and parabolic variations of elastic foundation, although the value of K_{Total} is equal for three cases. Fig. 5 shows the variation of nonlinear frequency versus dimensionless amplitude for different distributions according to Eq. (29). Once again, it is observed sinusoidal variation yields higher frequency although the value of

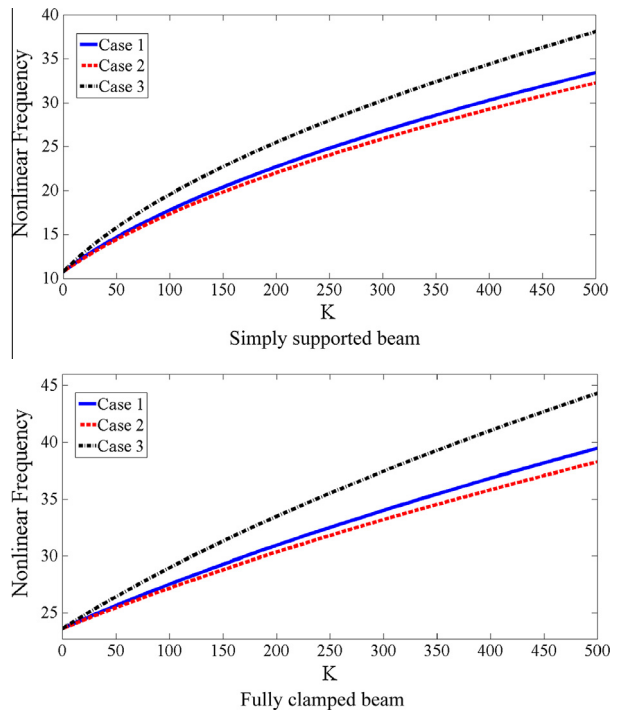


Figure 4 Effect of K on nonlinear frequency for different distributions of elastic foundation with same total stiffness ($A = 0.5$).

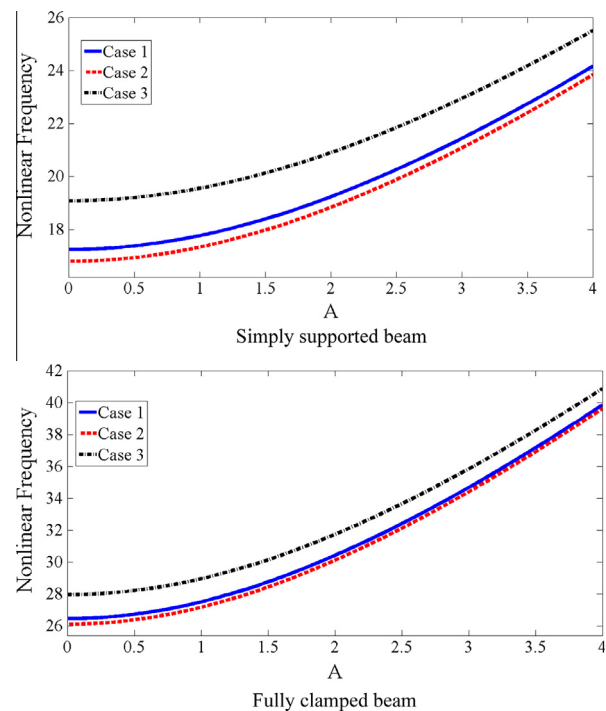


Figure 5 Effect of dimensionless amplitude on nonlinear frequency for different distributions of elastic foundation with same total stiffness ($K = 100$).

K_{Total} is equal for three cases. Also, it is almost true, if we say beam with clamp boundary conditions is more sensitive to variation of initial amplitude rather than beam with simply

Table 5 Effect of axial load, dimensionless amplitude and distribution type of elastic foundation on nonlinear frequency.

		Simply supported beam			Fully clamped beam		
		Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
$A = 0.1$	$F = 5$	15.75575	15.26582	17.745714	25.29042	24.923337	26.858019
	$F = 10$	14.103038	13.55350	16.29608	24.043613	23.65719	25.687413
	$F = 15$	12.228967	11.59092	14.704228	22.728514	22.319335	24.46085
$A = 1$	$F = 5$	16.317876	15.84524	18.246923	26.37416	26.022271	27.881258
	$F = 10$	14.727973	14.20247	16.840276	25.180702	24.811868	26.75518
	$F = 15$	12.94396	12.34260	15.304789	23.927703	23.539214	25.579525
$A = 5$	$F = 5$	26.312323	26.01669	27.569376	44.911757	44.70195	45.829805
	$F = 10$	25.338942	25.03096	26.644937	44.20747	43.994044	45.140898
	$F = 15$	24.32353	24.00141	25.68505	43.490854	43.273605	44.440517

boundary conditions. Effect of axial load on nonlinear frequency for different distributions according to Eq. (29) is presented in Table 5 and plotted in Fig. 6. It should be noted, in Fig. 6, anywhere nonlinear frequency becomes zero, the value of axial load is known as nonlinear post-buckling load.

Similar to the previous results in Figs. 4 and 5, it is observed in the presence of axial load sinusoidal distribution yields higher frequency and higher post-buckling load. Also, it is noticed frequency slope of variation versus axial load is the same for three cases of distribution when axial load is not near to post-buckling load.

It is seen that the sinusoidal distribution of Winkler foundation has greater effect on nonlinear frequency rather than linear and parabolic distributions while total stiffness of variable Winkler foundation is equal for three cases. It is almost

true if we say this phenomenon happened because sinusoidal distribution is more close to fundamental mode shape of simply support and clamped beams. In better words, for maximizing nonlinear frequency of beam resting on variable Winkler foundation, the distribution of foundation must be close to fundamental mode shape of beam.

5. Conclusion

In this study, large amplitude free vibration of axially loaded Euler–Bernoulli beams resting on variable elastic foundation is investigated. The Winkler model is applied to elastic foundation. It is assumed beam has immovable boundary conditions which lead to mid-plane stretching during vibrations. Simply support and fully clamp boundary conditions are used as immovable ends. Energy method and Hamilton's principle are used to derive equation of motion and partial differential equation in time and space obtained which reduced to ordinary differential equation with cubic nonlinear term after decomposition. Cubic nonlinear term is induced by mid-plane stretching. The second order homotopy perturbation method is applied to achieve explicit amplitude-frequency relation for equation of motion. Derived solution yields high level of accuracy with relative error less than 0.07% for all amplitude and is suitable for practical problems due to high accuracy and convenience for application. After definition concept of total stiffness of variable Winkler foundation, effect of distribution of elastic foundation along beam length is analyzed. Results show when distribution of elastic foundation has mathematical function near to fundamental eigenmode of beam, nonlinear frequency and post-buckling load increased. This problem handled for first time and results are new, and also, proposed formulation is capable to dealing with any arbitrary distribution of elastic foundation.

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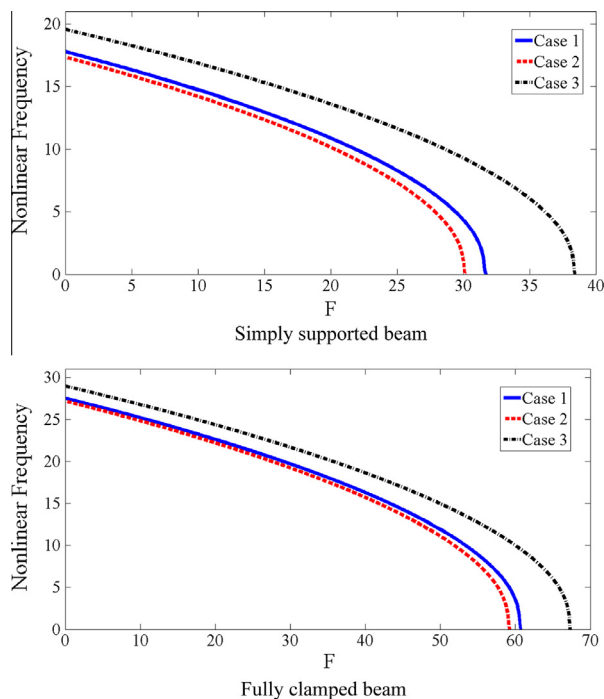


Figure 6 Effect of axial load on nonlinear frequency for different distributions of elastic foundation with same total stiffness ($K = 100$, $A = 1$).

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