JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 163 (2001) 301–318

www.elsevier.com/locate/jpaa

Group rings and semigroup rings over Strong Mori domains

Mi Hee Park

Department of Mathematics, Yonsei University, Seoul 120–749, South Korea

Received 28 December 1999; received in revised form 3 July 2000
Communicated by C.A. Weibel

Abstract

In this paper we study the transfer of the property of being a Strong Mori domain. In particular we give the characterizations of Strong Mori domains in certain types of pullbacks. We show that if R is a Strong Mori domain which is not a field, then the polynomial ring $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ is also a Strong Mori domain and $w\text{-}dim\, R[\{X_\lambda\}_{\lambda\in\Lambda}] = w\text{-}dim\, R$. We also determine necessary and sufficient conditions in order that the group ring $R[X;G]$ or the semigroup ring $R[X;S]$ should be a Strong Mori domain with *w-dimension* ≤ 1.

MSC: 13A15; 13F05; 13E99; 13B99

1. Introduction

Throughout this paper we shall use R to denote a commutative integral domain with quotient field K. Let $F(R)$ be the set of nonzero fractional ideals of R. A star operation on R is a mapping $I \to I^*$ of $F(R)$ into $F(R)$ such that for all $A, B \in F(R)$ and for all $a \in K \setminus \{0\}$,

(i) $(a)^* = (a)$ and $(aA)^* = aA^*$,
(ii) $A \subseteq A^*$ and $A \subseteq B$ implies $A^* \subseteq B^*$, and
(iii) $(A^*)^* = A^*$.

An ideal $A \in F(R)$ is called a $*$-ideal if $A = A^*$ and A is called a $*$-ideal of finite type if there exists a finitely generated $B \in F(R)$ such that $A = B^*$. A star operation is said

E-mail address: mhpark@euclid.postech.ac.kr (M.H. Park).

0022-4049/01/$ - see front matter
PII: S0022-4049(00)00160-2

to be of finite character if $A^* = \bigcup\{B^* \mid B$ is a finitely generated ideal contained in $A\}$ for each $A \in F(R)$.

For $A \in F(R)$, the operation $A \to A_v = (A^{-1})^{-1}$ is called the v-operation whereas the operation $A \to A_t = \bigcup B_v$, where B ranges over finitely generated subideals of A, is called the t-operation. These are well-known examples of star operations and the t-operation has finite character while the v-operation need not have finite character. In many literatures a v-ideal is called a divisorial ideal.

An ideal J of R is called a Glaz–Vasconcelos ideal (GV-ideal), denoted by $J \in GV(R)$, if J is finitely generated and $J^{-1} = R$. For $A \in F(R)$, the operation $A \to A_w = \{x \in K \mid Jx \subseteq A$ for some $J \in GV(R)\}$ is called the w-operation and it gives another example of a star operation of finite character. In [16] a w-ideal is called a semi-divisorial ideal and in [18] an F_∞-ideal.

In Section 2 we show that the w-operation is a star operation induced by overrings.

Recall that a Mori domain is a domain satisfying ACC on integral v-ideals and a Strong Mori (SM) domain is a domain satisfying ACC on integral w-ideals [11]. It is obvious that an SM domain is a Mori domain.

In Section 3 we give necessary and sufficient conditions for certain pullback type constructions to be SM domains. Using this characterization we show that the w-analogue of the converse of Eakin's Theorem does not hold.

In Section 4 we study the polynomial ring $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ and the formal power series ring $R[\![\{X_\lambda\}_{\lambda\in\Lambda}]\!]_1$, where $\{X_\lambda\}_{\lambda\in\Lambda}$ is an arbitrary set of indeterminates over R. If R is a Noetherian domain, then $R[X]$ is also a Noetherian domain by the Hilbert Basis Theorem. But if Λ is infinite, then $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ is not Noetherian. In Proposition 4.3 we show that $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ is a Mori domain. More generally, using the Hilbert Basis Theorem for SM domains we show that if R is an SM domain then $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ is also an SM domain [Theorem 4.7]. We also show that if R is a Noetherian domain then $R[\![\{X_\lambda\}_{\lambda\in\Lambda}]\!]_1$ is an SM domain [Proposition 4.9].

In the final section we study the group ring $R[X;G]$ and the semigroup ring $R[X;S]$, where G is a torsion-free abelian group and S is a torsion-free cancellative additive semigroup containing 0. In Theorems 5.8 and 5.9 we determine necessary and sufficient conditions under which $R[X;G]$ (resp., $R[X;S]$) is an SM domain with $w\text{-}dim\, R[X;G] \le 1$ (resp., $w\text{-}dim\, R[X;S] \le 1$). Since every Krull domain is an SM domain with w-*dimension* ≤ 1, those are generalizations of [24, Proposition 3.3] and [3, Proposition 5.11], respectively.

2. The v-, t-, w-operations

Given a star operation $*$ on R, a proper integral $*$-ideal maximal with respect to being a $*$-ideal is called a maximal $*$-ideal and a maximal $*$-ideal is prime. We denote the set of all maximal $*$-ideals of R by $*$-max(R).

Suppose that $*$ is of finite character. Then any proper $*$-ideal is contained in a maximal $*$-ideal (so the set $*$-max(R) is always nonempty) and any prime ideal minimal over a $*$-ideal is a $*$-ideal.

Recall that the w- and t-operations have finite character and for $A \in F(R)$, $A \subseteq A_w \subseteq A_t \subseteq A_v$.

Lemma 2.1. *Let R be an integral domain. Then* $w\text{-max}(R) = t\text{-max}(R)$.

Proof. Let Q be a maximal w-ideal of R. Then $Q \subseteq Q_t \subseteq R$. Since every t-ideal is a w-ideal, $Q = Q_t$ or $Q_t = R$. Suppose $Q_t = R$. Then there exists a finitely generated ideal J of R such that $J \subseteq Q$ and $J_v = R$. Thus since $J \in GV(R)$ and $J \subseteq Q$, $Q_w = R$. A contradiction! Therefore $Q = Q_t$, i.e., Q is a prime t-ideal of R. Since every t-ideal is a w-ideal and Q is a maximal w-ideal, Q is a maximal t-ideal.

Conversely let Q be a maximal t-ideal of R. Then Q is a w-ideal. Let M be a maximal w-ideal of R containing Q. Then by the above argument M is a t-ideal. Therefore by maximality of Q, $Q = M$, i.e., Q is a maximal w-ideal of R. □

Proposition 2.2. *The* $*$*-operation induced by the mapping* $A \to A^* = \bigcap\{AR_P \mid P \in t\text{-max}(R)\}$ *is just the* w*-operation.*

Proof. Since the w-operation has finite character, $A_w = \bigcap\{A_w R_P \mid P \in w\text{-max}(R)\}$ for all $A \in F(R)$ [17, Proposition 4]. Let $x \in A_w R_P$. Then there is an $s \in R \setminus P$ such that $sx \in A_w$. So for some $J \in GV(R)$, $Jsx \subseteq A$. Now since P is a w-ideal, $J \nsubseteq P$, and hence $sx \in A_P$, i.e., $x \in A_P$. Thus since $A_w R_P = AR_P$ and $w\text{-max}(R) = t\text{-max}(R)$, $A_w = A^*$. □

The reader may consult [2] for the star operations induced by overrings. By Proposition 2.2, we can say that the equivalent conditions in [2, Theorem 5] (resp., [2, Theorem 6]) are just the necessary and sufficient conditions for $w = t$ in a Mori domain (resp., an integrally closed domain).

Theorem 2.3 (Anderson [2, Theorem 5])**.** *Let R be a Mori domain. Then the following statements are equivalent*:

(1) $A_t = \bigcap\{AR_P \mid P \in t\text{-max}(R)\}$ *for each* $A \in F(R)$.
(2) $(A \cap B)_t = A_t \cap B_t$ *for all* $A, B \in F(R)$.
(3) $(A \cap B)_t = A_t \cap B_t$ *for all nonzero finitely generated integral ideals A and B of R.*
(4) $(A :_R B)_t = (A_t :_R B_t)$ *for all* $A \in F(R)$ *and for all nonzero finitely generated fractional ideals B of R.*
(5) $(A :_R B)_t = (A_t :_R B_t)$ *for all nonzero finitely generated integral ideals A and B of R.*
(6) *For each maximal t-ideal P of R,* R_P *is a one-dimensional Gorenstein domain.*
(7) *For each height one prime ideal P of R,* R_P *is Gorenstein and* $R = \bigcap\{R_P \mid ht\, P = 1\}$.
(8) $A_t = \bigcap\{AR_P \mid ht\, P = 1\}$ *for each* $A \in F(R)$.

Theorem 2.4 (Anderson [2, Theorem 6])**.** *Let R be an integrally closed domain. Then the following statements are equivalent*:

(1) *R is a Prüfer v-multiplication domain.*
(2) $(A \cap B)_t = A_t \cap B_t$ *for all* $A, B \in F(R)$.

(3) $(A \cap B)_t = A_t \cap B_t$ *for all nonzero finitely generated integral ideals* A *and* B *of* R.
(4) $(A :_R B)_t = (A_t :_R B_t)$ *for all* $A \in F(R)$ *and for all nonzero finitely generated fractional ideals* B *of* R.
(5) $(A :_R B)_t = (A_t :_R B_t)$ *for all nonzero finitely generated integral ideals* A *and* B *of* R.
(6) $A_t = \bigcap \{AR_P \mid P \in t\text{-max}(R)\}$ *for all* $A \in F(R)$.

Corollary 2.5. *In a Krull domain,* $w = t = v$.

A fractional ideal A of R is said to be $*$-invertible if there exists a fractional ideal B with $(AB)^* = R$ and in this case we can take $B = A^{-1}$. An integral domain R is said to be a Prüfer v-multiplication domain (PVMD) if each nonzero finitely generated ideal is t-invertible, or equivalently, if R_P is a valuation domain for all $P \in t\text{-max}(R)$. In [12], a w-multiplication domain is defined to be a domain in which each nonzero finitely generated ideal is w-invertible. It is clear that a w-multiplication domain is a PVMD. In fact since a PVMD is integrally closed, Theorem 2.4 implies $w = t$ in a PVMD. So they are the same concepts, which also follows from the next lemma.

Lemma 2.6. *Let* $A \in F(R)$. *Then* A *is* w*-invertible if and only if* A *is* t*-invertible.*

Proof. It follows from Lemma 2.1. Indeed, A is w-invertible $\Leftrightarrow AA^{-1}$ is contained in no maximal w-ideal $\Leftrightarrow AA^{-1}$ is contained in no maximal t-ideal $\Leftrightarrow A$ is t-invertible. □

Therefore we can replace "t-invertibility" by "w-invertibility" in all statements concerning t-invertibility. For results on t-invertibility, see [23,21,4].

Corollary 2.7. *A PVMD is the same as a* w*-multiplication domain.*

3. Pullbacks and SM domains

In [11,12] Fanggui and McCasland introduced an SM domain, which is a domain satisfying ACC on integral w-ideals, and they proved w-operation analogues of several theorems holding in a Noetherian domain. In this section we characterize SM domains in certain types of pullback constructions.

Recall first some terminology. Let M be a torsion-free R module. M is called a w-module if $J \in GV(R)$, $x \in M \otimes K$, and $Jx \subseteq M$ imply $x \in M$. M is a w-ideal if M is an ideal of R and is also a w-module. The w-envelope of M is the set given by $M_w = \{x \in M \otimes K \mid Jx \subseteq M \text{ for some } J \in GV(R)\}$. Let T be an overdomain of R. If T is a w-module (as an R-module) then we say that T is a w-overdomain of R. It is clear that for any overdomain T of R, T_w is a w-overdomain of R. A w-module M is called a Strong Mori module (SM module) if M satisfies ACC on w-submodules.

R is an SM domain if R is an SM module. The w-dimension of R (denoted by w-$dim(R)$) is defined by $\sup\{ht\, P \mid P \in w\text{-max}(R)\}$.

Below, we list for easy reference several facts which we shall need in the sequel.

Theorem 3.1 (Fanggui and McCasland [11,12]). *Let R be an integral domain.*

(1) *R is an SM domain if and only if R_p is Noetherian for every $P \in w$-max(R) and each nonzero element of R lies in only finitely many maximal w-ideals. Furthermore, if R is an SM domain, then $R = \bigcap\{R_p \mid P \in w\text{-max}(R)\}$.*
(2) (*The Hilbert Basis Theorem for SM domains*) *Let R be an SM domain, then $R[X]$ is likewise an SM domain.*
(3) (*The Cohen Theorem for SM domains*) *R is an SM domain if and only if each prime w-ideal of R is of finite type.*
(4) (*Generalized PIT for SM domains*) *Let R be an SM domain and let $I = (a_1, \ldots, a_n)_w$ be a w-ideal of R. If P is a prime ideal of R minimal over I, then $ht\, P \le n$.*
(5) *R is an SM domain if and only if every finite type torsion-free w-module over R is an SM module.*

Corollary 3.2. *Let R be an SM domain and let T be a finite type w-overdomain of R. Then T is an SM domain.*

Proof. Let Q be a prime w-ideal of T. Then by [12, Lemma 3.1] Q is a w-module over R. Since T is an SM module (Theorem 3.1(5)), Q is a finite type w-module over R, that is, there exists a finitely generated R-module A such that $Q = A_w = \{x \in A \otimes K \mid Jx \subseteq A \text{ for some } J \in GV(R)\}$. We claim that $Q = (AT)_w = \{x \in qf(T) \mid Jx \subseteq AT \text{ for some } J \in GV(T)\}$. If $x \in Q$, then $Jx \subseteq A$ for some $J \in GV(R)$, which implies $JTx \subseteq AT$. Since by [12, Lemma 3.1] $JT \in GV(T)$, $x \in (AT)_w$, and hence $Q \subseteq (AT)_w$. Since Q is a w-ideal of T, the opposite inclusion is clear. Thus each prime w-ideal of T is of finite type, so that T is an SM domain (Theorem 3.1(3)). □

It is well known that if $R \subset T$ are rings with T Noetherian and T a finitely generated R-module, then R is Noetherian. But its w-analogue, i.e., the converse of Corollary 3.2 does not hold. We will construct a counter example by using the next proposition.

Proposition 3.3. *Let T be a quasi-local domain with maximal ideal $M \neq (0)$, let $k(T) = T/M$ be the residue field, let $\phi : T \to k(T)$ be the natural projection, and let D be a proper subring of $k(T)$. Let $R = \phi^{-1}(D)$ be the domain arising from the following pullback diagram of canonical homomorphisms*:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\ \phi\ } & k(T) \end{array}$$

Then the following statements are equivalent:
(1) *R is an SM domain.*
(2) *R is Noetherian.*
(3) *T is Noetherian*, *D is a field*, *and* $[k(T):D]<\infty$.

Proof. Assume that R is an SM domain. Then since R is a Mori domain, D is a field [5, Proposition 3.4] and so M is the unique maximal ideal of R. Moreover $M=(R:T)$ is a divisorial ideal of R, so that M is the unique maximal w-ideal of R. Therefore by Theorem 3.1(1), $R=R_M$ is Noetherian. The equivalence of (2) and (3) follows from [13, Theorem 2.3]. □

Example 3.4. Consider the following pullback diagram:

$$\begin{array}{ccc} R=\mathbb{R}+M & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ T=\mathbb{C}[\{X_i\}_{i=1}^{\infty}]_{(\{X_i\}_{i=1}^{\infty})} & \longrightarrow & \mathbb{C}, \end{array} \quad \text{where } M=(\{X_i\}_{i=1}^{\infty})\mathbb{C}[\{X_i\}_{i=1}^{\infty}]_{(\{X_i\}_{i=1}^{\infty})}.$$

Since $T\subseteq M^{-1}=(R:M)=(M:M)\subseteq R''(=$ the complete integral closure of $R)=T$, $T=M^{-1}$ is a divisorial ideal of R, and hence it is a w-module over R. Since $[\mathbb{C}:\mathbb{R}]=2<\infty$, T is a finitely generated R-module, so T is a finite-type w-overdomain of R. Since T is a UFD, T is clearly an SM domain. But T is not Noetherian, and hence Proposition 3.3 says that R is not an SM domain.

We will extend Proposition 3.3 to the general case.

Lemma 3.5 (Gabelli and Houston [14, Theorem 4.18])**.** *Let T be a domain with a nonzero maximal ideal M*, *let $k=T/M$ be the residue field*, *let $\phi:T\to k$ be the natural projection*, *and let D be a proper subring of k. Let $R=\phi^{-1}(D)$ be the domain arising from the following pullback of canonical homomorphisms*:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\ \phi\ } & k \end{array}$$

Then R is a Mori domain if and only if T is a Mori domain and D is a field.

It is well known that if $\{R_i\}_{i\in I}$ is a defining family of overrings of R of finite character and each R_i is Mori, then so is R [28, Corollary 4]. But we do not know whether a similar result holds for SM domains. However we can say at least the following holds.

Lemma 3.6. *Let $\{S_i\}_{i\in I}$ be a family of multiplicative subsets of R such that $R=\bigcap R_{S_i}$ has the finite character and R_{S_i} is an SM domain for all $i\in I$. Then R is also an SM domain.*

Proof. Since R_{S_i} is a Mori domain for all $i \in I$, so is $R = \bigcap R_{S_i}$ by Zafrullah [28, Corollary 4]. Let $P \in w\text{-max}(R) = t\text{-max}(R)$. Since in a Mori domain, $t = v, P$ is divisorial. By [19, Proposition 1.1], PR_{S_i} is divisorial in R_{S_i} for all $i \in I$. Since for each $x \in q.f(R)$ such that $P \subseteq xR$, $\bigcap PR_{S_i} \subseteq \bigcap xR_{S_i} = xR$, we have $\bigcap PR_{S_i} \subseteq P_v = P$. Therefore PR_{S_i} is proper for some $i \in I$. Assume that $PR_{S_{i_0}}$ is proper. Put $S = S_{i_0}$. We claim that $PR_S \in w\text{-max}(R_S)$. Let N be a maximal w-ideal of R_S containing PR_S. Put $M = N \cap R$. Then since N is a divisorial ideal of R_S, M is also divisorial in R (see the proof of [19, Proposition 1.1(v)]). Since a divisorial ideal is a w-ideal, by maximality of P, $P = M$, and so $N = MR_S = PR_S$. Since R_S is an SM domain, $R_P = (R_S)_{PR_S}$ is Noetherian (Theorem 3.1(1)). From the above argument, we can see that $w\text{-max}(R) \subseteq \{P \in Spec(R) \mid PR_{S_i} \in w\text{-max}(R_{S_i})$ for some $i \in I\}$. Therefore it follows from the finite characterness of $R = \bigcap R_{S_i}$ and $R_{S_i} = \bigcap\{(R_{S_i})_{PR_{S_i}} \mid PR_{S_i} \in w\text{-max}(R_{S_i})\}$ for each $i \in I$ that $R = \bigcap\{R_P \mid P \in w\text{-max}(R)\}$ has the finite character. Thus by Theorem 3.1(1), R is an SM domain. □

Proposition 3.7. *With the notation of Lemma* 3.5, *R is an SM domain if and only if T is an SM domain*, *T_M is Noetherian*, *D is a field*, *and* $[k : D] < \infty$.

Proof. (⇒) Assume that R is an SM domain. Then since R is a Mori domain, T is a Mori domain and D is a field, so M is a maximal ideal of R. Moreover $M = (R : T)$ is a divisorial ideal of R, so that $M \in w\text{-max}(R)$. By Theorem 3.1(1), R_M is Noetherian. Since the following diagram of canonical homomorphisms

$$\begin{array}{ccc} R_M & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ T_M & \xrightarrow{\ \phi\ } & k \end{array}$$

is a pullback, T_M is Noetherian and $[k : D] < \infty$.

Now let Q be a maximal w-ideal of T which is not contained in M and let $P = Q \cap R$. Then since $M \nsubseteq Q$, $T_Q = R_P$. Since $w\text{-max}(T) = t\text{-max}(T)$ and T is a Mori domain, Q is divisorial in T, and so QT_Q is divisorial in T_Q, i.e., PR_P is divisorial in R_P. By [19, Proposition 1.1], $P = PR_P \cap R$ is divisorial in R, and hence P is a w-ideal of R. Let P' be a maximal w-ideal of R containing P. Suppose that $P' = M$. Choose $x \in Q \setminus M$. Then since $M + xT = T$, $m + xt = 1$ for some $m \in M$, $t \in T$. So $xt = 1 - m \in Q \cap R = P \subseteq P' = M$, whence $1 = m + xt \in M$. This contradiction implies that $M \nsubseteq P'$. Therefore, there is a unique prime ideal Q' of T such that $Q' \cap R = P'$ and $T_{Q'} = R_{P'}$. By the same argument as above, we can show that Q' is a w-ideal of T. But since $Q \subseteq Q'$ and $Q \in w\text{-max}(T)$, $Q = Q'$, which implies $P = P'$, i.e., $P \in w\text{-max}(R)$.

Now since R is an SM domain, R_P is Noetherian and $R = \bigcap_{P \in w\text{-max}(R)} R_P$ has the finite character. It follows that T_Q is Noetherian and $\bigcap\{T_Q \mid Q \in w\text{-max}(T)$ and $Q \nsubseteq M\}$ has the finite character. Since $T = \bigcap_{Q \in w\text{-max}(T)} T_Q = (\bigcap\{T_Q \mid Q \in w\text{-max}(T)$ and $Q \subseteq M\}) \cap (\bigcap\{T_Q \mid Q \in w\text{-max}(T)$ and $Q \nsubseteq M\}) = T_M \cap (\bigcap\{T_Q \mid Q \in w\text{-max}(T)$

and $Q \nsubseteq M\}$) and the last expression has the finite character, T is an SM domain by Lemma 3.6.

($\Leftarrow$) By Proposition 3.3 and Lemma 3.5, R is a Mori domain and R_M is Noetherian. Now let $P(\neq M) \in w\text{-max}(R)$. Since $M \nsubseteq P$, there is a unique prime ideal Q of T such that $Q \cap R = P$ and $R_P = T_Q$. Again by the same argument as above, we can show that $Q \in w\text{-max}(T)$. It follows from the assumption T is an SM domain that R_P is Noetherian and $R = \bigcap_{P \in w\text{-max}(R)} R_P$ has the finite character. Therefore by Theorem 3.1(1), R is an SM domain. □

Proposition 3.8. *Let $M_1, \ldots, M_r$ be finitely many maximal ideals of a domain T, let D be a domain contained in T/M_i, $i = 1, \ldots, r$, and let $\phi : T \to T/I$ be the natural projection, where $I = \bigcap_{i=1}^{r} M_i$. Let $R = \phi^{-1}(D)$ be the domain arising from the following pullback of canonical homomorphisms*:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & T/I \cong T/M_1 \oplus \cdots \oplus T/M_r \end{array}$$

Then R is an SM domain if and only if T is an SM domain, T_{M_i} is Noetherian for all $i = 1, \ldots, r$, D is a field, and T/I is a finite D-module.

Proof. ($\Rightarrow$) Assume that R is an SM domain. Let F be the quotient field of D. Then since $F \subseteq T/M_i$ for all $i = 1, \ldots, r$, $F \subseteq T/I$. Let $S = \phi^{-1}(F)$. Then the diagram

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ S & \longrightarrow & F \cong S/I \end{array}$$

is a pullback. Therefore by Proposition 3.7, D is a field, so that $I \in w\text{-max}(R)$. Since the diagram

$$\begin{array}{ccc} R_I & \longrightarrow & D \\ \downarrow & & \downarrow \\ T_{R \setminus I} & \longrightarrow & T_{R \setminus I}/IT_{R \setminus I} \cong T/I \end{array} \qquad (*)$$

is a pullback and R_I is Noetherian, $T_{R \setminus I}$ is Noetherian and T/I is a finite D-module. Clearly $T_{M_i} = (T_{R \setminus I})_{M_i T_{R \setminus I}}$ is Noetherian for all $i = 1, \ldots, r$.

Now let Q be a maximal w-ideal of T which is not contained in any M_i, $i = 1, \ldots, r$. Then since $I \nsubseteq Q$, $T_Q = R_P$ where $P = Q \cap R$. Thus since $\bigcap \{T_Q \mid Q \in w\text{-max}(T)$ and $Q \nsubseteq M_i$ for all $i = 1, \ldots, r\}$ is a generalized quotient ring of a Mori domain R, it is a Mori domain by [25, Section 2, Theorem 2]. Therefore $T = \bigcap_{Q \in w\text{-max}(T)} T_Q =$

$(\bigcap\{T_Q \mid Q \in w\text{-max}(T)$ and $Q \subseteq M_i$ for some $i = 1,\dots,r\}) \cap (\bigcap\{T_Q \mid Q \in w\text{-max}(T)$ and $Q \nsubseteq M_i$ for all $i = 1,\dots,r\}) = (\bigcap_{i=1}^{r} T_{M_i}) \cap (\bigcap\{T_Q \mid Q \in w\text{-max}(T)$ and $Q \nsubseteq M_i$ for all $i = 1,\dots,r\})$ is a Mori domain by [28, Corollary 4].

We claim that $P = Q \cap R \nsubseteq I$. Suppose not. Choose $x \in Q \setminus (\bigcup_{i=1}^{r} M_i)$. Then since $I + xT = T$, $a + xt = 1$ for some $a \in I$, $t \in T$. So $xt = 1 - a \in Q \cap R = P \subseteq I$, whence $1 = a + xt \in I$. This contradiction implies that $P \nsubseteq I$. So as in the proof of Proposition 3.7, we can show that $P \in w\text{-max}(R)$.

Since R is an SM domain, R_P is Noetherian and $R = \bigcap_{P \in w\text{-max}(R)} R_P$ has the finite character. It follows that T_Q is Noetherian and $\bigcap\{T_Q \mid Q \in w\text{-max}(T)$ and $Q \nsubseteq M_i$ for all $i = 1,\dots,r\}$ has the finite character. Therefore since $T = (\bigcap_{i=1}^{r} T_{M_i}) \cap (\bigcap\{T_Q \mid Q \in w\text{-max}(T)$ and $Q \nsubseteq M_i$ for all $i = 1,\dots,r\})$ has the finite character, T is an SM domain by Lemma 3.6.

($\Leftarrow$) Since T_{M_i} is Noetherian for all $i = 1,\dots,r$ and $T_{R \setminus I} = \bigcap_{i=1}^{r} T_{M_i}$, $T_{R\setminus I}$ is Noetherian [22, Section 2–3, Exercise 10]. Since the diagram $(*)$ is a pullback and T/I is a finite D-module, R_I is Noetherian.

Now let $P(\neq I) \in w\text{-max}(R)$. Then since $I \nsubseteq P$, there is a unique prime ideal Q of T such that $Q \cap R = P$ and $R_P = T_Q$. Thus since $\bigcap\{R_P \mid P \in w\text{-max}(R)$ and $P \neq I\}$ is a generalized quotient ring of a Mori domain T, it is a Mori domain. So as in the proof of Proposition 3.7, we can show that $Q \in w\text{-max}(T)$.

Now since T is an SM domain, $R_P = T_Q$ is Noetherian and $\bigcap\{R_P \mid P \in w\text{-max}(R)$ and $P \neq I\}$ has the finite character. Therefore $R = \bigcap_{P \in w\text{-max}(R)} = R_I \cap (\bigcap\{R_P \mid P \in w\text{-max}(R)$ and $P \neq I\})$ is an SM domain by Theorem 3.1(1) or Lemma 3.6. □

4. Polynomial rings and formal power series rings

Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary set of indeterminates over R.

Lemma 4.1. *If $*$ denotes either the v-, the t-, or the w-operations, then* $(IR[\{X_\lambda\}_{\lambda \in \Lambda}])^* = I^* R[\{X_\lambda\}_{\lambda \in \Lambda}]$ *for each* $I \in F(R)$.

Proof. This result is stated in [18, Proposition 4.3] for a single indeterminate, and the proofs for the multi-variable case are identical to those for the single-variable case. □

In [16] Glaz and Vasconcelos introduce the concept of an H-domain: a domain R in which every ideal A with $A^{-1} = R$ has a finitely generated subideal J such that $J^{-1} = A^{-1}$. They then prove that if R is an H-domain, then $R[X]$ is an H-domain [16, (3.2c)].

Proposition 4.2. *If R is an H-domain, then so is* $R[\{X_\lambda\}_{\lambda \in \Lambda}]$.

Proof. Let Q be a maximal t-ideal of $R[\{X_\lambda\}]$. By [20, Proposition 2.4], it suffices to show that Q is divisorial. Since $Q \neq \{0\}$, there exists a finite subset $\{\lambda_1,\dots,\lambda_n\}$ of Λ such that $Q \cap R[X_{\lambda_1},\dots,X_{\lambda_n}] \neq \{0\}$. Since $R[X_{\lambda_1},\dots,X_{\lambda_n}]$ is an H-domain, we may assume that $Q \cap R \neq \{0\}$. Suppose that Q is not divisorial. Then since every divisorial ideal is a t-ideal and Q is a maximal t-ideal, $Q_v = R[\{X_\lambda\}]$. Let $A = \sum_{f \in Q} A_f$, where A_f is the ideal of R generated by the coefficients of f. Then $Q \subseteq A[\{X_\lambda\}]$. Therefore $Q_v \subseteq (A[\{X_\lambda\}])_v = A_v[\{X_\lambda\}]$, so $A_v = R$. Since R is an H-domain, there exists a finitely generated ideal J of R such that $J \subseteq A$ and $J_v = R$. Therefore there exists an element $f \in Q$ such that $(A_f)_v = R$. Choose $a \in Q \cap R \setminus \{0\}$. We claim that $(a, f)^{-1} = R[\{X_\lambda\}]$. Let $g \in (a, f)^{-1} \subseteq K[\{X_\lambda\}]$. Then $gf \in R[\{X_\lambda\}]$. By Dedekind–Mertens theorem, there exists a positive integer m such that $A_g A_f^m = A_{gf} A_f^{m-1}$. Therefore $R \supseteq (A_{gf} A_f^{m-1})_v = (A_g A_f^m)_v = (A_g (A_f)_v^m)_v = (A_g)_v$, which implies $g \in R[\{X_\lambda\}]$, thus $(a, f)^{-1} = R[\{X_\lambda\}]$, i.e., $(a, f)_v = R[\{X_\lambda\}]$. But since Q is a t-ideal and $(a, f) \subseteq Q$, $R[\{X_\lambda\}] = (a, f)_v \subseteq Q_t = Q$, a contradiction. □

In [27] Roitman showed that there exists a Mori domain R such that $R[X]$ is not Mori using the following equivalent conditions: R is a Mori domain if and only if for any $a \in R \setminus \{0\}$, the ring R/Ra has $CC^\perp$ [26, Theorem 2.2]. We will also use this theorem in proving that if R is a Noetherian domain, then $R[\{X_\lambda\}_{\lambda \in \Lambda}]$ and $R[\![\{X_\lambda\}_{\lambda \in \Lambda}]\!]_1$ are Mori domains. Recall the condition $CC^\perp$ means the descending chain condition on annihilators, or equivalently, the ascending chain condition on annihilators. It is well known and easy that the $CC^\perp$ property is hereditary, i.e., subrings of $CC^\perp$-rings are also $CC^\perp$-rings (cf. [8,9]).

Proposition 4.3. *Let R be a Noetherian domain. Then $R[\{X_\lambda\}_{\lambda \in \Lambda}]$ is a Mori domain.*

Proof. By [26, Theorem 2.2], it suffices to show that for any $f \in R[\{X_\lambda\}] \setminus \{0\}$, the ring $R[\{X_\lambda\}]/fR[\{X_\lambda\}]$ has $CC^\perp$. Let $f \in R[\{X_\lambda\}] \setminus \{0\}$. Then there exists a finite subset $\{\lambda_1, \ldots, \lambda_n\}$ of Λ such that $f \in R[X_{\lambda_1}, \ldots, X_{\lambda_n}]$. Since $R[X_{\lambda_1}, \ldots, X_{\lambda_n}]$ is Noetherian, $fR[X_{\lambda_1}, \ldots, X_{\lambda_n}]$ has a reduced primary decomposition $fR[X_{\lambda_1}, \ldots, X_{\lambda_n}] = Q_1 \cap \cdots \cap Q_k$. Let $P_i = \sqrt{Q_i}$. Then $fR[\{X_\lambda\}] = Q_1 R[\{X_\lambda\}] \cap \cdots \cap Q_k R[\{X_\lambda\}]$ and $Q_i R[\{X_\lambda\}]$ is a $P_i R[\{X_\lambda\}]$-primary ideal. It is clear that $R[\{X_\lambda\}]/fR[\{X_\lambda\}] \subseteq R[\{X_\lambda\}]/Q_1 R[\{X_\lambda\}] \oplus \cdots \oplus R[\{X_\lambda\}]/Q_k R[\{X_\lambda\}] \subseteq T(R[\{X_\lambda\}]/Q_1 R[\{X_\lambda\}]) \oplus \cdots \oplus T(R[\{X_\lambda\}]/Q_k R[\{X_\lambda\}])$, where $T(R[\{X_\lambda\}]/Q_i R[\{X_\lambda\}])$ is the total quotient ring of $R[\{X_\lambda\}]/Q_i R[\{X_\lambda\}]$. Since $T(R[\{X_\lambda\}]/Q_i R[\{X_\lambda\}]) \cong (R[\{X_\lambda\}]/Q_i R[\{X_\lambda\}])_{P_i R[\{X_\lambda\}]/Q_i R[\{X_\lambda\}]} \cong (R[\{X_\lambda\}]_{P_i R[\{X_\lambda\}]})/(Q_i R[\{X_\lambda\}]_{P_i R[\{X_\lambda\}]})$ is a 0-dimensional quasi-local ring and its unique prime ideal $P_i T(R[\{X_\lambda\}]/Q_i R[\{X_\lambda\}])$ is finitely generated, $T(R[\{X_\lambda\}]/Q_i R[\{X_\lambda\}])$ is Noetherian. Therefore obviously it has $CC^\perp$. Now since $R[\{X_\lambda\}]/fR[\{X_\lambda\}]$ is a subring of the $CC^\perp$-ring $T(R[\{X_\lambda\}]/Q_1 R[\{X_\lambda\}]) \oplus \cdots \oplus T(R[\{X_\lambda\}]/Q_k R[\{X_\lambda\}])$, $R[\{X_\lambda\}]/fR[\{X_\lambda\}]$ has also $CC^\perp$. □

Corollary 4.4. *Let R be a Noetherian domain. Then every integral divisorial ideal of $R[\{X_\lambda\}_{\lambda \in \Lambda}]$ is finitely generated and it has a primary decomposition.*

Proof. Let A be an integral divisorial ideal of $R[\{X_\lambda\}]$. Then $A = I_v$ for some finitely generated ideal I of $R[\{X_\lambda\}]$. There exist a finite subset $\{\lambda_1, \ldots, \lambda_n\}$ of Λ and a finitely generated ideal J of $R[X_{\lambda_1}, \ldots, X_{\lambda_n}]$ such that $I = JR[\{X_\lambda\}]$. Therefore $A = I_v = J_v R[\{X_\lambda\}]$.

Since J_v is an ideal of a Noetherian ring $R[X_{\lambda_1},\dots,X_{\lambda_n}]$, it is finitely generated and has a primary decomposition. Therefore A is also finitely generated and has a primary decomposition. □

Proposition 4.5. *Let R be a Noetherian domain. Then $R[\![\{X_\lambda\}_{\lambda\in\Lambda}]\!]_1$ is a Mori domain.*

Proof. The proof is essentially the same as that for Proposition 4.3. All we have to check is that if Q is a P-primary ideal in R, then $QR[\![\{X_\lambda\}]\!]_1$ is a $PR[\![\{X_\lambda\}]\!]_1$-primary ideal. Since R is Noetherian, $QR[\![\{X_\lambda\}]\!]_1 = Q[\![\{X_\lambda\}]\!]_1$ and $PR[\![\{X_\lambda\}]\!]_1 = P[\![\{X_\lambda\}]\!]_1$. If k is a positive integer such that $P^k \subseteq Q$, then $(P[\![\{X_\lambda\}]\!]_1)^k \subseteq Q[\![\{X_\lambda\}]\!]_1$. Let $f,g\in R[\![\{X_\lambda\}]\!]_1$ with $fg\in Q[\![\{X_\lambda\}]\!]_1$. Assume that $f\notin Q[\![\{X_\lambda\}]\!]_1$. Then there exists a finite subset $\{\lambda_1,\dots,\lambda_n\}$ of Λ such that $f,g\in R[\![X_{\lambda_1},\dots,X_{\lambda_n}]\!]$. So $fg\in Q[\![X_{\lambda_1},\dots,X_{\lambda_n}]\!]$ and $f\notin Q[\![X_{\lambda_1},\dots,X_{\lambda_n}]\!]$. Since by [6, Theorem 8], $Q[\![X_{\lambda_1},\dots,X_{\lambda_n}]\!]$ is a $P[\![X_{\lambda_1},\dots,X_{\lambda_n}]\!]$-primary ideal, $g\in P[\![X_{\lambda_1},\dots, X_{\lambda_n}]\!]$, and hence $g\in P[\![\{X_\lambda\}]\!]_1$. □

Corollary 4.6. *Let R be a Noetherian domain. Then every integral divisorial ideal of $R[\![\{X_\lambda\}_{\lambda\in\Lambda}]\!]_1$ is finitely generated and it has a primary decomposition.*

Proof. Let A be an integral divisorial ideal of $R[\![\{X_\lambda\}]\!]_1$. Then $A=I_v$ for some finitely generated ideal I of $R[\![\{X_\lambda\}]\!]_1$. There exist a finite subset $\{\lambda_1,\dots,\lambda_n\}$ of Λ and a finitely generated ideal J of $R[\![X_{\lambda_1},\dots,X_{\lambda_n}]\!]$ such that $I=JR[\![\{X_\lambda\}_{\lambda\in\Lambda\setminus\{\lambda_1,\dots,\lambda_n\}}]\!]_1 = J[\![\{X_\lambda\}_{\lambda\in\Lambda\setminus\{\lambda_1,\dots,\lambda_n\}}]\!]_1$. Therefore $A=I_v=(J[\![\{X_\lambda\}_{\lambda\in\Lambda\setminus\{\lambda_1,\dots,\lambda_n\}}]\!]_1)_v = J_v[\![\{X_\lambda\}_{\lambda\in\Lambda\setminus\{\lambda_1,\dots,\lambda_n\}}]\!]_1$ by [10, Proposition 2.1] (due to Anderson and Kang). Since J_v is an ideal of Noetherian ring $R[\![X_{\lambda_1},\dots,X_{\lambda_n}]\!]$, it is finitely generated, and so is $A=J_v[\![\{X_\lambda\}_{\lambda\in\Lambda\setminus\{\lambda_1,\dots,\lambda_n\}}]\!]_1 = J_vR[\![\{X_\lambda\}_{\lambda\in\Lambda\setminus\{\lambda_1,\dots,\lambda_n\}}]\!]_1$. Let $J_v=Q_1\cap\cdots\cap Q_m$ be a primary decomposition. Then as we said in the proof of Proposition 4.5, each $Q_iR[\![\{X_\lambda\}]\!]_1$ is a primary ideal, hence A has a primary decomposition $A=J_vR[\![\{X_\lambda\}]\!]_1 = Q_1R[\![\{X_\lambda\}]\!]_1\cap\cdots\cap Q_mR[\![\{X_\lambda\}]\!]_1$. □

Theorem 4.7. *Let R be an SM domain. Then $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ is also an SM domain.*

Proof. Let Q be a prime w-ideal of $R[\{X_\lambda\}]$. Then there exists a maximal w-ideal M of $R[\{X_\lambda\}]$ containing Q. By Lemma 2.1, M is a maximal t-ideal of $R[\{X_\lambda\}]$. Since $M\neq\{0\}$, there exists a finite subset Λ_0 of Λ such that $M\cap R[\{X_\lambda\}_{\lambda\in\Lambda_0}]\neq\{0\}$. Since $R[\{X_\lambda\}_{\lambda\in\Lambda_0}]$ is an SM domain by Theorem 3.1(2), we may assume that $M\cap R\neq\{0\}$. Since R is a Mori domain, M is divisorial by Proposition 4.2, and hence $M=(M\cap R)R[\{X_\lambda\}]$ by [26, Theorem 3.6]. Since R is an SM domain and $M\cap R$ is a w-ideal of R (Lemma 4.1), there exists a finite subset $\{a_1,\dots,a_m\}$ of $M\cap R$ such that $M\cap R=(a_1,\dots,a_m)_w$. So for each finite subset Λ_1 of Λ, $(M\cap R)R[\{X_\lambda\}_{\lambda\in\Lambda_1}] = (a_1,\dots,a_m)_wR[\{X_\lambda\}_{\lambda\in\Lambda_1}] = ((a_1,\dots,a_m)R[\{X_\lambda\}_{\lambda\in\Lambda_1}])_w$, and hence $ht(M\cap R)R[\{X_\lambda\}_{\lambda\in\Lambda_1}]\leq m$ by Theorem 3.1(4). Therefore $ht\,M\leq m<\infty$. Let $ht\,Q=k<\infty$. Then there exists a chain of prime ideals $(0)\subsetneq Q_1\subsetneq Q_2\subsetneq\cdots\subsetneq Q_k=Q$. For each $i=1,\dots,k$, choose $f_i\in Q_i\setminus Q_{i-1}$. Then there exists a finite subset $\{\lambda_1,\dots,\lambda_n\}$ of

Λ such that $f_i \in R[X_{\lambda_1},\dots,X_{\lambda_n}]$ for all $i=1,\dots,k$. Let $P_i = Q_i \cap R[X_{\lambda_1},\dots,X_{\lambda_n}]$. Then $P_iR[\{X_\lambda\}_{\lambda\in\Lambda}] \in Spec(R\{X_\lambda\}_{\lambda\in\Lambda}])$ and $(0) \subsetneq P_1R[\{X_\lambda\}] \subsetneq \cdots \subsetneq P_kR[\{X_\lambda\}] \subseteq Q$. Since $ht\,Q = k$, $Q = P_kR[\{X_\lambda\}]$. Since P_k is a w-ideal of an SM domain $R[X_{\lambda_1},\dots,X_{\lambda_n}]$, $P_k = I_w$ for some finitely generated ideal I of $R[X_{\lambda_1},\dots,X_{\lambda_n}]$. Therefore by Lemma 4.1, $Q = (IR[\{X_\lambda\}])_w$. Thus every prime w-ideal of $R[\{X_\lambda\}]$ is of finite type, so that $R[\{X_\lambda\}]$ is an SM domain (Theorem 3.1(3)). □

Theorem 4.8. *Let R be an SM domain and Q a maximal w-ideal of $R[\{X_\lambda\}_{\lambda\in\Lambda}]$. Then*

$$ht\,Q = \begin{cases} ht(Q\cap R) & \text{if } Q\cap R \neq \{0\},\\ 1 & \text{otherwise.}\end{cases}$$

Therefore

$$w\text{-}dim R[\{X_\lambda\}_{\lambda\in\Lambda}] = \begin{cases} w\text{-}dim\, R & \text{if } R \text{ is not a field},\\ 1 & \text{if } R \text{ is a field and } \Lambda \text{ is nonempty.}\end{cases}$$

Proof. Let Q be a maximal w-ideal of $R[\{X_\lambda\}]$. Since $R[\{X_\lambda\}]$ is an SM domain, it is an H-domain, and hence Q is a divisorial ideal of $R[\{X_\lambda\}]$.

Case I: $Q\cap R = \{0\}$. Since $R[\{X_\lambda\}]$ is a Mori domain, $Q_{R\setminus\{0\}}$ is a divisorial ideal of $R[\{X_\lambda\}]_{R\setminus\{0\}} = K[\{X_\lambda\}]$. Since $K[\{X_\lambda\}]$ is a UFD, $ht\,Q_{R\setminus\{0\}} = 1$, so $ht\,Q = 1$.

Case II: $Q\cap R \neq \{0\}$. Since R is a Mori domain, $Q = (Q\cap R)R[\{X_\lambda\}]$ and so $ht\,Q \geq ht(Q\cap R)$. Since R is an SM domain and $Q\cap R$ is a prime w-ideal of R (Lemma 4.1), $R_{Q\cap R}$ is Noetherian. Therefore $ht(Q\cap R)R_{Q\cap R} < \infty$. Let $ht(Q\cap R)R_{Q\cap R} = n < \infty$. Then by [22, Theorem 153], there exist elements $a_1,\dots,a_n$ in R such that $(Q\cap R)R_{Q\cap R}$ is minimal over $(a_1,\dots,a_n)R_{Q\cap R}$. It is clear that $Q = (Q\cap R)R[\{X_\lambda\}]$ is minimal over $(a_1,\dots,a_n)_wR[\{X_\lambda\}] = ((a_1,\dots,a_n)R[\{X_\lambda\}])_w$. Since $R[\{X_\lambda\}]$ is an SM domain, $ht\,Q \leq n$ by Theorem 3.1(4). Thus $ht\,Q = ht(Q\cap R)(<\infty)$. The last statement follows directly. □

Proposition 4.9. *Let R be a Noetherian domain. Then every prime w-ideal of $R[\![\{X_\lambda\}_{\lambda\in\Lambda}]\!]_1$ is finitely generated, and so $R[\![\{X_\lambda\}_{\lambda\in\Lambda}]\!]_1$ is an SM domain.*

Proof. Let Q be a prime w-ideal of $R[\![\{X_\lambda\}]\!]_1$ and M a maximal w-ideal containing Q. Then M is a maximal t-ideal of $R[\![\{X_\lambda\}]\!]_1$. As in Theorem 4.7, we may assume that $M\cap R \neq \{0\}$. Since $R[\![\{X_\lambda\}]\!]_1$ is a Mori domain (Proposition 4.5), M is divisorial, and hence $M = (M\cap R)[\![\{X_\lambda\}]\!]_1$ by [26, Theorem 3.7]. It is easy to check that $ht\,M = ht(M\cap R) < \infty$. Let $ht\,Q = k < \infty$. Then there exists a chain of prime ideals $(0) \subsetneq Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_k = Q$. For each $i=1,\dots,k$, choose $f_i \in Q_i\setminus Q_{i-1}$. Then there exists a finite subset $\{\lambda_1,\dots,\lambda_n\}$ of Λ such that $f_i \in R[\![X_{\lambda_1},\dots,X_{\lambda_n}]\!]$ for all $i=1,\dots,k$. Let $P_i = Q_i \cap R[\![X_{\lambda_1},\dots,X_{\lambda_n}]\!]$. Then $P_iR[\![\{X_\lambda\}_{\lambda\in\Lambda\setminus\{\lambda_1,\dots,\lambda_n\}}]\!]_1 = P_i[\![\{X_\lambda\}_{\lambda\in\Lambda\setminus\{\lambda_1,\dots,\lambda_n\}}]\!]_1 \in Spec(R[\![\{X_\lambda\}_{\lambda\in\Lambda}]\!]_1)$ and $(0) \subsetneq P_1R[\![\{X_\lambda\}]\!]_1 \subsetneq \cdots \subsetneq P_kR[\![\{X_\lambda\}]\!]_1 \subseteq Q$. Since $ht\,Q = k$, $Q = P_kR[\![\{X_\lambda\}]\!]_1$ and it is finitely generated. Thus every prime w-ideal of $R[\![\{X_\lambda\}]\!]_1$ is finitely generated, so that $R[\![\{X_\lambda\}]\!]_1$ is an SM domain (Theorem 3.1(3)). □

Remark 4.10. (1) Since a Noetherian domain is an SM domain and an SM domain is a Mori domain, Proposition 4.3 follows from Theorem 4.7.

(2) Let R be a nonintegrally closed Noetherian domain. Then for any infinite set Λ, $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ (or $R[\![\{X_\lambda\}_{\lambda\in\Lambda}]\!]_1$) is an example of a nonKrull, nonNoetherian, SM domain.

(3) Question: If R is an SM domain, is $R[\![X]\!]$ an SM domain?

5. Group rings and semigroup rings

We now consider group rings and semigroup rings over SM domains. We begin with a generalization of Lemma 4.1.

Lemma 5.1. *Let R be an integral domain, and let S be a torsion-free cancellative additive semigroup. Let I be a nonzero fractional ideal of R. Then*
(1) $(IR[X;S])^{-1} = I^{-1}R[X;S]$,
(2) $(IR[X;S])_v = I_vR[X;S]$,
(3) $(IR[X;S])_t = I_tR[X;S]$, *and*
(4) $(IR[X;S])_w = I_wR[X;S]$.

Proof. (1) Let $a\in I^{-1}$. Then $aI\subseteq R$, so $aIR[X;S]\subseteq R[X;S]$. Thus $a\in(IR[X;S])^{-1}$, i.e., $I^{-1}R[X;S]\subseteq(IR[X;S])^{-1}$. Conversely let $f\in(IR[X;S])^{-1}$. Then $fIR[X;S]\subseteq R[X;S]$. Note that $f\in K[X;S]$, where $K=q\cdot f(R)$. Let A_f be the ideal of R generated by the coefficients of f. Then $A_fI\subseteq R$, i.e., $A_f\subseteq I^{-1}$. Thus $f\in I^{-1}R[X;S]$, i.e., $(IR[X;S])^{-1}\subseteq I^{-1}R[X;S]$.

(2) $(IR[X;S])_v=((IR[X;S])^{-1})^{-1}=(I^{-1}R[X;S])^{-1}=I_vR[X;S]$.

(3) Let J be a finitely generated ideal of R contained in I. Then $JR[X;S]$ is a finitely generated ideal of $R[X;S]$ contained in $IR[X;S]$. Therefore $J_v\subseteq(JR[X;S])_v\subseteq(IR[X;S])_t$. Hence $I_tR[X;S]\subseteq(IR[X;S])_t$. Conversely let J' be a finitely generated ideal of $R[X;S]$ contained in $IR[X;S]$. Then there exists a finitely generated ideal J of R contained in I such that $J'\subseteq JR[X;S]$. Since $J'_v\subseteq(JR[X;S])_v=J_vR[X;S]\subseteq I_tR[X;S]$, we have $(IR[X;S])_t\subseteq I_tR[X;S]$.

(4) Assume that I is an integral ideal of R. Let $a\in I_w$. Then $Ja\subseteq I$ for some $J\in GV(R)$. Since $JR[X;S]a\subseteq IR[X;S]$ and $JR[X;S]\in GV(R[X;S])$ by (2), $a\in(IR[X;S])_w$. Thus $I_wR[X;S]\subseteq(IR[X;S])_w$. For the opposite inclusion, it suffices to show that $I_wR[X;S]$ is a w-ideal of $R[X;S]$. Suppose $u(f_1,\dots,f_n)\subseteq I_wR[X;S]$, $u\in q\cdot f(R[X;S])$ and $(f_1,\dots,f_n)\in GV(R[X;S])$. Then $uR[X;S]=u(f_1,\dots,f_n)_v\subseteq(I_wR[X;S])_v\subseteq R[X;S]$, so $u\in R[X;S]$. Since $(f_1,\dots,f_n)\subseteq(A_{f_1}+\cdots+A_{f_n})R[X;S]$, $(f_1,\dots,f_n)_v\subseteq(A_{f_1}+\cdots+A_{f_n})_vR[X;S]\subseteq R[X;S]$, thus $(A_{f_1}+\cdots+A_{f_n})_v=R$. By [16, Theorem 4.3], there is a positive integer m such that $A_uA_{f_i}^m=A_{uf_i}A_{f_i}^{m-1}$ for all $i=1,\dots,n$. Since $uf_i\in I_wR[X;S]$, $A_{uf_i}\subseteq I_w$. Therefore $A_u(A_{f_1}^m+\cdots+A_{f_n}^m)\subseteq I_w$. Since $(A_{f_1}+\cdots+A_{f_n})_v=R$, $(A_{f_1}^m+\cdots+A_{f_n}^m)_v=R$, i.e., $(A_{f_1}^m+\cdots+A_{f_n}^m)\in GV(R)$. Therefore $A_u\subseteq(I_w)_w=I_w$, i.e., $u\in I_wR[X;S]$. Hence $I_wR[X;S]$ is a w-ideal. The proof for the case when I is a fractional ideal follows easily. □

Proposition 5.2. *Let T be an integral extension domain of R with T a free R-module. Then w-$\dim T = w$-$\dim R$.*

Proof. Note that since T is faithfully flat and integral over R, GD, GU, LO and INC hold between T and R.

Let P be a maximal w-ideal of R. We claim that $(PT)_w \neq T$. Suppose not. Then there exists an ideal $J = (b_1,\dots,b_m) \in GV(T)$ such that $J \subseteq PT$. Let $\{e_\alpha\}$ be an R-basis for T. Then we can write $1 = c_1 e_{\alpha_1} + \cdots + c_n e_{\alpha_n}$, $c_i \in R$ and $b_i = a_{i1} e_{\alpha_1} + \cdots + a_{in} e_{\alpha_n}$, $a_{ij} \in P$, $i = 1,\dots,m$. Put $I = (\{a_{ij}\})$. We claim that $I^{-1} = R$. Let $x = a/b$, $a(\neq 0)$, $b \in R$ such that $xI \subseteq R$. Then $xIT \subseteq T$. Since $J \subseteq IT$, $xJ \subseteq T$, i.e., $x \in J^{-1} = T$. Write $x = r_1 e_{\alpha_1} + \cdots + r_n e_{\alpha_n}$, $r_i \in R$. Then $ac_1 e_{\alpha_1} + \cdots + ac_n e_{\alpha_n} = a = bx = br_1 e_{\alpha_1} + \cdots + br_n e_{\alpha_n}$. Since $e_{\alpha_1},\dots,e_{\alpha_n}$ are linearly independent over R, $ac_i = br_i$, $i = 1,\dots,n$. Meanwhile, since T is integral over R and $(c_1,\dots,c_n)T = T$, $(c_1,\dots,c_n) = R$. Therefore there exist $d_1,\dots,d_n \in R$ such that $1 = c_1 d_1 + \cdots + c_n d_n$. Thus $a = a(c_1 d_1 + \cdots + c_n d_n) = b(r_1 d_1 + \cdots + r_n d_n)$, and hence $x = a/b = r_1 d_1 + \cdots + r_n d_n \in R$, i.e., $I^{-1} = R$. Therefore $I \in GV(R)$. But since $I \subseteq P$ and P is a w-ideal, we reach a contradiction. So $(PT)_w \neq T$. Let Q be a maximal w-ideal of T containing PT. Then since $ht\, Q \geq ht\, P$, w-$\dim T \geq w$-$\dim R$.

Conversely let Q be a maximal w-ideal of T. Suppose $P_w = R$, where $P = Q \cap R$. Then there exists an ideal $I \in GV(R)$ such that $I \subseteq P$. Since I is finitely generated and T is flat over R, $(T : IT) = (R : I)T = T$. Thus $IT \in GV(T)$. But since $IT \subseteq PT \subseteq Q$ and Q is a w-ideal, a contradiction. Therefore $P_w \neq R$, so by [12, Proposition 1.1], P is a w-ideal of R. Since $ht\, Q = ht\, P \leq w$-$\dim R$, w-$\dim T \leq w$-$\dim R$. □

Corollary 5.3. *Let R be an SM domain which is not a field, and let G be a torsion-free abelian group. Then w-$\dim R = w$-$\dim R[X; G]$.*

Proof. Let F be a free subgroup of G such that G/F is torsion. Then by [7, Lemma 1], $R[X; G]$ is an integral extension domain of $R[X; F]$ and a free $R[X; F]$-module. Note that $R[X; F] \cong R[\{X_\lambda, X_\lambda^{-1}\}]$. Put $Y_\lambda = X_\lambda + X_\lambda^{-1}$. Then $R[X; F]$ is an integral extension of $R[\{Y_\lambda\}]$ and a free $R[\{Y_\lambda\}]$-module. Therefore, $R[X; G]$ is an integral extension domain of $R[\{Y_\lambda\}]$ and a free $R[\{Y_\lambda\}]$-module. So by Proposition 5.2, w-$\dim R[X; G] = w$-$\dim R[\{Y_\lambda\}]$. Since in case R is an SM domain, w-$\dim R[\{Y_\lambda\}] = w$-$\dim R$ by Theorem 4.8, we get w-$\dim R[X; G] = w$-$\dim R$. □

Remark 5.4. In the proof of Corollary 5.3, if G/F is finitely generated (which holds if G is finitely generated), then G/F is a finite abelian group, and so $R[X; G]$ is a finite type w-module over $R[X; F]$. Since $R[X; F] \cong R[\{X_\lambda, X_\lambda^{-1}\}] = R[\{X_\lambda\}]_T$, where T is the multiplicative subset of $R[\{X_\lambda\}]$ generated by $\{X_\lambda\}$, and $R[\{X_\lambda\}]_T$ is an SM domain [11, Proposition 4.7], $R[X; G]$ is an SM domain by Corollary 3.2.

Proposition 5.5. *Let R be an SM domain, and let G be a torsion-free abelian group such that each element of G is of type $(0,0,0,\dots)$. Then $R[X; G]$ is an H-domain.*

Proof. Let F be a free subgroup of G such that G/F is torsion. Then $R[X; F] \cong R[\{X_\lambda, X_\lambda^{-1}\}]$. Set $Y_\lambda = X_\lambda + X_\lambda^{-1}$. Then the ring extension $R[\{Y_\lambda\}] \subseteq R[X; G]$ is integral with $R[X;G]$ a free $R[\{Y_\lambda\}]$-module. Note that $R[\{Y_\lambda\}]$ is an SM domain by Theorem 4.7. Let Q be a maximal w-ideal of $R[X;G]$. Set $P=Q\cap R[\{Y_\lambda\}]$. Then as we can see from the proof of Proposition 5.2, P is a w-ideal of $R[\{Y_\lambda\}]$. Let P' be a maximal w-ideal of $R[\{Y_\lambda\}]$ containing P. Then since $w\text{-max}(R[\{Y_\lambda\}])=t\text{-max}(R[\{Y_\lambda\}])$ (Lemma 2.1) and $R[\{Y_\lambda\}]$ is an H-domain, P' is divisorial. Moreover since $R[\{Y_\lambda\}]$ is a Mori domain, $P'=I_v$ and $(R[\{Y_\lambda\}]:I)=J_v$ for some finitely generated ideals I and J of $R[\{Y_\lambda\}]$. Then $P'R[X;G]=I_vR[X;G]=(R[\{Y_\lambda\}]:(R[\{Y_\lambda\}]:I))R[X;G]=(R[\{Y_\lambda\}]:J_v)R[X;G]=(R[\{Y_\lambda\}]:J)R[X;G]=(R[X;G]:JR[X;G])$. Thus $P'R[X;G]$ is a divisorial ideal of $R[X;G]$. By GU, there exists a prime ideal Q' of $R[X;G]$ such that $Q\subseteq Q'$ and $Q'\cap R[\{Y_\lambda\}]=P'$. By INC, Q' is minimal over $P'R[X;G]$. Since $P'R[X;G]$ is a w-ideal, Q' is also a w-ideal. So by maximality of Q, $Q=Q'$. Thus $P=P'$, i.e., P is a maximal w-ideal of $R[\{Y_\lambda\}]$ and a divisorial ideal of $R[\{Y_\lambda\}]$. Now we claim that Q is a divisorial ideal of $R[X;G]$.

Case I: $Q\cap R=\{0\}$. Since $R[\{Y_\lambda\}]$ is a Mori domain, $P_{R\setminus\{0\}}$ is a divisorial ideal of $K[\{Y_\lambda\}]$. Since $K[\{Y_\lambda\}]$ is a Krull domain, $ht\,P_{R\setminus\{0\}}=1$. Therefore $ht\,Q=ht\,P=1$. Since $K[X;G]$ is a UFD by [15, Theorem 7.12], $Q_{R\setminus\{0\}}$ is principal. Let $Q_{R\setminus\{0\}}=fK[X;G]$, $f\in Q$. Then $Q=fK[X;G]\cap R[X;G]$. Set $A=\sum_{g\in Q}A_g$. Since $Q\subsetneq AR[X;G]$ and $Q\in w\text{-max}(\mathrm{R[X;G]})$, $A_wR[X;G]=(AR[X;G])_w=R[X;G]$ (Lemma 5.1), whence $A_w=R$. Since the operation w has the finite character, there exists a finite subset $\{g_1,\dots,g_m\}$ of Q such that $(A_{g_1}+\cdots+A_{g_m})_w=R$. Recall that since G is a torsion-free abelian group, it admits a total order $<$ compatible with the group structure. (See the proof of [15, Lemma 4.1].) So there exists an element $g\in(g_1,\dots,g_m)\subseteq Q$ such that $(A_g)_w=R$.

Let $h\in Q$. Then there is an element $a\in R\setminus\{0\}$ such that $ah\in(f)$. We claim that $(a,g)_v=R[X;G]$. Let $h'\in(a,g)^{-1}$. Then $h'\in K[X;G]$. By [15, Lemma 4.3], there exists a positive integer k such that $A_g^kA_{h'}=A_g^{k-1}A_{gh'}$. Then since $(A_g)_v=((A_g)_w)_v=R$ and $gh'\in R[X;G]$, $(A_{h'})_v=(A_g^kA_{h'})_v=(A_g^{k-1}A_{gh'})_v\subseteq R$, hence $h'\in R[X;G]$. Thus $(a,g)^{-1}=R[X;G]$. Since $h(a,g)\subseteq(f,g)$. $h(a,g)_v\subseteq(f,g)_v$, so $h\in(f,g)_v$. Meanwhile, since $w\text{-max}(R[X;G])=t\text{-max}(R[X;G])$, Q is a t-ideal, so $(f,g)_v\subseteq Q$. Thus $Q=(f,g)_v$ is divisorial.

Case II: $Q\cap R\neq\{0\}$. Note that $Q\subseteq AR[X;G]$. Suppose that $Q\subsetneq AR[X;G]$ or $AR[X;G]$ is not divisorial. Then $A_v=R$. Since R is an H-domain, there exists a finite subset $\{g_1,\dots,g_m\}\subseteq Q$ such that $(A_{g_1}+\cdots+A_{g_m})_v=R$. By the same reason as above, there exists an element $g\in(g_1,\dots,g_m)\subseteq Q$ such that $(A_g)_v=R$. Choose $a\in Q\cap R\setminus\{0\}$. Then $(a,g)_v=R[X;G]$. But since Q is a t-ideal, $(a,g)_v\subseteq Q$, a contradiction. Therefore $Q=AR[X;G]=(Q\cap R)R[X;G]$ is divisorial. (Since R is a Mori domain, $Q\cap R=I_v$ for some finitely generated ideal I of R. Therefore $Q=(Q\cap R)R[X;G]=I_vR[X;G]=(IR[X;G])_v$. Thus Q is a divisorial ideal of finite type.) □

Corollary 5.6. *Let R be an SM domain, and let G be a torsion-free abelian group such that each element of G is of type* $(0,0,0,\dots)$. *Then every maximal w-ideal of $R[X;G]$ is of finite type.*

Proof. We will use the same notation as in the proof of Proposition 5.5.

Case I: $Q \cap R = \{0\}$. Let $h \in Q$. Then there is an element $a \in R \setminus \{0\}$ such that $ah \in (f)$. So $h(a,g) \subseteq (f,g)$ and then $h(a,g)_w \subseteq (f,g)_w$. Since $(a,g)_v = R[X;G]$, $1 \in (a,g)_w$, i.e., $(a,g)_w = R[X;G]$. Therefore $h \in (f,g)_w$, thus $Q \subseteq (f,g)_w$. Since the opposite inclusion is clear, we have $Q = (f,g)_w$.

Case II: $Q \cap R \neq \{0\}$. Then $Q = (Q \cap R)R[X;G]$. Since $Q \cap R$ is a w-ideal of the SM domain R, $Q \cap R = I_w$ for some finitely generated ideal I of R. Therefore $Q = (Q \cap R)R[X;G] = I_w R[X;G] = (IR[X;G])_w$ is of finite type. □

Corollary 5.7. *Let R be an SM domain with w-$\dim R \leq 1$, and let G be a torsion-free abelian group such that each element of G is of type* $(0,0,0,\ldots)$. *Then $R[X;G]$ is an SM domain with w-$\dim R[X;G] \leq 1$.*

Proof. If w-$\dim R = 0$, then R is a field. By [15, Theorem 7.12], $R[X;G]$ is a UFD. Since an Krull domain is an SM domain and its w-dimension is at most 1, the conclusion follows. Now assume that w-$\dim R = 1$. Then since w-$\dim R[X;G] = w$-$\dim R = 1$ by Corollary 5.3 and every maximal w-ideal of $R[X;G]$ is of finite type by Corollary 5.6, every prime w-ideal of $R[X;G]$ is of finite type. Therefore by Theorem 3.1(3), $R[X;G]$ is an SM domain. □

The following theorem generalizes [24, Proposition 3.3]: *$R[X;G]$ is a Krull domain if and only if R is a Krull domain and each element of G is of type* $(0,0,0,\ldots)$.

Theorem 5.8. *Let R be an integral domain, and let G be a torsion-free abelian group. Then $R[X;G]$ is an SM domain with w-$\dim R[X;G] \leq 1$ if and only if R is an SM domain with w-$\dim R \leq 1$ and each element of G is of type* $(0,0,0,\ldots)$.

Proof. $(\Leftarrow)$ See Corollary 5.7.

$(\Rightarrow)$ Let I be a w-ideal of R. Since $R[X;G]$ is an SM domain, there exists a finitely generated ideal J of R such that $J \subseteq I$ and $(IR[X;G])_w = (JR[X;G])_w$. Since $(IR[X;G])_w = I_w R[X;G] = IR[X;G]$ and $(JR[X;G])_w = J_w R[X;G]$, $I = J_w$. Thus every w-ideal of R is of finite type, and hence R is an SM domain. Therefore by Corollary 5.3, R is a field or w-$\dim R = w$-$\dim R[X;G]$, thus w-$\dim R \leq 1$. Finally, since $R[X;G]$ is an SM domain, it is a Mori domain and so it satisfies the ascending chain condition for principal ideals (a.c.c.p.). Therefore, by [15, Lemma 7.8, Theorem 7.9], each element of G is of type $(0,0,0,\ldots)$. □

Now we generalize [3, Proposition 5.11]: *Let R be an integral domain with quotient field K, and let S be a torsion-free cancellative additive semigroup containing* 0 *with quotient group G. Then the semigroup ring $R[X;S]$ is a Krull domain if and only if R and $K[X;S]$ are Krull domains.*

Theorem 5.9. *Let R be an integral domain with quotient field K, and let S be a torsion-free cancellative additive semigroup containing* 0 *with quotient group G. Then $R[X;S]$ is an SM domain with w-$\dim R[X;S] \leq 1$ if and only if R and $K[X;S]$ are SM domains with w-dimension ≤ 1.*

Proof. ($\Rightarrow$) Since $R[X;G] = R[X;S]_T$, where $T = \{X^s \mid s \in S\}$, is an SM domain with w-$\dim R[X;G] \leq w$-$\dim R[X;S] \leq 1$ by [11, Propositions 4.7 and 2.5], R is an SM domain with w-$\dim R \leq 1$ by Theorem 5.8. Similarly since $K[X;S] = R[X;S]_{R\setminus\{0\}}$, $K[X;S]$ is an SM domain with w-$\dim K[X;S] \leq w$-$\dim R[X;S] \leq 1$.

($\Leftarrow$) Note that $R[X;S] = R[X;G] \cap K[X;S]$. Since $R[X;G] = R[X;S]_T$, where $T = \{X^s \mid s \in S\}$, is an SM domain by Theorem 5.8 and $K[X;S] = R[X;S]_{R\setminus\{0\}}$ is an SM domain by assumption, $R[X;S] = R[X;S]_T \cap R[X;S]_{R\setminus\{0\}}$ is also an SM domain by Lemma 3.6. As we can see from the proof of Lemma 3.6, w-$\dim R[X;S] \leq \max(w$-$\dim R[X;S]_T$, w-$\dim R[X;S]_{R\setminus\{0\}}) = \max(w$-$\max R[X;G]$, w-$\dim K[X;S]) \leq 1$ by Theorem 5.8 and our assumption. □

Remark 5.10. Recall that $K[X;S]$ is a Krull domain if and only if each element of $G = \langle S \rangle$ is of type $(0,0,0,\ldots)$ and S is a Krull semigroup, i.e., S satisfies the ascending chain condition on v-ideals and satisfies the following property: $g \in S$, $h \in G$, and $g + nh \in S$ for all $n \geq 1$ implies $h \in S$ [3, Proposition 5.11].

It is natural to ask whether a similar characterization holds regarding SM domains. But we are unable to answer this question.

We close with one more observation which gives other examples of SM domains.

Proposition 5.11. *Let R be an SM domain which is not a field, and let S be a nonzero subsemigroup of $\mathbb{Z}$ containing* 0. *Then $R[X;S]$ is an SM domain with w-$\dim R[X;S] = w$-$\dim R$.*

Proof. If S is a group, then $S = d\mathbb{Z} \cong \mathbb{Z}$ ($d \in \mathbb{Z}$), so the conclusion follows from Corollary 5.3 and Remark 5.4. Assume that S is not a group. Choose $d \in S \setminus \{0\}$. Then by [1, Lemma 2.4], $R[X;S]$ is integral over $R[X;d\mathbb{Z} \cap S]$ and $d\mathbb{Z} \cap S = d\mathbb{N} \cong \mathbb{N}$. Since $S/(d\mathbb{Z} \cap S) \subseteq \mathbb{Z}/d\mathbb{Z}, R[X;S]$ is a free $R[X;d\mathbb{Z} \cap S]$-module of finite rank. Thus $R[X;S]$ is a finite type w-module over $R[X;d\mathbb{Z} \cap S]$. Since $R[X;d\mathbb{Z} \cap S] \cong R[X;\mathbb{N}] \cong R[X]$ is an SM domain with w-$\dim R[X] = w$-$\dim R, R[X;S]$ is also an SM domain and w-$\dim R[X;S] = w$-$\dim R[X;d\mathbb{Z} \cap S] = w$-$\dim R$ by Proposition 5.2. □

Acknowledgements

I would like to thank heartily Professor Byung Gyun Kang of POSTECH for his advice and the referee for several helpful suggestions. This work was supported by the Brain Korea 21 Project in 2000.

References

[1] S. Ameziane, D.E. Dobbs, S. Kabbaj, On the prime spectrum of commutative semi-group rings, Comm. Algebra 26 (8) (1998) 2559–2589.
[2] D.D. Anderson, Star-operations induced by overrings, Comm. Algebra 16 (12) (1988) 2535–2553.
[3] D.D. Anderson, D.F. Anderson, Divisibility properties of graded domains, Canad. J. Math. 34 (1) (1982) 196–215.
[4] D.D. Anderson, M. Zafrullah, On t-invertibility III, Comm. Algebra 21 (4) (1993) 1189–1201.
[5] V. Barucci, On a class of Mori domains, Comm. Algebra 11 (17) (1983) 1989–2001.
[6] J.W. Brewer, Power Series Over Commutative Rings, Marcel Dekker, New York, 1981.
[7] J.W. Brewer, D.L. Costa, E.L. Lady, Prime ideals and localization in commutative group rings, J. Algebra 34 (1975) 300–308.
[8] D.L. Costa, Some remarks on the ACC on annihilators, Comm. Algebra 18 (3) (1990) 635–658.
[9] D. Costa, M. Roitman, A lifting approach to v- and t-ideals, Comm. Algebra 18 (11) (1990) 3725–3742.
[10] D.E. Dobbs, E.G. Houston, On $t\text{-}Spec(R[\![X]\!])$, Canad. Math. Bull. 38 (2) (1995) 187–195.
[11] W. Fanggui, R.L. McCasland, On w-modules over Strong Mori domains, Comm. Algebra 25 (1997) 1285–1306.
[12] W. Fanggui, R.L. McCasland, On Strong Mori domains, J. Pure Appl. Algebra 135 (1999) 155–165.
[13] M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura Appl. 123 (4) (1980) 331–355.
[14] S. Gabelli, E. Houston, Coherentlike conditions in pullbacks, Michigan Math. J. 44 (1997) 99–123.
[15] R. Gilmer, T. Parker, Divisibility properties in semigroup rings, Michigan Math. J. 21 (1974) 65–85.
[16] S. Glaz, W.V. Vasconcelos, Flat ideals II, Manuscripta Math. 22 (1977) 325–341.
[17] M. Griffin, Some results on v-multiplication rings, Canad. J. Math. 19 (1967) 710–722.
[18] J.R. Hedstrom, E.G. Houston, Some remarks on star-operations, J. Pure Appl. Algebra 18 (1980) 37–44.
[19] E.G. Houston, T.G. Lucas, T.M. Viswanathan, Primary decomposition of divisorial ideals in Mori domains, J. Algebra 117 (1988) 327–342.
[20] E. Houston, M. Zafrullah, Integral domains in which each t-ideal is divisorial, Michigan Math. J. 35 (1988) 291–300.
[21] E. Houston, M. Zafrullah, On t-invertibility II, Comm. Algebra 17 (8) (1989) 1955–1969.
[22] I. Kaplansky, Commutative Rings, University of Chicago Press, Chicago, 1974.
[23] S. Malik, J.L. Mott, M. Zafrullah, On t-invertibility, Comm. Algebra 16 (1) (1988) 149–170.
[24] R. Matsuda, On algebraic properties of infinite group rings, Bull. Fac. Sci. Ibaraki Univ. Ser. A Math. 7 (1975) 29–37.
[25] J. Querrè, Intersections d'anneaux integres, J. Algebra 43 (1976) 55–60.
[26] M. Roitman, On Mori domains and commutative rings with $CC^{\perp}$I, J. Pure Appl. Algebra 56 (1989) 247–268.
[27] M. Roitman, On polynomial extensions of Mori domains over countable fields, J. Pure Appl. Algebra 64 (1990) 315–328.
[28] M. Zafrullah, Two characterizations of Mori domains, Math. Japonica 33 (4) (1988) 645–652.