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Towards a dichotomy for the Possible Winner problem in elections based on scoring rules ☆

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ABSTRACT

To make a joint decision, agents (or voters) are often required to provide their preferences as linear orders. To determine a winner, the given linear orders can be aggregated according to a voting protocol. However, in realistic settings, the voters may often only provide partial orders. This directly leads to the Possible Winner problem that asks, given a set of partial votes, whether a distinguished candidate can still become a winner. In this work, we consider the computational complexity of Possible Winner for the broad class of voting protocols defined by scoring rules. A scoring rule provides a score value for every position which a candidate can have in a linear order. Prominent examples include plurality, k-approval, and Borda. Generalizing previous NP-hardness results for some special cases, we settle the computational complexity for all but one scoring rule. More precisely, for an unbounded number of candidates and unweighted voters, we show that Possible Winner is NP-complete for all pure scoring rules except plurality, veto, and the scoring rule defined by the scoring vector $(2, 1, \ldots, 1, 0)$, while it is solvable in polynomial time for plurality and veto.

1. Introduction

Voting scenarios arise whenever the preferences of different parties (*voters*) have to be aggregated to form a joint decision. This is what happens in political elections, group decisions, web site rankings, or multiagent systems. Often, the voting process is executed in the following way: each voter provides his preference as a ranking (linear order) of all the possible alternatives (*candidates*). Given these rankings as an input, a *voting rule* produces a subset of the candidates (*winners*) as an output. However, in realistic settings, the voters may often only provide partial orders (or partial votes) instead of linear ones: For example, it might be impossible for the voters to provide a complete preference list because the set of candidates is too large, as it is the case for web page ranking. In addition, not all voters might have given their preferences yet during the aggregation process, or new candidates might be introduced after some voters already have given their rankings. Moreover, one often has to deal with partial votes due to incomparabilities: for some voters it might not be possible to compare two candidates or certain groups of candidates, be it because of lack of information or due to personal reasons. Hence, the study of partial voting profiles is natural and essential. One question that immediately comes to mind is whether any information on a possible outcome of the voting process can be given in the case of incomplete votes. More specifically, in this

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paper, we study the Possible Winner problem: Given a partial order for each of the voters, can a distinguished candidate c win for at least one extension of the partial orders into linear ones?

Of course, the answer to this question depends on the voting rule that is used. In this work, we will stick to the broad class of *scoring rules*. A scoring rule provides a score value for every position that a candidate can take within a linear order, given as a *scoring vector* of length m in the case of m candidates. The scores of the candidates are then added over all votes and the candidates with the highest score win. Famous examples are Borda, defined by the scoring vectors $(m-1, m-2, \ldots, 0)$ and k-approval, defined by $(1, \ldots, 1, 0, \ldots, 0)$ starting with k ones. Two relevant special cases of k-approval are plurality, defined by $(1, 0, \ldots, 0)$, and veto, defined by $(1, \ldots, 1, 0)$. Typically, k-approval can be used in political elections whenever the voters can express their preference for k candidates within the set of all candidates. Another example is the Formula 1 scoring, which until the year 2009 used the scoring rule defined by the vector $(10, 8, 6, 5, 4, 3, 2, 1, 0, \ldots, 0)$ and since 2010 uses $(25, 18, 15, 12, 10, 8, 6, 4, 2, 1, 0, \ldots, 0)$.

The study of the computational complexity of voting problems is an active area of research (see the surveys [9,19]). The Possible Winner problem was introduced by Konczak and Lang [26] and has been further investigated since then for many types of voting systems [7,27,31,33,34]. Note that the related Necessary Winner problem (Given a set of partial orders, does a distinguished candidate c win for every extension of the partial orders into linear ones?) can be solved in polynomial time for all scoring rules [34]. A prominent special case of Possible Winner is Manipulation (see e.g. [8,13, 25,36,37]). Here, the given set of partial orders consists of two subsets; one subset contains linearly ordered votes and the other one completely unordered votes. Clearly, all NP-hardness results would carry over from Manipulation to Possible Winner. However, whereas the case of *weighted voters* is settled by a full dichotomy for Manipulation for scoring rules [25], so far, for *unweighted voters* we are only aware of one NP-hardness result for a specially constructed scoring rule [35]. Indeed, the NP-hardness of Manipulation for Borda is a prominent open question [35,36]. There are NP-hardness results for Manipulation in the unweighted voter case for several common voting rules which are not scoring rules [20,21,36]. Another closely related problem is Preference Elicitation (see e.g. [11,12]). Here, the idea is to avoid that each voter has to report his whole preference list, but to ask only for some part of the information that suffices to determine a winner.

Now, let us briefly summarize the known results for Possible Winner for scoring rules. Correcting Konczak and Lang [26] who claimed polynomial-time solvability for all scoring rules, Xia and Conitzer [34] provided NP-completeness results for a class of scoring rules, more specifically, for all scoring rules that have four “equally decreasing score values” followed by another “strictly decreasing score value”; we will provide a more detailed discussion later. Betzler et al. [7] studied the multivariate complexity of Possible Winner for scoring rules and other types of voting systems, providing an NP-hardness proof for k-approval in case of only two partial votes. However, this NP-hardness result holds only if k is part of the input and does not carry over for fixed values of k. Furthermore, whereas the corresponding many-one reduction relies on two partial votes, the construction used in this work makes use of an unbounded number of partial votes and thus is completely different.

Until now, the computational complexity of Possible Winner was still open for a large number of naturally appearing scoring rules. One such open case has been k-approval for small values of k which is motivated as follows. A common way of voting for a board consisting of a small number, for example, of five members, is that every voter awards one point each to five of the candidates (5-approval). A second example is given by voting systems in which each voter is allowed to specify a (small) group of favorites and a (small) group of most disliked candidates. As final example, we mention scoring rules that have decreasing differences between successive score values as, for example, the scoring vector $(2^m, 2^{m-1}, \ldots, 0)$.

This work aims at a computational complexity dichotomy for *pure* scoring rules. The class of pure scoring rules covers all of the common scoring rules. It only constitutes some restrictions in the sense that for different numbers of candidates the corresponding scoring vectors cannot be chosen completely independently (see Section 2). Our results can also be extended to broad classes of “non-pure” scoring rules, see Section 7. Altogether, we settle the computational complexity of Possible Winner for all pure scoring rules except the scoring rule defined by $(2, 1, \ldots, 1, 0)$. For plurality and veto, we provide polynomial-time algorithms whereas for the remaining cases we show NP-completeness. Surprisingly, this includes the NP-hardness of Possible Winner even for 2-approval. Our NP-hardness result for 2-approval has also been used to settle the complexity of the Swap Bribery problem [16].

2. Preliminaries

Let $C = \{c_1, \ldots, c_m\}$ be the set of *candidates*. A *vote* is a linear order (i.e., a transitive, antisymmetric, and total relation) on C. An n-voter profile P on C consists of n votes $(v_1, \ldots, v_n)$ on C. A *voting rule* r is a function from the set of all profiles on C to the power set of C, that is, $r(P)$ denotes the set of winners. (*Positional*) *scoring rules* are a special kind of voting rules. They are defined by scoring vectors $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ with integers $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_m$, the *score values*. More specifically, we define that a scoring rule r consists of a sequence of scoring vectors $s_1, s_2, \ldots$ such that for any $i \in \mathbb{N}_{>0}$ there is a scoring vector s_i for i candidates which can be computed in time polynomial in i.[2] Here, we focus our attention on *pure* scoring rules, that is, for every $i \geqslant 2$, the scoring vector for i candidates can be obtained from the scoring vector for $i-1$ candidates by inserting an additional score value at an arbitrary position (respecting the described monotonicity).

[2] For scoring rules that are defined for a constant number of candidates, the Possible Winner problem can be decided in polynomial time, see [13,33].

This definition includes all of the common protocols like Borda or k-approval. We further assume that $\alpha_m = 0$ and that there is no integer greater than one that divides all score values. This does not constitute a restriction since for every other voting system there must be an equivalent one that fulfills these constraints [25, Observation 2.2]. Moreover, we only consider *non-trivial* scoring rules, that is, scoring rules with $\alpha_1 \neq 0$ for scoring vectors of every size.

For a vote $v \in P$ and a candidate $c \in C$, let the *score* $s(v, c)$ be defined by $s(v, c) := \alpha_j$ where j is the position of c in v. For any profile $P = \{v_1, \ldots, v_n\}$, let $s(P, c) := \sum_{i=1}^{n} s(v_i, c)$. Whenever it is clear from the context which P we refer to, we will just write $s(c)$. A scoring rule selects all candidates c as winners with maximum $s(P, c)$ over all candidates.

A *partial vote* on C is a transitive and antisymmetric relation on C. We use $>$ to denote the relation given between candidates in a linear order and $\succ$ to denote the relation given between candidates in a partial vote. Sometimes, we specify a whole subset of candidates in a partial vote, e.g., $e \succ D$ for a candidate $e \in C$ and a subset of candidates $D \subseteq C$. Unless stated otherwise, this notation means that $e \succ d$ for all $d \in D$ and there is no specified order among the candidates in D. In contrast, writing $e > D$ in a linear order means that $e > d_1 > \cdots > d_l$ for an arbitrary but fixed order of $D = \{d_1, \ldots, d_l\}$. A linear order v' *extends* a partial vote v if $v \subseteq v'$, that is, for any $i, j \leqslant m$, from $c_i \succ c_j$ in v it follows that $c_i > c_j$ in v'. Given a profile of partial votes $P = (v_1, \ldots, v_n)$ on C, a candidate $c \in C$ is a *possible winner* if there exists an extension $P' = (v'_1, \ldots, v'_n)$ such that each v'_i extends v_i and $c \in r(P')$. The corresponding decision problem is defined as follows.

Possible Winner
Given: A set of candidates C, a profile of partial votes $P = (v_1, \ldots, v_n)$ on C, and a distinguished candidate $c \in C$.
Question: Is there an extension profile $P' = (v'_1, \ldots, v'_n)$ such that each v'_i extends v_i and $c \in r(P')$?

This definition allows that multiple candidates obtain the maximal score and we end up with a whole set of winners. If the possible winner c has to be unique, one speaks of a possible *unique winner*, and the corresponding decision problem is defined analogously. All our results hold for both cases.

Several of our NP-hardness proofs rely on reductions from the NP-complete Exact Cover By 3-Sets (X3C) problem [24] defined as follows. Given a set of elements $E = \{e_1, \ldots, e_q\}$, a family of subsets $\mathcal{S} = \{S_1, \ldots, S_t\}$ with $|S_i| = 3$ and $S_i \subseteq E$ for $1 \leqslant i \leqslant t$, it asks whether there is a subset $\mathcal{S}' \subseteq \mathcal{S}$ such that for every element $e_j \in E$ there is exactly one $S_i \in \mathcal{S}'$ with $e_j \in S_i$. In our NP-hardness proofs we need to describe the consequence of extending partial votes for specific candidates. To this end, we say that a candidate c_i is *shifted to the left* (*right*) by another candidate c_j when adding the constraint $c_i \succ c_j$ ($c_j \succ c_i$) to a partial vote.

In some of our theorems, we will need functions that map each instance of a certain problem $\mathcal{P}$ to some natural number and in some sense behave like a polynomial. For this sake, we call

$$f : \{I \mid I \text{ is an instance of } \mathcal{P}\} \to \mathbb{N}$$

a *poly-type function for* $\mathcal{P}$ if the function value $f(I)$ is bounded by a polynomial in $|I|$ for every input instance I of $\mathcal{P}$.

3. General strategy

This work aims at providing a dichotomy for Possible Winner for practically relevant scoring rules. To this end, we will show the following.

Theorem. Possible Winner *is NP-complete for all non-trivial pure scoring rules except plurality, veto, and scoring rules for which there is a constant z such that the produced scoring vector is* $(2, 1, \ldots, 1, 0)$ *for every number of candidates greater than z. For plurality and veto,* Possible Winner *is solvable in polynomial time.*

The proof consists of several parts, see Table 1 for an overview. The polynomial time results for plurality and veto are based on flow computations. Regarding the NP-hardness results, we give many-one reductions that work for scoring rules that produce specific “types of scoring vectors” for an appropriate number of candidates. We combine the single results to obtain the main result in Section 7. To this end, we have to take into account that, in general, a scoring rule might produce different types of scoring vectors for different numbers of candidates.

The basic observation to classify the scoring vectors is that a scoring vector of unbounded size must have an unbounded number of different score values or an unbounded number of equal score values. This leads to the following strategy. First, we show NP-hardness for all scoring vectors having an unbounded number of different score values. To this end, we generalize a many-one reduction due to Xia and Conitzer [34]. Second, we deal with scoring vectors having an unbounded number of equal score values. Here, we consider two subcases, i.e., scoring vectors of type $\alpha_1 > \alpha_2 = \alpha_{m-1} > 0$ but $\alpha_1 \neq 2 \cdot \alpha_2$, and all remaining scoring vectors with an unbounded number of equal score values.

Before stating the specific results, we give a construction scheme that is used in all many-one reductions in this work.

3.1. A general scheme to construct linear votes

In all many-one reductions presented in this work, one constructs a partial profile P consisting of a set of linear orders V^l and a set of partial votes V^p. The position of the distinguished candidate c is already determined in every vote from V^p,

Table 1
Overview of results and outline of the work. Basically, we partition the scoring rules into five different types according to the types of algorithms or many-one reductions that are used to achieve the results. By "different-type" we denote all scoring vectors with an unbounded number of different score values. By "equal-type" we denote all scoring vectors with an unbounded number of equal score values if not listed explicitly in another type. Reductions are from Exact Cover By 3-Sets (X3C) or Multicolored Clique (MC).

Scoring rule	Result	
Plurality and veto	in P	Proposition 1, Section 4
Different-type	NP-c (X3C)	Theorem 1, Section 5
Equal-type	NP-c (MC/X3C)	Theorem 2, Lemmata 3–6, Section 6.1
$\alpha_1 > \alpha_2 = \alpha_{m-1} > 0$ and $\alpha_1 \neq 2 \cdot \alpha_2$	NP-c (X3C)	Theorem 4, Section 6.2
$(2, 1, \ldots, 1, 0)$	?	

$$
\begin{array}{lllllllll}
v_1: & c_1 & > & c_2 & > & \cdots & > & c_{m-1} & > c_m \\
v_2: & c_2 & > & c_3 & > & \cdots & > & c_m & > c_1 \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots \\
v_{m-1}: & c_{m-1} & > & c_m & > & \cdots & > & c_{m-3} & > c_{m-2} \\
v_m: & c_m & > & c_1 & > & \cdots & > & c_{m-2} & > c_{m-1}
\end{array}
$$

Fig. 1. Circular block for $c_1, c_2, \ldots, c_m$.

that is, $s(P', c)$ is the same in every extension P' and thus is fixed. The "interesting" part of the reductions is given by the partial votes of V^p in combination with upper bounds for the scores which the non-distinguished candidates can make in V^p. For every candidate $c' \in C \setminus \{c\}$, the *maximum partial score* $s_p^{\max}(c')$ is the maximum number of points c' may make in V^p without beating c in P. More precisely, for the unique winner case, $s_p^{\max}(c') = s(P', c) - s(V^l, c') - 1$ and, for the winner case, $s_p^{\max}(c') = s(P', c) - s(V^l, c')$ for any extension P' of P. Since the maximum partial scores can be adjusted to the unique and to the winner case, all results hold for both cases.

In the following, we show that for all our reductions, there is an easy way to cast the linear votes such that the maximum partial scores that are required in the reductions are *realized*. For every many-one reduction of this work, it will be easy to verify that the underlying partial profile fulfills the following two properties.[3]

Property 1. There is a "dummy" candidate d which cannot beat the distinguished candidate in any extension, that is, $s_p^{\max}(d) \geqslant \alpha_1 \cdot |V^p|$.

Property 2. For every $c' \in C \setminus \{c\}$, the maximum partial score $s_p^{\max}(c')$ can be written as a sum of at most $|V^p|$ integers from $\{\alpha_1, \ldots, \alpha_m\}$. Formally, the definition of $s_p^{\max}(c')$ will be of the form $s_p^{\max}(c') = \sum_{j=1}^{m} n_j \alpha_j$ where $n_j \in \mathbb{N}_0$ denotes how often the score value α_j is added. We will always have that $\sum_{j=1}^{m} n_j \leqslant |V^p|$, that is, the total number of summands is at most the number of partial votes.

The sets of linear votes which are necessary for the reductions given in this paper can be obtained according to the following lemma.

Lemma 1. *Given a scoring rule r, a set of candidates C with distinguished candidate $c \in C$, a set of partial votes V^p in which c is fixed, and $s_p^{\max}(c')$ for all $c' \in C \setminus \{c\}$, a set of linear votes that realizes the maximum partial scores for all candidates can be constructed in time polynomial in $|V^p|$ and m if Properties* 1 *and* 2 *hold.*

Proof. We are interested in "setting" relative score difference between the distinguished candidate c and every other candidate. By inserting one linear order we change the relative score difference between c and all other candidates. To be able to change the relative score difference only for c and one specific candidate while keeping the relative score difference of c and all other candidates, we will build V^l by sets of circular shifts instead of single votes. More precisely, for a set of candidates $\{c_1, c_2, \ldots, c_m\}$, a *circular block* consists of m linear orders as given in Fig. 1. Clearly, all candidates have the same score within a circular block.

We start with the construction for the winner case and then explain how to adapt it for the unique winner case. For the winner case ($s_p^{\max}(c') = s(P', c) - s(V^l, c')$ for any extension P'), for each candidate $c' \in C \setminus \{c, d\}$ where d denotes a dummy as specified in Property 1, add the following votes to the set of linear votes V^l. For each $n_j \neq 0$ as specified in Property 2, construct n_j circular blocks over C such that in one of the linear orders of every block, c' sits on position j and d sits on position m. Exchange the places of c' and d in this linear order and add the modified circular block to V^l. Then, for

[3] The only exception appears in the proof of Theorem 4 and will be discussed there.

$v_1 : a \succ c \succ d, b \succ c$
$v_2 : c \succ a \succ b \qquad \Rightarrow c > a > b > d$
$v_3 : a \succ d \succ b \qquad \Rightarrow c > a > d > b$
$v_4 : a \succ b \succ c$
$v_5 : a \succ c, b \succ d$

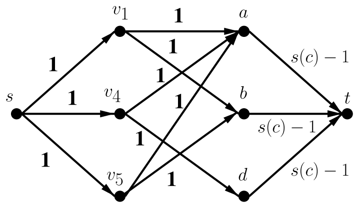

Fig. 2. Possible Winner for plurality: The left-hand side shows an example for an election and the right-hand side the corresponding flow network. The votes v_2 and v_3 can be extended such that c takes the first position. The position of the remaining candidates in v_2 and v_3 is not relevant; one possibility how to extend these votes is shown in the picture.

one block, c' has lost α_j points and gained $\alpha_m = 0$ points relative to c. Thus, in total, one has the situation that c and c' have exactly the same score if c' makes $s_p^{\max}(c')$ points in V^p. This settles the winner case. For the unique-winner case, we additionally decrease the score of c' by the minimum of $\{\alpha_i - \alpha_j \mid \alpha_i > \alpha_j \text{ and } i, j \in \{1, 2, \ldots, m\}\}$. This can be achieved by adding a circular block such that in one of the linear orders of the block, c' sits on position α_i and d sits on position α_j, and by exchanging the places of c' and d in this linear order. Then, c beats c' if c' makes at most $s_p^{\max}(c')$ points in V^p and c' beats c, otherwise.

Altogether, due to Property 2, we add at most $|V^p|$ summands for each candidate. Hence, so far, the number of linear votes is bounded by $m^2 \cdot (|V^p| + 1)$ and can be constructed in polynomial time. It remains to adjust the maximum partial score of d. Until now, we added at most $m \cdot (|V^p| + 1)$ circular blocks. Thus, d can make at most $\alpha_1 \cdot m \cdot |V^p|$ points more than c. By adding $m(|V^p| + 1) + |V^p|$ further circular blocks for candidates from $C \setminus \{d\}$ that are inserted in the first $m - 1$ positions, while d is put on the last position in these votes, $s_p^{\max}(d)$ can be realized in polynomial time. □

4. Plurality and veto

Employing network flows turned out to be useful to design algorithms for several voting problems (see e.g. [17,18]). Here, by using some flow computations very similar to [7, Theorem 6], we show the following.

Proposition 1. Possible Winner *can be solved in polynomial time for plurality and veto.*

Proof. First, we give an algorithm for plurality. Let P on C denote a Possible Winner-instance with distinguished candidate c. Clearly, it is safe to set c to the first position in all votes in which this is possible. Then the score of c is fixed at the maximum possible value. We denote the partial votes of P in which the first position is not taken by c as P_1. Now, we can model the problem as network flow as follows (see Fig. 2): The flow network consists of a source node s, a target node t, one node for every vote of P_1, and one node for every candidate from $C \setminus \{c\}$. There are three layers of arcs:

1. an arc from s to every node corresponding to a vote in P_1 with capacity one,
2. an arc from a node corresponding to $v_j \in P_1$ to a node corresponding to a candidate $c' \in C \setminus \{c\}$ with capacity one if and only if c' can take the first position in an extension of v_j, and
3. an arc from every node corresponding to $c' \in C \setminus \{c\}$ to target t with capacity $s(c) - 1$.

Now, c is a possible winner if and only if there is a flow of size $|P_1|$: The first layer simulates that the first position of every partial vote from P_1 has to be taken, the second layer that it can only be taken by appropriate candidates, and the last one that the score of every candidate will be lower than the score of c. Clearly, the flow network can be constructed in time polynomial in $|P_1|$ and an integral flow computation can be done in polynomial time [14].

For veto, we first fix c at the best (leftmost) possible position in every vote. This fixes the maximum score of c. Then for every candidate $c' \in C \setminus \{c\}$, let $z(c')$ denote the minimum number of last positions that c' must take such that it does not beat c. Let P_1 denote the set of partial votes in which c does not take the last position. Again, we model the problem by a flow network with source node s, target node t, one node for every candidate from $C \setminus \{c\}$, and one node for every vote of P_1. The arcs are as follows:

1. an arc from s to every node corresponding to $c' \in C \setminus \{c\}$ with capacity $z(c')$,
2. an arc from a node corresponding to $c' \in C \setminus \{c\}$ to a node corresponding to $v_j \in P_1$ with capacity one if and only if c' can take the last position in an extension of v_j, and
3. an arc from every node corresponding to $v_j \in P_1$ to target t with capacity 1.

By similar arguments as for plurality, it follows that c is a possible winner if and only if there is a flow of size $\sum_{c' \in C \setminus \{c\}} z(c')$. □

5. An unbounded number of positions with different score values

Xia and Conitzer [34] developed a many-one reduction from Exact Cover By 3-Sets showing that Possible Winner is NP-complete for any scoring rule with scoring vectors which contain four consecutive, "equally decreasing" score values, followed by another strictly decreasing score value. Using some additional gadgetry, we extend their proof to work for scoring vectors with an unbounded number of different, not necessarily equally decreasing score values.

We start by describing the basic idea employed in [34] (using a slightly modified construction). Given an X3C-instance $(E, \mathcal{S})$, construct a partial profile $P := V^l \cup V^p$ on a set of candidates C where V^l denotes a set of linear orders and V^p a set of partial votes. To describe the basic idea, assume that there is a scoring vector with $\alpha_1 > \alpha_2$ and the differences between the four following score values are *equally decreasing*, that is, $\alpha_2 - \alpha_3 = \alpha_3 - \alpha_4 = \alpha_4 - \alpha_5$. Then, $C := \{c, x, w\} \cup E$ where E is the universe from the X3C-instance. The distinguished candidate is c. The candidates whose element counterparts belong to the set S_i are denoted by e_{i1}, e_{i2}, e_{i3}. The partial votes V^p consist of one partial vote v_i^p for every $S_i \in \mathcal{S}$ which is given by

$$x \succ e_{i1} \succ e_{i2} \succ e_{i3} \succ C', \quad w \succ C'$$

with $C' := C \setminus \{x, e_{i1}, e_{i2}, e_{i3}, w\}$. Note that in v_i^p, the positions of all candidates except w, x, e_{i1}, e_{i2}, e_{i3} are fixed. More precisely, w has to be inserted between positions 1 and 5 maintaining the partial order $x \succ e_{i1} \succ e_{i2} \succ e_{i3}$. By setting the linear votes, the maximum partial scores are realized such that the following three conditions hold.

- For every *element candidate* $e \in E$ one has the following. Inserting w behind e in two partial votes has the effect that e would beat c, whereas when w is inserted behind e in at most one partial vote, c still beats e (Condition 1). Note that e may occur in several votes at different positions, e.g. e might be identical with e_{i1} and e_{j3} for $i \neq j$. However, due to the condition of "equally decreasing" scores, "shifting" e increases its score by the same value in all of the votes.
- The maximum partial score of x is set such that if takes more than $|V^p| - |E|/3$ times the first position, then it would beat c. That is, w must be inserted before x at least $|V^p| - |E|/3$ times (Condition 2).
- We set $s_p^{\max}(w) = (|V^p| - |E|/3) \cdot \alpha_1 + |E|/3 \cdot \alpha_5$. This implies that if w is inserted before x in $|V^p| - |E|/3$ votes, then it must be inserted at the last possible position, that is, position 5, in all remaining votes (Condition 3).

Having an exact 3-cover for $(E, \mathcal{S})$, extend the partial votes as follows:

$$v_i^p: \quad x > e_{i1} > e_{i2} > e_{i3} > w > \cdots \quad \text{if } S_i \text{ is in the exact 3-cover,}$$
$$v_i^p: \quad w > x > e_{i1} > e_{i2} > e_{i3} > \cdots \quad \text{if } S_i \text{ is not in the exact 3-cover.}$$

Then, every element candidate e is shifted exactly once (in v_i^p for $e \in S_i$, if S_i is in the exact 3-cover) and thus is beaten by c. It is easy to verify that c beats w and x as well. In a yes-instance for (C, P, c), it follows directly from Condition 2 and 3 that w must have the position 5 in exactly $|E|/3$ votes and the first position in all remaining partial votes. Since there are $|E|/3$ partial votes such that three element candidates are shifted in each of them, due to Condition 1, every element candidate must appear in exactly one of these votes. Hence, c is a possible winner in P if and only if there exists an exact 3-cover of E.

By inserting further candidates, one can pad the construction such that is also works if the equally decreasing score differences appear at other positions [34]. Now, we consider the situation in which no such equally decreasing score differences appear at all. More precisely, we show how to extend the reduction to scoring vectors with strictly, but not equally decreasing scoring values. The problem we encounter is the following: By sending candidate w to the last possible position in the partial vote v_i^p, each of the candidates e_{i1}, e_{i2}, e_{i3} improves by one position and therefore improves its score by the difference given between the corresponding positions. In [34], these differences all had the same value, but now we have to deal with varying differences. Since the same candidate $e \in E$ may appear in several votes at different positions, e.g. e might be identical with e_{i1} and e_{j3} for $i \neq j$, it is not clear how to set the maximum partial score of e. Basically, to cope with this situation, we construct three partial votes v_i^1, v_i^2, and v_i^3 for every set $S_i \in \mathcal{S}$ and permute the positions of the candidates e_{i1}, e_{i2}, e_{i3} such that each of them takes a different position in v_i^1, v_i^2, v_i^3. For example:

$$v_i^1: \quad \cdots \succ x \succ e_{i1} \succ e_{i2} \succ e_{i3} \succ \cdots,$$
$$v_i^2: \quad \cdots \succ x \succ e_{i2} \succ e_{i3} \succ e_{i1} \succ \cdots,$$
$$v_i^3: \quad \cdots \succ x \succ e_{i3} \succ e_{i1} \succ e_{i2} \succ \cdots.$$

In this way, if the candidate w is sent to the last possible position in all three partial votes of a set S_i, each of the candidates e_{i1}, e_{i2}, e_{i3} improves its score by the same value. We only have to guarantee that whenever w is sent back in the partial vote v_i^1, then it has to be sent back v_i^2 and v_i^3 as well. This is realized by a gadget construction, which is the main technical contribution of the following theorem.

Table 2
Maximum partial scores. Recall that $t = |\mathcal{S}|$, $q = |E|$, and $n_e = |\{S_i \in \mathcal{S} \mid e \in S_i\}|$.

	$s_p^{\max}(w)$	$= (3t - q) \cdot \alpha_1 + q \cdot \alpha_{5+2t}$
	$s_p^{\max}(x)$	$= q \cdot \alpha_1 + (3t - q) \cdot \alpha_2$
$\forall e \in E$	$s_p^{\max}(e)$	$= (\alpha_2 + \alpha_3 + \alpha_4) + (n_e - 1) \cdot (\alpha_3 + \alpha_4 + \alpha_5) + \mathrm{fixed}(e)$
$\forall d_i$	$s_p^{\max}(d_i)$	$= q/3 \cdot \alpha_{4+i} + (t - q/3) \cdot \alpha_{5+i} + \mathrm{fixed}(d_i)$
$\forall h_i$	$s_p^{\max}(h_i)$	$= q/3 \cdot \alpha_{4+i} + (t - q/3) \cdot \alpha_{5+i} + \mathrm{fixed}(h_i)$
$\forall d'_i$	$s_p^{\max}(d'_i)$	$= q/3 \cdot \alpha_{4+i+t} + (t - q/3) \cdot \alpha_{5+i+t} + \mathrm{fixed}(d'_i)$
$\forall h'_i$	$s_p^{\max}(h'_i)$	$= q/3 \cdot \alpha_{4+i+t} + (t - q/3) \cdot \alpha_{5+i+t} + \mathrm{fixed}(h'_i)$
$\forall l_i$	$s_p^{\max}(l_i)$	$= 2t \cdot \alpha_1 + \mathrm{fixed}(l_i)$

Theorem 1. *An X3C-instance I can be reduced to a* Possible Winner*-instance for a scoring rule which produces a scoring vector having $f(I)$ positions with different score values. A suitable poly-type function f can be computed in polynomial time.*

Proof. Given an X3C-instance $(E, \mathcal{S})$ with $\mathcal{S} = \{S_1, \ldots, S_t\}$ and $S_i = \{e_{i1}, e_{i2}, e_{i3}\}$ for $i \in \{1, \ldots, t\}$, construct a partial profile P on C as follows. The set of candidates is defined as $C := \{x, w, c\} \uplus E \uplus D_{12} \uplus D_{13} \uplus L$ (where $\uplus$ denotes the disjoint union), where E is the set of candidates that represent the elements of the universe of the X3C-instance, $D_{12} := \{d_1, \ldots, d_t, h_1, \ldots, h_t\}$, $D_{13} := \{d'_1, \ldots, d'_t, h'_1, \ldots, h'_t\}$, and $L := \{l_1, \ldots, l_t\}$. We define $f((E, \mathcal{S})) := |C|$. To ease the presentation, we first assume that we have a strictly decreasing scoring vector of size $f((E, \mathcal{S}))$ and describe how to generalize this at the end of the proof. The partial profile consists of a set of partial votes V^p and a set of linear votes V^l. The partial votes are $V^p := \{v_i^1, v_i^2, v_i^3 \mid 1 \leqslant i \leqslant t\}$ with, for $1 \leqslant i \leqslant t - 1$,

$$\begin{array}{lllllllllllllllll} v_i^1\colon & x & e_{i1} & e_{i2} & e_{i3} & d_1 & \ldots & d_i & h_{i+1} & \ldots & h_t & d'_1 & \ldots & d'_i & h'_{i+1} & \ldots & h'_t \succ C_i^1, \quad w \succ C_i^1, \\ v_i^2\colon & x & e_{i2} & e_{i3} & e_{i1} & h_1 & \ldots & h_i & d_{i+1} & \ldots & d_t & l_1 & \ldots & & \ldots & \ldots & l_t \succ C_i^2, \quad w \succ C_i^2, \\ v_i^3\colon & x & e_{i3} & e_{i1} & e_{i2} & l_1 & \ldots & & \ldots & \ldots & l_t & h'_1 & \ldots & h'_i & d'_{i+1} & \ldots & d'_t \succ C_i^3, \quad w \succ C_i^3, \end{array}$$

and

$$\begin{array}{llllllllll} v_t^1\colon & x & e_{t1} & e_{t2} & e_{t3} & d_1 & \ldots & d_t & d'_1 & \ldots d'_t \succ C_t^1, \quad w \succ C_t^1, \\ v_t^2\colon & x & e_{t2} & e_{t3} & e_{t1} & h_1 & \ldots & h_t & l_1 & \ldots l_t \succ C_t^2, \quad w \succ C_t^2, \\ v_t^3\colon & x & e_{t3} & e_{t1} & e_{t2} & l_1 & \ldots & l_t & h'_1 & \ldots h'_t \succ C_t^3, \quad w \succ C_t^3, \end{array}$$

where "$\succ$" signs are partially omitted and C_i^1, C_i^2, and C_i^3 denote the remaining candidates that are fixed in an arbitrary order, respectively. Now, we give some notation needed to define the maximum partial scores. For $c' \in C \setminus \{c\}$, let $\mathrm{fixed}(c')$ denote the number of points which c' makes in the partial votes in which the position of c' is already fixed. Let n_e denote the number of subsets with $e \in S_i$ and $q = |E|$. Due to Lemma 1, one can set the maximum partial scores as given in Table 2. The particular partial scores will be explained within the proof of the following claim.

Claim: Candidate c is a possible winner of P if and only if there is an exact 3-cover for $(E, \mathcal{S})$.

"$\Leftarrow$": Given an exact 3-cover $\mathcal{S}' \subseteq \mathcal{S}$, complete the votes in V^p in the following way: For each $S_i \in \mathcal{S}'$, place w in the last possible position (i.e., position $5 + 2t$) in the partial votes v_i^1, v_i^2, and v_i^3, and on the first position in the remaining partial votes. Since $|\mathcal{S}'| = q/3$, in the extension of the votes from V^p ones has $s(w) = (3t - q) \cdot \alpha_1 + q \cdot \alpha_{5+2t} = s_p^{\max}(w)$ and $s(x) = q \cdot \alpha_1 + (3t - q) \cdot \alpha_2 = s_p^{\max}(x)$. Furthermore, it is easy to see that $s(l_i) < s_p^{\max}(l_i)$ for every i. Every element candidate e is shifted to the left in exactly three partial votes. More precisely, in the three votes that correspond to $S_i \in \mathcal{S}'$ with $e \in S_i$, it makes $\alpha_2 + \alpha_3 + \alpha_4$ points and $(n_e - 1) \cdot (\alpha_3 + \alpha_4 + \alpha_5) + \mathrm{fixed}(e)$ points in the remaining votes and thus does not beat c. Every candidate from D_{12} is not "fixed" in exactly one vote of every triple corresponding to an S_i. More precisely, it can be shifted either in v_i^1 or in v_i^2 and never in v_i^3. Due to the insertion of w, it is shifted to position $4 + i$ in $q/3$ of the votes and takes position $5 + i$ in the remaining $t - q/3$ non-fixed votes. Thus, it does not beat c. Analogously, every candidate from D_{13} makes α_{4+i+t} points in $q/3$ of the non-fixed votes and α_{5+i+t} in the remaining $t - q/3$ votes and hence does not beat c. Altogether, c beats every other candidate and wins.

"$\Rightarrow$": Consider an extension of P in which c wins. Due to its maximum partial score, candidate x can take the first position only q times. Thus, it must be shifted $3t - q$ times to position 2. Clearly, this is only possible if w is placed on the first position in $3t - q$ votes. Then due to its maximum partial score, w can only be set to position $5 + 2t$ in the remaining q votes. In the following, we will show that for every i, w takes position $5+2t$ in v_i^1 if and only if it takes position $5+2t$ in v_i^2 if and only if it takes position $5+2t$ in v_i^3 (Observation I). Then it follows that in the votes in which w takes position $5+2t$, the corresponding element candidates are shifted to the left and obtain $\alpha_2 + \alpha_3 + \alpha_4$ points each, whereas they obtain $\alpha_3 + \alpha_4 + \alpha_5$ points in the remaining corresponding vote triples. Since each element candidate e_j can only obtain $\alpha_2 + \alpha_3 + \alpha_4$ points exactly once (and the scoring values are strictly decreasing), the set $\mathcal{S}' := \{S_i \mid w \text{ is placed on position } 5 + 2t \text{ in } v_i^1\}$ must be an exact 3-cover of E.

It remains to settle Observation I, which says that w behaves equally in the votes corresponding to one subset. First, we argue that w must be inserted at position $5+2t$ in exactly $q/3$ votes of $V_1^p := \{v_i^1 \mid 1 \leqslant i \leqslant t\}$, $V_2^p := \{v_i^2 \mid 1 \leqslant i \leqslant t\}$, and $V_3^p := \{v_i^3 \mid 1 \leqslant i \leqslant t\}$, respectively. Assume that w is inserted at position $5+2t$ in more than $q/3$ votes of V_1^p. Then, d_1, which is not fixed in every vote of V_1^p, would beat c. Analogously, if w was inserted at position $5+2t$ in more than $q/3$ votes of V_2^p or V_3^p, then c would be beaten by h_1 or h_1', respectively. Now, we have that w must take position $5+2t$ in q votes and can take this position in at most $q/3$ votes from V_i^p, for every $i \in \{1,2,3\}$ and thus must take this position in exactly $q/3$ votes of V_1^p, V_2^p, and V_3^p.

Second, we show that the candidates from D_{12} ensure that w takes position $5+2t$ in v_i^1 if and only if w takes position $5+2t$ in v_i^2. The proof is by contradiction. Assume that there is an extension in which w takes position $5+2t$ in v_i^1 and another position in v_i^2 for any i. Since d_i and h_{i+1} have been shifted to the left in v_i^1, each of them can only be shifted to the left in at most $q/3-1$ further votes. By construction, v_i^2 is the only vote of $V_1^p \cup V_2^p$ in which neither d_i nor h_{i+1} is shifted to the left by setting w to position $5+2t$. However, since w can either take the first or position $2t+5$ in an extension (as argued above), it must take the first position in v_i^2. Now, w has to take the position $5+2t$ in $2q/3-1$ further votes from $V_1^p \cup V_2^p$ and thus in each of these votes w will either shift d_i or h_{i+1}. Hence, either d_i or h_{i+1} must be shifted to the left in more than $q/3-1$ further votes and will beat c, a contradiction. The other case (w takes position $5+2t$ in v_i^2 and another position in v_i^1) follows in complete analogy by considering h_i and d_{i+1}. One can show analogously that the candidates of D_{13} ensure that w takes position $5+2t$ in v_i^1 if and only if it takes the same position in v_i^3. Thus, Observation I follows.

Now, one has that Possible Winner is NP-hard for all scoring rules with a scoring vector of size $f((E,\mathcal{S}))$ with strictly decreasing score values. By using some simple padding, we extend the result for the remaining cases, that is, for scoring vectors of size $m' > f((E,\mathcal{S}))$ and $f((E,\mathcal{S}))$ different score values. To this end, we introduce a set of $m' - f((E,\mathcal{S}))$ new dummy candidates and cast the linear votes such they cannot beat the distinguished candidate in any extension. The original candidates from C are placed on positions endued with strictly decreasing points, whereas the new candidates are placed on the remaining positions. Then, if the positions of candidates get shifted (when w is inserted), the "old" candidates are affected in the same manner as in the above construction and the theorem follows. $\square$

6. An unbounded number of positions with equal score values

In the previous section, we showed NP-hardness for scoring vectors with an unbounded number of different score values. In this section, we discuss scoring vectors with an unbounded number of positions with equal score value. In the first subsection, we show NP-hardness for Possible Winner for scoring vectors that fulfill $\alpha_2 \neq \alpha_{m-1}$, and, in the second subsection, we consider the special case that $\alpha_1 > \alpha_2 = \cdots = \alpha_{m-1} > 0$. Note that these two cases cover all scoring vectors with an unbounded number of equal score values (except plurality and veto): There are three ways to "violate" $\alpha_1 > \alpha_2 = \cdots = \alpha_{m-1} > 0$. First, if one requires $\alpha_1 = \alpha_2$, then one ends up with veto. Second, requiring $\alpha_{m-1} = 0$, one arrives at plurality. Third, requiring $\alpha_2 \neq \alpha_{m-1}$, then one ends up with the other case that includes the famous examples 2-approval and $(m-2)$-approval.

6.1. An unbounded number of equal score values and $\alpha_2 \neq \alpha_{m-1}$

The scoring vectors considered in this subsection divide into two classes. First, there are at least two score values that are greater than the "equal score value". Second, there are at least two score values that are smaller than the "equal score value". Formally, a size-m scoring vector for the second class looks as follows: there is an i, with $i < m-2$ and an "unbounded" integer x such that $\alpha_{i-x} = \alpha_i > \alpha_{i+1}$. This property can be used to construct a basic "logical" tool used in the many-one reductions of this subsection: For two candidates c, c', having $c \succ c'$ in a partial vote implies that setting c such that it makes less than α_i points implies that also c' makes less than α_i points whereas all candidates placed in the range between $i-x$ and i make exactly α_i points. This can be used to model some implication of the type "$c \Rightarrow c'$" in a vote. For $(m-2)$-approval, which will play a prominent role for stating our results, this condition means that c only has the possibility to make zero points in a vote if also c' makes zero points in this vote whereas all other candidates make one point.

Most of the reductions of this subsection are from the NP-complete Multicolored Clique (MC) problem [22]:

Given: An undirected graph $G = (X_1 \cup X_2 \cup \cdots \cup X_k, E)$ with $X_i \cap X_j = \emptyset$ for $1 \leqslant i < j \leqslant k$ and the vertices of X_i induce an independent set for $1 \leqslant i \leqslant k$.
Question: Is there a complete subgraph (clique) of size k?

Here, $1,\dots,k$ are considered as different colors. Then, the problem is equivalent to ask for a *multicolored clique*, that is, a clique that contains one vertex for every color. To ease the presentation, for any $1 \leqslant i \neq j \leqslant k$, we interpret the vertices of X_i as red vertices and write $r \in X_i$, and the vertices of X_j as green vertices and write $g \in X_j$.

Reductions from MC are often used to show parameterized hardness results [22]. Intuitively, the different colors give some useful structure to the instance. The general idea is to construct different types of gadgets. Here, the partial votes realize four kinds of gadgets. First, gadgets that choose a vertex of every color (vertex selection). Second, gadgets that choose an edge of every ordered pair of colors, for example, one edge from green to red and one edge from red to green (edge selection). Third, gadgets that check the consistency of two selected ordered edges, e.g. does the chosen red–green candidate refer to the same edge as the choice of the green–red candidate (edge–edge match)? Finally, gadgets that check whether all edges starting from the same color start from the same vertex (vertex–edge match). Though reductions from MC have become a standard tool to obtain hardness results, the reduction given here is not straightforward. For example, we are not aware of any reduction in the literature for which it is necessary to employ vertex–edge match gadgets.

We start by giving a reduction from MC that settles the NP-hardness of Possible Winner for $(m-2)$-approval. Then we describe how the given construction can be generalized to work for most of the cases considered in this subsection. The NP-hardness of the remaining cases will be shown by reductions from Exact Cover By 3-Sets.

Lemma 2. Possible Winner *is NP-hard for* $(m-2)$*-approval.*

Proof. Given an MC-instance $G=(X,E)$ with $X=X_1\cup X_2\cup\cdots\cup X_k$. Let $E(i,j)$ denote all edges from E between X_i and X_j. Without loss of generality, we can assume that there are integers s and t such that $|X_i|=s$ for $1\leqslant i\leqslant k$, $|E(i,j)|=t$ for all i, j, and that k is odd since every other instance can be padded easily in this way. We construct a partial profile P on a set C of candidates such that the distinguished candidate $c\in C$ is a possible winner if and only if there is a size-k clique in G. The set of candidates $C:=\{c\}\uplus C_X\uplus C_E\uplus D$, where $\uplus$ denotes the disjoint union, is specified as follows:

- For $i\in\{1,\dots,k\}$, let $C_X^i:=\{r_1,\dots,r_{k-1}\mid r\in X_i\}$ and $C_X:=\bigcup_i C_X^i$.
- For $i,j\in\{1,\dots,k\}$, $i\neq j$, let

  $$C_{i,j}:=\bigl\{rg\bigm|\{r,g\}\in E(i,j),\ r\in X_i,\ \text{and } g\in X_j\bigr\}$$

  and

  $$C'_{i,j}:=\bigl\{rg'\bigm|\{r,g\}\in E(i,j),\ r\in X_i,\ \text{and } g\in X_j\bigr\}.$$

  Then, $C_E:=(\bigcup_{i\neq j}C_{i,j})\uplus(\bigcup_{i\neq j}C'_{i,j})$, i.e., for every edge $\{r,g\}\in E(i,j)$, the set C_E contains the four candidates rg,rg',gr,gr'.
- The set $D:=D_X\uplus D_1\uplus D_2$ is defined as follows. For $i\in\{1,\dots,k\}$, $D_X^i:=\{c_1^r,\dots,c_{k-2}^r\mid r\in X_i\}$ and $D_X:=\bigcup_i D_X^i$. For $i\in\{1,\dots,k\}$, one has $D_1^i:=\{d_1^i,\dots,d_{k-2}^i\}$ and $D_1:=\bigcup_i D_1^i$. The set D_2 is defined as $D_2:=\{d^i\mid i\in\{1,\dots,k\}\}$.

We refer to the candidates of C_X as *vertex-candidates*, to the candidates of C_E as *edge-candidates*, and to the candidates of D as *dummy-candidates*.

The partial profile P consists of a set of linear votes V^l and a set of partial votes V^p. In each extension of P, the distinguished candidate c gets one point in every vote from V^p (see definition below). Thus, according to Lemma 1, we can set the maximum partial scores as follows. For every candidate $d^i\in D_2$, $s_p^{\max}(d^i)=|V^p|-s+1$, that is, d^i must get zero points (*take a zero position*) in at least $s-1$ of the partial votes. For every remaining candidate $c'\in C\setminus(\{c\}\cup D_2)$, $s_p^{\max}(c')=|V^p|-1$, that is, c' must get zero points in at least one of the partial votes.

In the following, we define $V^p:=V_1\cup V_2\cup V_3\cup V_4$. For all our gadgets only the last positions of the votes are relevant. Hence, in the partial votes it is sufficient to explicitly specify the "relevant candidates". More precisely, we define for all partial votes that each candidate that does not appear explicitly in the description of a partial vote is positioned before all candidates that appear in this vote.

The partial votes of V_1 realize the **edge selection gadgets**. Basically, selecting an ordered edge (r,g) with $\{r,g\}\in E$ means to select the corresponding *pair of edge-candidates* rg and rg'. The candidate rg is used for the vertex–edge match check and rg' for the edge–edge match check. Now, we give the definition of V_1. For every ordered color pair (i,j), $i\neq j$, V_1 has $t-1$ copies of the partial vote $\{rg\succ rg'\mid\{r,g\}\in E(i,j)\}$, that is, one partial vote contains the constraint $rg\succ rg'$ for every $\{r,g\}\in E(i,j)$. The idea of this gadget is as follows. For every ordered color pair we have t edges and $t-1$ corresponding votes. Within one vote, one pair of edge-candidates can get the two available zero positions. Thus, it is possible to set all but one, namely the selected pair of edge-candidates, to zero positions.

The partial votes of V_2 realize the **vertex selection gadgets**. Here, we will use the $k-1$ candidates corresponding to a selected vertex to do the vertex–edge match for all edges that are incident in a multicolored clique. Formally, we set $V_2:=V_2^a\cup V_2^b$ as further defined in the following. Intuitively, in V_2^a we select a vertex and in V_2^b, by a cascading effect, we achieve that all $k-1$ candidates that correspond to this vertex are selected. In V_2^a, for every color i, we have $s-1$ copies of the partial vote $\{r_1\succ c_1^r\mid r\in X_i\}$. In V_2^b, for every color i and for every vertex $r\in X_i$, we have the following $k-2$ votes:

For all *odd* $z\in\{1,\dots,k-4\}$, $\quad v_z^{r,i}$: $\bigl\{c_z^r\succ c_{z+1}^r,\ r_{z+1}\succ r_{z+2}\bigr\}$.

For all *even* $z\in\{2,\dots,k-3\}$, $\quad v_z^{r,i}$: $\bigl\{c_z^r\succ c_{z+1}^r,\ d_{z-1}^i\succ d_z^i\bigr\}$,

$v_{k-2}^{r,i}$: $\bigl\{c_{k-2}^r\succ d_{k-2}^i,\ r_{k-1}\succ d^i\bigr\}$.

$V_1:$		$\cdots > rg > rg'$	for $i, j \in \{1, \ldots, k\}, i \neq j, r \in X_i \backslash Q,$ and $g \in X_j \backslash Q$
$V_2^a:$		$\cdots > r_1 > c_1^r$	for $1 \leq i \leq k$ and $r \in X_i \backslash Q$
$V_2^b:$	$v_z^{r,i}$	$\cdots > r_{z+1} > r_{z+2}$	for $1 \leq i \leq k, r \in X_i \backslash Q$ for all $z \in \{1, 3, 5, \ldots, k-4\}$
	$v_z^{r,i}$	$\cdots > c_z^r > c_{z+1}^r$	for $1 \leq i \leq k, r \in X_i \backslash Q$ for all $z \in \{2, 4, 6, \ldots, k-3\}$
	$v_{k-2}^{r,i}$	$\cdots > r_{k-1} > d^i$	for $1 \leq i \leq k, r \in X_i \backslash Q$
	$v_z^{r,i}$	$\cdots > c_z^r > c_{z+1}^r$	for $1 \leq i \leq k, r \in X_i \cap Q$ for all $z \in \{1, 3, 5, \ldots, k-4\}$
	$v_z^{r,i}$	$\cdots > d_{z-1}^i > d_z^i$	for $1 \leq i \leq k, r \in X_i \cap Q$ for all $z \in \{2, 4, 6, \ldots, k-3\}$
	$v_{k-2}^{r,i}$	$\cdots > c_{k-2}^r > d_{k-2}^i$	for $1 \leq i \leq k, r \in X_i \cap Q$
$V_3:$		$\cdots > rg > r_j$	for $i, j \in \{1, \ldots, k\}, j < i, r \in X_i \cap Q,$ and $g \in X_j \cap Q$
		$\cdots > rg > r_{j-1}$	for $i, j \in \{1, \ldots, k\}, j > i, r \in X_i \cap Q,$ and $g \in X_j \cap Q$
$V_4:$		$\cdots > rg' > gr'$	for $i, j \in \{1, \ldots, k\}, i \neq j, r \in X_i \cap Q, g \in X_j \cap Q$

Fig. 3. Extension of the partial votes for the MC-instance. Extensions in which candidates that do not correspond to the solution set Q take the zero positions are highlighted.

The partial votes of V_3 realize the **vertex–edge match gadgets**. For $i, j \in \{1, \ldots, k\}$, for $j < i$, V_3 contains the vote $\{rg \succ r_j \mid \{r, g\} \in E,\ r \in X_i,\ \text{and } g \in X_j\}$ and, for $j > i$, V_3 contains the vote $\{rg \succ r_{j-1} \mid \{r, g\} \in E,\ r \in X_i,\ \text{and } g \in X_j\}$.

The partial votes of V_4 realize the **edge–edge match gadgets**. For every unordered color pair $\{i, j\}$, $i \neq j$ there is the partial vote $\{rg' \succ gr' \mid \{r, g\} \in E(i, j),\ r \in X_i,\ \text{and } g \in X_j\}$.

This completes the description of the partial profile. Now, we verify a property of the construction that is crucial to see the correctness: In total, the number of zero positions available in the partial votes is exactly equal to the sum of the minimum number of zero position the candidates of $C \setminus \{c\}$ must take such that c is a winner. We denote this property of the construction as *tightness*. To see the tightness property, we first compute the number of partial votes:

$$\begin{aligned} |V_1| + |V_2| + |V_3| + |V_4| &= k(k-1)(t-1) + k(s-1) + ks(k-2) + k(k-1) + k(k-1)/2 \\ &= t\bigl(k^2 - k\bigr) + s\bigl(k^2 - k\bigr) + k^2/2 - 3k/2. \end{aligned} \tag{1}$$

Regarding the number of zero positions that must be taken, we first compute the number of candidates for each subset:

- $|C_X| = sk(k-1)$,
- $|C_E| = 2tk(k-1)$,
- $|D_X| = sk(k-2)$, $|D_1| = k(k-2)$, and $|D_2| = k$.

The candidates of D_2 must take at least $s-1$ zero positions and all other candidates at least one. Thus, in total the number of zero positions must be at least

$$sk^2 - sk + 2tk^2 - 2tk + sk^2 - 2ks + k^2 - 2k + k(s-1) = 2s\bigl(k^2 - k\bigr) + 2t\bigl(k^2 - k\bigr) + k^2 - 3k. \tag{2}$$

Furthermore, there are two zero positions for every partial vote. It is easy to verify that (1) times two equals (2). Hence, the tightness of the construction is shown. It directly follows that if there is a candidate that takes more zero positions than desired, then c cannot win in this extension since then at least one zero position must be "missing" for another candidate.

We can now show the following claim to complete the proof.

Claim: The graph G has a clique of size k if and only if c is a possible winner in P.

"$\Rightarrow$": Given a multicolored clique Q of G of size k. We refer to the vertices and edges belonging to Q as solution vertices and solution edges, respectively, and to the corresponding candidates as solution candidates. Then, extend the partial profile P as given in Fig. 3. In the following we argue that in the given extension every candidate takes the required number of zero positions.

In V_1, for every ordered color pair, all pairs of edge-candidates except the pair of solution edge-candidates are set to the last two positions in one of the $t-1$ votes.

In V_2^a for every color i, we set all candidates r_1 that do not belong to the solution vertices and the corresponding c_1^r to zero positions in one of the votes. In V_2^b for every non-solution vertex $r \in X_i \setminus Q$ we set the corresponding candidates r_{z+1} and r_{z+2} at zero positions in the votes $v_z^{r,i}$ with odd index $z \in \{1, \ldots, k-4\}$. In the votes with even index $z \in \{2, \ldots, k-3\}$, we set the corresponding dummy candidates c_z^r, c_{z+1}^r at zero positions. We further set the candidate r_{k-1} at a zero position in votes $v_z^{r,i}$ for all the $s-1$ non-solution vertices of color i, which implies that the dummy candidate d^i is placed at $s-1$ zero positions. Thus, we have "enough" zero positions for all the copies of the non-solution candidates, the corresponding dummy candidates $\{c_1^r, \ldots, c_{k-2}^r \mid r \in X_i \setminus Q\}$, and d^i. The remaining votes of V_2^b "correspond" to the gadgets for the solution vertices. Here, we set the candidate pairs $c_z^r > c_{z+1}^r$ in the votes with odd index $z \in \{1, \ldots, k-4\}$ at position zero and the candidate pairs with candidates d_p^i for $p = 1, \ldots, k-2$ to zero positions in the votes with even index. Thus, in V_2, we have improved c upon all dummy candidates and upon all candidates corresponding to non-solution vertices, whereas each candidate corresponding to a solution vertex must still take a zero position.

Now, it remains to set every candidate corresponding to a solution vertex or a solution edge to a zero position in at least one vote. Due to construction, for a solution edge $\{r, g\} \in E$, the two corresponding candidates rg' and gr' can be set to zero in the corresponding vote of V_4. And, in V_3 the $k-1$ vertex-candidates belonging to every solution vertex can be set to a zero position in combination with the corresponding edge-candidate. Thus, the distinguished candidate c is the winner of the described extension.

"$\Leftarrow$": Given an extension of P in which c is a winner, we show that the "selected" candidates must correspond to a size-k clique. Recall that the number of zero positions that each candidate must take is "tight" in the sense that if one candidate gets an unnecessary zero position, then for another candidate there are not enough zero positions left.

First (edge selection), for $i, j \in \{1, \dots, k\}$, $i \neq j$, we consider the candidates of $C_{i,j}$. The candidates of $C_{i,j}$ can take zero positions in one vote of V_3 and in $t-1$ votes of V_1. Since $|C_{i,j}| = t$ and in the considered votes at most one candidate of $C_{i,j}$ can take a zero position, every candidate of $C_{i,j}$ must take one zero position in one of these votes. We refer to a candidate that takes the zero position in V_3 as solution candidate rg_{sol}. For every non-solution candidate $rg \in C_{i,j} \setminus \{rg_{\mathrm{sol}}\}$, its placement in V_1 also implies that rg' gets a zero position, whereas rg'_{sol} still needs to take one zero position (which is only possible in V_4).

Second, we consider the vertex selection gadgets. Here, analogously to the edge selection, for every color i, we can argue that in V_2^a, out of the set $\{r_1 \mid r \in X_i\}$, we have to set all but one candidate to a zero position. The corresponding *solution vertex* is denoted as r_{sol}. For every vertex $r \in X_i \setminus \{r_{\mathrm{sol}}\}$, this implies that the corresponding dummy-candidate c_1^r also takes a zero position in V_2^a. Now, we show that in V_2^b we have to set all candidates that correspond to non-solution vertices to a zero position whereas all candidates corresponding to r_{sol} must appear only at one-positions. Since for every vertex $r \in X_i \setminus \{r_{\mathrm{sol}}\}$, the vertex c_1^r has already a zero position in V_2^a, it cannot take a zero position within V_2^b anymore without violating the tightness. In contrast, for the selected solution candidate r_{sol}, the corresponding candidates $c_1^{r_{\mathrm{sol}}}$ and r_{sol_1} still need to take one zero position. The only possibility for $c_1^{r_{\mathrm{sol}}}$ to take a zero position is within vote $v_1^{r_{\mathrm{sol}},i}$ by setting $c_1^{r_{\mathrm{sol}}}$ and $c_2^{r_{\mathrm{sol}}}$ to the last two positions. Thus, one cannot set r_{sol_2} and r_{sol_3} to a zero position within V_2. Hence, the only remaining possibility for r_{sol_2} and r_{sol_3} to get zero points remains within the corresponding votes in V_3. This implies for every non-solution vertex r that r_2 and r_3 cannot get zero points in V_3 and thus we have to choose to put them on zero positions in the vote $v_1^{r,i}$ from V_2^b. The same principle leads to a cascading effect in the following votes of V_2^b: One cannot choose to set the candidates $c_p^{r_{\mathrm{sol}}}$ for $p \in \{1, \dots, k-2\}$ to zero positions in votes of V_2^b with even index z and thus has to improve upon them in the votes with odd index z. This implies that all vertex-candidates belonging to r_{sol} only appear in one-positions within V_2^b and that all dummy candidates d_p^i for $p \in \{1, \dots, k-2\}$ are set to one zero position. In contrast, for every non-solution vertex r, one has to set the candidates c_p^r, $p \in \{2, \dots, k-2\}$, to zero positions in the votes with even index z, and thus in the votes with odd index z, one has to set all vertex-candidates belonging to r to zero positions. This further implies that for every non-solution vertex in the last vote of V_2^b one has to set d^i to a zero position, and since there are exactly $s-1$ non-solution vertices, d^i takes the required number of zero positions. Altogether, all vertex-candidates belonging to a solution vertex still need to be placed at a zero position in the remaining votes $V_3 \cup V_4$, whereas all dummy candidates of D and the candidates corresponding to the other vertices must have taken enough zero positions.

Third, consider the vertex–edge match realized in V_3. For $i, j \in \{1, \dots, k\}$, $i \neq j$, there is only one remaining vote in which rg_{sol} with $r \in X_i$ and $g \in X_j$ can take a zero position. Hence, rg_{sol} must take this zero-position. This implies that the corresponding incident vertex-candidate x is also set to a zero-position in this vote. If $x \neq r_{\mathrm{sol}_i}$, then x has already a zero-position in V_2. Hence, this would contradict the tightness and rg_{sol} and the corresponding vertex must "match". Furthermore, the construction ensures that each of the $k-1$ candidates corresponding to one vertex appears exactly in one vote of V_3 (for each of the $k-1$ candidates, the vote corresponds to edges from different colors). Hence, c can only be a possible winner if a selected vertex matches with all selected incident edges.

Finally, we discuss the edge–edge match gadgets. In V_4, for $i, j \in \{1, \dots, k\}$, $i \neq j$, one still needs to set the solution candidates from $C_{i,j}$ to zero positions. We show that this can only be done if the two "opposite" selected edge-candidates match each other. For two such edges rg_{sol} and gr_{sol}, $r \in X_i$, $g \in X_j$, there is only one vote in V_4 in which they can get a zero position. If rg_{sol} and gr_{sol} refer to different edges, then in this vote only one of them can get zero points, and thus the other one still beats c. Altogether, if c is a possible winner, then the selected vertices and edges correspond to a multicolored clique of size k. □

By generalizing the reduction used for Lemma 2, one can show the following.

Theorem 2. *An MC-instance I can be reduced to a* Possible Winner*-instance for a scoring rule which produces a size-m scoring vector that fulfills the following. There is an $i \leqslant m-1$ such that $\alpha_{i-x} = \cdots = \alpha_{i-1} > \alpha_i$ with $x = f(I)$. A suitable poly-type function f can be computed in polynomial time.*

Proof. We describe how to modify the reduction given in the proof of Lemma 2 to work for the considered cases. For this, let P on C denote a partial profile as constructed in the proof of Lemma 2. Since $i \leqslant m-1$, the position $i+1$ must exist. We set $x = f(I) := |C| - 2$ and fill all positions smaller than $i - x$ and all positions greater than $i+1$ with dummy candidates that are different from candidates in C and that are beaten by c in every extension. We distinguish the two subcases $\alpha_i = \alpha_{i+1}$ (1a) and $\alpha_i \neq \alpha_{i+1}$ (1b).

For the case (1a), one can argue in complete analogy to Lemma 2 by "identifying" the two zero positions of Lemma 2 with position i and $i+1$ and setting the maximum partial score as follows (which can be done without changing the partial votes due to Lemma 1). For all $d^i \in D_2$, $s_p^{\max}(d^i) = (s-1)\cdot\alpha_i + (|V^p| - s + 1)\cdot\alpha_{i-1}$ and for all $c' \in C \setminus (\{c\} \cup D_2)$, $s_p^{\max}(c') = \alpha_i + (|V^p| - 1)\cdot\alpha_{i-1}$.

For (1b), we need to argue that the tightness argument still holds. For this, we set the maximum partial scores as follows (which can be done without changing the partial votes due to Lemma 1). For all $d^i \in D_2$, $s_p^{\max}(d^i) = (s-1)\cdot\alpha_{i+1} + (|V^p| - s + 1)\cdot\alpha_{i-1}$ and, for all $c' \in C \setminus (\{c\} \cup D_2)$, $s_p^{\max}(c') = \alpha_i + (|V^p| - 1)\cdot\alpha_{i-1}$. Now, in any extension in which c wins, each candidate in D_2 must be placed at least $s-1$ times on position $i+1$, and each of the other candidates must be placed on position i or $i+1$ at least once. Then again, the number of positions i and $i+1$ that still have to be assigned to candidates is exactly equal to the number of candidates that need to take these positions, hence, the tightness argument still holds. Thus, the correctness of the modified reduction can be shown in complete analogy to Lemma 2. □

In the following, we consider scoring rules with an unbounded number x of equal positions for which it holds that there is an $i \geqslant 2$ such that $\alpha_i > \alpha_{i+1} = \cdots = \alpha_{i+x}$. Parts of the results are based on further extensions of the MC-reduction used to prove Lemma 2. After that there still remain some cases for which it seems even more complicated to adapt the MC-reduction. However, for these cases we can make use of other properties of the scoring rules and settle them by less involved reductions from Exact Cover by 3-Sets. As we will see in Section 7, the following Lemmata 3–6 cover all scoring vectors with $i \geqslant 2$ such that $\alpha_i > \alpha_{i+1} = \cdots = \alpha_{i+x}$.

Lemma 3. *An MC-instance I can be reduced to a* Possible Winner*-instance for a scoring rule which produces a size-m scoring vector that fulfills the following. There is an $i \geqslant 2$ such that $\alpha_i > \alpha_{i+1} = \cdots = \alpha_{i+x}$ with $x = f(I)$* ***and*** *there is a position $j < i$ with $\alpha_j < 2\alpha_{j+1}$. A suitable poly-type function f can be computed in polynomial time.*

Proof. We describe how to modify the MC-reduction given in the proof of Lemma 2 to work for the considered case. For this, let P on C denote a partial profile as constructed in the proof of Lemma 2. First, we describe the construction for $j = i-1$, that is, one has $\alpha_{i-1} < 2\alpha_i$. We construct a partial profile $\widetilde{P}$ as follows. We set $x = f(I) = |C| - 2$ and all positions $< i-1$ and $> i+x$ are filled with dummy candidates that are beaten by c in every extension. The positions not filled with dummies "contain" the partial votes of P in "reverse" order: In P all relative orders are given for pairs of candidates. In $\widetilde{P}$ we just "flip" every pair, for example, instead of having $rg \succ rg'$ we have $rg' \succ rg$ in V_1. We define that all candidates that are not given explicitly are worse than the given candidates in a vote (instead of being better). By flipping the order of a pair, we adapt the "logical implication", for example, instead of having "if rg makes zero points, then also rg' makes zero points" in P, we have "if rg makes α_i points, then also rg' makes at least α_i points" in $\widetilde{P}$. Furthermore, we set the maximum partial scores to $s_p^{\max}(d^i) = (s-1)\cdot\alpha_{i-1} + (|V^p| - s + 1)\cdot\alpha_{i+1}$ for all $d^i \in D_2$ and $s_p^{\max}(c') = \alpha_{i-1} + (|V^p| - 1)\cdot\alpha_{i+1}$ for all $c' \in C \setminus (\{c\} \cup D_2)$. Note that since $\alpha_{i-1} < 2\alpha_i$, every candidate c' can take either position i or position $i-1$ in one of the partial votes. Then, we can use a "reverse" tightness argument: Since the positions i and $i-1$ must be taken by two candidates in every vote and every candidate can take at most one such position (or at most $s-1$ such positions for candidates in D_2, respectively), by counting candidates and positions it holds that if every candidate of D_2 must make α_{i-1} points exactly $(s-1)$ times, then every other candidate must make α_{i-1} or α_i points exactly once. Thus, it remains to show that every $d^i \in D$ must take position $i-1$ in $s-1$ of the votes. Assume this is not the case, then there must be two votes $v_{k-2}^{r,i}$ and $v_{k-2}^{r',i}$ with $r \neq r'$ in which d^i does not take position $i-1$. Due to construction, the only remaining candidate that can take this position in these votes is d_{k-2}^i, but this is not possible due to $s_p^{\max}(d_{k-2}^i)$. Hence, we can use a tightness argument analogously to Lemma 2. Since we also adapted the logical implication, the correctness follows in complete analogy to Lemma 2.

The remaining cases $(j < i-1)$ follow by padding positions within the gadgets. More precisely, replace each specified pair, e.g. $rg' \succ rg$ by $rg' \succ rg \succ H$ with a dummy set H of size $i - (j+1)$ and replace α_{i-1} by α_j in the new definitions of the maximum partial scores. □

So far, we settled the NP-hardness for scoring vectors with $i \geqslant 2$ such that $\alpha_i > \alpha_{i+1} = \cdots = \alpha_{i+x}$ if there is a position $j < i$ with $\alpha_j < 2\alpha_{j+1}$. Without the constraint $\alpha_j < 2\alpha_{j+1}$, it seems pretty complicated to adapt the tightness property which is crucial for the MC-reduction. Fortunately, the remaining cases have some different properties that allow to settle them by less complicated reductions from Exact Cover By 3-Sets. More precisely, in the following, we give three reductions with increasing difficulty. (Although all three reductions are self-contained, they might be easier to understand when reading them in the given order.)

Lemma 4. *An X3C-instance I can be reduced to a* Possible Winner*-instance for a scoring rule which produces a size-m scoring vector that fulfills the following. There is an $i \geqslant 2$ such that $\alpha_i > \alpha_{i+1} = \cdots = \alpha_{i+x}$ with $x = f(I)$* ***and*** *there is a position $j < i$ with $\alpha_j \geqslant 3\alpha_i$. A suitable poly-type function f for X3C can be computed in polynomial time.*

Proof. Let $(E, \mathcal{S})$ denote an X3C-instance. Construct a partial profile P on a set of candidates C. The set C of candidates is defined by $C := \{c\} \uplus S \uplus E \uplus H \uplus D$ where c denotes the distinguished candidate c, $S := \{s_z \mid S_z \in \mathcal{S}\}$, E the set of candidates that represent the elements of the universe, and H and D contain disjoint candidates such that the following hold. We define $H := \biguplus_{z=1}^{|\mathcal{S}|} H_z$ with $|H_z| = i - j$ for all $z \in \{1, \ldots, |\mathcal{S}|\}$ needed to "pad" some positions relevant to the construction and $|D| = m - |S| - |E| - |H| - 1$ needed to pad irrelevant positions. We refer to the candidates from S as *subset candidates* and to the candidates from E as *element candidates*. Set $f((E, \mathcal{S})) := |C \setminus D| - (i - j)$. For $1 \leqslant z \leqslant |\mathcal{S}|$, let $S_z = \{e_{z1}, e_{z2}, e_{z3}\}$. The partial profile P consists of a set of linear votes and a set of partial votes V^p. In all votes of V^p, we pad all irrelevant positions, i.e. all positions smaller than j and greater than $j - 1 + |C \setminus D|$ by fixing candidates from D (omitted in the further description). The set V^p consists of $|\mathcal{S}| - |E|/3$ copies of the vote

$$s_1 \succ H_1 \succ C \setminus (S \cup H), \quad s_2 \succ H_2 \succ C \setminus (S \cup H), \quad \ldots, \quad s_{|\mathcal{S}|} \succ H_{|\mathcal{S}|} \succ C \setminus (S \cup H)$$

denoted as V_1^p and the following three votes, denoted as $V_2^p(z)$, for every $s_z \in S$

$$v_z^1: \quad H_1 \succ \{s_z, e_{z1}\} \succ C \setminus (\{s_z, e_{z1}\} \cup H_1),$$
$$v_z^2: \quad H_1 \succ \{s_z, e_{z2}\} \succ C \setminus (\{s_z, e_{z2}\} \cup H_1), \quad \text{and}$$
$$v_z^3: \quad H_1 \succ \{s_z, e_{z3}\} \succ C \setminus (\{s_z, e_{z3}\} \cup H_1).$$

The basic idea of this construction is that in V_1^p one has to set all but $|E|/3$ "subset" candidates to position j whereas the remaining candidates will be able to take a position greater than i in all votes from V_1^p. Therefore, the remaining $|E|/3$ subset candidates can make $\alpha_j - \alpha_{i+1}$ points more than the other candidates within the remaining votes. This will enable them to shift their corresponding element candidates to position $i + 1$ by taking position i. Since $\alpha_j > 3 \cdot \alpha_i$, they will be able to shift all three element candidates, respectively. To realize the basic idea, we adapt the maximum partial scores appropriately. For $e \in E$, let n_e denote the number of subsets in $\mathcal{S}$ which contain e. Then according to Lemma 1, we can cast the linear votes such that the following hold:

- $s_p^{\max}(s_z) = \alpha_j + (|V^p| - 1) \cdot \alpha_{i+1}$, for all $s_z \in S$,
- $s_p^{\max}(e) = (n_e - 1) \cdot \alpha_i + (|V^p| - n_e + 1) \cdot \alpha_{i+1}$, for all $e \in E$, and
- all other candidates are beaten by c in every extension.

We show that c is a possible winner in P if and only if there is an exact 3-cover for $(E, \mathcal{S})$:

Assume there is an exact 3-cover Q. Then one extends P by setting each s_z with $S_z \notin Q$ at position j in one vote from V_1^p and the corresponding candidates from H_z to the positions $j + 1, \ldots, i$ in the same vote. Furthermore, set s_z to position $i + 1$ in v_z^1, v_z^2, and v_z^3. Now, we have that every s_z with $S_z \notin Q$ takes position j in one vote and a position greater than i in all remaining votes and thus is beaten by c. This also means that in V_1^p all positions $\leqslant i$ are filled and thus every candidate s_z with $S_z \in Q$ takes a position greater than i in all votes from V_1^p. Thus, the remaining votes can be extended by setting every s_z with $S_z \in Q$ to position i in v_z^1, v_z^2, and v_z^3. Since $\alpha_j \geqslant 3\alpha_i$, the maximum partial score of s_z is not exceeded. Because Q is an exact 3-cover, all element candidates are shifted to position $i + 1$ in one vote and thus are beaten by c. Hence, c is a winner in the described extension.

For the other direction, consider an extension of P in which c wins. Due to construction, in V_1^p only subset candidates from S can take position j. Because of the maximum partial scores, position j must be taken by different candidates from S in the $|\mathcal{S}| - |E|/3$ votes of V_p^1. We denote these candidates as non-solution candidates and the remaining $|E|/3$ candidates from S as solution candidates. Due to $s_p^{\max}(s_z)$, every non-solution candidate must take position $i + 1$ in all remaining votes and thus the corresponding element candidates must make α_i points in the corresponding votes. Hence, there remain only $|E|/3$ solution candidates that have to "shift" the $|E|$ element candidates to position $i + 1$. Since every solution candidate can shift at most 3 candidates, the solution candidates must correspond to an exact 3-cover. □

In the following lemma, we consider a more specific type of scoring vector in the sense that there are only two score values greater than zero. This restriction allows us to find an easy way to "lift" the condition "$\alpha_j \geqslant 3 \cdot \alpha_i$" for two special types of scoring rules that will be sufficient for the proof of the main result in Section 7. Compared to the reduction from the previous lemma, for the following cases we also choose a set of "solution subset candidates" within the first part of the partial votes, but we will need some additional gadgetry to be able to "shift" the corresponding element candidates.

Lemma 5. *An X3C-instance I can be reduced to a* Possible Winner-*instance for a scoring rule which produces a size-m scoring vector $(\alpha_1, \alpha_2, 0, \ldots, 0)$ with $3\alpha_2 > \alpha_1 > 2\alpha_2$ and $m = f(I) + 2$. A suitable poly-type function f can be computed in polynomial time.*

Proof. Let $(E, \mathcal{S})$ denote an X3C-instance. Construct a partial profile P on a set of candidates C as follows. The set of candidates consists of a distinguished candidate c, a set $S := \{s_i \mid S_i \in \mathcal{S}\}$ (the subset candidates), a set $D := \{d_i \mid S_i \in \mathcal{S}\}$, the set E (the element candidates), a candidate x, and $H := \{h_1, \ldots, h_{|\mathcal{S}|}\}$. Set $f((E, \mathcal{S})) := |C| - 2$. For $1 \leqslant i \leqslant |\mathcal{S}|$, let

$S_i = \{e_{i1}, e_{i2}, e_{i3}\}$. The partial profile P consists of a set of linear votes and a set of partial votes V^p. The set V^p consists of $|\mathcal{S}| - |E|/3$ copies of the vote

$$s_1 \succ h_1 \succ C \setminus (S \cup H), \quad s_2 \succ h_2 \succ C \setminus (S \cup H), \quad \ldots, \quad s_{|\mathcal{S}|} \succ h_{|\mathcal{S}|} \succ C \setminus (S \cup H)$$

denoted as V_1^p and the following three votes for every $S_i \in \mathcal{S}$

$$v_i^1\colon\ d_i \succ e_{i1} \succ C \setminus \{d_i, e_{i1}, s_i\}, \quad s_i \succ C \setminus \{d_i, e_{i1}, s_i\},$$
$$v_i^2\colon\ x \succ \{d_i, e_{i2}\} \succ C \setminus \{d_i, e_{i2}, x\},$$
$$v_i^3\colon\ x \succ \{d_i, e_{i3}\} \succ C \setminus \{d_i, e_{i3}, x\}.$$

Let n_e denote the number of subsets in which e occurs. Then, due to Lemma 1, we can set the maximum partial scores as follows:

- $s_p^{\max}(s_i) = \alpha_1$ for all $s_i \in S$,
- $s_p^{\max}(d_i) = 3 \cdot \alpha_2$ for all $d_i \in D$,
- $s_p^{\max}(e) = (n_e - 1) \cdot \alpha_2$ for all $e \in E$,
- all other candidates are beaten by c in every extension.

We show that c is a possible winner in P if and only if there is an exact 3-cover for $(E, \mathcal{S})$:

Assume there is an exact 3-cover Q for $(E, \mathcal{S})$. Then we extend P as follows. For every $S_i \notin Q$, s_i takes position 1 and h_i takes position 2 in one vote from V_1^p and s_i takes position 3 in v_i^1. The corresponding d_i takes position 3 in v_i^2 and v_i^3. Clearly, for $S_i \notin Q$, $s_p^{\max}(s_i)$ is not exceeded, $s_p(d_i) = \alpha_1 < 3\alpha_2 = s_p^{\max}(d_i)$, and within V_1^p all first positions are fixed. For every solution set $S_i \in Q$, we set s_i to a position greater than 2 in all votes from V_1^p and to the first position in v_i^1. Since this implies that d_i takes the second position in v_i^1, this enables us to set d_i to the second position in v_i^2 and v_i^3 without violating $s_p^{\max}(d_i)$. Since Q is an exact 3-cover, all corresponding element candidates are shifted to the third position once and for every element candidate the maximum partial score is not exceeded. Hence, c is a winner.

To see the other direction, assume there is an extension in which c wins. In V_1^p, the first positions can only be taken by candidates from S. Since each $s_i \in S$ can get α_1 points exactly once, $|\mathcal{S}| - |E|/3$ different subset candidates from S have to be placed on the first position. Let the set consisting of these candidates be denoted by S'. Every candidate s_i from S' has exploited its maximum partial score and therefore has to be placed on the third position in v_i^1. This implies that the corresponding candidate d_i takes the first position in v_i^1. Since $\alpha_1 > 2\alpha_2$ and $s_p^{\max}(d_i) = 3\alpha_2$, d_i has to take the third position in both v_i^2 and v_i^3. Hence, for $s_i \in S'$, the corresponding element candidates e_{i1}, e_{i2}, e_{i3} receive α_2 points each. However, each of the element candidates from E has to be placed on position 3 at least once due to its maximum partial score. This can only be in the remaining partial votes, that is, all v_i^1, v_i^2, v_i^3 with $s_i \in S \setminus S'$. Since $|S \setminus S'| = |E|/3$, one must shift one element candidate in each of these votes. For this, the only possibility is to set every $s_i \in S \setminus S'$ to position 1 in v_i^1, and the corresponding candidate d_i takes the second position in v_i^2 and v_i^3. Since c wins, all $|E|$ element candidates must get shifted to position 3. Hence, $S \setminus S'$ corresponds to an exact 3-cover of $(E, \mathcal{S})$. □

Finally, we settle the NP-hardness for a specific scoring vector.

Lemma 6. *An X3C-instance I can be reduced to a* Possible Winner*-instance for a scoring rule which produces a size-m scoring vector* $(2, 1, 0, \ldots, 0)$ *for $m = f(I) + 2$. A suitable poly-type function f can be computed in polynomial time.*

Proof. Let $(E, \mathcal{S})$ denote an X3C-instance. Construct a partial profile P on a set of candidates C as follows. The set of candidates consists of a distinguished candidate c, a set $S := \{s_i \mid S_i \in \mathcal{S}\}$ (the subset candidates), $D := \{d_i \mid S_i \in \mathcal{S}\}$, $T := \{t_i \mid S_i \in \mathcal{S}\}$, E (the element candidates), a candidate y, and $X := \{x_1, \ldots, x_{|\mathcal{S}|-|E|/3}\}$. Set $f((E, \mathcal{S})) := |C| - 2$. For $1 \leqslant i \leqslant |\mathcal{S}|$, let $S_i = \{e_{i1}, e_{i2}, e_{i3}\}$. The partial profile P consists of a set of linear votes and a set of partial votes V^p. The set $V^p := V_1^p \cup V_2^p \cup V_3^p$ is further defined as follows. The set V_1^p consists of $|\mathcal{S}| - |E|/3$ copies of the partial vote

$$s_1 \succ t_1 \succ C \setminus (S \cup T), \quad s_2 \succ t_2 \succ C \setminus (S \cup T), \quad \ldots, \quad s_{|\mathcal{S}|} \succ t_{|\mathcal{S}|} \succ C \setminus (S \cup T).$$

The set V_2^p consists of $|\mathcal{S}| - |E|/3$ copies of the partial vote

$$y \succ T \succ C \setminus \big(T \cup \{y\}\big)$$

and V_3^p contains the following three votes for every $S_i \in \mathcal{S}$

$$v_i^1\colon\ d_i \succ e_{i1} \succ C \setminus \{d_i, e_{i1}, s_i\}, \quad s_i \succ C \setminus \{d_i, e_{i1}, s_i\},$$
$$v_i^2\colon\ y \succ \{d_i, e_{i2}\} \succ C \setminus \{d_i, e_{i2}, y\},$$
$$v_i^3\colon\ \{t_i, e_{i3}\} \succ C \setminus \big(\{t_i, e_{i3}\} \cup X\big).$$

Table 3
Extension for the X3C-reduction for the case $(2, 1, 0, \ldots)$. The remark "different q" means that for $i \neq i'$ with $S_i \notin Q$ and $S_{i'} \notin Q$ one chooses two different candidates from X. Extensions corresponding to non-solution candidates are highlighted.

V_1^p :		$s_i > t_i > \ldots$	for $S_i \notin Q$
V_2^p :		$y > t_i > \ldots$	for $S_i \notin Q$
V_3^p :	v_i^1	$d_i > e_{i1} > s_i > \ldots$	for $S_i \notin Q$
	v_i^2	$y > e_{i2} > d_i > \ldots$	for $S_i \notin Q$
	v_i^3	$e_{i3} > x_q > t_i \ldots$	for $S_i \notin Q$ and different q
	v_i^1	$s_i > d_i > e_{i1} > \ldots$	for $S_i \in Q$
	v_i^2	$y > d_i > e_{i2} > \ldots$	for $S_i \in Q$
	v_i^3	$t_i > e_{i3} > \ldots$	for $S_i \in Q$

Let n_e denote the number of subsets in which e occurs and $n_{e,3}$ the number of subsets in which e is denoted as e_{i3} for $i \in \{1, \ldots, |S|\}$. Then, using Lemma 1, we set the maximum partial scores as follows:

- $s_p^{\max}(s_i) = s_p^{\max}(t_i) = s_p^{\max}(d_i) = 2$ for $i \in \{1, \ldots, |\mathcal{S}|\}$,
- $s_p^{\max}(x_i) = 1$ for $i \in \{1, \ldots, |\mathcal{S}| - |E|/3\}$,
- $s_p^{\max}(e) = 2n_{e,3} + (n_e - n_{e,3}) - 1$ for $e \in E$,
- the candidate y is beaten by c in every extension.

We show that c is a possible winner in P if and only if there is an exact 3-cover for $(E, \mathcal{S})$:

Assume there is an exact 3-cover Q for $(E, \mathcal{S})$. Then we extend P as given in Table 3. For every $S_i \notin Q$, s_i takes the first position in one vote from V_1^p and makes zero points in all remaining votes. The corresponding t_i takes the second position in one vote from V_1^p and one vote from V_2^p and makes zero points in all remaining votes. Hence, c beats these s_i and t_i and the votes from V_1^p and V_2^p are fixed. For every $S_i \notin Q$, we extend v_i^3 by setting a different candidate from X at the second position such that none of them is put on this position twice, and hence c also beats every candidate from X. For every $S_i \in Q$, d_i, t_i and s_i make exactly 2 points in V_3^p and thus are beaten by c as well. It remains to consider the element candidates. To this end, note that a candidate $e \in E$ is beaten by c if there is an i such that e takes position 3 in v_i^1 or v_i^2 or takes position 2 in v_i^3. Since Q is an exact 3-cover and all candidates corresponding to subsets from Q are shifted to the right in one vote, c wins in the given extension.

To see the other direction, assume there is an extension in which c wins. Let $G^1 := \{v_i^1 \mid 1 \leqslant i \leqslant |\mathcal{S}|\}$, $G^2 := \{v_i^2 \mid 1 \leqslant i \leqslant |\mathcal{S}|\}$, and $G^3 := \{v_i^3 \mid 1 \leqslant i \leqslant |\mathcal{S}|\}$. We start by arguing that at most $2/3 \cdot |E|$ candidates from E can make zero points in a vote from $G^1 \cup G^2$. For any i, at most two element candidates, namely e_{i1} and e_{i2} can make zero points in $G^1 \cup G^2$. More precisely, due to $s_p^{\max}(d_i)$, if s_i takes the first position in v_i^1, then e_{i1} and e_{i2} can take the third position and if s_i takes the second position, then only e_{i1} can be shifted to the third position, since d_i takes the first position in v_i^1 and has exploited its maximum partial score. Thus, the number of points that all candidates from S can make within V_3^p is an upper bound for the number of element candidates that can be shifted. Since only candidates from S can take the first positions in V_1^p, $|V_1^p| = |\mathcal{S}| - |E|/3$, and $s_p^{\max}(s_i) = 2$, the candidates from S can make at most $2/3|E|$ points in V_3^p. Thus, there are at most $2/3|E|$ element candidates that can take a position with zero points in $G^1 \cup G^2$. Thus, due to $s_p^{\max}(e)$, in G^3 one must shift (at least) $|E|/3$ candidates to the second position (**Observation 1**). In the following, we show that the only way to do so leads to an extension in which exactly $|E|/3$ candidates s_i from S make zero points in V_1^p and the corresponding t_i make zero points in $V_1^p \cup V_2^p$ whereas all other candidates from $S \cup T$ have already accomplished their maximum partial score in $V_1^p \cup V_2^p$ (**Claim 1**). This means that the element candidates that are shifted to the right correspond to exactly $|E|/3$ subsets $S_i \in \mathcal{S}$. Since every element candidate must be shifted at least once, these subsets must form an exact 3-cover in $(E, \mathcal{S})$.

We use a tightness criterion (analogously to the MC-reduction from Lemma 2) to prove Claim 1. To this end, we show that the score of all positions that must be filled equals the sum of the maximum partial scores of all candidates. Again, it directly follows that a candidate $c' \in C \setminus \{c\}$ cannot make less than $s_p^{\max}(c')$ points since otherwise there must be another candidate that beats c. Now, we show the tightness. The total number of votes is

$$\left|V_1^p\right| + \left|V_2^p\right| + \left|V_3^p\right| = |\mathcal{S}| - |E|/3 + |\mathcal{S}| - |E|/3 + 3|\mathcal{S}| = 5|\mathcal{S}| - 2/3|E|.$$

In V_2^p and V_3^p, candidate y is already fixed at the first position in $2|\mathcal{S}| - 1/3|E|$ votes and since in every vote 3 points have to be given, there are $3 \cdot (5|\mathcal{S}| - 2/3|E|) - 2 \cdot (2|\mathcal{S}| - 1/3|E|) = 11|\mathcal{S}| - 4/3|E|$ points for the remaining candidates left. The sum of the maximum partial scores from all candidates from $S \cup T \cup D \cup X \cup E$ is

$$3 \cdot 2 \cdot |\mathcal{S}| + |\mathcal{S}| - |E|/3 + 2|\mathcal{S}| + 2|\mathcal{S}| - |E| = 11|\mathcal{S}| - 4/3|E|.$$

To see this, note that clearly $\sum_{e \in E} n_{e,3} = |\mathcal{S}|$ and $\sum_{e \in E} n_e = 3|\mathcal{S}|$. Thus, the tightness follows.

Now, we finally show the correctness of Claim 1. Due to the tightness, the $|\mathcal{S}| - |E|/3$ candidates from X must take position 2 in $|\mathcal{S}| - |E|/3$ votes from G^3. Thus, there remain $|E|/3$ second positions in G^3 that are not fixed. Note that due to tightness, a candidate e_{i3} cannot take the third position in v_i^3. Hence, if the remaining second positions are not taken by candidates from E, we shift less than $|E|/3$ candidates in G^3, a contradiction to Observation 1. Hence, these positions must be taken by candidates from E and thus all second positions within G^3 are fixed. This implies that every candidate t_i from T must take either the first or the third position in v_i^3. More precisely, since $|E|/3$ candidates from E take a second position there must be $|E|/3$ candidates from T that take the first positions within the corresponding votes. However, a candidate from T can only take the first position if it makes zero points in $V_1^p \cup V_2^p$. Hence, there must be $|E|/3$ candidates from T, denoted as T', that make zero points in $V_1^p \cup V_2^p$ and, due to tightness, all remaining candidates from T must make 2 points in $V_1^p \cup V_2^p$. A candidate $t_i \in T$ can make at most one point in V_1^p since due to the condition "$s_i \succ t_i$" it shifts s_i to the first position (and $s_p^{\max}(s_i) = 2$). Hence, making two points within $V_1^p \cup V_2^p$ implies that t_i must make one point in V_1^p and one point in V_2^p and that the corresponding s_i must make 2 points in V_1^p. This fixes all positions in $V_1^p \cup V_2^p$ and since a candidate s_i with $t_i \in T'$ clearly makes zero points in $V_1^p \cup V_2^p$, the correctness of Claim 1 follows. Altogether, we have that $\{S_i \mid t_i \in T'\}$ forms an exact 3-cover for $(E, \mathcal{S})$. $\square$

6.2. Scoring vectors with $\alpha_1 > \alpha_2 = \cdots = \alpha_{m-1} > 0$

In this subsection, we consider scoring rules defined by scoring vectors that fulfill $\alpha_1 > \alpha_2 = \cdots = \alpha_{m-1} > 0$. Although quite special, these rules might be of interest of their own. They can be considered as a direct combination of the very common plurality and veto rules where one allows to weight the contribution of the plurality or veto part. For example, by using $(10, 1, \ldots, 1, 0)$ the "plurality" part would have more influence to the outcome, whereas for $(10, 9, \ldots, 9, 0)$ the "veto" part would be more important. To show NP-hardness, we give two types of many-one reductions from X3C; one for the case $\alpha_1 < 2 \cdot \alpha_2$ and one for the case $\alpha_1 > 2 \cdot \alpha_2$. As mentioned before, the case $\alpha_1 = 2 \cdot \alpha_2$ remains open. Intuitively, for all other cases we make use of the "asymmetry" of the differences of the score values, that is, by shifting a candidate from the first to the second position one decreases its score by a different amount than by shifting it from the last but one to the last position. In the two following proofs, the position in a linear order in which a candidate gets α_1 points is denoted as *top position*, a position in which a candidate gets α_2 points as *middle position*, and the position in which a candidate gets zero points as *last position*.

Theorem 3. *An X3C-instance I can be reduced to a* Possible Winner*-instance for a scoring rule which produces a size-m scoring vector satisfying the conditions $\alpha_1 > \alpha_2 = \alpha_{m-1} > \alpha_m = 0$ and $\alpha_1 < 2 \cdot \alpha_2$ for $m = f(I) + 2$. A suitable poly-type function f can be computed in polynomial time.*

Proof. Let $(E, \mathcal{S})$ denote an X3C-instance. We construct a partial profile P for which the distinguished candidate $c \in C$ is a possible winner if and only if $(E, \mathcal{S})$ is a yes-instance. The set of candidates is $C := \{c, h\} \uplus \{s_i, d_i, t_i \mid S_i \in \mathcal{S}\} \uplus E$. The partial profile P consists of a set of partial votes V^p and a set of linear orders V^l. For $1 \leqslant i \leqslant |\mathcal{S}|$, let $S_i = \{e_{i1}, e_{i2}, e_{i3}\}$. Then the set of partial votes $V^p := V_1^p \cup V_2^p$ is given by the following subsets. The set V_1^p consists of $|E|/3$ copies of the partial vote

$$h \succ C \setminus \{h, s_1, \ldots, s_{|\mathcal{S}|}\} \succ \{s_1, \ldots, s_{|\mathcal{S}|}\}.$$

For every $i \in \{1, \ldots, |\mathcal{S}|\}$, the set V_2^p contains the three votes

$$v_i^1: \quad h \succ C \setminus \{h, s_i, d_i\} \succ \{s_i, d_i\},$$

$$v_i^2: \quad e_{i1} \succ C \setminus \{e_{i1}, t_i, d_i\} \succ t_i, \quad \text{and}$$

$$v_i^3: \quad e_{i2} \succ C \setminus \{e_{i2}, e_{i3}, t_i\} \succ e_{i3}.$$

Now, we pass on to the definitions of the maximum partial scores. To this end, for a candidate e corresponding to an element $e \in E$ (referred to as element candidate), let $n_{e,1+2}$ denote the number of subsets from $\mathcal{S}$ in which e is identical with e_{i1} or e_{i2}. Due to Lemma 1, we can cast the linear votes such that the following hold:

- $s_p^{\max}(s_i) = (|V^p| - 1) \cdot \alpha_2$,
- $s_p^{\max}(d_i) = s_p^{\max}(t_i) = \alpha_1 + (|V^p| - 2) \cdot \alpha_2$,
- $s_p^{\max}(e) = (|V^p| - n_{e,1+2} + 1) \cdot \alpha_2 + (n_{e,1+2} - 1) \cdot \alpha_1$,
- h is beaten by c in every extension.

The maximum partial scores of the element candidates are set such that every element candidate has to be "shifted" to the right at least once. More precisely, if a candidate e took the first position in all votes in which it is identical with e_{i1}

V_1^p :		$h >$	$\ldots$	$> s_i$	$S_i \in Q$
V_2^p :	v_i^1	$h >$	$\ldots$	$> s_i > d_i$	$S_i \in Q$
	v_i^2	$d_i > e_{i1} >$	$\ldots$	$> t_i$	$S_i \in Q$
	v_i^3	$t_i > e_{i2} >$	$\ldots$	$> e_{i3}$	$S_i \in Q$
	v_i^1	$h >$	$\ldots$	$> d_i > s_i$	$S_i \notin Q$
	v_i^2	$e_{i1} >$	$\ldots$	$> t_i > d_i$	$S_i \notin Q$
	v_i^3	$e_{i2} >$	$\ldots$	$> e_{i3} > t_i$	$S_i \notin Q$

Fig. 4. Extension for the case $\alpha_1 > \alpha_2 = \alpha_{m-1} > 0$ and $\alpha_1 < 2 \cdot \alpha_2$. Extensions for candidates that do not correspond to subsets belonging to the solution set Q are highlighted.

or e_{i2} and the second position in all remaining votes (including the votes in which it is identical with e_{i3}), then $s(e) = (|V^p| - n_{e,1+2}) \cdot \alpha_2 + n_{e,1+2} \cdot \alpha_1 > s_p^{\max}(e)$ since $\alpha_1 > \alpha_2$. However, if, for any i, t_i or d_i are inserted at the first position in one of the votes in which e appears, then e makes at least $\alpha_1 - \alpha_2$ points less and thus is beaten by c. We denote this as **Observation 2**. Now, we show the correctness of the construction.

Claim: Candidate c is a possible winner in P if and only if $(E, \mathcal{S})$ is a yes-instance.

"$\Leftarrow$": Let Q denote an exact 3-cover for $(E, \mathcal{S})$. Then extend P as displayed in Fig. 4. More precisely, within V_1^p every candidate s_i with $S_i \in Q$ takes the last position in exactly one of the $|E|/3$ votes. Then, the candidates make the following points within the extension of the partial votes. Every s_i takes the last position in one vote and middle positions in all other votes and thus makes exactly $s_p^{\max}(s_i)$ points. For $S_i \in Q$, every candidate t_i and every candidate d_i takes one first and one last position, and thus, $s(d_i) = s(t_i) = \alpha_1 + (|V^p| - 2) \cdot \alpha_2 = s_p^{\max}(d_i) = s_p^{\max}(t_i)$. In the corresponding votes every element candidate is shifted once since Q is an exact 3-cover and thus is beaten by c due to Observation 2. Clearly, for $S_i \notin Q$, s_i is beaten by c as well. It remains to consider d_i and t_i with $S_i \notin Q$. Here, one has $s(d_i) = (|V^p| - 1) \cdot \alpha_2 < s_p^{\max}(d_i)$ and $s(t_i) = (|V^p| - 1) \cdot \alpha_2 < s_p^{\max}(t_i)$. Hence, c beats all other candidates and wins.

"$\Rightarrow$": Consider an extension in which c wins. Due to $s_p^{\max}(s_i)$, every candidate s_i must take the last position in at least one of the votes. Since $|V_1^p| = |E|/3$, at most $|E|/3$ candidates can take a last position in V_1^p; denote the set of them by S'. Hence at least $|\mathcal{S}| - |E|/3$ candidates s_i must take the last position in v_i^1. Now, we show that for these candidates the corresponding element candidates cannot be shifted to the right in v_i^2 or v_i^3. Since s_i takes the last position in v_i^1, d_i already makes $(|V^p| - 1) \cdot \alpha_2$ in the extended partial votes without v_i^2. Hence, d_i must take the last position in v_i^2 since otherwise $s(d_i) = |V^p| \cdot \alpha_2 > s_p^{\max}(d_i)$ because $\alpha_1 < 2\alpha_2$. This implies that e_{i1} is not shifted and that t_i takes a middle position in v_i^2. Now, for t_i it follows analogously that t_i must take the last position in v_i^3 and thus neither e_{i2} nor e_{i3} is shifted. Altogether, this means that all element candidates must be shifted by candidates from S'. Every $s_i \in S'$ can shift three candidates by setting s_i in the last position in v_i^1 and d_i and t_i to the first positions in v_i^2 and v_i^3, respectively. Since there are $|E|$ element candidates, it follows that $|S'| = |E|/3$ and that all $s_i \in S'$ must shift disjoint sets of element candidates. Hence, S' corresponds to an exact 3-cover for $(E, \mathcal{S})$. $\square$

In the remainder of this subsection, we consider the case that $\alpha_1 > 2 \cdot \alpha_2$. We also give a reduction from X3C. Note that the previous proof cannot be transferred directly and thus we give a modified construction for which it will be more laborious to show the correctness.

Theorem 4. *An X3C-instance I can be reduced to a* Possible Winner*-instance for a scoring rule which produces a size-m-scoring vector satisfying the conditions $\alpha_1 > \alpha_2 = \alpha_{m-1} > \alpha_m = 0$ and $\alpha_1 > 2 \cdot \alpha_2$ for $m = f(I) + 2$. A suitable poly-type function f can be computed in polynomial time.*

Proof. Let $(E, \mathcal{S})$ denote an X3C-instance. Let k denote the size of a solution for $(E, \mathcal{S})$, that is, $k := |E|/3$, and $t := |\mathcal{S}|$. We construct a partial profile P for which the distinguished candidate $c \in C$ is a possible winner if and only if $(E, \mathcal{S})$ is a yes-instance. The set of candidates is $C := S \uplus D \uplus E \uplus \{c, h\}$ with $S := \{s_i \mid 1 \leqslant i \leqslant t\}$ (the subset candidates) and $D := \{d_i \mid 1 \leqslant i \leqslant t\}$, and E (the element candidates).

Very roughly, the basic idea of the reduction is as follows. There are three subsets of partial votes, in the first subset V_1^p one "selects" $t - k$ subset candidates from S that do not correspond to an exact 3-cover and in the second subset V_2^p one selects k subset candidates that correspond to an exact 3-cover. Selecting hereby means that a solution subset candidate gets zero points in one vote of V_2^p whereas every non-solution candidate gets α_1 points in a vote of V_1^p. Hence, a solution candidate can make more points than a non-solution candidate in the third subset V_3^p. Thus, a solution candidate can take a top position in V_3^p which yields a cascading effect that makes it possible to shift the corresponding element candidates such that they do not beat the distinguished candidate c.

Formally, the partial profile P consists of a set of partial votes V^p and a set of linear orders V^l. For $1 \leqslant i \leqslant t$, let $S_i = \{e_{i1}, e_{i2}, e_{i3}\}$, then the set of partial votes $V^p := V_1^p \cup V_2^p \cup V_3^p$ is given by the following subsets:

V_1^p :		$s_i > C\backslash\{s_i, h\} > h$	$\forall s_i$ with $S_i \notin \mathcal{S}'$
V_2^p :		$h > C\backslash\{s_i, h\} > s_i$	$\forall s_i$ with $S_i \in \mathcal{S}'$
V_3^p :	w_1^i	$d_i > C\backslash\{s_i, d_i\} > s_i$	$\forall s_i$ with $S_i \notin \mathcal{S}'$
	w_2^i	$h > C\backslash\{d_i, h\} > d_i$	$\forall s_i$ with $S_i \notin \mathcal{S}'$
	w_3^i	$h > C\backslash\{d_i, h\} > d_i$	$\forall s_i$ with $S_i \notin \mathcal{S}'$
	w_1^i	$s_i > C\backslash\{s_i, e_{i1}\} > e_{i1}$	$\forall s_i$ with $S_i \in \mathcal{S}'$
	w_2^i	$h > C\backslash\{e_{i2}, h\} > e_{i2}$	$\forall s_i$ with $S_i \in \mathcal{S}'$
	w_3^i	$h > C\backslash\{e_{i3}, h\} > e_{i3}$	$\forall s_i$ with $S_i \in \mathcal{S}'$

Fig. 5. Extension of V^p for an exact 3-cover $\mathcal{S}' \subseteq \mathcal{S}$. The middle positions are not given explicitly since the order of the candidates is irrelevant. Extensions for candidates which do not belong to the solution set $\mathcal{S}'$ are highlighted.

V_1^p: $t-k$ copies of the partial vote $\quad S \succ C \setminus \big(S \cup \{h\}\big) \succ h$,

V_2^p: k copies of the partial vote $\quad h \succ C \setminus \big(S \cup \{h\}\big) \succ S$,

V_3^p: for $1 \leqslant i \leqslant t$ the three partial votes

w_1^i: $d_i \succ C \setminus \{d_i, e_{i1}, s_i\} \succ e_{i1}$,

w_2^i: $h \succ C \setminus \{d_i, e_{i2}, h\} \succ \{e_{i2}, d_i\}$,

w_3^i: $h \succ C \setminus \{d_i, e_{i3}, h\} \succ \{e_{i3}, d_i\}$.

Note that in w_1^i, candidate s_i can be inserted at any position. The distinguished candidate c makes α_2 points in every partial vote from V^p. Hence, according to Lemma 1, we can set the linear orders of V^l such that the following holds. For $i = 1, \ldots, t$,

$$s_p^{\max}(s_i) = \big(\big|V^p\big| - 2\big) \cdot \alpha_2 + \alpha_1,$$
$$s_p^{\max}(d_i) = \big(\big|V^p\big| - 2\big) \cdot \alpha_2 + \alpha_1 - z$$

with $z = \alpha_1 \bmod \alpha_2$ if $\alpha_1 < 3\alpha_2$, and $z = \alpha_2$, otherwise.[4] Note that it holds that $\alpha_2 \geqslant z$ and

$$\alpha_1 - z \geqslant 2\alpha_2. \tag{3}$$

For all $e \in E$, $s_p^{\max}(e) = (|V^p| - 1) \cdot \alpha_2$, that is, e must have the last position in one of the partial votes. And, $s_p^{\max}(h) \geqslant |V^p| \cdot \alpha_1$, that is, h can beat c in no extension.

We now prove the following claim.

Claim: Candidate c is a possible winner of (V, C) if and only if $(E, \mathcal{S})$ is a yes-instance for X3C.

"$\Leftarrow$": Let $S' \subseteq \mathcal{S}$ denote an exact 3-cover for $(E, \mathcal{S})$. Then, we extend the partial profile as follows (Fig. 5). If $S_i \in S'$, then s_i is placed at the last position in one vote of V_2^p and at a middle position in all other votes from $V_1^p \cup V_2^p$. If $S_i \notin S'$, then s_i is placed at the first position in one of the votes in V_1^p and at a middle position in all other votes from $V_1^p \cup V_2^p$. This is possible since there are $t-k$ top position and k last positions that can be taken by candidates from S in $V_1^p \cup V_2^p$. In V_3^p, every candidate s_i with $S_i \in \mathcal{S}'$ is placed at the top position and the corresponding element candidates e_{i2}, e_{i3} at the last position in the respective votes. Every candidate s_i with $S_i \notin \mathcal{S}'$ is placed at the last position and the corresponding element candidates e_{i2}, e_{i3} are placed at a middle position.

In the described extension, the candidates make the following points in V^p. Every candidate $s_i \in S$ takes exactly one top position and exactly one last position in V^p. Hence $s(s_i) = s_p^{\max}(s_i)$. For the candidates of D one has to distinguish two cases. First, if $S_i \notin S$, then, $s(d_i) = (|V^p| - 3) \cdot \alpha_2 + \alpha_1 \leqslant s_p^{\max}(d_i)$ since $\alpha_2 \geqslant z$. Second, if $S_i \in S$, then $s(d_i) = |V^p| \cdot \alpha_2 = (|V^p| - 2) \cdot \alpha_2 + 2\alpha_2 \leqslant (|V^p| - 2) \cdot \alpha_2 + \alpha_1 - z = s_p^{\max}(d_i)$ because of Inequality (3). Finally, we have to consider the candidates from E. Since for every S_i in the 3-cover, the corresponding element candidates e_{i1}, e_{i2}, and e_{i3} get at the last position, every candidate of E takes one last and $|V^p| - 1$ middle positions and thus makes $(|V^p| - 1) \cdot \alpha_2$ points. It follows that c wins in the considered extension.

"$\Rightarrow$": In an extension of V in which c is the winner, every element candidate from E must take the last position in one vote of V^p. This is only possible in V_3^p since every element candidate is already fixed at a middle position in $V_1^p \cup V_2^p$. More precisely, for every i, e_{i1} gets a last position if s_i is inserted at a middle or the top position in the corresponding vote w_1^i and e_{i2}/e_{i3} can get a last position only if d_i takes a middle position in the corresponding vote w_2^i/w_3^i.

To find out what this means for the other candidates, we have to go into details here. For $i = 1, \ldots, t$, let b_i denote the "benefit", i.e., the maximum number of element candidates that can be put at a last position in V_3^p depending on where s_i is placed in w_1^i. Then, we can show the following.

[4] Note that this maximum partial score does not exactly fulfill the conditions of Lemma 1 if $z \neq \alpha_2$. However, the construction can be easily extended to work for this case as well. More precisely, in this case $z = \alpha_1 - \lfloor \alpha_1/\alpha_2 \rfloor \cdot \alpha_2$ and $\lfloor \alpha_1/\alpha_2 \rfloor \leqslant 3$. Thus, in the construction given in the proof of Lemma 1 one can add α_1 and "subtract" α_2 as often as required. The subtraction can be accomplished by changing the role of the dummy "d" and d_i within a block.

Observation 3.

1. $b_i = 3$ if s_i is placed in a top position in w_1^i.
2. $b_i = 1$ if s_i is placed in a middle position in w_1^i.
3. $b_i = 0$ if s_i is placed in a last position in w_1^i.

To see Observation 3, note that if s_i is on the top position in w_1^i, then d_i can take the middle position in w_2^i or w_3^i since the corresponding score $s(d_i) = |V^p| \cdot \alpha_2 \leqslant s_p^{\max}(d_i)$. Thus, all three element candidates can be shifted to the last position. If s_i is not placed on the top, but in the middle position, then e_{i1} is still shifted to the last position, but d_i must take the last position in w_2^i or w_3^i and thus neither e_{i2} nor e_{i3} can have a last position in w_2^i or w_3^i. To see this, assume that d_i has the top position in w_1^i and a middle position in w_2^i or w_3^i, then

$$s(d_i) \geqslant \left|V_1^p \cup V_2^p\right| \cdot \alpha_2 + \left(\left|V_3^p\right| - 2\right) \cdot \alpha_2 + \alpha_1 = \left(\left|V^p\right| - 2\right) \cdot \alpha_2 + \alpha_1 > s_p^{\max}(d_i),$$

a contradiction. If s_i is placed on the last position in w_1^i, then e_{i1} cannot take the last position in V_3^p, and neither can e_{i2} and e_{i3}, because d_i takes the first position in w_1^i and gets α_1 points and has to take the last position in both w_2^i and w_3^i by the same argument as before.

In the following, we show that in an extension in which c wins, in V_1^p there must be $t - k$ different subset candidates s_i that take the top position and each of the remaining k (solution) candidates of S must take one last position in V_2^p. It directly follows by Observation 3 that for all non-solution candidates we must have that $b_i = 0$ and thus every solution candidate must shift the three corresponding element candidates that must be different from the element candidates corresponding to the other solution candidates.

For every i, let t_i denote the number of top positions that s_i takes within V_1^p and l_i the number of last positions that s_i takes within V_2^p. Observe that the following conditions must hold:

$$\sum_{i=1}^{t} l_i = k,$$

$$\sum_{i=1}^{t} t_i = t - k, \quad \text{since every position must be taken,}$$

$$\sum_{i=1}^{t} b_i \geqslant 3k, \quad \text{since there are } 3k \text{ element candidates and each one must take at least one last position.} \tag{4}$$

In the following, our strategy consists of three steps:

- We first investigate the dependencies of l_i, t_i, and b_i upon each other. For that sake, we distinguish the cases $l_i = 0$, $l_i = 1$, and $l_i \geqslant 2$.
- Second, based on these case distinctions, we can show that the case $l_i \geqslant 2$ is not possible, that is, every s_i can have at most one last position in V_2^i. This will need the most technical effort and will directly imply $t_i \leqslant 1$ for all i.
- Third, we show that there is no candidate s_i with $l_i = t_i = 1$, which will imply that only candidates with $l_i = 1$ contribute with a positive benefit and can place their element candidates at a last position. Since there are only k such candidates, they must correspond to an exact 3-cover.

First step. We show some dependencies of l_i, t_i, and b_i by systematically enumerating all possible cases. (In the argumentation that follows the case distinction we are only interested in upper bounds of b_i. Hence, we omit to show lower bounds.)

Case I: $l_i = 0$ **a)** if $t_i = 0$, then $b_i \leqslant 1$,
b) if $t_i = 1$, then $b_i = 0$,
c) $t_i \geqslant 2$ is not possible.

Proof. Ia) ($l_i = t_i = 0$): Assume $b_i = 3$, i.e., s_i is on the top position in w_1^i due to Observation 3. Then $s(s_i) = (|V^p| - 1)\alpha_2 + \alpha_1 > s_p^{\max}(s_i)$, a contradiction, hence $b_i \leqslant 1$.

Ib) ($l_i = 0$, $t_i = 1$): Assume $b_i = 1$, i.e., s_i is on a middle position in w_1^i due to Observation 3. Then $s(s_i) = (|V^p| - 1)\alpha_2 + \alpha_1 > s_p^{\max}(s_i)$, a contradiction, hence $b_i = 0$.

Ic) ($l_i = 0$, $t_i \geqslant 2$): Assume s_i takes the last position in w_1^i, that is, s_i makes as few points as possible within this case. Then,

$$
\begin{aligned}
s(s_i) &= \big(|V^p| - t_i - 1\big)\alpha_2 + t_i\alpha_1 \\
&> \big(|V^p| - t_i - 1 + 2(t_i - 1)\big)\alpha_2 + \alpha_1 \\
&> s_p^{\max}(s_i),
\end{aligned}
$$

a contradiction, hence this case is not possible. □

Case II: $l_i = 1$ **a)** if $t_i = 0$, then $b_i \leqslant 3$,
b) if $t_i = 1$, then $b_i \leqslant 1$,
c) $t_i \geqslant 2$ is not possible.

Proof. IIa) ($l_i = 1$, $t_i = 0$), trivial upper bound.

IIb) ($l_i = t_i = 1$) Assume $b_i = 3$, i.e., s_i is on the top position in w_1^i due to Observation 3. Then $s(s_i) = (|V^p| - 3)\alpha_2 + 2\alpha_1 > s_p^{\max}(s_i)$, a contradiction, hence $b_i \leqslant 1$.

IIc) ($l_i = 1$, $t_i \geqslant 2$): Even if s_i takes the last position in w_1^i one has

$$
\begin{aligned}
s(s_i) &= \big(\big|V^p\big| - t_i - 2\big)\alpha_2 + t_i\alpha_1 \\
&> \big(\big|V^p\big| - t_i - 2 + 2(t_i - 1)\big)\alpha_2 + \alpha_1 \\
&= \big(\big|V^p\big| + t_i - 4\big)\alpha_2 + \alpha_1 \\
&\geqslant s_p^{\max}(s_i),
\end{aligned}
$$

a contradiction, hence this case is not possible. □

Case III: $l_i \geqslant 2$ **a)** if $t_i = l_i$, then $b_i = 0$,
b) if $t_i = l_i - 1$, then $b_i \leqslant 1$,
c) if $t_i \leqslant l_i - 2$, then $b_i \leqslant 3$,
d) $t_i > l_i$ is not possible.

Proof. IIIa) ($l_i \geqslant 2$, $t_i = l_i$): Assume $b_i = 1$, i.e., s_i is on a middle position in w_1^i due to Observation 3. Then

$$
\begin{aligned}
s(s_i) &= \big(\big|V^p\big| - t_i - l_i\big)\alpha_2 + t_i\alpha_1 \\
&= \big(\big|V^p\big| - 2t_i\big)\alpha_2 + t_i\alpha_1 \\
&> \big(\big|V^p\big| - 2t_i + 2(t_i - 1)\big)\alpha_2 + \alpha_1 \\
&= \big(\big|V^p\big| - 2\big)\alpha_2 + \alpha_1 \\
&= s_p^{\max}(s_i),
\end{aligned}
$$

a contradiction, hence $b_i = 0$.

IIIb) ($l_i \geqslant 2$, $t_i = l_i - 1$): Assume $b_i = 3$, i.e., s_i is on the top position in w_1^i due to Observation 3, then

$$
\begin{aligned}
s(s_i) &= \big(\big|V^p\big| - t_i - l_i - 1\big)\alpha_2 + (t_i + 1)\alpha_1 \\
&= \big(\big|V^p\big| - 2t_i - 2\big)\alpha_2 + (t_i + 1)\alpha_1 \\
&> \big(\big|V^p\big| - 2t_i - 2 + 2t_i\big)\alpha_2 + \alpha_1 \\
&= \big(\big|V^p\big| - 2\big)\alpha_2 + \alpha_1 \\
&= s_p^{\max}(s_i),
\end{aligned}
$$

a contradiction, hence $b_i \leqslant 1$.

IIIc) ($l_i \geqslant 2$, $t_i \leqslant l_i - 2$): trivial upper bound.

IIId) ($l_i \geqslant 2$, $t_i > l_i$): Then

$$
\begin{aligned}
s(s_i) &= \big(\big|V^p\big| - t_i - l_i - 1\big)\alpha_2 + t_i\alpha_1 \\
&> \big(\big|V^p\big| - t_i - l_i - 1 + 2(t_i - 1)\big)\alpha_2 + \alpha_1 \\
&= \big(\big|V^p\big| + t_i - l_i - 3\big)\alpha_2 + \alpha_1 \\
&\geqslant s_p^{\max}(s_i),
\end{aligned}
$$

a contradiction, hence this case is not possible. □

Second step. Using the previous case distinctions, we show that no subset candidate s_i can take more than one last position in V_2^p. For this, without loss of generality, we assume that the candidates s_i are sorted in decreasing order according to their corresponding l_i, i.e.,

$$\underbrace{s_1, \ldots, s_j}_{l_i \geqslant 2}, \underbrace{s_{j+1}, \ldots, s_r}_{l_i=1}, \underbrace{s_{r+1}, \ldots, s_t}_{l_i=0}.$$

Claim 1: In an extension in which c wins, it holds that $l_i \leqslant 1$ for all i.

To prove Claim 1, we show that $j = 0$. More specifically, we prove that $j > 0$ implies that the total benefit $B := \sum_{i=1}^t b_i$ is less than $3k$. This means that not all $3k$ element candidates can take a last position and thus c cannot win.

Assume that $j > 0$. We start to show how to distribute the last and the first positions of V_1^p and V_2^p in order to maximize B. For that sake, let $T_j := \sum_{i=1}^j t_i$ denote the number of top positions that were taken by the first j candidates $s_1, \ldots, s_j$. Now, we consider the remaining indices $i \in \{j+1, \ldots, t\}$. Since for all of them $l_i \leqslant 1$, it must also hold $t_i \leqslant 1$ (see Cases I and II). Thus and because of Eq. (4), there must be at least $t - k - T_j$ candidates from $s_{j+1}, \ldots, s_t$ with $t_i = 1$. For both remaining cases ($l_i = 1$ and $l_i = 0$), the benefit b_i is greater for the case $t_i = 0$ than it is for the case $t_i = 1$ (cf. Cases I and II). Hence, to maximize the total benefit B, it is desirable to minimize the number of candidates having $t_i = 1$. Since there are $t - j$ indices greater than j and t_i must be equal to one for at least $t - k - T_j$ indices, there are at most $t - j - (t - k - T_j) = k + T_j - j$ indices with $t_i = 0$ (**Observation 4**). Furthermore, for every index from $\{j+1, \ldots, s_r\}$, by setting t_i to zero or one, one can "choose" between $b_i = 1$ and $b_i = 3$ (Case II). For the remaining indices, one can choose between $b_i = 0$ and $b_i = 1$ by setting t_i to zero or one (Case I). We show by contradiction that choosing Case IIa (which results in $b_i = 3$) as often as possible is the way to maximize B:

Assume that Case IIa holds, that is, $l_i = 1$ and $t_i = 0$ is not chosen as often as possible. Then, first, there must be an index $i \in \{j+1, \ldots, r\}$ with $t_i = 1$ and hence with $b_i = 1$ (Case IIb). Second, there must be an index $x > r$ with $t_x = 0$ and hence $b_x = 1$ (Case Ia).

Then setting $t_i = 1$ and $t_x = 0$ does not violate Eq. (4) and has the following effect.

- b_i is increased by 2 (from 1 to 3),
- b_x is decreased by 1 (from 1 to 0).

Thus, $B = \sum_{i=1}^t b_i$ was not maximal.

Now, we have argued that to maximize B, one has to choose Case IIa as often as possible (**Observation 5**). Using this, we can compute the maximal value $\max B$ of B (showing that it must be less than $3k$). For that sake, we first consider the benefit coming from the first j candidates $s_1, \ldots, s_j$, which we denote by $B_j := \sum_{i=1}^j b_i$. Let B_j^0 denote the set of indices $i \in \{1, \ldots, j\}$ with $b_i = 0$, let B_j^1 denote the set of indices $i \in \{1, \ldots, j\}$ with $b_i = 1$, and let B_j^3 denote the set of indices $i \in \{1, \ldots, j\}$ with $b_i = 3$. Then, Case III directly gives the following bound for the number of top positions assumed by the first j candidates:

$$T_j \leqslant \sum_{i \in B_j^0} l_i + \sum_{i \in B_j^1} (l_i - 1) + \sum_{i \in B_j^3} (l_i - 2) = \sum_{i=1}^j l_i - |B_j^1| - 2|B_j^3|, \tag{5}$$

which will be needed in the following.

Due to the previous discussion we know that in the remaining positions, we have to choose $t_i = 0$ for $k + T_j - j$ indices (cf. Observation 4) and one should choose Case IIa, that is, $l_i = 1$ and $t_i = 0$, as often as possible (cf. Observation 5). Clearly, $l_i = 1$ must be chosen $k - \sum_{i=1}^j l_i$ times whereas there are $k + T_j - j$ indices with $t_i = 0$. Hence, to compute a total upper bound on B, we have to distinguish two cases: First, $k - \sum_{i=1}^j l_i \leqslant k + T_j - j$, and, second, $k - \sum_{i=1}^j l_i > k + T_j - j$.

For the first case, we obtain

$$\begin{aligned} \max B &= \underbrace{|B_j^1| + 3|B_j^3|}_{B_j} + \underbrace{3\left(k - \sum_{i=1}^j l_i\right)}_{l_i=1,\, t_i=0} + \underbrace{k + T_j - j - \left(k - \sum_{i=1}^j l_i\right)}_{l_i=0,\, t_i=0} \\ &= |B_j^1| + 3|B_j^3| + 3k - 2 \cdot \sum_{i=1}^j l_i + T_j - j \end{aligned}$$

$$\overset{(5)}{\leqslant} |B_j^1| + 3|B_j^3| + 3k - 2 \cdot \sum_{i=1}^{j} l_i + \sum_{i=1}^{j} l_i - |B_j^1| - 2|B_j^3| - j$$
$$= 3k - \sum_{i=1}^{j} l_i - j + |B_j^3|.$$

Since $|B_j^3| \leqslant j$ it holds that the maximal value of B is strictly less than $3k$ for $j \geqslant 1$. Thus, at least one element candidate does not take a last position and hence beats c, a contradiction.

For the second case, we obtain

$$\max B = \underbrace{|B_j^1| + 3|B_j^3|}_{B_j} + 3\underbrace{(k + T_j - j)}_{l_i=1,\, t_i=0} + \underbrace{k - \sum_{i=1}^{j} l_i - (k + T_j - j)}_{l_i=1,\, t_i=1}$$
$$= |B_j^1| + 3|B_j^3| + 3k + 2T_j - 2j - \sum_{i=1}^{j} l_i$$
$$\overset{(5)}{\leqslant} |B_j^1| + 3|B_j^3| + 3k + \sum_{i=1}^{j} l_i - |B_j^1| - 2|B_j^3| + T_j - 2j - \sum_{i=1}^{j} l_i$$
$$= 3k + |B_j^3| + T_j - 2j.$$

Furthermore, in this case it follows directly from $k - \sum_{i=1}^{j} l_i > k + T_j - j$ that $\sum_{i=1}^{j} l_i + T_j < j$. For $j > 0$ this means that $T_j < j$. By definition, we have $|B_j^3| \leqslant j$, and thus $\max B$ is less than $3k$. This completes the proof of Claim 1. We therefore have $j = 0$ which means $l_i \leqslant 1$ for all $i \in \{1, \ldots, t\}$ and thus also $t_i \leqslant 1$ for all i (Cases I and II).

Third step. We now show that there cannot be any candidate s_i which takes one last position and one first position in $V_1 \cup V_2$, i.e. we cannot have $t_i = l_i = 1$ for any s_i. Assume that the set of candidates $Q := \{s_i \mid t_i = l_i = 1\}$ is not empty. Then, due to Observation 3, the maximum value of B is

$$\max B = \underbrace{1 \cdot |Q|}_{l_i=t_i=1} + \underbrace{3 \cdot \big(k - |Q|\big)}_{l_i=1,\, t_i=0} + \underbrace{0}_{l_i=0,\, t_i=1} + \underbrace{1 \cdot |Q|}_{l_i=t_i=0} = 3k - |Q|,$$

a contradiction. Thus, $t - k$ many of the subset candidates s_i take a top position in V_1^p, and the remaining k subset candidates take a last position in V_2^p. Now, each of these k candidates must place its corresponding element candidates at the last positions in V_3^p. Since c can only be a winner if each of the $3k$ element candidates takes a last position in a vote from V_3^p and in total at most $3k$ element candidates can take a last position in V_3^p, every element candidate must take exactly one last position. Thus, for $i \neq j$ such that s_i and s_j take a last position in V_2^p, $\{e_{i1}, e_{i2}, e_{i3}\}$ and $\{e_{j1}, e_{j2}, e_{j3}\}$ must be disjoint. It follows that $\{S_i \mid s_i$ takes a last position in $V_2^p\}$ forms an exact 3-cover. $\square$

7. Putting all together

We are now ready to combine the many-one reductions from the previous sections to one general reduction. Basically, the problem we encounter by using one specific reduction from the previous sections is that such a reduction produces a Possible Winner-instance with a certain number m of candidates. Thus, one needs to ensure that the size-m scoring vector provides a sufficient number of positions with equal/different scores. This seems not to be possible in general. However, for every specific instance of Exact Cover By 3-Sets or Multicolored Clique, we can compute a number of positions with equal or different scores that is sufficient for the corresponding reduction, and we can use the maximum of all these numbers for the combined reduction. This is the underlying idea for the following proof.

Theorem 5. Possible Winner *is NP-complete for a scoring rule r if there is a constant z such that all scoring vectors produced by r for more than z candidates are different from* $(0, \ldots, 0)$, $(1, 0, \ldots, 0)$, $(1, \ldots, 1, 0)$, *and* $(2, 1, \ldots, 1, 0)$.

Proof. We give a reduction from X3C restricted to instances of size greater than z to Possible Winner for r. Let I with $|I| > z$ denote an X3C-instance. Since X3C and MC are NP-complete, there is a polynomial-time reduction from X3C to MC. Hence, let I' denote an MC-instance whose size is polynomial in $|I|$ and which is a yes-instance if and only if I is a yes-instance.

Table 4
Subcases for scoring rules having an unbounded number of equal score values.

Case I	$\exists i \leqslant m-1$ s.t.	$\alpha_{i-x} = \cdots = \alpha_{i-1} > \alpha_i$	Theorem 2
Case IIa	$\exists i \geqslant 2$, $\exists j < i$ s.t.	$\alpha_i > \alpha_{i+1} = \cdots = \alpha_{i+x}$ and $\alpha_j < 2\alpha_{j+1}$	Lemma 3
Case IIb	$\exists i \geqslant 2$, $\exists j < i$ s.t.	$\alpha_i > \alpha_{i+1} = \cdots = \alpha_{i+x}$ and $\alpha_j \geqslant 3\alpha_i$	Lemma 4
Case IIc		$(\alpha_1, \alpha_2, 0, \ldots, 0)$ and $3\alpha_2 > \alpha_1 > 2\alpha_2$	Lemma 5
Case IId		$(2, 1, 0, \ldots, 0)$	Lemma 6
Case III		$\alpha_1 > \alpha_2 = \alpha_{m-1} > \alpha_m = 0$ and $\alpha_1 \neq 2 \cdot \alpha_2$	Theorem 4

Let f_1 denote a poly-type function to compute the number of different score values as stated for Theorem 1, f_1' as for Theorem 2, f_2' as for Lemma 3, f_2 as for Lemma 4, f_3 as for Lemma 5, f_4 as for Lemma 6, and f_5 as for Theorem 4. Define $x := \max\{f_1(I), f_1'(I'), f_2'(I'), f_2(I), f_3(I), f_4(I), f_5(I)\}$ and consider the scoring vector $\vec{\alpha}$ of size $x \cdot (x+1)$ produced by r. Then we show the following.

Claim: For $\vec{\alpha}$ it holds that $|\{i \mid \alpha_i > \alpha_{i+1}\}| \geqslant x$ or that $\alpha_i = \cdots = \alpha_{i+x}$ for some position i.

The correctness of the claim can be seen as follows. First, assume that $\vec{\alpha}$ does not fulfill $\alpha_i > \alpha_j$ for x different positions i. Then consider $x \cdot (x+1)$ indices of $\vec{\alpha}$. Since they can have at most x different score values, there must be a single score value that is assigned to at least $x+1$ indices, that is, there is an index i with $\alpha_i = \cdots = \alpha_{i+x}$. Second, if there is no index i such that $\alpha_i = \cdots = \alpha_{i+x}$ for a position i, then again consider $x \cdot (x+1)$ indices of $\vec{\alpha}$. Since each score value can be assumed at most x times, there must be at least x different score values.

Now, due to the Claim, we can distinguish two main cases. If $\vec{\alpha}$ has at least x different score values, then we apply the X3C-reduction given in Theorem 1. Otherwise, we have an unbounded number of equal score values. In this case we distinguish the subcases given in Table 4. For all these subcases, there are many-one reductions used in the corresponding lemmata/theorems. Hence, it remains to show that each scoring vector can be handled by at least one of these cases. Clearly, $\vec{\alpha}$ must have the form $\alpha_{i-x} = \cdots = \alpha_{i-1} > \alpha_i$ for an $i \leqslant m-1$ (Case I), or $\alpha_i > \alpha_{i+1} = \cdots = \alpha_{i+x}$ for $i \geqslant 2$ (Case II), or $\alpha_1 > \alpha_2 = \alpha_{m-1} > \alpha_m = 0$ and $\alpha_1 \neq 2 \cdot \alpha_2$ (Case III). For Cases I and III, the existence of many-one reductions follows immediately from the corresponding Theorems 2 and 4. Thus, it remains to discuss Case II, the case that $\vec{\alpha}$ has the form $\alpha_i > \alpha_{i+1} = \cdots = \alpha_{i+x}$ for $i \geqslant 2$.

To this end, we start with the case $i > 2$. Clearly, there must be at least three scoring values which are not equal to zero, namely, $\alpha_{i-2}, \alpha_{i-1}$, and α_i. If one has $\alpha_{i-1} < 2\alpha_i$ or $\alpha_{i-2} < 2\alpha_{i-1}$, then NP-hardness follows directly from Lemma 3. Otherwise, one must have $\alpha_{i-1} \geqslant 2\alpha_i$ and $\alpha_{i-2} \geqslant 2\alpha_{i-1}$. Hence, it follows that $\alpha_{i-2} \geqslant 4\alpha_i$ and NP-hardness follows directly from Lemma 4. It remains to consider all scoring rules of type $(\alpha_1, \alpha_2, 0, \ldots, 0)$. Here, we can distinguish the following four cases:

- $\alpha_1 < 2\alpha_2$: NP-hardness follows from Lemma 3,
- $\alpha_1 = 2\alpha_2$: NP-hardness follows from Lemma 6,
- $2\alpha_2 < \alpha_1 < 3\alpha_2$: NP-hardness follows from Lemma 5, and
- $\alpha_1 \geqslant 3\alpha_2$: NP-hardness follows from Lemma 4.

Since the membership in NP is obvious, the main theorem follows. □

Pure scoring rules. Based on all previous considerations, for pure scoring rules we almost arrive at a dichotomy. More precisely, we can state the following.

Theorem 6. Possible Winner *is NP-complete for all non-trivial pure scoring rules except plurality, veto, and scoring rules for which there is a constant z such that the produced scoring vector is* $(2, 1, \ldots, 1, 0)$ *for every number of candidates greater than z. For plurality and veto it is solvable in polynomial time.*

Proof. Plurality and veto are polynomial-time solvable due to Proposition 1. Having any non-trivial scoring vector different from $(1, 0, \ldots, 0)$, $(1, \ldots, 1, 0)$, and $(2, 1, \ldots, 1, 0)$ for m candidates, it is not possible to obtain a scoring vector of one of these three types (or $(0, \ldots, 0)$) for $m' > m$ by inserting scoring values. Hence, since we only consider pure scoring rules, the scoring rule does not produce a scoring vector of type plurality, veto, $(0, \ldots, 0)$, or $(2, 1, \ldots, 1, 0)$ for all $m \geqslant z$. Then the statement follows by Theorem 5. □

"Non-pure" scoring rules. We end this section with a brief informal discussion about the problem of classifying non-pure scoring rules in general. As stated in Theorem 5, we can show NP-hardness for non-pure scoring rules if (starting from a constant number of candidates) all produced scoring vectors are "difficult". Clearly, it is possible to extend the range of NP-hardness results to scoring rules that produce only few "easy" vectors; for example, having a difficult vector for all odd numbers of candidates and an easy vector for all even ones. However, this is not possible in general. Roughly speaking, if the underlying difficult part of the language becomes too sparse, then there cannot be a many-one reduction from an NP-complete problem since the densities of the problems are not polynomially related (see e.g. [30]). Note that this situation

does not appear for the dichotomy result from Hemaspaandra and Hemaspaandra [25] for MANIPULATION for weighted voters. The intuitive reason for this is that their reductions for the NP-hardness in the case of weighted voters already hold for a constant number of candidates (and all scoring rules except plurality are NP-hard in this case).

8. Conclusion and outlook

In this work, we settled the computational complexity for POSSIBLE WINNER for almost all pure scoring rules. More precisely, the only case that was left open regards the scoring rule defined by the scoring vector $(2, 1, \dots, 1, 0)$, whereas for all other rules except plurality and veto, we obtained NP-completeness results. In a very recent work, Baumeister and Rothe [2] completed the dichotomy by showing the NP-completeness of POSSIBLE WINNER for the case of $(2, 1, \dots, 1, 0)$.

A natural next step of research is to investigate algorithmic approaches that deal with NP-hard problems like approximation algorithms or "efficient" exponential-time algorithms. Here, an interesting approach is to consider the parameterized complexity [15,23,28] and its sequel multivariate algorithmics [29]. There are first considerations for several voting rules [7] as well as fixed-parameter tractability results for POSSIBLE WINNER for k-approval with respect to the combined parameter "number of partial votes" and k [3]. A parameter of general interest is the "number of candidates". In this case, POSSIBLE WINNER is shown to be fixed-parameter tractable for several voting systems using a powerful classification framework based on integer linear programming but still lacks efficient combinatorial fixed-parameter algorithms [7]. Furthermore, multivariate complexity analysis might offer a way to tackle the POSSIBLE WINNER problem for voting systems for which the "normal" winner determination is already NP-hard. For example, there are recent studies for Kemeny, Dodgson, and Young elections that contain parameterized algorithms with respect to several parameters [4–6,32]. It is open whether such results can be achieved for the POSSIBLE WINNER problem.

The POSSIBLE WINNER problem not only generalizes the MANIPULATION problem but also comprises other relevant special cases. For example, very recently, Chevaleyre et al. [10] investigated the computational complexity of the following problem: Given a set of linear votes, an integer s, and a distinguished candidate c, can one add s candidates such that c becomes a winner? There is reasonable hope to achieve more positive algorithmic results for this and other relevant special cases of POSSIBLE WINNER.

A further direction of future research regards the counting version of POSSIBLE WINNER [1]. Here, one wants to find out in how many extensions a distinguished candidate wins. Answering this question allows to compare two candidates that are possible winners.

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