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Potential Theory on Hilbert Space

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1. INTRODUCTION

The Laplacian of a real-valued function u whose domain is an open subset of a real Hilbert space H is defined as the trace of the second Frechet derivative of u when the latter exists and is a trace class operator. This definition coincides with the usual one when H is finite-dimensional. We shall consider the Dirichlet problem for the equation $\Delta u = f$ when H is separable and infinite-dimensional and Δu denotes the Laplacian of u. When the Dirichlet problem is suitably formulated the existence and uniqueness of generalized solutions is an immediate consequence of the theory of Markov processes. It is our object to investigate regularity properties of the solutions.

The dissimilarities with the finite-dimensional case originate in two ways. First of all the open set in which it is appropriate to seek a solution to the equation $\Delta u = f$ is not actually a subset of H but rather a subset of a topological space which contains H as a dense subset. The reason for this is discussed in the following paragraphs. The second dissimilarity lies in the fact that even if the second Frechet derivative of the generalized solution u at the point x, $D^2u(x)$, exists as a bounded operator on H it may not be of trace class and one may ask whether or not it is in fact of trace class, or of Hilbert-Schmidt type, or compact, etc. This type of question does not arise when H is finite-dimensional and represents, therefore, a new kind of regularity problem. We shall show that even if a potential u is harmonic in a region in a generalized sense then $D^2u(x)$ need not be of trace class but will in general be of Hilbert-Schmidt type. Thus it appears that the generalized Laplacian of a function u involves some summability method applied to the series of eigenvalues of the Hilbert-Schmidt

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operators $D^2u(x)$. In this article we shall give sufficient conditions on a function f which ensure that the second Frechet derivative of the potential of f is, respectively, Hilbert-Schmidt, or trace class. Regularity properties of harmonic functions will be studied in another paper.

In order to motivate our formulation of the Dirichlet problem let us consider for each real number $t > 0$ Gauss measure p_t on H. p_t is a cylinder set measure on H corresponding to the normal distribution with variance parameter t. (See [7], [10], [11], or [20] for expositions of these concepts.) If H is finite-dimensional then the measures p_t determine the transition probabilities for Brownian motion in H in a well-known manner. In case H is infinite-dimensional, a Brownian motion with values in H (in so far as this concept is meaningful) might similarly be defined with $p_t(A - x)$ as transition probability from the point x to the tame set A. But for all $t > 0$ the outer p_t measure of the ball of radius r is zero so that a Brownian motion with values in H would have the property that a particle starting at the origin instantly leaves the ball of radius r with probability one. In particular the sample paths are not continuous, nor right-continuous, when the norm topology is taken on H. In view of the intimate and well-known connection between Brownian motion in E_n and potential theory on E_n the preceding heuristic considerations indicate that the Dirichlet problem for Laplace's equation in a region $\Omega \subset H$ is not likely to be reasonable when Ω is chosen to be such a simple set as the unit ball or for that matter any other bounded set if one wishes to use arbitrary bounded strongly continuous boundary data.

On the other hand these same considerations point the way to a reasonable formulation of the Dirichlet problem. As is known there are various topological linear spaces [11] which contain H as a dense subset and on which the set functions p_t can be realized as countably additive measures. Such a space B is the completion of H in a suitable topology $\mathscr{T}$ weaker than the norm topology of H. The measures p_t on B may now be used to construct a Brownian motion with values in B and having continuous sample paths. The characteristic operator of the resulting Markov process is an extension of the previously defined Laplacian to less smooth functions. By utilizing this Markov process in a standard way generalized solutions of the Dirichlet problem for Laplace's equation may be obtained for reasonable $\mathscr{T}$-open subsets of B and $\mathscr{T}$-continuous boundary data. In this connection we mention that a further dissimilarity from the finite dimensional case arises in that the transition probabilities $p_t(A - x)$ are continuous neither in t nor in x ($\mathscr{T}$ topology) for some Borel sets A in B.

Perhaps the best known example of such a triple (H, B, p_t) is that in which B is Wiener space and H is the subset of Wiener space consisting of the absolutely continuous functions on [0, 1] vanishing at zero and having square-integrable first derivative. In this case p_t is simply Wiener measure on B with variance parameter t. We shall use in this article a simple abstraction of this example. H may be any real separable Hilbert space and B will be the completion of H with respect to a measurable norm on H. Although greater generality in the choice of B is possible it does not seem worthwhile pursuing it at this early stage of development of the subject. Thus B will be a Banach space and in fact any real separable Banach space can arise in this fashion.

Although neither the fundamental solution $c_n r^{2-n}$ in E_n nor Lebesgue measure dx makes sense when $n = \infty$ their product makes sense and may simply be defined as the Green's measure $G(A) = \int_0^\infty p_t(A)\, dt$. The potential of a real-valued function f on B is the convolution of G with f. We elaborate here on a feature that distinguishes sharply between finite and infinite-dimensional potential theory. As is known, the potential u of a bounded function f (with bounded support say) on E_n is analytic in a neighborhood of a point if f is zero in a neighborhood of the point. Consequently local smoothness behavior of u in a neighborhood of a point depends only on the local behavior of f near the point. In infinite dimensions u is again analytic near a point x if f is zero in a neighborhood of x (and is bounded and has bounded support). However the rate of decay of the eigenvalues of $D^2u(x)$ is a new and meaningful local property of u in infinite dimensions and it develops that it does not depend only on the local behavior of f at x. Thus if f is Lip 1 on B then $D^2u(x)$ is a trace class operator (Theorem 3). But if f is merely bounded and continuous with bounded support then $D^2u(x)$ need not be of trace class even if f is zero in a neighborhood of x (Theorem 2). However it is of Hilbert-Schmidt type in this case.

An infinite-dimensional analog of Poisson's equation has also been studied over a period of many years by Levy [*14*]-[*17*]. His Laplacian is defined by $\Delta' u = \lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \partial^2 u/\partial x_j^2$ where the differentiations refer to an orthonormal coordinate system. In particular if u is a twice-differentiable function that depends on only finitely many coordinates then $\Delta' u = 0$. There appears to be no relation between Levy's investigations and the work in the present paper.

When the manuscript for this paper was nearly complete we learned of the recent work of Daletzki [*2*] on the existence and uniqueness of solutions of parabolic equations with variable coefficients in infinitely many variables. Although technically disjoint from the main results of the present paper it is closely related in point of view.

There are several motives behind the present work. Firstly, classical potential theory has deserved and received extension to a wide variety of different contexts and abstraction in many directions. Secondly, recent progress has shown analysis over infinite-dimensional (nonlinear) manifolds to be a potentially rich field for investigation. We regard the present work as a step toward the study of elliptic and parabolic partial differential equations over such manifolds. In particular, an extension of Hodge's theorem to suitable infinite-dimensional manifolds seems at the present time to be a reasonable goal.

In order to make this work more accessible to those with a background primarily in functional analysis we have included several remarks containing proofs of facts which are well known in the theory of Markov processes but which are particularly simple to prove in the present case. The results of Sections 2 and 3 are largely probabilistic in nature while Section 4 which contains the main results is entirely functional-analytic.

2. Continuity Properties of the Transition Functions

We begin this section with a brief review of some background material and, at the same time, establish notation. Let B be a real Banach space and B^* its topological dual space. A *tame* set in B (also known as a measurable cylinder set) is a set of the form $C = \{x \in B : (\langle y_1, x\rangle, \ldots, \langle y_n, x\rangle) \in D\}$ where $y_1, \ldots, y_n$ are in B^* and D is a Borel set in R_n. If K is a finite-dimensional subspace of B^* containing $y_1, \ldots, y_n$ then C is said to be *based on* K. The collection of tame sets in B is a ring $\mathscr{R}$ and the collection of tame sets based on a fixed finite-dimensional subspace K of B^* is a σ-ring $\mathscr{S}_K$. A non-negative set function μ on $\mathscr{R}$ is called a *cylinder set measure* on B if $\mu(B) = 1$ and μ is countably additive on $\mathscr{S}_K$ for each finite-dimensional subspace $K \subset B^*$. If H is a real Hilbert space then every tame set is of the form $C = \{x \in H : Px \in D\}$ where P is a finite-dimensional projection in H and D is a Borel set in PH. For $t > 0$ Gauss measure on H with variance parameter t is the cylinder set measure μ_t defined by

$$\mu_t(C) = (2\pi t)^{-k/2} \int_D \exp\left[-\frac{|x|^2}{(2t)}\right] dx, \tag{1}$$

where $C = \{x \in H : Px \in D\}$, k is the dimension of PH and dx is

Lebesque measure in PH. Henceforth H will denote a real separable Hilbert space and $|\cdot|$ its norm. A *measurable norm* on H is a norm $\|x\|$ on H with the property that for every strictly positive number ϵ there is a finite-dimensional projection P_0 such that, for every finite-dimensional projection P orthogonal to P_0, $\mu_1(\{x \in H : \|Px\| > \epsilon\}) < \epsilon$. It is a consequence of the definition of measurable norm that there exists a constant a such that $\|x\| \leqslant a\,|x|$ for all x in H. Denote by B the completion of H with respect to the measurable norm $\|\cdot\|$. Then B is a separable Banach space and, as has been pointed out in [*11*], any real separable Banach space can arise (up to a linear isometry) as the completion of H with respect to a suitable measurable norm. If y is a non zero element of B^* then the restriction of y to H is a nonzero element of H^*. Thus restriction is a (continuous) embedding of B^* into H^* and we shall identify B^* with its image in H^*. Since B^* separates points of H, B^* is dense in H^*. Now μ_t induces a cylinder set measure m_t in B as follows. If $y_1, \ldots, y_n$ are in B^* and D is a Borel set in R_n define

$$\begin{aligned} &m_t(\{x \in B : (\langle y_1, x\rangle, \ldots, \langle y_n, x\rangle) \in D) \\ &= \mu_t(\{x \in H : (\langle y_1, x\rangle, \ldots, \langle y_n, x\rangle) \in D\}). \end{aligned} \tag{2}$$

m_t is well-defined. It is established in [*11*] that m_t is countably additive on the ring of tame sets of B. Hence it has a unique countably additive extension p_t to the σ-ring $\mathscr{S}$ generated by all tame sets in B. $\mathscr{S}$ is exactly the Borel field of B.

The triple (H, B, i) where $i : H \to B$ is the natural injection is called an abstract Wiener space. The measure p_t is Wiener measure on B with variance parameter t.

In concluding this review of background material we define for x in B and for a Borel subset A of B

$$p_t(x, A) = p_t(A - x) \qquad \text{when} \qquad t > 0.$$

PROPOSITION 1. *For positive s and t and for x and y in B $p_s(x, \cdot)$ and $p_t(y, \cdot)$ are equivalent measures if and only if $s = t$ and $x - y$ is in H. Otherwise they are mutually singular.*

Proof. By translating by y we may assume $y = 0$. The functionals in B^* are Gaussianly distributed with respect to the measures $p_t(x, \cdot)$ and $p_t(0, \cdot)$ and determine the σ-ring $\mathscr{S}$. According to a theorem of Feldman [*6*], these measures are either equivalent or singular and a necessary condition for equivalence is that the L^2 norms defined by these measures on the space of functions spanned by B^* and the constant functions be equivalent. If z is in B^* and a is a constant then

$$\int (z(u) + a)^2 p_s(x, du) = s \mid z \mid^2 + (a + z(x))^2$$

and

$$\int (z(u) + a)^2 p_t(du) = t \mid z \mid^2 + a^2,$$

where $\mid z \mid$ denotes the H^* norm of z. The first L^2 norm is dominated by a multiple of the second L^2 norm if and only if x is in H. Thus if $p_s(x, du)$ and $p_t(du)$ are equivalent then x is in H. But for x in H, $p_s(x, du)$ and $p_s(du)$ are equivalent by [*22*, Theorem 3]. Hence the equivalence of $p_s(x, \cdot)$ and $p_t(\cdot)$ implies the equivalence of $p_s(\cdot)$ and $p_t(\cdot)$ which, by [*22*, Theorem 3], implies $s = t$. The converse is also implied by [*22*, Theorem 3].

PROPOSITION 2. *If x_0 in in B and $t > 0$ then there exists a closed set A such that $p_t(x, A)$ is discontinuous in the B topology as a function of x at $x = x_0$.*

Proof. Choose y in B such that $x_0 - y$ is not in H. Then there exists a Borel set A_1 such that $p_t(x_0, A_1) \neq 0$ and $p_t(y, A_1) = 0$ by Proposition 1. Since $p_t(x_0, \cdot)$ is regular there exists a closed set $A \subset A_1$ such that $p_t(x_0, A) \neq 0$. Of course $p_t(y, A) = 0$. Moreover, $p_t(y + h, A) = 0$ for all h in H. Since $y + H$ is dense in B there exists a sequence x_n in B such that $x_n \to x_0$ in B norm and $p_t(x_n, A) = 0$. Thus $\lim_{n\to\infty} p_t(x_n, A) \neq p_t(x_0, A)$.

We note here the equation

$$p_{ts}(E) = p_t(s^{-1/2}E) \tag{3}$$

valid for any Borel set E and strictly positive real numbers t and s. Such an equation is valid for the Gauss measure μ_t and tame sets in H by (1) and therefore also for the measures m_t and tame sets in B by (2). Thus (3) holds for tame sets E and therefore for all sets in the monotone class generated by the tame sets, i.e., for all Borel sets.

PROPOSITION 3. *For any x_0 in B and $t_0 > 0$ there is a closed set A such that $p_t(x_0, A)$ is a discontinuous function of t at $t = t_0$.*

Proof. We construct an example of A. We may clearly take $x_0 = 0$. Let $y_3, y_4, y_5, \ldots$ be an orthonormal basis of H^* with $y_j \in B^*$, $j = 3, 4, 5, \ldots$. Let b_j be an increasing sequence of positive numbers.

We shall later choose them to go to infinity in a suitable way. Let $A_n = \{x \in B : | y_j(x) | \leqslant b_j\ j = 3, 4,..., n\}$ for $n = 3, 4,...$. Then the sets A_n decrease and if $A = \bigcap_{n=3}^{\infty} A_n$ we shall show that $p_t(A)$ is discontinuous at $t = t_0$ for a suitable sequence b_n . The functions y_j are independent with respect to p_t and normally distributed with mean zero and variance t. Hence

$$\begin{aligned} p_t(A) &= \lim_{n\to\infty} p_t(A_n) \\ &= \lim_{n\to\infty} \prod_{j=3}^{n} (2\pi t)^{-1/2} \int_{-b_j}^{b_j} \exp\left[-\frac{s^2}{(2t)}\right] ds \\ &= \exp\left[\sum_{j=3}^{\infty} \log \left\{2(2\pi t)^{-1/2} \int_0^{b_j} \exp\left[-\frac{s}{(2t)}\right] ds\right\}\right]. \end{aligned} \tag{4}$$

Thus $p_t(A) > 0$ if and only if the infinite sum in the last line converges.

Using the inequalities

$$2((x^2 + 4)^{1/2} + x)^{-1/2} < e^{x^2/2} \int_x^{\infty} e^{-t^2/2}\, dt < x^{-1}$$

and estimating $\log (1 - a)$ by $- a$ for small a it follows that $p_t(A) > 0$ if and only if the sum

$$\sum_{j=3}^{\infty} b_j^{-1} \exp\left[-\frac{b_j^2}{(2t)}\right]$$

converges. Now take $b_j^2 = 2t_0 \log [j(\log j)^{3/4}]$. Then the sum converges if and only if

$$\sum_{j=3}^{\infty} (2t_0 \log \{ j (\log j)^{3/4}\})^{-1/2} [j (\log j)^{3/4}]^{-t_0/t}$$

converges. This is easily seen to converge if and only if $t \leqslant t_0$ by using the inequality $\sum_j j^{-1}(\log j)^{-P} < \infty$ when $P > 1$. Thus $p_t(A) > 0$ when $t \leqslant t_0$ and $p_t(A) = 0$ when $t > t_0$. Hence $p_t(A)$ is discontinuous at $t = t_0$.

Remark 2.1. The example used in the preceding proposition may also be used to settle a point left open in [9]. If $\|\cdot\|_n$ is an increasing sequence of measurable seminorms on H such that the corresponding sequence of measurable functions $\|\cdot\|_n^{\sim}$ converges in probability with respect to the normal distribution to a function h with essential lower bound zero then $\|\cdot\|_n$ converges on H to a measurable seminorm by Theorem 4 of [9]. We shall show the indispensability of the assumption that the essential lower bound of h is zero. Indeed, resuming the notation of Proposition 3, and choosing $t_0 = 1$, we let $\| x \|_n = \sup \{| y_j(x) | b_j^{-1} : j = 3, 4, ..., n\}$ for x in H. Then $\| x \|_n$ is an increasing sequence of measurable (in fact tame) semi-norms on H. In view of the choice of the numbers b_j the limit $\| x \|_0 = \lim_{n\to\infty} \| x \|_n$ exists for each x in H. If $\| x \|_n^{\sim}$ denotes the corresponding random variable defined with respect to the normal distribution with variance parameter one [e.g., we may and shall take

$$\| x \|_n^{\sim} = \sup \{| y_j(x) | b_j^{-1} : j = 3, ..., n\}$$

defined for x in B], then the $\| \cdot \|_n^{\sim}$ form an increasing sequence. It is easily seen that the sequence converges almost everywhere (a.e.) $[p_1]$ if and only if for every strictly positive number ϵ there is a number N such that $p_1(\{x : \| x \|_n^{\sim} \leqslant N\}) > 1 - \epsilon$ uniformly in n. Putting $N = t^{-1/2}$, this condition may be written $\inf_n p_t(\{x : \| x \|_n^{\sim} \leqslant 1\}) \to 1$ as $t \to 0$ by (3). However, since $\{x : \| x \|_n^{\sim} \leqslant 1\} = A_n$ and $A_n \downarrow A$, this condition may be rewritten $p_t(A) \to 1$ as $t \to 0$. That $p_t(A)$ does indeed approach one as $t \to 0$ follows from the last expression for $p_t(A)$ in Eq. (4) because the infinite sum in the exponent converges for $t \leqslant 1$, as we have shown, and each term in the sum increases monotonically to zero as t decreases to zero. Thus $h \equiv \lim_{n\to\infty} \| x \|_n^{\sim}$ exists as a measurable function with respect to the normal distribution. However,

$$\begin{aligned} p_1(h < \tfrac{1}{2}) &\leqslant \lim_{n\to\infty} p_1(\| x \|_n^{\sim} \leqslant \tfrac{1}{2}) \\ &= \lim_{n\to\infty} p_4(\| x \|_n^{\sim} \leqslant 1) \\ &= p_4(A) \\ &= 0. \end{aligned}$$

Thus only the hypothesis that essential inf $h = 0$ fails in the present example. And indeed $\| x \|_0$ is not a measurable seminorm. For if it were then denoting by $x_3, x_4, ...$ the orthonormal basis of H dual to $y_3, y_4, ...$ and by P_n the projection onto span $(x_3, ..., x_n)$ the sequence $\| P_n x \|_0^{\sim}$ would converge in probability with respect to the measure p_1 by [9, Corollary 5.3] and by [9, Theorem 1] the limit must have essential inf equal to zero. But $\| P_n x \|_0 = \| x \|_n$ so that the limit function in question is h. Since the essential lower bound of h is not zero $\| \cdot \|_0$ is not a measurable seminorm.

DEFINITION 1. A Borel set $A \subset B$ is called an *absolute null set* if $p_t(x, A) = 0$ for all $t > 0$ and all x in B.

PROPOSITION 4. *If A is a closed (open) set in B then for $t > 0$ $p_t(x, A)$ is upper (lower) semicontinuous in x in the topology of B. If A is a Borel set and if the boundary of A (∂A) is an absolute null set; then, for $t > 0$, $p_t(x, A)$ is continuous in x in the topology of B.*

Proof. If f is a bounded continuous real-valued function on B then the function $(p_t f)(x) = \int f(y)\, p_t(x, dy) = \int f(z + x)\, p_t(dz)$ is also bounded and continuous. Thus if $f_n(x) = \exp[-n\delta(x, A)]$ where $\delta(x, A)$ is the distance from x to A in the B norm then for closed A, $p_t(x, A) = \lim_{n\to\infty} (p_t f_n)(x)$ is the limit of a decreasing sequence of continuous functions. Hence $p_t(x, A)$ is upper-semicontinuous. If A is open then $p_t(x, A) = 1 - p_t(x, B - A)$ is lower-semicontinuous. If ∂A is a null set then $p_t(x, A) = p_t(x, \bar{A}) = p_t(x, A^0)$ where $\bar{A}$ and A^0 are the closure and interior of A, respectively. Thus $p_t(x, A)$ is both upper- and lower-semicontinuous and hence is continuous.

Remark 2.2. In view of the natural isomorphism of H^* with H we may and shall identify B^* with a subset of B via the injections $B^* \to H^* \to H \to B$. This leads to a useful product decomposition of Wiener measure p_t as follows. Let K be a finite-dimensional subspace of B^*. Let L be its annihilator in B. If $y_1, y_2, \ldots, y_n$ is an orthonormal basis of K then the equation $Qx = \sum_{j=1}^{n} \langle y_j, x\rangle y_j$ defines a continuous operator on B and since $\langle y_j, x\rangle = (y_j, x)$ for x in H Q is the continuous extension to B of the orthogonal projection of H onto K. Hence Q is a projection and it is clear that its range is K and null space is L. Consequently $B = K \oplus L$. If $K^\perp$ is the orthogonal complement of K in H then $K^\perp \subset L$ and moreover L is the closure in B of $K^\perp$. For if x is in L then there is a sequence x_n in H converging to x. Then $(I - Q)\, x_n$ lies in $K^\perp$ and also converges to x. It is a simple consequence of the definition of a measurable norm that the restriction of a measurable norm to a closed subspace is again a measurable norm. Hence analogous to the measures p_t on B there is a Wiener measure p_t' on the space L. If μ_t' denotes Gauss measure in K then we assert that in the Cartesian product decomposition $B = K \times L$ there holds,

$$p_t = \mu_t' \times p_t',$$

for p_t is characterized by the property that for any y in B^* the function $x \to \langle y, x\rangle$ on B is normally distributed with mean zero and variance $t \mid y \mid^2$ where $\mid \cdot \mid$ is the H norm. p_t' is similarly characterized. But writing $y = u + v$ where u is in K and v is in $K^\perp$ we see that the functions $x \to \langle u, x\rangle$ and $x \to \langle v, x\rangle$ are independent with respect to $\mu_t' \times p_t'$ and have mean zero and variance $t \mid u \mid^2$ and $t \mid v \mid^2$, respectively with respect to this measure. Hence the function $x \to \langle y, x \rangle$ is also normally distributed with respect to $\mu_t' \times p_t'$ with mean zero and variance $t(\mid u \mid^2 + \mid v \mid^2)$ which equals $t \mid y \mid^2$. Thus $\mu_t' \times p_t' = p_t$.

DEFINITION 2. An open set $V \subset B$ has a differentiable boundary if for each point x in ∂V there is a neighborhood U of x in B and a continuously differentiable real-valued function g defined in U such that $g'(x) \neq 0$ and such that $V \cap U = \{y : g(y) > 0\}$.

We note that g' is the derivative of g as a function defined in B, i.e., $dg(y + sz)/ds \mid_{s=0}$ exists uniformly for $\| z \| \leqslant 1$.

PROPOSITION 5. *Let V be an open set in B which is either convex or has a differentiable boundary. Then ∂V is an absolute null set.*

Proof. Since B is a separable metric space so is ∂V and the latter is therefore a Lindelöf space. Hence it suffices to show that every point in ∂V has a neighborhood in ∂V which is an absolute null set for then ∂V can be covered by a countable family of absolute null sets. Using the natural isomorphism of H^* with H we may and shall identify B^* with a subset of H and hence with a subset of B.

First assume V is a convex nonempty open set. Let x be a point in ∂V. Then the translated set $V - x$ is open and since B^* is dense in B there exists a nonzero vector v in $B^* \cap (V - x)$. Then $x + v$ is in V and there exists a strictly positive real number ϵ such that the ball $S \equiv \{y : \| y - (x + v) \| < \epsilon\}$ is contained in V. We may choose $\epsilon < \| v \|/2$. We show that the set U which is the intersection of ∂V with the ball $T \equiv \{y : \| y - x \| < \epsilon\}$ is an absolute null set. Thus for any vector x_0 in B it must be shown that $p_t(x_0 , U) = 0$ for $t > 0$. By translating V if necessary we may assume $x_0 = 0$. Let M be the null space in B of the linear functional v and let K be the line in B spanned by v. Then $B = M \oplus K$ and p_t is a product measure in this Cartesian product decomposition of B. Thus $p_t = p_t' \times \mu_t'$ where p_t' is Wiener measure in M and μ_t' is Gauss measure on the line K. We assert that any line k parallel to K intersects U in at most one point. For if x_1 and x_2 are two distinct points on k lying in U then, since k is parallel to v and intersects T, it also intersects S. Hence there is a third point y on k lying in V. But it is easily seen that x_1 or x_2 lies between the other two points since S and T are disjoint. Since x_1 and x_2 are in ∂V and y is not this is impossible. Thus $k \cap U$ consists of at most one point and therefore $\mu_t'(k \cap U) = 0$. Hence by Fubini's theorem $p_t(U) = 0$.

Now suppose that V has a differentiable boundary. If x is in ∂V and U' is a neighborhood of x such that $V \cap U'$ is given by $g > 0$ where g is a differentiable function defined in U' then letting $v = \operatorname{grad} g(x)$ and $M = \operatorname{kernel} v$ we have, upon identifying H^* with H, $B = M \oplus \operatorname{span} v$. Moreover, for y in M we have $\partial g(\alpha v + y)/\partial\alpha \neq 0$ at $\alpha v + y = x$. Hence by the implicit function theorem there is a neighborhood U'' of x such that for each y in the projection of U'' into M along v the equation $g(\alpha v + y) = 0$ has exactly one solution α. In particular any line k parallel to v intersects $U'' \cap \partial V$ in at most one point. The remainder of the proof is the same as that for convex sets.

COROLLARY 5.1. *The distribution function of any measurable norm with respect to the normal distribution is continuous.*

Proof. $p_1(\{x : \| x \| = \lambda\}) = 0$ by Proposition 5 for any $\lambda > 0$ and any measurable norm.

If f is a bounded measurable complex valued function on B we denote by $p_t f$ the function

$$(p_t f)(x) = \int_B f(u)\, p_t(x, du) = \int_B f(x + u)\, p_t(du)$$

when $t > 0$. We put $p_0 f = f$.

It is clear that if f is bounded and continuous then so is $p_t f$.

PROPOSITION 6. *The operators p_t, $t \geqslant 0$ form a strongly continuous contraction semigroup on the Banach space $\mathfrak{A}$ consisting of bounded uniformly continuous complex valued functions on B.*

Proof. It is clear from the definition of $p_t f$ that if f is in $\mathfrak{A}$ so is $p_t f$ and that $| p_t f |_\infty \leqslant | f |_\infty$. Let E be a tame set in B. Suppose that E is based on the finite dimensional subspace K of B^*. Adopting the notation of Remark 2.2 we may write $B = K \times L$ where L is the annihilator of K. Then E is a product: $E = F \times L$ where F is a Borel set in K. Consequently, for any x in B, $E - x$ depends only on the component of x in $K : E - x = (F - Qx) \times L$. Thus $p_t(E - x) = \mu_t'(F - Qx)$ from which it follows that $p_t(E - x)$ is continuous in x in the strong B topology and, moreover,

$$\int_B p_t(E - x)\, p_s(dx) = \int_K \mu_t'(F - y)\, \mu_s'(dy) = \mu_{t+s}'(F) = p_{t+s}(E)$$

since, as is well-known, the measures μ_t' in K form a semigroup. Now the collection $\mathscr{F}$ of Borel sets E for which $p_t(E - x)$ is a Borel-measurable function on B is clearly closed under monotone limits and complements and as has been noted includes all tame sets. Therefore $\mathscr{F}$ consists of all Borel sets in B. The same argument applies to the collection of all Borel sets for which the equation

$$\int_B p_t(E - x)\, p_s(dx) = p_{t+s}(E)$$

holds since both sides are countably additive set functions of E. Hence the last equation holds for all Borel sets E in B. As in well known this implies that the operators p_t form a semi group acting in the space of all bounded measurable functions on B as well as in $\mathfrak{A}$.

In order to establish strong continuity we note that from (3) it follows that for any real number $a > 0$,

$$p_t(\{x : \| x \| \geqslant a\}) = p_1(\{x : \| x \| \geqslant at^{-1/2}\}) = o(1) \qquad \text{as} \qquad t \to 0.$$

Hence if f is in $\mathfrak{A}$ and if $\| y \| < \delta$ implies $| f(x + y) - f(x) | < \epsilon$ for all x then

$$\begin{aligned} | (p_t f)(x) - f(x) | &= \left| \int_B (f(x+y) - f(x))\, p_t(dy) \right| \\ &\leqslant \int_{\|y\| < \delta} \epsilon p_t(dy) + \int_{\|y\| \geqslant \delta} | f(x+y) - f(x) |\, p_t(dy) \\ &\leqslant \epsilon + 2 \, | f |_\infty \, p_t(\{y : \| y \| \geqslant \delta\}). \end{aligned}$$

Thus $| p_t f - f |_\infty \leqslant 2\epsilon$ for all sufficiently small t and strong continuity is established.

Remark 2.3. It follows from [*18*, Theorem 2] that if the measures p_t satisfy

$$p_t(\{x : \| x \| \geqslant \delta\}) = o(t), \qquad t \downarrow 0, \tag{5}$$

for all real numbers $\delta > 0$, then there exists a Markov process with state-space B and transition functions $p_t(x, A)$ and having continuous sample paths and which starts at the origin of B. In one of our main results (Theorem 3) and elsewhere we shall have to assume that the B norm is in $L^2(p_1)$. This implies (5) since

$$\lambda^2 p_1(\| x \| \geqslant \lambda) \leqslant \int_{\|x\| \geqslant \lambda} \| x \|^2 \, p_1(dx) = o(1) \qquad \text{as} \qquad \lambda \to \infty,$$

so that by (3)

$$p_t(\|x\| \geqslant \delta) = p_1(\|x\| \geqslant \delta t^{-1/2}) = o((\delta t^{-1/2})^{-2}) = o(t) \qquad \text{as} \qquad t \downarrow 0.$$

All measurable norms known to us (see [*9*]) are in $L^2(p_1)$. In any case, Varadhan has pointed out to us that methods established in [*23*] can be modified to prove that (5) holds for any measurable norm. In the remainder of this paper we shall therefore use the validity of (5) for the measurable norm $\|\cdot\|$. Thus the measures $p_t(x, \cdot)$ are the transition functions for a Markov process with continuous sample paths starting at the origin in B. In view of the translation invariance of the $p_t(x, A)$, the process may be described as follows. Let Ω be the space of continuous functions ω on $[0, \infty)$ with values in B and such that $\omega(0) = 0$. Then there is a unique probability measure $\mathscr{P}$ on the σ-field in Ω generated by the functions $\omega \to \omega(t)$ for $t > 0$ with the property that if $0 = t_0 < t_1 < \cdots < t_n$ then $\omega(t_j) - \omega(t_{j-1})$, $j = 1,\dots, n$ are independent and the jth one is distributed in B according to $p_{t_j - t_{j-1}}(\cdot)$. We denote expectation with respect to $\mathscr{P}$ by $E[\ ]$ and put $W(t)(\omega) = \omega(t)$. Thus $W(t)$ may be called a Wiener process with state space B. The corresponding Brownian motion starting at x is $x + W(t)$. Thus $\mathscr{P}(x + W(t) \in A) = p_t(x, A)$ when A is a Borel set in B. We use the notation established in this remark in the remainder of the paper.

Let V be an open set in B. We denote by $\tau_x(\omega)$ the first exit time from V starting at x, i.e.,

$$\tau_x(\omega) = \inf\{t \geqslant 0 : x + W(t)(\omega) \notin V\}$$

and by $\tau'_x(\omega)$ the first exit time from V after time $+0$, i.e.,

$$\tau'_x(\omega) = \inf\{t > 0 : x + W(t)(\omega) \notin V\}$$

Then τ and τ' are measurable functions from $B \times \Omega$ to $[0, \infty]$ (cf. [*12*]). A point x in the boundary of V is called a ***regular point*** if $\mathscr{P}(\tau'_x = 0) = 1$.

Many of the standard theorems concerning the relation of Markov processes to potential theory are derived under the assumption that the operators p_t take the bounded measurable functions into continuous functions. This assumption does not hold for the process under consideration in view of Proposition 2. Nevertheless under mild regularity assumptions on the boundary of V it is easy to show (cf. Corollary 1.1 below) that generalized solutions of the Dirichlet problem $\Delta u = 0$ for V exist where Δ denotes here the characteristic operator of the process. The regularity assumption on ∂V that will be made is the infinite-dimensional analog of Poincaré's cone condition.

DEFINITION 3. A *cone* in B *with vertex* x is the closed convex hull of the set consisting of x and a ball $B_r(y) = \{z : \| z - y \| \leqslant r\}$ of positive radius r, not containing x. An open set $V \subset B$ is called *strongly regular* at a point x in ∂V if there is a cone K in B with vertex x such that $V \cap K$ is empty. V is *strongly regular* if it is strongly regular at each of its boundary points.

PROPOSITION 7. *Let x be a point in the boundary of an open set $V \subset B$. If V is strongly regular at x then x is a regular point of V. Moreover for any number $\delta > 0$, $\mathscr{P}(x + W(t) \in \bar{V},\ 0 < t \leqslant \delta) = 0$.*

Proof. The proof of the first assertion is identical to one of the standard finite-dimensional proofs given some background facts. We include it here for completeness. We have $\mathscr{P}(\tau_x' > 0)$ is equal to zero or one by the zero-one law. Since $\mathscr{P}(\tau_x' > 0) = \lim_{t \downarrow 0} \mathscr{P}(\tau_x' > t)$, it suffices to show that $\lim_{t \downarrow 0} \mathscr{P}(\tau_x' > t) < 1$. But for $t > 0$,

$$\{\omega : \tau_x'(\omega) > t\} \subset \{\omega : x + W(t) \in V\}$$

Hence

$$\begin{aligned} \mathscr{P}(\tau_x' > t) &\leqslant \mathscr{P}(x + W(t) \in V) \\ &\leqslant \mathscr{P}(x + W(t) \notin K) \\ &= 1 - \mathscr{P}(x + W(t) \in K) \\ &= 1 - p_t(x, K). \end{aligned}$$

Therefore it suffices to prove that $\limsup_{t \to 0} p_t(x, K) > 0$. But by Eq. (3) $p_t(x, K) = p_1((K - x)\, t^{-1/2}) \geqslant p_1(K - x)$ for $t \leqslant 1$. Since $K - x$ has a nonempty interior $p_1(K - x) > 0$ by Corollary 4 of [*11*]. This establishes the first part of the proposition.

In order to prove the second assertion we suppose that K is the convex hull of $\{y : \| y - x_0 \| \leqslant r\} \cup \{x\}$. Let $0 < r_1 < r$ and let K_1 be the convex hull of $\{y : \| y - x_0 \| \leqslant r_1\} \cup \{x\}$. Let V_1 be the complement of K_1. Then x is in ∂V_1 and is the vertex of the cone K_1 which is itself disjoint from V_1. Hence by the first part of this proposition almost every path starting at x goes outside of V_1, i.e., enters K_1 at some time strictly between 0 and δ. But K_1 is contained in the complement of $\bar{V}$ except for the point x. Hence the second assertion of the proposition will follow once it is shown that almost no path returns to x up to time δ. This follows however from the corresponding fact in two dimensions. For if y_1 and y_2 are in B^* and are orthonormal as elements of H^* then the map

$$t, \omega \to (\langle y_1, x + W(t)(\omega)\rangle, \langle y_2, x + W(t)(\omega)\rangle)$$

is a two-dimensional Brownian motion which as is well known (see, e.g., [*4*]) returns to $(\langle y_1, x\rangle, \langle y_2, x\rangle)$ with probability zero.

Remark 2.4. Instead of using a cone K with a nonempty interior in the first part of Proposition 7 one can also use certain compact cones. For let $\|\cdot\|_1$ be a measurable norm on H which dominates the B norm. Then the closure, C, in B of $\{x \in H : \| x \|_1 \leqslant r\}$ is a set of positive p_1 measure by Corollary 2 of [*11*]. If K' is the closed convex hull of $C \cup \{y\}$ where y is a point not in C then any translate of K' can be used in place of K in Proposition 7 to establish regularity of a boundary point. There is no change in the proof. Now if $\| \|_1$ is sufficiently strong then C and hence K' will be compact in B and will therefore have an empty interior. Such strong measurable norms $\|\cdot\|_1$ always exist by Lemma 2 of [*11*]. We shall not use such cones in this paper since our methods will require the cone to have a nonempty interior.

THEOREM 1. *Let V be an open set in B which is strongly regular. Suppose also that ∂V is an absolute null set. If f is a bounded continuous function on B then so is the function*

$$(q_t f)(x) = E[f(x + W(t))\chi_{\tau_x > t}], \qquad t \geqslant 0.$$

The operators q_t form a semigroup acting in the space of bounded measurable functions on B. The semigroup leaves invariant the Banach space $\mathfrak{A}_0$ of bounded continuous functions on B which vanish on the complement of V. If f is in $\mathfrak{A}_0$ and is in addition uniformly continuous then $q_t f \to f$ uniformly as $t \to 0$.

LEMMA 1.1. *If V is a strongly regular open set in B then for each point x_0 in B there is a null set $N_{x_0} \subset \Omega$ such that $\tau_x(\omega)$ is continuous at x_0 for each ω not in N_{x_0}.*

Proof. Even with no restrictions on the open set V $\tau_x(\omega)$ is lower semicontinuous for every continuous path ω since if $\tau_{x_0}(\omega) = t > 0$ then for a given number $\epsilon > 0$ with $\epsilon < t$ the path $\{x_0 + W(s, \omega) : 0 \leqslant s \leqslant t - \epsilon\}$ is a compact set contained in V

and is therefore bounded away from the complement of V by δ, say. Hence if $\| x - x_0 \| < \delta$ then the set $\{x + W(s, \omega) : 0 \leqslant s \leqslant t - \epsilon\}$ is contained in V. Thus $\tau_x(\omega) \geqslant t - \epsilon$ when $\| x - x_0 \| < \delta$ so that $\liminf_{x \to x_0} \tau_x(\omega) \geqslant \tau_{x_0}(\omega)$. If $t = 0$ then the last equation holds trivially and if $t = \infty$ then a similar argument shows that $\liminf_{x \to x_0} \tau_x(\omega) \geqslant M$ for all M. Let

$$N_{x_0} = \bigcup \{\omega \in \Omega' : x_0 + W(s + \tau_{x_0}(\omega), \omega) \in \bar{V}, 0 \leqslant s \leqslant \epsilon\}$$

where the union is taken over all rational numbers $\epsilon > 0$ and $\Omega' = \{\omega : \tau_{x_0}(\omega) < \infty\}$. If ω is not in N_{x_0} then either $\tau_{x_0}(\omega) = \infty$, in which case $\limsup_{x \to x_0} \tau_x(\omega) \leqslant \tau_{x_0}(\omega)$ so that $\tau_x(\omega)$ is continuous at x_0 (in the usual sense for $[0, \infty]$ valued functions), or else for every positive rational number ϵ there is a number s with $0 \leqslant s \leqslant \epsilon$ such that the point $x_0 + W(s + \tau_{x_0}(\omega), \omega) \notin \bar{V}$. If δ is the distance of this point from $\bar{V}$ then $\| x - x_0 \| < \delta$ implies that $x + W(s + \tau_{x_0}(\omega), \omega) \notin \bar{V}$. Hence $\tau_x(\omega) \leqslant \tau_{x_0}(\omega) + \epsilon$ whenever $\| x - x_0 \| < \delta$. Therefore $\limsup_{x \to x_0} \tau_x(\omega) \leqslant \tau_{x_0}(\omega)$ and $\tau_x(\omega)$ is continuous at x_0. Thus it remains to show that $\mathscr{P}(N_{x_0}) = 0$.

Now if $\mathscr{P}(\Omega') = 0$ then $\mathscr{P}(N_{x_0}) = 0$. So assume $\mathscr{P}(\Omega') > 0$. Let $\mathscr{P}'(A) = \mathscr{P}(A)/\mathscr{P}(\Omega')$ when A is a measurable subset of Ω'. Then $(\Omega', \mathscr{P}')$ is a probability space and we assert that the random variable $Y(\omega) = x_0 + W(\tau_{x_0}(\omega), \omega)$ over Ω' is independent of the process $W'(s, \omega) = W(s + \tau_{x_0}(\omega), \omega) - W(\tau_{x_0}(\omega), \omega)$ which is also defined over Ω'. That is, for any finite set $s_1, \ldots, s_n$ Y is independent of $(W'(s_1), \ldots, W'(s_n))$ which takes values in $B \times B \times \cdots \times B$ (n factors). Moreover $W'(s)$ is a Wiener process in B. The proof of these assertions is exactly the same as the proof of Theorem 2.5 of [*12*] which deals with the case in which H (and consequently B) is finite-dimensional. We need only remark that the set A used in Section 2.9 of the proof of Theorem 2.5 should be taken to be a set of the form

$$\{\omega \in \Omega : \beta_{ij} \leqslant \langle y_j, \omega(\tau_i) \rangle \leqslant \gamma_{ij}, 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n, \tau_i > 0\}$$

where n is arbitrary and $y_1, \ldots, y_n$ are in B^* and are orthonormal in the H^* inner product.

Now for every positive rational number ϵ

$$\begin{aligned} &\mathscr{P}(\Omega', x_0 + W(s + \tau_{x_0}(\omega), \omega) \in \bar{V}, 0 \leqslant s \leqslant \epsilon) \\ &\quad = \mathscr{P}(\Omega')\, \mathscr{P}'(Y + W'(s) \in \bar{V}, 0 \leqslant s \leqslant \epsilon). \end{aligned}$$

Let $H(x_0, A) = \mathscr{P}'(Y \in A)$ be defined for the Borel sets of ∂V. Y

clearly takes its values in ∂V. Then standard arguments involving the separability of the process W' and independence of it from Y yield

$$\mathscr{P}'(Y + W'(s) \in \bar{V}, 0 \leqslant s \leqslant \epsilon)$$
$$= \int_{\partial V} \mathscr{P}'(u + W'(s) \in \bar{V}, 0 \leqslant s \leqslant \epsilon)\, H(x_0, du).$$

However by Proposition 7 and the hypotheses of this Lemma,

$$\mathscr{P}'(u + W'(s) \in \bar{V}, 0 \leqslant s \leqslant \epsilon) = 0 \qquad \text{for every} \quad u \quad \text{in} \quad \partial V.$$

Hence $\{\omega \in \Omega' : x_0 + W(s + \tau_{x_0}(\omega), \omega) \in \bar{V}, 0 \leqslant s \leqslant \epsilon\}$ is a set of $\mathscr{P}$ measure zero and since N_{x_0} is a countable union of such sets $\mathscr{P}(N_{x_0}) = 0$. This concludes the proof of Lemma 1.1.

Proof of Theorem. Suppose that f is bounded and continuous on B and x_0 is a point of B. For $t > 0$

$$\mathscr{P}(\tau_{x_0}(\omega) = t) \leqslant \mathscr{P}(x_0 + W(t) \in \partial V) = p_t(x_0, \partial V) = 0.$$

Thus if N_{x_0} is the set given in Lemma 1.1 then $\mathscr{P}(N_{x_0} \cup \{\tau_{x_0} = t\}) = 0$. On the complement of $N_{x_0} \cup \{\tau_{x_0} = t\}$ the function $\chi_{\tau_x > t}(\omega)$ is easily seen to be continuous in x at $x = x_0$. Hence if $x_n \to x_0$ then

$$(q_t f)(x_n) = E[f(x_n + W(t))\, \chi_{\tau_{x_n} > t}] \to (q_t f)(x_0) \quad \text{as} \qquad n \to \infty$$

by the dominated convergence theorem, showing $q_t f$ is continuous on B.

It is well known that the q_t, $t \geqslant 0$, form a semigroup. If f is a bounded continuous function on B then $q_t f$ is zero on the complement of V for $t \geqslant 0$ since $\tau_x = 0$ for x not in V. Finally if f is bounded, uniformly continuous and zero on the complement of V then $q_t f \to f$ uniformly as $t \downarrow 0$ since

$$|(q_t f)(x) - f(x)| \leqslant |E[\{f(x + W(t)) - f(x)\}\, \chi_{\tau_x > t}]|$$
$$+ |f(x)\, E[\chi_{\tau_x > t} - 1]|$$
$$\leqslant E[|f(x + W(t)) - f(x)|] + |f(x)|\, E[\chi_{\tau_x \leqslant t}]$$

The first term goes to zero uniformly as $t \downarrow 0$ by Proposition 6. Given a number $\epsilon > 0$ choose a number $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $\|x - y\| < \delta$. Thus if the distance from x to ∂V is less than δ then $|f(x)| < \epsilon$. If x is in V and the distance from x to ∂V is greater or equal to δ then writing $S_\delta(x) = \{z : \| z - x \| < \delta\}$ we have $\mathscr{P}(\tau_x \leqslant t) \leqslant \mathscr{P}(x + W(s) \notin S_\delta(x)$ for some s in $[0, t]) = \mathscr{P}(W(s) \notin S_\delta(0)$ for some s in $[0, t]) \to 0$ uniformly in x as $t \downarrow 0$. Thus in either case $| f(x) | \, E[\chi_{\tau_x \leqslant t}] < \epsilon$ for all sufficiently small t and all x in V. This concludes the proof of the theorem.

The following is a corollary of Lemma 1.1.

Corollary 1.1. *Let V be an open set in B. Let a be in ∂V and suppose V is strongly regular at a. Let φ be a bounded measurable function on ∂V. If φ is continuous at a and if*

$$u(x) = E[\varphi(x + W(\tau_x)) \, \chi_{\tau_x < \infty}], \qquad x \in V,$$

then

$$\lim_{x \to a, x \in V} u(x) = \varphi(a).$$

Proof. Let K be a cone with vertex a which is disjoint from V. Let T_x be the first exit time from the complement K' of K. Since $V \subset K'$ there holds $\tau_x(\omega) \leqslant T_x(\omega)$ for all ω. It is easily seen that K' is strongly regular. Since $T_a(\omega) = 0$ for all ω it is a consequence of Lemma 1.1 that $\lim_{x \to a} T_x(\omega) = 0$ with probability one. Hence $\lim_{x \to a} \tau_x(\omega) = 0$ with probability one also. Thus if x_n is a sequence in V with $x_n \to a$ then $x_n + W(\tau_{x_n}) \to a$ with probability one and $\chi_{\tau x_n < \infty} \to 1$ with probability one and the corollary now follows from the dominated convergence theorem.

Corollary 1.2. *If V is strongly regular and φ is a bounded-continuous function on ∂V then the function $u(x)$ given in the preceding corollary is continuous on V and equals φ on ∂V.*

Proof. The corollary follows immediately from the a.e. continuity of τ_x.

Remark 2.5. Theorem 1 shows that, when V is strongly regular, the semigroup q_t acts in the space $C(\bar{V})$ of bounded continuous functions on V vanishing on ∂V. For a function f in $C(\bar{V})$ it is easily seen that $q_t f \to f$ pointwise as $t \downarrow 0$. However it need not necessarily converge uniformly and q_t is therefore not strongly continuous on $C(\bar{V})$. The largest subspace, $\mathscr{M}$, of $C(\bar{V})$ on which q_t is strongly continuous includes the uniformly continuous functions of $C(\bar{V})$ along with some others (e.g., functions which are uniformly continuous with respect to a strictly stronger measurable norm than the B

norm.) Although $\mathcal{M}$ is invariant under the semigroup q_t it seems doubtful that the space of uniformly continuous functions is invariant under the q_t without further restrictions on V.

Remark 2.6. Corollary 1.2 shows that the usual stochastic solution of the Dirichlet problem actually assumes its correct boundary values and is continuous in V.

3. The Generalized Laplacian and B-Smooth Functions

Consider a real valued function u defined in an open set V of the abstract Wiener space B. If x is a point of V there are three relevant senses in which u may possess a Frechet derivative at x. As a function defined in B its Frechet derivative at x is the element y of B^* determined by $|u(x+z) - u(x) - \langle y, z\rangle| = o(\|z\|)$ for small z in B. If, however, one restricts u to the coset $x + H$ of H then one obtains a function $v(h) = u(x+h)$ defined in a neighborhood of the origin in H. The Frechet derivative of v at 0 is an element y' of H^* such that $|u(x+h) - u(x) - \langle y', h\rangle| = o(|h|)$ for small h in H. We shall refer to y' as the derivative of u at x *in H directions* and we shall say that u is H differentiable at x if such a vector y' exists. Clearly, since the H norm is stronger than the B norm, the derivative of u at x in H directions will exist when the derivative of u at x in B directions exists and they will then be equal. Finally, by virtue of the existence of continuous injections $B^* \to H^* \to H \to B$, we may also regard B^* as a subset of B, in which case the restriction of u to $x + B^*$ defines a function on a neighborhood of the origin in B^*. Its Frechet derivative is an element of B^{**}. Differentiability in this sense is weaker than in the other two senses. We shall be primarily interested in differentiability in H and B directions. Similar considerations apply to higher derivatives. The second H-derivative of u at x will be denoted by $D^2u(x)$. $D^2u(x)$ will always denote a bounded operator from H into H.

Suppose that u is B differentiable in a neighborhood V of a point x. Its derivative u' is then a function from V to B^*. Suppose further that u' is B differentiable at x. Its derivative $u''(x)$ is a bounded linear operator from B to B^*. Since $H \subset B$ and $B^* \subset H^*$, the restriction of $u''(x)$ to H may be regarded, upon identifying H^* with H, as a bounded linear operator from H into H and it is clear that the operator so obtained is exactly $D^2u(x)$. By Corollary 5 of [*11*] the symmetric part of $D^2u(x)$ is therefore a trace class operator on H and since $D^2u(x)$ is symmetric it is trace class. We shall use this fact in the next proposition.

Not only are there several senses of differentiability for a function on B but also there are several senses of continuity of which we shall need two. If x is in B but not in H, put $| x | = + \infty$. Then in addition to the B topology on B, whose basic neighborhoods of x are of the form $\{z \in B : \| z - x \| < \epsilon\}$, there is also the H topology whose basic open neighborhoods of a point x in B are of the form $\{z \in B : | z - x | < \epsilon\}$. Clearly, the last neighborhood is contained in the coset $x + H$ of H. A function which is continuous with respect to one of these topologies will be called B-continuous or H-continuous, respectively. Of course a function which is H-differentiable at x is H-continuous at x but not necessarily B-continuous at x.

Denote by $L(B, B^*)$ the space of all bounded linear operators from B to B^*. Define the weak operator topology on $L(B, B^*)$ to be the weakest topology which for every y and z in B makes the function $A \to \langle Ay, z\rangle$ continuous. Let L_s be the subspace of $L(B, B^*)$ consisting of those operators A whose restriction to H is symmetric. L_s is a closed subspace of $L(B, B^*)$ in operator norm since the $L(B, B^*)$ norm of A dominates the $L(H, H)$ norm of $A \mid H$. Since $A \mid H$ is a trace class operator for every operator A in L_s a simple application of the closed graph theorem shows that the trace class norm of $A \mid H$ is dominated by a constant times the $L(B, B^*)$ norm of A.

Proposition 8. *Assume $\| \cdot \|$ is in $L^2(p_1(dx))$. Let u be a bounded measurable function on B. Suppose that in a neighborhood V of a point x u' and u'' exist, u'' is bounded and the map $u'' : V \to L(B, B^*)$ is B continuous at x in the weak operator topology. Then*

$$\lim_{t \downarrow 0} \frac{(p_t u)(x) - u(x)}{t} = \frac{1}{2} \operatorname{trace} [D^2 u(x)]. \tag{6}$$

Moreover, if $V = B$, u is uniformly B continuous on B, and $u'' : B \to L(B, B^*)$ is uniformly B continuous in the weak operator topology, then u is in the domain of the infinitesimal generator of the semigroup p_t and the limit in (6) holds uniformly.

Proof. Let T be a closed ball of radius a with center x and contained in V. The function $u(x + sy)$ is a twice differentiable function of s on the interval $[0, 1]$ when $\| y \| \leqslant a$ with

$$\frac{du(x + sy)}{ds} = \langle u'(x + sy), y\rangle \qquad \text{and} \qquad \frac{d^2u(x + sy)}{d^2s} = \langle u''(x + sy)\, y, y\rangle.$$

Since the second derivative is bounded, the first derivative is absolutely continuous and two integrations by parts yield

$$u(x+y) = u(x) + \langle u'(x), y\rangle + \int_0^1 (1-s)\,\langle u''(x+sy)\,y, y\rangle\,ds \tag{7}$$

for $\| y \| \leqslant a$. Now since u is bounded, say $| u(z) | \leqslant M$, there holds

$$\begin{aligned}\frac{p_t u(x) - u(x)}{t} &= \frac{1}{t}\int_B \{u(x+y) - u(x)\}\,p_t(dy)\\ &= \frac{1}{t}\int_{\|y\|\leqslant a} \{u(x+y) - u(x)\}\,p_t(dy)\\ &\qquad + \frac{1}{t}\int_{\|y\|>a} \{u(x+y) - u(x)\}\,p_t(dy).\end{aligned}$$

The second term is dominated by $2Mt^{-1}p_t(\| y \| > a)$, which goes to zero as $t \downarrow 0$ and uniformly in x when relevant. (See Remark 2.3.) Apply (7) to the integrand in the first term of the last line. Since the measure p_t is even and $\{y : \| y \| \leqslant a\}$ is symmetric while $\langle u'(x), y\rangle$ is an odd function,

$$\int_{\|y\|\leqslant a} \langle u'(x), y\rangle\, p_t(dy) = 0.$$

Hence

$$\begin{aligned}\lim_{t\downarrow 0} \frac{p_t u(x) - u(x)}{t} &= \lim_{t\downarrow 0} \int_0^1 (1-s) \int_{\|y\|\leqslant a} \langle u''(x+sy)\,y, y\rangle\, t^{-1}p_t(dy)\,ds\\ &= \frac{1}{2}\lim_{t\downarrow 0}\int_{\|y\|\leqslant a} \langle u''(x)\,y,y\rangle\, t^{-1}p_t(dy)\\ &\quad + \lim_{t\downarrow 0}\int_0^1 (1-s)\int_{\|y\|\leqslant a} \langle \{u''(x+sy) - u''(x)\}\,y,y\rangle\, t^{-1}p_t(dy)\,ds\\ &= \frac{1}{2}\lim_{t\downarrow 0}\int_{\|y\|\leqslant at^{-1/2}} \langle u''(x)\,y,y\rangle\, p_1(dy)\\ &\quad + \lim_{t\downarrow 0}\int_0^1 (1-s)\int_{\|y\|\leqslant at^{-1/2}} \langle \{u''(x+st^{1/2}y) - u''(x)\}\,y,y\rangle\, p_1(dy)\,ds,\end{aligned}$$

where the last equality is obtained by replacing y by $t^{1/2}y$ and using (3). If $\sup_{z\in V} \| u''(z) \| = N$, then the integrand in the last term is dominated by $2N \| y \|^2$, which is integrable over all of B with respect to p_1. For each y the integrand converges to zero as $t \to 0$, by the weak continuity of u'' at x. Thus the last term converges to zero as $t \to 0$.

Moreover, if $u''(z)$ is uniformly continuous on B into $L(B, B^*)$ with the weak operator topology, then the convergence to zero is uniform in x because, for each y in B and each s, and any countable dense set $\{x_n\}$ in B,

$$\sup_n | \langle \{u''(x_n + st^{1/2}y) - u''(x_n)\} y, y \rangle |$$

$$= \sup_{x \in B} | \langle \{u''(x + st^{1/2}y) - u''(x)\} y, y \rangle |$$

by continuity in x, so that the last sup is a measurable function of y and s and converges to zero for each y and s as $t \downarrow 0$ while remaining dominated by $2N \| y \|^2$.

Finally,

$$\lim_{t \downarrow 0} \int_{\|y\| \leqslant a t^{-1/2}} \langle u''(x) y, y \rangle \, p_1(dy) = \int_B \langle u''(x) y, y \rangle \, p_1(dy)$$

by dominated convergence. By Corollary 3 of [*11*] and Lemma 1.2 of [*8*] applied to the positive and negative parts of $D^2u(x)$, this integral is trace $D^2u(x)$. Moreover, the convergence is uniform on B when u'' is bounded since the integral for positive t differs from the limit by at most

$$N \int_{\|y\| > a t^{-1/2}} \| y \|^2 \, p_1(dy).$$

Thus, for the second part of the proposition, the convergence of all terms has been established to be uniform on B and thus, in particular, trace $[D^2u(x)]$ is a bounded uniformly continuous function and u is in the domain of the infinitesimal generator of the semigroup p_t.

Remark 3.1. In a general Banach space, the functions which are twice continuously differentiable do not constitute a very large set of functions. The works [*1*], [*13*], [*24*] show in particular that for many separable Banach spaces B the bounded continuously differentiable functions on B are not dense in the space of bounded uniformly continuous functions. In fact for some spaces, including classical Wiener space, there exists no nonzero differentiable function with bounded support.

DEFINITION 4. Let $\tau_x^{(r)}$ be the first exit time for the Brownian motion starting at x from the open ball of radius r in B with center x.

Let f be a Borel-measurable function defined in a neighborhood of the point x. The generalized Laplacian of f at the point x is defined by

$$\Delta f(x) = 2 \lim_{r \downarrow 0} \frac{E[f(x + W(\tau_x^{(r)}))] - f(x)}{E[\tau_x^{(r)}]}$$

when it exists.

Remark 3.2. The existence of $\Delta f(x)$ means, in particular, that $E[f(x + W(\tau_x^{(r)}))]$ is finite for all sufficiently small r.

Remark 3.3. $E[\tau_x^{(r)}]$ is clearly not zero since $\tau_x{}^r(\cdot)$ is a strictly positive function on Ω. Moreover, $E[\tau_x^{(r)}]$ is also finite. In fact, for any bounded open set V in B and any x in B, $E(\tau_x) < \infty$ when τ_x is the first exit time from V starting at x. This follows by a standard technique (see [5, p. 112]) once it is known that for some t $\sup_{x \in B} \mathscr{P}(\tau_x > t) < 1$. But for x not in V $\mathscr{P}(\tau_x > t) = 0$ and for x in V $\mathscr{P}(\tau_x > t) \leqslant p_t(x, V) \leqslant p_t(\{y : \| y \| \leqslant d\}) < 1$ where d is the diameter of V.

We shall also need the dependence of $E[\tau_x^{(r)}]$ on r. By (3), the process $W'(t) = r^{-1} W(tr^2)$ has the same transition functions as the process $W(t)$ and is clearly a Wiener process. Hence

$$\begin{aligned} \mathscr{P}(\tau_x^{(r)} > t) &= \mathscr{P}(\| W(s) \| < r, 0 \leqslant s \leqslant t) \\ &= \mathscr{P}\left(\| W(sr^2) \| < r, 0 \leqslant s \leqslant \frac{t}{r^2}\right) \\ &= \mathscr{P}\left(\| W'(s) \| < 1, 0 \leqslant s \leqslant \frac{t}{r^2}\right) = \mathscr{P}\left(\tau_x^{(1)} > \frac{t}{r^2}\right). \end{aligned}$$

Thus if $\varphi_r(t) = \mathscr{P}(\tau_x^{(r)} > t)$ then $\varphi_r(t) = \varphi_1(t/r^2)$. Hence

$$E(\tau_x^{(r)}) = -\int_0^\infty t \, d\varphi_r(t) = -r^2 \int_0^\infty t \, d\varphi_1(t) = r^2 E(\tau_x^{(1)}).$$

Denote by S_r the set $\{y \in B : \| y \| = r\}$. Let A be a Borel set in S_r and put

$$\pi_r(A) = \mathscr{P}(W(\tau_0^{(r)}) \in A),$$

$$c = E[\tau_0^{(1)}].$$

Then π_r is a probability measure on the Borel sets of S_r and the generalized Laplacian of a function u at x may be written

$$\Delta u(x) = 2c^{-1} \lim_{r\to 0} r^{-2} \left[\int_{S_r} u(x+y)\, \pi_r(dy) - u(x) \right].$$

Remark 3.4. The function u in Corollary 1.2 satisfies $\Delta u = 0$ in V. Although this type of result is standard in the theory of Markov processes we shall give a short proof here suitable for the present setting. If τ_x is the first exit time from V and x is a point in V choose r so small that x is at distance greater than r from ∂V. Now since the paths are continuous there holds $\tau_x^{(r)} < \tau_x$ since a path starting at x must cross the surface of the sphere with radius r and center x before it reaches ∂V. But $u(x) = E[\varphi(x + W(\tau_x^{(r)}) + \{W(\tau_x) - W(\tau_x^{(r)})\}]$. Since $W'(t) = W(t + \tau_x^{(r)}) - W(\tau_x^{(r)})$ is a Wiener process independent of $W(\tau_x^{(r)})$, the term in braces on the right of the last equation may be regarded as W' stopped on ∂V when started at $x + W(\tau_x^{(r)})$. Taking the expectation first with respect to W' we therefore get

$$u(x) = E[u(x + W(\tau_x^{(r)})],$$

which is the general Markov-process analog for the mean-value property of harmonic functions. It follows now from Definition 4 that $\Delta u = 0$.

Thus by Corollary 1.2 the Dirichlet problem can be solved in a generalized sense for reasonable regions in B. It remains to see to what extent the generalized Laplacian of a function u agrees with trace $D^2u(x)$ for a smooth function u. In Corollary 8.1 below we consider the simplest case in which they agree.

DEFINITION 5. The *Green's measure* on B is the measure

$$G(A) = \int_0^\infty p_t(A)\, dt$$

defined on Borel sets A. The *potential* of a Borel function f on B is the function u defined by

$$u(x) = \int_B f(x+y)\, G(dy) \equiv (Gf)(x)$$

whenever it exists.

Remark 3.5. The function $G(A - x)$ is bounded on B whenever A is a bounded set. For let e_1, e_2, e_3 be elements of B^* which are orthonormal in H^*. Then each e_j is bounded on A—say, $|\langle e_j, y\rangle| \leqslant M$ for y in A. Let $T = \{y \in B : |\langle e_j, y\rangle| \leqslant M, j = 1, 2, 3\}$. Then T is a tame set based on the span K of e_1, e_2, e_3. In the notation of Remark 2.2, T is a product: $T = C \times L$ where C is a cube in K of side $2M$. If Q is the projection of B onto K then for any x in B

$$T - x = C \times L - x = (C - Qx) \times L.$$

Thus

$$G(A - x) \leqslant G(T - x) = \int_0^\infty p_t(T - x)\, dt = \int_0^\infty \mu_t'(C - Qx)\, dt$$
$$= \left(\frac{1}{4\pi}\right) \int_{C-Qx} r^{-1}\, dy$$

where $r = |y|$ and dy is Lebesgue measure in the three-dimensional space K. The last expression is clearly bounded. We have used the well known fact that, in n dimensions,

$$\int_0^\infty \mu_t'(A)\, dt = ((n-2)\,\omega_n)^{-1} \int_A r^{2-n}\, dy,$$

provided $n \geqslant 3$ where ω_n is the surface area of $S^{(n-1)}$.

Remark 3.6. Let f be a bounded Borel function on B such that $G|f|$ is bounded on B. Let $u = Gf$. If f is continuous at x then $\Delta u(x) = -f(x)$.

Proof.

$$u(x) = E\left[\int_0^\infty f(x + W(t))\, dt\right]$$
$$= E\left[\int_0^{\tau_x^{(r)}} f(x + W(t))\, dt\right] + E\left[\int_{\tau_x^{(r)}}^\infty f(x + W(t))\, dt\right]$$
$$= E\left[\int_0^{\tau_x^{(r)}} f(x + W(t))\, dt\right]$$
$$+ E\left[\int_0^\infty f(x + W(\tau_x^{(r)}) + \{W(t + \tau_x^{(r)}) - W(\tau_x^{(r)})\}\, dt\right]$$
$$= E\left[\int_0^{\tau_x^{(r)}} f(x + W(t))\, dt\right] + E[u(x + W(\tau_x^{(r)}))],$$

where we have used again the fact that $W'(t) = W(t + \tau_x^{(r)}) - W(\tau_x^{(r)})$ is a Wiener process independent of $W(\tau_x^{(r)})$. Thus

$$\begin{aligned} \Delta u(x) &= \lim_{r \downarrow 0} \frac{E[u(x + W(\tau_x^{(r)}))] - u(x)}{E(\tau_x^{(r)})} \\ &= -\lim_{r \downarrow 0} \frac{E\left[\int_0^{\tau_x^{(r)}} f(x + W(t))\, dt\right]}{E[\tau_x^{(r)}]} \\ &= -f(x) - \lim_{r \downarrow 0} \frac{E\left[\int_0^{\tau_x^{(r)}} \{f(x + W(t)) - f(x)\}\, dt\right]}{E(\tau_x^{(r)})} . \end{aligned}$$

The numerator after the last limit sign is $E(\tau_x^{(r)}) \cdot o(1)$ as $r \to 0$ because of the assumed continuity of f at x. Hence $\Delta u(x) = f(x)$.

Remark 3.7. We give here an example which will also be useful later. Let A be a bounded operator from B to B^*. Then $A \mid H$ is a Hilbert-Schmidt operator on H by Corollary 5 of [*11*]. However since H is real only the symmetric part of $A \mid H$ enters into the expression $\sum_n (Ae_n, e_n)$ for the trace of A and since the symmetric part of $A \mid H$ is of trace class $A \mid H$ has a well-defined trace. Let $u(y) = (\frac{1}{2}) \langle Ay, y \rangle$ be defined on B. Assume that the B norm is twice continuously B-differentiable away from the origin and that its second Frechet B-derivative is bounded on the annulus $C: 1 \leqslant \| x \| \leqslant 2$. Then $\Delta u(0) = \text{trace } (A \mid H)$.

For the proof we let g be an infinitely differentiable function on $[0, \infty)$ such that $g(r) = 1$ for $0 \leqslant r \leqslant 1$ and $g(r) = 0$ for $2 \leqslant r < \infty$. Let $f(x) = u(x)\, g(\| x \|)$. One verifies easily that $f''(x)$ exists for every x in B, is bounded and continuous into the weak operator topology and $f''(0) = \frac{1}{2} [A + A^* \mid B]$. Consequently by Proposition 8 $\lim_{t \downarrow 0} t^{-1}[(p_t f)(x) - f(x)]$ exists for each x and equals $(\frac{1}{2})$ trace $f''(x)$. By the techniques used in Proposition 8 it follows that the limit exists boundedly (although not necessarily uniformly) and that the limit is B-continuous. Thus f is in the domain of the weak infinitesimal operator (see [*5*]) and by [*5*, Theorem 5.2, p. 133]

$$(\Delta f)(0) = \text{trace}\, [f''(0) \mid H] = \text{trace}\, (A \mid H).$$

Since $(\Delta u)(0) = \Delta f(0)$ the proof is concluded.

Remark 3.8. We conjecture that the preceding remark remains true without any differentiability assumption on the B norm or any assumption concerning the existence of smooth functions on B with bounded support. However a proof is lacking.

COROLLARY 8.1. *Let x be a point of an open set V in B. Let u be a real valued function on V such that*

(a) *u'' exists at each point of V.*

(b) *$u''(z)$ is continuous at x into $L(B, B^*)$ with the weak operator topology.*

(c) *$u''(z)$ is uniformly bounded on V.*

Assume that the B norm is twice continuously B-differentiable away from the origin and that its second Frechet derivative is bounded on the annulus $1 \leqslant \| x \| \leqslant 2$. Assume also that $\| \cdot \|$ is in $L^2(B, p_1)$.

Then $\Delta u(x)$ exists, $D^2u(x)$ is trace class and

$$\Delta u(x) = \operatorname{trace} D^2 u(x).$$

Proof. Equation (7) is applicable and, since $\pi_r(dy)$ is an even probability measure while $\langle u'(x), y\rangle$ is an odd function of y, we have, for sufficiently small r,

$$\int [u(x+y) - u(x)]\, \pi_r(dy) = \int_0^1 (1-s) \int \langle u''(x+sy)\, y, y\rangle\, \pi_r(dy)\, ds.$$

Putting $y = rz$ the term on the right becomes

$$r^2 \int_0^1 (1-s) \int \langle u''(x+rsy)\, y, y\rangle\, \pi_1(dy)\, ds.$$

Hence by dominated convergence,

$$\begin{aligned} &\lim_{r\downarrow 0} (r^2c)^{-1} \int [u(x+y) - u(x)]\, \pi_r(dy) \\ &= c^{-1} \int_0^1 \int (1-s) \langle u''(x)\, y, y\rangle\, \pi_1(dy)\, ds = \left(\frac{1}{(2c)}\right) \int \langle u''(x)\, y, y\rangle\, \pi_1(dy) \\ &= \lim_{r\downarrow 0} (\tfrac{1}{2})\, (cr^2)^{-1} \int \langle u''(x)\, y, y\rangle\, \pi_r(dy) = \tfrac{1}{2}\, \Delta \langle u''(x)\, y, y\rangle \,|_{y=0} \\ &= \operatorname{trace}\, [u''(x) \mid H] \end{aligned}$$

by Remark 3.7.

4. REGULARITY OF POTENTIALS

We recall (cf. [*11*]) that if h is an element of H^* then there is a measurable function on B which may be denoted by $y \to \langle h, y\rangle$

and which is defined first for h in B^* as the linear functional h itself and then defined for other h in H^* as follows. One observes that the map $h \to h$ from B^* into $L^2(B, p_t)$ satisfies $\int_B \langle h, y\rangle^2 p_t(dy) = t \mid h \mid^2$ so that it is continuous as a densely defined linear map from H^* into $L^2(B, p_t)$. Consequently it extends uniquely to a continuous linear map from H^* into $L^2(B, p_t)$ which assigns to h the function $y \to \langle h, y\rangle$ in question.

Thus $\langle h, \cdot\rangle$ is defined only up to a set of p_t measure zero when h is not in B^*. Since the measures p_t are mutually singular for different values of t the choice of a representative function for the element $\langle h, \cdot\rangle$ of $L^2(p_t)$ may depend significantly on t. Actually it is possible to find a Borel measurable function f on B which simultaneously defines the correct element of $L^2(B, p_t)$ for all t, i.e., a function f with the property that if h_n is in B^* for $n = 1, 2,\ldots$ and $h_n \to h$ in H^* norm then $h_n \to f$ in $L^2(B, p_t)$ for all $t \geqslant 0$. Such a function f may be constructed, for example, by defining f_1 to be any measurable function satisfying $h_n \to f_1$ in $L^2(B, p_1)$, setting $f_t(y) = t^{1/2} f_1(t^{-1/2} y)$, and putting $f(y) = f_{\varphi(y)}(y)$ where φ is the function defined in Eq. (29). However we shall not need to consider more than one value of t at a time so we omit further details and simply allow $\langle h, \cdot\rangle$ to depend on t.

For any h in H there is also a function $y \to (h, y)$ defined a.e. $[p_t]$, obtainable from the preceding discussion by identifying H with H^* in the usual way.

In the remainder of this section we shall identify H^* with H. Thus the three spaces B^*, H, B are related by $B^* \subset H \subset B$.

Definition 6. A *test operator* is a bounded operator T of finite rank from B to B whose range is contained in B^*.

If T is a test operator then its restriction to H is a bounded operator on H. We shall denote this operator on H by $T \mid H$. If A is a bounded operator on H then for such a test operator T we shall write "trace $[TA]$" instead of trace $[(T \mid H) A]$ since there is no danger of confusion. In particular we shall write "trace $[T]$" instead of trace $[T \mid H]$. It is easily verified that if T is a test operator then T has the form

$$Tx = \sum_{j=1}^{m} \langle e_j, x\rangle f_j$$

where $\{e_j\}_{j=1}^{n}$ is an orthonormal (o.n.) basis of the annihilator (in B^*) of the null space of T and $f_j = Te_j$. The span M of $e_1, \ldots, e_n, f_1, \ldots, f_n$ is called the *carrier* of T. Note that M depends only on T and that $M \subset B^*$.

We recall that a bounded operator A on H is said to be of *Hilbert-Schmidt* type or simply a Hilbert-Schmidt operator if

$$\| A \|_2^2 = \operatorname{trace}[A^*A] < \infty,$$

(cf. [21].) The Hilbert-Schmidt operators form a Hilbert space with the norm $\| \cdot \|_2$ and inner product $(A, C) = \operatorname{trace}[C^*A]$. A bounded operator A on H is said to be of *trace class* if

$$\| A \|_1 = \operatorname{trace}[(A^*A)^{1/2}] < \infty.$$

The trace class operators form a Banach space in this norm dual to the space of completely continuous operators on H under the pairing $\langle A, C\rangle = \operatorname{trace}[C^*A]$ where A is trace class [*21*, Theorem 5.11].

The set of restrictions of test operators to H is dense in the space of completely continuous operators as well as in the space of Hilbert-Schmidt operators. In fact if $e_1, e_2, \ldots$ is an o.n. basis of H lying in B^* and P_n is the orthogonal projection onto span $(e_1, \ldots, e_n)$ then P_n is the restriction to H of the test operator Q_n given by

$$Q_n x = \sum_{j=1}^{n} \langle e_j, x\rangle e_j$$

for x in B. If A is a bounded operator on H then $T_n = P_n A Q_n$ is easily seen to be a test operator and $T_n \mid H = P_n A P_n$. Now if A is completely continuous then since the sequence $|(I - P_n) x|$ decreases to zero uniformly on compact sets the sequence $|(A - P_n A) x|$ decreases to zero uniformly on the unit ball in H, i.e., $\| A - P_n A \| \to 0$. Hence the operator norm of

$$A - P_n A P_n = (A - P_n A) + P_n (A^* - P_n A^*)^*$$

approaches zero as $n \to \infty$. Finally if A is of Hilbert-Schmidt type and has matrix (a_{ij}) on the basis $e_1, e_2, \ldots$ then

$$\| A \|_2^2 = \sum_{i,j=1}^{\infty} | a_{ij} |^2 < \infty$$

and

$$\| A - P_n A P_n \|_2^2 = \sum_{\max(i,j) > n} | a_{ij} |^2$$

which converges to zero as $n \to \infty$.

PROPOSITION 9. *Let f be a bounded measurable function on B and let t be a fixed number greater than zero. The function $x \to (p_t f)(x)$ is infinitely H differentiable on B with first and second derivatives given by*

$$((Dp_t f)(x), h) = t^{-1} \int_B f(x+y)(h, y)\, p_t(dy), \tag{8}$$

$$((D^2 p_t f)(x)\, k, h) = t^{-1} \int_B f(x+y)\{t^{-1}(h, y)(k, y) - (h, k)\}\, p_t(dy), \tag{9}$$

where h and k are in H. If T is a test operator then

$$\operatorname{trace}\,[T(D^2 p_t f)(x)] = t^{-1} \int_B f(x+y)\{t^{-1}\langle Ty, y\rangle - \operatorname{trace} T\}\, p_t(dy). \tag{10}$$

Moreover, $(D^2 p_t f)(x)$ is of Hilbert-Schmidt type and

$$\| (D^2 p_t f)(x) \|_2 \leqslant t^{-1} \left(\int_B f(x+y)^2\, p_t(dy) \right)^{1/2}. \tag{11}$$

Finally for each x in B the functions $t \to Dp_t f(x)$ and $t \to D^2 p_t f(x)$ from $(0, \infty)$ into H and into the Hilbert space of Hilbert-Schmidt operators respectively are Borel-measurable.

Proof. Put $g(x) = (p_t f)(x)$. Then for all h in H

$$g(x+h) = \int_B f(x+y)\, p_t(h, dy).$$

Now $p_t(h, \cdot)$ is absolutely continuous with respect to p_t by Theorem 3 of [*22*]. This also follows in the present context directly from the construction of p_t which may be used to verify that for a bounded tame function v on B there holds

$$\int_B v(y)\, p_t(h, dy) = \int_B v(y) \exp\left[- \frac{|h|^2 + 2(h, y)}{2t} \right] p_t(dy).$$

This equation remains valid under the passage boundedly to pointwise limits and therefore holds for all bounded Borel measurable functions v on B. In particular we have

$$g(x+h) = \int_B f(x+y) \exp\left[- \frac{|h|^2 + 2(h, y)}{2t} \right] p_t(dy) \tag{12}$$

for all h in H. All the derivatives of g may be obtained by differentiating (12) under the integral sign with respect to h. We show that the expressions so obtained actually represent the Frechet derivatives. Let

$$J(h, y) = \exp\left[-\frac{|h|^2 + 2(h, y)}{2t}\right].$$

Then

$$\left(\frac{d}{ds}\right) J(sh, y) = t^{-1}\{(h, y) - s\,|h|^2\}\, J(sh, y)$$

and

$$g(x + h) - g(x) = \int_B \int_0^1 f(x + y)\, t^{-1}\{(h, y) - s\,|h|^2\}\, J(sh, y)\, ds\, p_t(dy)$$

so that

$$\left| g(x + h) - g(x) - \int_B f(x + y)\, t^{-1}(h, y)\, p_t(dy) \right|$$

$$\leqslant \left| \int_B f(x + y) \int_0^1 t^{-1}\{(h, y)\,(J(sh, y) - 1) - s\,|h|^2\, J(sh, y)\}\, ds\, p_t(dy) \right|$$

$$\leqslant \int_0^1 \int_B t^{-1}\,|f(x + y)|\,|(h, y)\,(J(sh, y) - 1)|\, p_t(dy)\, ds$$

$$+ t^{-1} \int_0^1 s\,|h|^2 \int_B |f(x + y)|\, p_t(sh, dy)\, ds$$

$$\leqslant t^{-1}\, \|f\|_\infty \left(\int_B (h, y)^2\, p_t(dy)\right)^{1/2} \int_0^1 \left(\int_B |J(sh, y) - 1|^2\, p_t(dy)\right)^{1/2} ds$$

$$+ \frac{1}{2t}\,|h|^2\, \|f\|_\infty .$$

Since (h, y) is normally distributed with respect to p_t with mean zero and variance $t\,|h|^2$, both integrals in the last line may be evaluated explicitly with the result

$$\int_B |J(sh, y) - 1|^2\, p_t(dy) = \exp\left[\frac{s^2\,|h|^2}{t}\right] - 1.$$

Consequently,

$$\left| g(x + h) - g(x) - \int_B f(x + y)\, t^{-1}(h, y)\, p_t(dy) \right|$$

$$\leqslant t^{-1}\,|h|\,\|f\|_\infty \left\{ t^{1/2} \int_0^1 \left(\exp\left[\frac{s\,|h|^2}{t}\right] - 1\right)^{1/2} ds + \left(\frac{1}{2}\right)|h| \right\}$$

$$= o(|h|)$$

This establishes that the first Frechet derivative of g is given by (8). We note that

$$\begin{aligned} |(Dg)(x)| &= \sup_{|h|=1} |((Dg)(x), h)| \\ &\leqslant t^{-1} \|f\|_\infty \sup_{|h|=1} \left(\int_B (h, y)^2 p_t(dy)\right)^{1/2} \\ &= t^{-1/2} \|f\|_\infty . \end{aligned}$$

For the second-order derivative we have

$$\begin{aligned} (Dg(x+k), h) &= t^{-1} \int_B f(x+k+y)(h, y)\, p_t(dy) \\ &= t^{-1} \int_B f(x+y)(h, y-k)\, J(k, y)\, p_t(dy) \end{aligned}$$

for k in H. Now the function $y \to (h, y)$ is not linear unless h is in B^*. Nevertheless,

$$(Dg(x+k), h) = t^{-1} \int_B f(x+y)\{(h, y) - (h, k)\}\, J(k, y)\, p_t(dy) \tag{13}$$

because this equation holds when h is in B^* and both sides are continuous in h in the H norm. Thus replacing k by sk in (13) one obtains

$$\begin{aligned} &\left| (Dg(x+k), h) - (Dg)(x), h) \right. \\ &\qquad \left. - t^{-1} \int_B f(x+y)\{t^{-1}(h, y)(k, y) - (h, k)\}\, p_t(dy) \right| \\ &= \left| \int_B t^{-1} \int_0^1 f(x+y) \left(\frac{d}{ds}\right) [\{(h, y) - (sk, y)\}\, J(sk, y)]\, ds\, p_t(dy) \right. \\ &\qquad \left. - t^{-1} \int_B f(x+y)\{t^{-1}(h, y)(k, y) - (h, k)\}\, p_t(dy) \right| \\ &= \left| t^{-1} \int_0^1 \int_B f(x+y)\{t^{-1}(h, y)(k, y) - (h, k)\}(J(sk, y) - 1)\, p_t(dy)\, ds \right. \\ &\quad \left. + t^{-2} \int_0^1 \int_B f(x+y)\, s\{(h, k)[s\,|k|^2 - (k, y)] - |k|^2 (h, y)\}\, J(sk, y)\, p_t(dy) \right| \\ &\leqslant t^{-1} \|f\|_\infty \left(\int_B \{t^{-1}(h, y)(k, y) - (h, k)\}^2 p_t(dy)\right)^{1/2} \int_0^1 \left(\exp\left[\frac{s\,|k|^2}{t}\right] - 1\right)^{1/2} ds \\ &\quad + t^{-2} \|f\|_\infty \int_0^1 s \left(\int_B \{(h, k)[s\,|k|^2 - (k, y)] - |k|^2 (h, y)\}^2 p_t(dy)\right)^{1/2} \\ &\qquad \times \exp\left[\frac{s^2\,|k|^2}{t}\right] ds \\ &= O(|h|)\, o(|k|). \end{aligned}$$

It follows that the Frechet derivative of Dg is given by (9). That (9) actually represents a bounded operator on H is a consequence of the fact that (h, y) and (k, y) are jointly Gaussianly distributed with respect to p_t with covariance $t(h, k)$. The operator norm may be estimated easily by performing a suitable two-dimensional integral.

The existence of the higher-order Frechet derivatives may be established in the same way. The validity of the necessary estimates reflect the integrability with respect to p_t of all polynomials in the functions $(h, \cdot)$ for several vectors h. It is not hard to see that the method used above for first and second derivatives yields estimates of the form

$$| (D^n g)(x + k)(h_1, \ldots, h_n) - (D^n g)(x)(h_1, \ldots, h_n) - D^{n+1}g(x)(h_1, \ldots, h_n, k) |$$
$$= O(| h_1 | | h_2 | \cdots | h_n |)\, o(| k |),$$

where $(D^n g)(x)(h_1, \ldots, h_n)$ is the multilinear form associated with the nth Frechet derivative of g. This estimate ensures the existence of the $(n + 1)$st H-derivative in the Frechet sense.

Now if T is a test operator whose restriction to H is symmetric then for some o.n. set $e_1, \ldots, e_n$ lying in B^* T is given by $Tx = \sum \lambda_j \langle e_j, x\rangle e_j$ for x in B. Then

$$\begin{aligned}
\operatorname{trace}[TD^2g(x)] &= \sum_{j=1}^{n} (TD^2g(x)\, e_j, e_j) \\
&= \sum_{j=1}^{n} \lambda_j (D^2g(x)\, e_j, e_j) \\
&= \sum_{j=1}^{n} t^{-1} \int_B f(x + y) \lambda_j \{t^{-1}\langle e_j, y\rangle \langle e_j, y\rangle - 1\} p_t(dy) \\
&= t^{-1} \int_B f(x + y) \{t^{-1}\langle Ty, y\rangle - \operatorname{trace}[T]\} p_t(dy).
\end{aligned}$$

Since $D^2g(x)$ is symmetric both sides of (10) depend only on the symmetric part of $T \mid H$ and consequently Eq. (10) is established.

It remains to prove that $D^2g(x)$ is of Hilbert-Schmidt type. The space of symmetric Hilbert-Schmidt operators is a Hilbert space in the inner product $(A, C) = \operatorname{trace}[AC]$ and to prove that $D^2g(x)$ is of Hilbert-Schmidt type, it suffices to establish an inequality of the form $| \operatorname{trace}[T(D^2g)(x)] | \leqslant a \| T \mid H \|_2$ for test operators T with symmetric restrictions since the restrictions of these to H are dense in the space of symmetric Hilbert-Schmidt operators on H. But from (10) there follows

$$| \operatorname{trace} [T(D^2g)(x)] |$$
$$\leqslant t^{-1} \left[\int_B f(x+y)^2 p_t(dy)\right]^{1/2} \left[\int_B \{t^{-1}\langle Ty, y\rangle - \operatorname{trace}[T]\}^2 p_t(dy)\right]^{1/2}.$$

But

$$\int_B \{t^{-1}\langle Ty, y\rangle - \operatorname{trace}[T]\}^2 p_t(dy) = \| T \mid H \|_2^2 .$$

This may be derived by writing $Ty = \sum_{j=1}^n \lambda_j \langle e_j , y\rangle e_j$ where the e_j are o.n. and lie in B^*, observing that

$$t^{-1}\langle Ty, y\rangle - \operatorname{trace}[T] = \sum_{j=1}^{n} \lambda_j [t^{-1}\langle e_j , y\rangle^2 - 1],$$

and that this is a sum of independent functions with respect to p_t with mean zero and variance λ_j^2. This establishes (11).

Finally if h is in B^* then from (8) we have

$$(Dp_t f(x), h) = t^{-1/2} \int_B f(x + t^{1/2}z) \langle h, z\rangle p_1(dz),$$

which is a measurable function of t. Hence since B^* is dense in H $(Dp_t f(x), h)$ is measurable for all h in H. Since H is separable $Dp_t f(x)$ is a strongly measurable function of t. Similarly Eq. (10) together with the denseness of test operators in the space of Hilbert-Schmidt operators establishes first weak measurability of $D^2 p_t f(x)$ and therefore strong measurability as a function of t into the space of Hilbert-Schmidt operators.

DEFINITION 7. A bounded measurable function f on B will be said to satisfy a *Dini condition* of order $p \geqslant 1$ at x if

$$\int_0^1 t^{-1} \left(\int_B | f(x+y) - f(x) |^p p_t(dy)\right)^{1/p} dt < \infty.$$

A Dini condition of order p will be said to hold uniformly on a set $U \subset B$ if

$$\lim_{\delta \downarrow 0} \int_0^\delta t^{-1} \left[\int_B | f(x+y) - f(x) |^p p_t(dy)\right]^{1/p} dt = 0$$

uniformly for x in U.

Remark 4.1. In n dimensions, a Dini condition of order 1 in the above sense at x is easily seen to be equivalent to the condition

$$\int_{|y|<1} |f(x+y)-f(x)|\,|y|^{-n}\,dy < \infty.$$

This is a condition which is slightly weaker than that actually used by Dini (for $n = 2$) [3, p. 200].

Remark 4.2. If f is a bounded measurable function on B and satisfies a Hölder condition at x, i.e., $|f(x+y)-f(x)| \leqslant C\|y\|^{\alpha}$, $0 < \alpha \leqslant 1$ for all y in a B neighborhood of x (and therefore for all y), then f satisfies a Dini condition at x of order p provided $\int_B \|y\|^{\alpha p}\,p_1(dy) < \infty$. If the Hölder condition holds uniformly for x in a set U then the Dini condition also holds uniformly in U.

THEOREM 2. *Let f be a bounded measurable function on B with bounded support and let $u = Gf$. Let x be a point in B. Assume that for some $p > 1$ one of the following two conditions holds:*

(a) *f satisfies a Dini condition of order p uniformly on some H-neighborhood U of x;*

(b) *f satisfies a Dini condition of order p at x and f is B-continuous at x.*

Then $D^2u(x)$ exists and is of Hilbert-Schmidt type. It is given by

$$D^2u(x) = \int_0^\infty (D^2p_tf)(x)\,dt, \tag{14}$$

where the integral converges in Hilbert-Schmidt norm.

There exists a bounded uniformly continuous function f on B with bounded support which is zero in a B-neighborhood of the origin and such that $D^2Gf(0)$ is not a trace class operator.

LEMMA 2.1. *Let f be a bounded measurable real-valued function on B. Let λ be a nonnegative real number and assume that either $\lambda > 0$ or that f has bounded support. Let*

$$u(x) = \int_0^\infty e^{-\lambda t}(p_tf)(x)\,dt.$$

Then

$$Du(x) = \int_0^\infty e^{-\lambda t}\,Dp_tf(x)\,dt, \tag{15}$$

where the integral converges in H norm.

If in addition f satisfies hypothesis (a) *of the theorem then*

$$D^2u(x) = \int_0^\infty e^{-\lambda t} D^2 p_t f(x)\, dt, \tag{16}$$

where the integral converges in Hilbert-Schmidt norm.

Proof. To justify the differentiation under the integral sign in (15) it suffices to show that $\exp[-\lambda t] \mid Dp_t f(x) \mid$ is dominated by an integrable function of t on $(0, \infty)$ uniformly in x. Now from Proposition 9,

$$\begin{aligned} \mid (Dp_t f(z), h) \mid &\leqslant t^{-1} \left[\int_B f(z+y)^2 p_t(dy)\right]^{1/2} \left[\int_B (h, y)^2 p_t(dy)\right]^{1/2} \\ &= \mid h \mid t^{-1/2} \left[\int_B f(z+y)^2 p_t(dy)\right]^{1/2}. \end{aligned}$$

Hence

$$\mid Dp_t f(z) \mid \leqslant t^{-1/2} \left[\int_B f(z+y)^2 p_t(dy)\right]^{1/2}.$$

This is bounded for large t and for small t it is $O(t^{-1/2})$. Thus if $\lambda > 0$ then (15) is established. If $\lambda = 0$ and f has bounded support in B we show that

$$\int_B f(z+y)^2 p_t(dy) = \|f\|_\infty^2\, O(t^{-n/2}), \qquad t \to \infty \tag{17}$$

for all positive integers n, uniformly in z. Let $y_1, \ldots, y_n$ be orthonormal vectors in H^* which lie in B^*. Let A be a bounded closed set in B outside of which f is zero. Then $f(z + \cdot)$ is supported in $A - z$ and $\int_B f(z+y)^2 p_t(dy) \leqslant \|f\|_\infty^2 p_t(A - z)$. Now $y_1, \ldots, y_n$ are bounded on A so that A is contained in a tame set C based on the subspace $K \equiv \text{span}\,(y_1, \ldots, y_n)$ and whose base is a bounded set C_0 in K^*. Thus

$$\begin{aligned} p_t(A - z) \leqslant p_t(C - z) &= (2\pi t)^{-n/2} \int_{C_0 - \pi(z)} \exp\left[-\sum_{j=1}^n \frac{s_j^2}{2t}\right] ds_1 \cdots ds_n \\ &= O(t^{-n/2}) \qquad t \to \infty \end{aligned} \tag{18}$$

uniformly in z where $\pi(z)$ denotes the natural projection of z into K^*. This establishes (15) in all cases.

Now let $g_\epsilon(z) = \int_\epsilon^\infty e^{-\lambda t} Dp_t f(z)\, dt$. Then for $\epsilon > 0$ it follows from (11) and from the preceding argument that $Dg_\epsilon(z)$ is given by $Dg_\epsilon(z) = \int_\epsilon^\infty e^{-\lambda t} D^2 p_t f(z)\, dt$ where the integral converges in Hilbert-Schmidt norm and that $Dg_\epsilon(z)$ is a Hilbert-Schmidt operator.

We estimate $\| D^2 p_t f(z) \|_2$ for small t under assumption (a) of the theorem. Let T be a test operator. Then since

$$\int_B \{t^{-1}\langle Ty, y\rangle - \text{trace}\,[T]\}\, p_t(dy) = 0,$$

Eq. (10) yields

$$| \text{trace}\,[TD^2 p_t f(z)] |$$

$$= \left| t^{-1} \int_B [f(z+y) - f(z)] \{t^{-1}\langle Ty, y\rangle - \text{trace}\, T\}\, p_t(dy) \right|$$

$$\leqslant t^{-1} \left\{ \int_B |f(z+y) - f(z)|^p\, p_t(dy) \right\}^{1/p}$$

$$\times \left\{ \int_B |t^{-1}\langle Ty, y\rangle - \text{trace}\, T|^q\, p_t(dy) \right\}^{1/q}$$

where $q = p/(p-1)$.

Now assume that T restricted to H is symmetric. We shall show that the last factor in the last inequality is dominated by $C \| T \|_2$ where C depends only on q.

As noted in the proof of Proposition 9, $\langle Ty, y\rangle - \text{trace}\,[T]$ may be written $\sum_{j=1}^m \lambda_j \xi_j$ where the λ_j are the nonzero eigenvalues of T and $\xi_j = (e_j, \cdot)^2 - 1$. Thus the ξ_j are independent random variables with respect to p_1 with mean zero and finite moments of all orders. Moreover they are identically distributed. If n is the smallest even integer satisfying $n \geqslant q$ then

$$\left\{ \int_B |t^{-1}\langle Ty, y\rangle - \text{trace}\, T|^q\, p_t(dy) \right\}^{1/q} = \left\{ \int_B |\langle Tv, v\rangle - \text{trace}\, T|^q\, p_1(dv) \right\}^{1/q}$$

$$\leqslant \left\{ \int_B |\langle Tv, v\rangle - \text{trace}\, T|^n\, p_1(dv) \right\}^{1/n} = \left\{ E\left[\sum_j \lambda_j \xi_j \right]^n \right\}^{1/n}.$$

Now

$$E\left(\left(\sum_{j=1}^m \lambda_j \xi_j \right)^n \right) = \sum_{j_1, \dots, j_n} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_n} E(\xi_{j_1} \xi_{j_2} \cdots \xi_{j_n}).$$

If any subscript occurs only once among the subscripts $j_1, \dots, j_n$, then $E(\xi_{j_1} \cdots \xi_{j_n}) = 0$ since the ξ_j have zero mean and are independent. Hence we may write

$$E\left(\left(\sum \lambda_j \xi_j \right)^n \right) = \sum \lambda_{j_1}^{n_1} \cdots \lambda_{j_k}^{n_k} E(\xi_{j_1}^{n_1} \cdots \xi_{j_k}^{n_k}),$$

where the sum is carried out over all partitions of n, i.e., $\sum_{j=1}^{k} n_j = n$ satisfying $2 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ and each j_i runs from 1 to m with no two j_i equal for a given partition. The numbers

$$E(\xi_{j_1}^{n_1} \cdots \xi_{j_k}^{n_k})$$

are bounded and therefore

$$E\left(\left(\sum_{j=1}^{m} \lambda_j \xi_j\right)^n\right) \leqslant K \sum | \lambda_{j_1}^{n_1} \cdots \lambda_{j_k}^{n_k} |$$

for some constant K, the sum being carried out over the same set as before. For all j there holds $| \lambda_j | \leqslant \| T \|_2$. Hence for any one of the above partitions $n_1, \ldots, n_k$ there holds

$$\begin{aligned} \sum_{j_i \text{ distinct}} | \lambda_{j_1}^{n_1} \cdots \lambda_{j_k}^{n_k} | &\leqslant \sum_{\text{all } j_i} \lambda_{j_1}^2 \cdots \lambda_{j_k}^2 \| T \|_2^{(n_1-2)+\cdots+(n_k-2)} \\ &= \left(\sum_{j=1}^{m} \lambda_j^2\right)^k \| T \|_2^{n-2k} \\ &= \| T \|_2^n . \end{aligned}$$

Since there are only a finite number of such partitions, there is a constant C such that $E((\sum_{j=1}^{m} \lambda_j \xi_j)^n) \leqslant C \| T \|_2^n$.

Thus

$$| \operatorname{trace} [T D^2 p_t f(z)] | \leqslant C t^{-1} \left\{ \int_B [f(z+y) - f(z)]^p \, p_t(dy) \right\}^{1/p} \| T \|_2$$

for all symmetric test operators T. Hence

$$\| D^2 p_t f(z) \|_2 \leqslant C t^{-1} \left\{ \int_B [f(z+y) - f(z)]^p \, p_t(dy) \right\}^{1/p} .$$

Thus by assumption (a) of the theorem,

$$\int_0^\epsilon e^{-\lambda t} \| D^2 p_t f(z) \|_2 \, dt \to 0 \qquad \text{as} \qquad \epsilon \to 0$$

uniformly for z in an H-neighborhood U of x. Consequently $Dg_\epsilon(z)$ converges uniformly on U in Hilbert-Schmidt norm and therefore also in operator norm. By Proposition 9 and the dominated convergence theorem $Dg_\epsilon(z)$ is H-differentiable and hence continuous on U with the operator norm on the range. Hence $\int_0^\infty \exp [-\lambda t] \, D^2 p_t f(z) \, dt$

is continuous on U in operator norm and represents the Frechet derivative of $Du(z)$ on U.

LEMMA 2.2. *If f is a bounded measurable function on B with bounded support and satisfies hypothesis* (b) *of the theorem then $D^2u(x)$ exists and is given by the integral (14) which converges in Hilbert-Schmidt norm.*

Proof. The proof in Lemma 2.1 that the integral on the right of (16) converges in Hilbert-Schmidt norm (with $\lambda = 0$) is applicable also under the present hypothesis. It must be shown that the operator so obtained actually represents $D^2u(x)$. We use an adaptation of methods of H. Petrini [*19*].

Let h be an arbitrary vector in H and k a unit vector in H. For $\epsilon \neq 0$ consider the difference quotient $\epsilon^{-1}[(Du(x + \epsilon k), h) - (Du(x), h)]$ which by (8) and (13) is given by

$$\epsilon^{-1} \int_0^\infty \int_B f(x+y)\, t^{-1} \Big\{[(h, y) - \epsilon(h, k)] \times \exp\left[\frac{2\epsilon(k, y) - \epsilon^2}{2t}\right] - (h, y)\Big\}\, p_t(dy)\, dt.$$

Put $y = \epsilon z$ and $t = \epsilon^2 s$. Then $p_t(dy) = p_s(dz)$ by (3), and the difference quotient may therefore be written

$$\begin{aligned} &\epsilon^{-1}[(Du(x + \epsilon k), h) - (Du(x), h)] \\ &= \int_0^\infty \int_B f(x + \epsilon z)\, s^{-1} \Big\{[(h, z) - (h, k)] \exp\left[\frac{2(k, z) - 1}{2s}\right] - (h, z)\Big\}\, p_s(dz) ds \\ &= \int_0^1 \int_B f(x + \epsilon z)\, \psi(s, z, h, k)\, p_s(dz)\, ds + \int_1^\infty \int_B f(x + \epsilon z)\, \psi(s, z, h, k)\, p_s(dz)\, ds \end{aligned} \tag{19}$$

where

$$\psi(s, z, h, k) = s^{-1} \Big\{[(h, z) - (h, k)] \exp\left[\frac{2(k, z) - 1}{2s}\right] - (h, z)\Big\}.$$

We assert that

$$\int_0^1 \int_B |\, \psi(s, z, h, k)\, |\, p_s(dz)\, ds < \infty \tag{20}$$

and

$$\lim_{r\to\infty} \int_0^1 \int_{\|z\| \geqslant r} |\, \psi(s, z, h, k)\, |\, p_s(dz)\, ds = 0$$

uniformly for $|k| = |h| = 1$. Indeed,

$$\int_0^1 \int_{\|z\| \geqslant r} |\psi(s, z, h, k)| \, p_s(dz) \, ds$$

$$\leqslant \int_0^1 s^{-1} \int_{\|z\| \geqslant r} |(h, z) - (h, k)| \exp\left[\frac{2(k, z) - 1}{2s}\right] p_s(dz)$$

$$+ \int_0^1 s^{-1} \int_{\|z\| \geqslant r} |(h, z)| \, p_s(dz) \, ds$$

$$= \int_0^1 s^{-1} \int_{\|z+k\| \geqslant r} |(h, z)| \, p_s(dz) \, ds + \int_0^1 s^{-1} \int_{\|z\| \geqslant r} |(h, z)| \, p_s(dz) \, ds.$$

$$\leqslant 2 \int_0^1 s^{-1} \int_{\|z\| \geqslant r-1} |(h, z)| \, p_s(dz) \, ds$$

$$= 2 \int_0^1 s^{-1/2} \int_{\|s^{1/2} y\| \geqslant r-1} |(h, y)| \, p_1(dy) \, ds$$

$$\leqslant 2 \int_0^1 s^{-1/2} \, ds \int_{\|y\| \geqslant r-1} |(h, y)| \, p_1(dy)$$

$$\leqslant 4 \, [p_1(\| y \| \geqslant r - 1)]^{1/2} \left[\int_B (h, y)^2 \, p_1(dy)\right]^{1/2}.$$

We have used here the fact that the exponential factor in the second line is the Radon-Nikodym derivative of $p_s(k, dz)$ with respect to $p_s(dz)$. The third line is then obtained from the second by translating by k, while the fifth line follows from the fourth by making the substitution $z = s^{1/2} y$. The last line is independent of the unit vector h and goes to zero as $r \to \infty$. Equation (20) follows by putting $r = 0$. Moreover, $\int_0^1 \int_B \psi(s, z, h, k) \, p_s(dz) \, ds = 0$ follows from again replacing z by $z + k$ in the terms involving the exponential factor. Consequently,

$$\left| \int_0^1 \int_B f(x + \epsilon z) \, \psi(s, z, h, k) \, p_s(dz) \, ds \right|$$

$$\leqslant \left| \int_0^1 \int_B (f(x + \epsilon z) - f(x)) \, \psi(s, z, h, k) \, p_s(dz) \, ds \right|$$

$$\leqslant \int_0^1 \int_{\|z\| < r} |f(x + \epsilon z) - f(x)| \, |\psi(s, z, h, k)| \, p_s(dz) \, ds$$

$$+ \int_0^1 \int_{\|z\| \geqslant r} |f(x + \epsilon z) - f(x)| \, |\psi(s, z, h, k)| \, p_s(dz) \, ds$$

$$\leqslant \sup_{\|z\| \leqslant r} |f(x + \epsilon z) - f(x)| \int_0^1 \int_B |\psi(s, z, h, k)| \, p_s(dz) \, ds.$$

$$+ 2 \, \|f\|_\infty \int_0^1 \int_{\|z\| \geqslant r} |\psi(s, z, h, k)| \, p_s(dz) \, ds.$$

Thus if $\delta > 0$ is given and if r is chosen large enough then for all sufficiently small $|\epsilon|$ the continuity of f at x implies that both of the last terms are less than δ uniformly for $|h| = |k| = 1$. Hence

$$\lim_{\epsilon\to 0} \int_0^1 \int_B f(x + \epsilon z)\, \psi(s, z, h, k)\, p_s(dz)\, ds = 0 \tag{21}$$

uniformly over the set $|h| = |k| = 1$.

We now consider the last member on the right of Eq. (19). Let

$$\varphi(s, z, h, k) = \psi(s, z, h, k) - s^{-1}[s^{-1}(h, z)\,(k, z) - (h, k)].$$

Thus

$$\varphi(s, z, h, k) = s^{-1} \left\{(h, z) \left[\exp\left(\frac{2(k, z) - 1}{2s}\right) - 1 - \frac{2(k, z) - 1}{2s} - \frac{1}{2s}\right]\right.$$

$$\left. + (h, k) \left[1 - \exp\left(\frac{2(k, z) - 1}{2s}\right)\right]\right\}.$$

From the mean value theorem it follows that, for any real number a, $|e^a - 1| \leqslant |a|\, e^{|a|}$ and from the equation

$$e^a = 1 + a + \int_0^a (a - t)\, e^t\, dt$$

it follows that

$$|e^a - 1 - a| \leqslant a^2 e^{|a|}.$$

Upon putting $a = (2(k, z) - 1)/(2s)$ in these two inequalities we obtain for φ the estimate

$$|\varphi(s, z, h, k)| \leqslant s^{-1} \left\{|(h, z)| \left[\left|\frac{2(k, z) - 1}{2s}\right|^2 \exp\left(\left|\frac{2(k, z) - 1}{2s}\right|\right) + \frac{1}{2s}\right]\right.$$

$$\left. + |(h, k)| \left|\frac{2(k, z) - 1}{2s}\right| \exp\left|\frac{2(k, z) - 1}{2s}\right|\right\}.$$

We assert that, for all numbers $r \geqslant 0$,

$$\int_1^\infty \int_{\|z\| \geqslant r} |\varphi(s, z, h, k)|\, p_s(dz)\, ds$$

is finite and goes to zero as $r \to \infty$ uniformly over the set $|h| = |k| = 1$. Upon making the substitution $z = s^{1/2}y$ and then estimating the exponential factors by setting $s = 1$ in them we obtain

$$\int_1^\infty \int_{\|z\| \geqslant r} | \varphi(s, z, h, k) | \, p_s(dz) \, ds$$

$$\leqslant \int_1^\infty \int_{\|s^{1/2}y\| \geqslant r} | \varphi(s, s^{1/2}y, h, k) | \, p_1(dy) \, ds$$

$$\leqslant \int_1^\infty s^{-3/2} \int_{s^{1/2}\|y\| \geqslant r} \Big\{ | (h, y) | \Big[\frac{| 2s^{1/2}(k, y) - 1 |^2}{4s} \exp (| (k, y) | + 1) + \frac{1}{2s^{1/2}} \Big]$$

$$+ | (h, k) | \, \Big| (k, y) + \frac{1}{2s^{1/2}} \Big| \exp (| (k, y) | + 1) \Big\} \, p_1(dy) \, ds.$$

Since (h, y) and (k, y) are jointly normally distributed with respect to $p_1(dy)$, it is clear that $\int_{s^{1/2}\|y\| \geqslant r} \{\cdots\} \, p_1(dy)$ is bounded as a function of s on $[1, \infty)$ uniformly over the set $| h | = | k | = 1$. In fact the expression in braces is dominated by

$$\chi(y, h, k) = \{ | (h, y) | \, [(| (k, y) | + 1)^2 \exp (| (k, y) | + 1) + 1]$$

$$+ | (h, k) | \exp (| (k, y) | + 1) \}$$

which is square-integrable with respect to $p_1(dy)$. Thus

$$\int_1^\infty \int_{\|z\| \geqslant r} | \varphi(s, z, h, k) | \, p_s(dz) \, ds$$

$$\leqslant \int_1^\infty s^{-3/2} [p_1(\| y \| \geqslant rs^{-1/2})]^{1/2} \Big[\int_B \chi(y, h, k)^2 \, p_1(dy) \Big]^{1/2} ds,$$

which approaches zero as $r \to \infty$ uniformly over the set $| h | = | k | = 1$. It follows from the boundedness of f and its continuity at x just as before that

$$\lim_{\epsilon \to 0} \int_1^\infty \int_B f(x + \epsilon z) \, \varphi(s, z, h, k) \, p_s(dz) \, ds = \int_1^\infty \int_B f(x) \, \varphi(s, z, h, k) \, p_s(dz) \, ds \tag{22}$$

uniformly on the set $| h | = | k | = 1$. It is clear from the definition of φ that $\int_B \varphi(s, z, h, k) \, p_s(dz) = 0$ so that the limit in (22) is zero.

Thus from (19) we have

$$\epsilon^{-1}(Du(x + \epsilon k) - Du(x), h) = \int_0^1 \int_B f(x + \epsilon z) \, \psi(s, z, h, k) \, p_s(dz) \, ds$$

$$+ \int_1^\infty \int_B f(x + \epsilon z) \, \varphi(s, z, h, k) \, p_s(dz) \, ds$$

$$+ \int_1^\infty \int_B s^{-1} f(x + \epsilon z) \, [s^{-1}(h, z) \, (k, z) - (h, k)] \, p_s(dz) \, ds. \tag{23}$$

Denoting the last term in (23) by K_ϵ we have established that

$$\lim_{\epsilon\to 0} [\epsilon^{-1}(Du(x+\epsilon k) - Du(x), h) - K_\epsilon] = 0 \tag{24}$$

uniformly over the set $|h| = |k| = 1$. But upon substituting $y = \epsilon z$ and $t = \epsilon^2 s$ into K_ϵ we have

$$\begin{aligned} K_\epsilon &= \int_{\epsilon^2}^{\infty} t^{-1} \int_B f(x+y)\,[t^{-1}(h,y)(k,y) - (h,k)]\, p_t(dy)\, dt \\ &= \left(\int_{\epsilon^2}^{\infty} t^{-1} D^2 p_t f(x)\, dt\, k, h\right) \end{aligned} \tag{25}$$

Since, as noted at the beginning of this proof, $\int_{\epsilon^2}^{\infty} t^{-1} D^2 p_t f(x)\, dt$ converges to $\int_0^{\infty} t^{-1} D^2 p_t f(x)\, dt$ in Hilbert-Schmidt norm and, *a fortiori*, in operator norm, as $\epsilon \to 0$, it follows that the last integral represents the Frechet derivative of Du at x.

Proof of Theorem. It follows from Lemmas 2.1 and 2.2 that, under either condition (a) or (b) of the theorem, $D^2u(x)$ is given by (14) which converges in Hilbert-Schmidt norm.

Let $\mathscr{C}$ be the Banach space consisting of all bounded real-valued uniformly continuous functions on B vanishing outside $\{x : 1 < \|x\| < 2\}$. We shall show that there exists a function f in $\mathscr{C}$ such that $(D^2Gf)(0)$ is not a trace class operator. Any f in $\mathscr{C}$ satisfies a Dini condition of order 2 at the origin because

$$\begin{aligned} \int_0^1 t^{-1} \left(\int_B |f(y) - f(0)|^2\, p_t(dy)\right)^{1/2} dt &\leqslant \|f\|_\infty \int_0^1 t^{-1} p_t(\|y\| > 1)^{1/2}\, dt \\ &\leqslant \text{constant } \|f\|_\infty \end{aligned} \tag{26}$$

since $p_t(\|y\|) > 1) = o(t)$ (see Remark 2.3).

Since f is continuous at 0, $(D^2Gf)(0)$ exists, is a Hilbert-Schmidt operator, and is given by (14).

Moreover, from (11),

$$\|(D^2Gf)(0)\|_2 \leqslant \int_0^{\infty} t^{-1} \left(\int_B f(y)^2\, p_t(dy)\right)^{1/2} dt$$

and from (26) and (17) it follows that

$$\|(D^2Gf)(0)\|_2 \leqslant \text{constant } \|f\|_\infty$$

for all f in $\mathscr{C}$. Thus the map $f \to (D^2Gf)(0)$ from $\mathscr{C}$ into the space of Hilbert-Schmidt operators is continuous. Now suppose that for every function f in $\mathscr{C}$ $(D^2Gf)(0)$ is a trace class operator. The map $f \to (D^2Gf)(0)$ from $\mathscr{C}$ into the Banach space of trace class operators on H is a closed operator for if $f_n \to f$ in $\mathscr{C}$ and the sequence $(D^2Gf_n)(0)$ converges in trace class norm to an operator L then since it also converges to L in Hilbert-Schmidt norm and converges also to $(D^2Gf)(0)$ in Hilbert-Schmidt norm it follows that $(D^2Gf)(0) = L$. Thus by the closed-graph theorem there is a constant K such that $\| (D^2Gf)(0) \|_1 \leqslant K \| f \|_\infty$ for f in $\mathscr{C}$ where $\| L \|_1$ denotes the trace class norm of L.

Hence for any bounded operator T on H,

$$| \operatorname{trace} [T(D^2Gf)(0)] | \leqslant K \| T \| \, \| f \|_\infty ,$$

where $\| T \|$ denotes the operator norm. In particular, if T is a test operator, then by (14) and (10),

$$\left| \int_0^\infty t^{-1} \int_A f(y) \{ t^{-1} \langle Ty, y \rangle - \operatorname{trace} [T] \} p_t(dy) \, dt \right| \leqslant K \| (T \mid H) \| \, \| f \|_\infty \tag{27}$$

for all f in $\mathscr{C}$ where $A = \{ x \in B : 1 < \| x \| < 2 \}$. Consider a fixed test operator T on B. Then

$$\begin{aligned} &\int_0^\infty t^{-1} \int_A | t^{-1} \langle Ty, y \rangle - \operatorname{trace} [T] | \, p_t(dy) \, dt \\ &= \int_0^\infty \int_B t^{-1} \chi_A(t^{1/2} z) \, | \langle Tz, z \rangle - \operatorname{trace} [T] | \, p_1(dz) \, dt \\ &= \int_B \left(\int_0^\infty t^{-1} \chi_A(t^{1/2} z) \, dt \right) | \langle Tz, z \rangle - \operatorname{trace} [T] | \, p_1(dz) \\ &= (\ln 4) \int_B | \langle Tz, z \rangle - \operatorname{trace} [T] | \, p_1(dz) < \infty. \end{aligned} \tag{28}$$

Thus by dominated convergence the set $\mathscr{E}$ of real Borel measurable functions f on A satisfying both $| f(y) | \leqslant 1$ on A and

$$\left| \int_0^\infty t^{-1} \int_A f(y) \{ t^{-1} \langle Ty, y \rangle - \operatorname{trace} [T] \} p_t(dy) \, dt \right| \leqslant K \| T \mid H \|$$

is closed under the operation of taking pointwise limit of sequences. By (27), $\mathscr{E}$ includes the restrictions to A of all elements of $\mathscr{C}$ of norm at most one. It follows from standard arguments that $\mathscr{E}$ consists of all Borel functions on A with sup norm at most one. By positive homogenity, (27) now holds for all bounded Borel functions on A. We shall show that this already implies that (27) also holds when f depends on t as well as y. We use a method applicable only to the case of an infinite-dimensional space H. Let $e_1, e_2, \ldots$ be an orthonormal basis of H^* lying in B^*. Let

$$\varphi(y) = \lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \langle e_j, y\rangle^2, \tag{29}$$

wherever the limit exists and is finite, and put φ equal to zero everywhere else. Then φ is a Borel function in B. Since the functions $\langle e_j, \cdot\rangle^2$ are identically distributed with respect to p_t and have mean t with respect to p_t the strong law of large numbers implies that $p_t(\{y : \varphi(y) = t\}) = 1$ for all $t \geqslant 0$. Now let $f(t, y)$ be a bounded measurable function on $[0, \infty) \times B$ which is zero off $[0, \infty) \times A$. Then $f(\varphi(y), y) = f(t, y)$ a.e. with respect to p_t. Thus

$$\begin{aligned}
&\left| \int_0^\infty \left[\int_B f(t,y)\, t^{-1}\{t^{-1}\langle Ty, y\rangle - \operatorname{trace}[T]\}\, p_t(dy)\right] dt \right| \\
&= \left| \int_0^\infty \left[\int_B f(\varphi(y),y)\, t^{-1}\{t^{-1}\langle Ty, y\rangle - \operatorname{trace}[T]\}\, p_t(dy)\right] dt \right| \\
&\leqslant K \,\| T \mid H \|\, \sup_{y\in A} | f(\varphi(y), y) | \\
&\leqslant K \,\| T \mid H \|\, \sup_{t,y} | f(t, y) | .
\end{aligned} \tag{30}$$

Now put $f(t, y) = \operatorname{sgn}\{t^{-1}\langle Ty, y\rangle - \operatorname{trace}[T]\}$ for y in A and zero otherwise. Thus (30) implies

$$\int_0^\infty \int_A t^{-1} \,| t^{-1}\langle Ty, y\rangle - \operatorname{trace}[T] |\, p_t(dy)\, dt \leqslant K \,\| T \mid H \| .$$

Thus by (28) we have

$$(\ln 4) \int_B | \langle Tz, z\rangle - \operatorname{trace}[T] |\, p_1(dz) \leqslant K \,\| T \mid H \|$$

for every test operator T. Now let T_n be the operator on B defined by $T_n x = \sum_{k=1}^n \langle e_k, x\rangle e_k$. Then $\| T_n \mid H \| = 1$ for all n and we have $\langle T_n z, z\rangle - \operatorname{trace}[T] = \sum_{k=1}^n [\langle e_k, z\rangle^2 - 1]$. Thus

$$(\ln 4) \int_B \left| \sum_{k=1}^{n} [\langle e_k, z\rangle^2 - 1] \right| p_1(dz) \leqslant K$$

for all n. We show that this is impossible by showing that $|\sum_{k=1}^{n}[\langle e_k, z\rangle^2 - 1]|$ diverges to $+\infty$ in probability with respect to p_1. The functions $\xi_k \equiv \langle e_k, z\rangle^2 - 1$ are identically distributed, have mean zero and possess finite moments of all orders with respect to p_1. They are independent and the central limit theorem is clearly applicable. Thus, putting

$$B_n = \sum_{k=1}^{n} \int_B \xi_k(z)^2 \, p_1(dz) = 2n,$$

we have, for any number N and real number $\alpha > 0$ and for sufficiently large n,

$$p_1\left(\left|\sum_{k=1}^{n} \xi_k\right| > N\right) \geqslant p_1\left(\left|\sum_{k=1}^{n} \xi_k\right| > B_n\alpha\right),$$

which converges as $n \to \infty$ to

$$(2\pi)^{-1/2} \int_{|x|>\alpha} \exp\left[-\frac{x^2}{2}\right] dx.$$

The last expression can be made arbitrarily close to one by choosing α sufficiently small. This concludes the proof of Theorem 2.

Remark 4.3. Other regularity properties may be established similar to those due to Petrini [*19*]. For example if f is bounded, has bounded support and u is its potential then each of the following statements is an immediate consequence of Eq. (23). K_ϵ is given by (25);

(a) $B_\epsilon \equiv \epsilon^{-1}(Du(x + \epsilon k) - Du(x), h) - K_\epsilon$ remains bounded as $\epsilon \to 0$

(b) If $\lim_{\epsilon \downarrow 0} f(x + \epsilon z)$ exists for almost all z with respect to p_1 then $\lim_{\epsilon \downarrow 0} B_\epsilon$ exists for each h and k in H.

We note that $\lim_{\epsilon \downarrow 0} f(x + \epsilon z)$ exists a.e. $[p_1]$ if and only if it exists a.e. $[p_s]$ for every s.

(c) If $f(x + \epsilon z)$ is continuous at $\epsilon = 0$ for almost all $z[p_1]$ then $\lim_{\epsilon \downarrow 0} B_\epsilon = 0$ for each h, k in H. If f is B-continuous at x then $\lim_{\epsilon \downarrow 0} B_\epsilon = 0$ uniformly over the set $|h| = |k| = 1$

(d) If $f(x + \epsilon z)$ is continuous in ϵ at $\epsilon = 0$ for almost all $z[p_1]$ and f satisfies a Dini condition at x of order greater than one, then $D^2u(x)$ exists as a Gateaux differential and is given by (14) which converges in Hilbert-Schmidt norm.

Remark 4.4. By Theorem 2 there exists a function f on B which is bounded and uniformly continuous on B, has bounded support, vanishes in a neighborhood V of the origin and whose potential u does not have a trace-class second Frechet derivative at zero. By Remark 3.5 the generalized Laplacian of u exists everywhere on B and has the value zero in V, i.e., u is harmonic in V in a generalized sense. Thus although the eigenvalues of the Hilbert-Schmidt operator $D^2u(0)$ do not converge absolutely the generalized Laplacian appears to provide some (orthogonally invariant) summability method to sum them to zero.

Remark 4.5. Unlike in finite dimensions, a Dini condition of order one at x is not sufficient in the infinite-dimensional case—even in the presence of continuity of f—to establish the existence of $D^2u(x)$ as a bounded operator.

DEFINITION 8. A function f on B is Lip 1 on B if there is a constant C such that $|f(x) - f(y)| \leqslant C \| x - y \|$ for all x and y in B.

THEOREM 3. *Assume $\| \cdot \|$ is in $L^2(p_1)$. If f is a bounded Lip 1 function on B then $D^2(p_t f)(x)$ is a trace class operator for each x and each $t > 0$. For each strictly positive number a the map $(t, x) \to D^2(p_t f)(x)$ is uniformly continuous on $[a, \infty) \times B$ into the Banach space of trace class operators on H. The function $v(t, x) = (p_t f)(x)$ is jointly uniformly continuous on $[0, \infty) \times B$ and satisfies*

$$\frac{\partial v}{\partial t} = \left(\frac{1}{2}\right) \operatorname{trace} [D^2 v(t, x)] \tag{31}$$

In particular $(\partial v / \partial t)$ is bounded and uniformly continuous on $[a, \infty) \times B$ for each number $a > 0$. For each $t > 0$ the derivative $\partial v / \partial t$ exists uniformly in x.

If in addition f has bounded support then $(D^2 G f)(x)$ is a trace class operator for each x and

$$(\tfrac{1}{2}) \operatorname{trace} [(D^2 G f)(x)] = -f(x).$$

LEMMA 3.1. *Let $t > 0$. Let S be a test operator on B. Let $T : H \to H$ be the restriction of S to H and let $L_\epsilon \equiv (I + \epsilon t^{-1} S)^{1/2}$ be defined for all sufficiently small real ϵ by a power series. If f is a bounded measurable function on B then*

$$\operatorname{trace} [T(D^2 p_t f)(x)] = 2 \frac{d}{d\epsilon} \int_B f(x + L_\epsilon y)\, p_t(dy), \tag{32}$$

where the derivative on the right is to be evaluated at $\epsilon = 0$. Moreover the derivative exists uniformly with respect to x.

Proof. The measure $E \to p_t(L_\epsilon^{-1}E)$ is absolutely continuous with respect to p_t. This is a very special case of Theorem 3 of [*22*] since the operator T is of finite rank and it may be derived directly along with the Radon-Nikodym derivative as follows. Let M be the carrier of S. Then as noted after Definition 6 $M \subset B^*$. Let P be the orthogonal projection of H onto M. Then P is continuous in B norm as may be seen by writing $Px = \sum_{j=1}^n \langle e_j , x\rangle e_j$ where $e_1 ,..., e_n$ is an o.n. basis of M. Let Q be the continuous extension of P to B. Then the function

$$R(\epsilon, x) = [\det (I + \epsilon t^{-1}T)]^{-1/2} \exp \left[- \frac{|(I + \epsilon t^{-1}T)^{-1/2} Qx|^2 - |Qx|^2}{2t}\right]$$

is a continuous tame function on B and is defined for all sufficiently small real ϵ. Consider the Cartesian product decomposition $B = M \times (I - Q) B$. The measure p_t is a product measure relative to this decomposition as noted in Remark 2.2. Since $Tx = 0$ for all x in H orthogonal to M the transformation L_ϵ acts as the identity on the factor $(I - Q) B$ and acts as $(I + \epsilon t^{-1}T)^{1/2}$ in M. Moreover, L_ϵ leaves M invariant. A straightforward transformation of Gauss measure μ_t in M by the transformation $(I + \epsilon t^{-1}T)^{1/2}$ shows that the Radon-Nikodym derivative $d\mu_t o(I + \epsilon t^{-1}T)^{-1/2}/d\mu_t$ is the restriction of $R(\epsilon, x)$ to M and consequently the derivative $dp_t oL_\epsilon^{-1}/dp_t$ is $R(\epsilon, x)$.

Thus

$$\int_B f(x + L_\epsilon y)\, p_t(dy) = \int_B f(x + y)\, p_t(L_\epsilon^{-1}dy)$$

$$= \int_B f(x + y)\, R(\epsilon, y)\, p_t(dy). \tag{33}$$

By choosing a basis of M on which T is triangular one readily computes

$$\frac{dR}{d\epsilon} = \left(\frac{1}{2}\right) t^{-1}\{t^{-1}(T(I + \epsilon t^{-1}T)^{-2} Qy, Qy) - \operatorname{trace} [T(I + \epsilon t^{-1}T)^{-1}]\}\, R(\epsilon, y).$$

For sufficiently small ϵ this is easily seen to be dominated by an integrable function of the form constant times $\exp [\alpha\, |Qy|^2/2t]$ for some $\alpha < 1$. Consequently, differentiation under the integral on the right in (33) is permissible and one obtains

$$\frac{d}{d\epsilon} \int_B f(x + L_\epsilon y)\, p_t(dy) \Big|_{\epsilon=0}$$

$$= \left(\frac{1}{2}\right) \int_B f(x + y)\, t^{-1}\{t^{-1}\langle Sy, y\rangle - \operatorname{trace} T\}\, p_t(dy)$$

which in view of (10) yields (32).

Moreover the uniform existence of the derivative follows from the inequalities

$$\left| \frac{(p_t o L_\epsilon^{-1}) f(x) - (p_t f)(x)}{\epsilon} - \left(\frac{1}{2}\right) \text{trace}\,[T(D^2 p_t f)(x)] \right|$$

$$= \left| \int_B f(x+y) \left[\frac{R(\epsilon, y) - 1}{\epsilon} - \left(\frac{1}{2}\right) t^{-1}\{t^{-1}\langle Sy, y\rangle - \text{trace}\, T\} \right] p_t(dy) \right|$$

$$\leqslant \| f \|_\infty \int_B \left| \frac{R(\epsilon, y) - 1}{\epsilon} - \left(\frac{1}{2}\right) t^{-1}\{t^{-1}\langle Sy, y\rangle - \text{trace}\, T\} \right| p_t(dy),$$

which approaches zero with ϵ, by virtue of the above remarks on the form of $dR/d\epsilon$.

Proof of Theorem. Suppose f is a bounded real valued function on B and satisfies $|f(x) - f(y)| \leqslant C \| x - y \|$ for all x and y in B. Let S be a test operator on B and T its restriction to H. Then, carrying over the notation from Lemma 3.1, we have from (32)

$$| \text{trace}\,[T(D^2 p_t f)(x)] | = 2 \lim_{\epsilon \to 0} \left| \epsilon^{-1} \int_B \{ f(x + L_\epsilon y) - f(x + y) \} p_t(dy) \right|$$

$$\leqslant 2 \limsup_{\epsilon \to 0} C \int_B | \epsilon |^{-1} \| L_\epsilon y - y \| p_t(dy). \tag{34}$$

Now the square root of $I + \epsilon t^{-1} S$ on B is given by a power series

$$I + 2^{-1} \epsilon t^{-1} S + \sum_{k=2}^{\infty} C_k (\epsilon t^{-1} S)^k$$

for small ϵ. Thus

$$| \epsilon |^{-1} \| L_\epsilon y - y \| \leqslant \frac{1}{2t} \| Sy \| + \sum_{k=2}^{\infty} C_k \epsilon^{k-1} t^{-k} \| S^k y \| .$$

Since $\| S^k y \| \leqslant \| S^k \| \| y \| \leqslant \| S \|^k \| y \|$, and $\| y \|$ is integrable with respect to $p_t(dy)$, it follows from dominated convergence that

$$| \text{trace}\,[T(D^2 p_t f)(x)] | \leqslant C t^{-1} \int_B \| Sy \| p_t(dy). \tag{35}$$

Now we wish to estimate the right side of the last inequality in terms of the operator norm of T (which is an operator on H) instead of the operator norm of S (which is an operator on B.) Since $\| Sy \|$ is a tame function based on the carrier, M, of S we have, for any finite number $p \geqslant 1$,

$$\int_B \| Sy \|^p p_t(dy) = \int_M \| Ty \|^p \mu_t(dy),$$

where μ_t is Gauss measure in M with variance parameter t. Let $m(\lambda) = \mu_t(\{y \in M : \| y \| > \lambda\})$. Let $n(\lambda) = \mu_t(\{y \in M : \| Ty \| > \lambda\})$. Then, by Theorem 5 of [9] applied to the finite-dimensional Hilbert space M, there holds $n(\lambda) \leqslant m(\lambda/\| T \|)$ where $\| T \|$ denotes the norm of T as an operator on H, i.e.,

$$\| T \| = \sup_{h \neq 0} \frac{| Th |}{| h |} = \sup_{h \in M, h \neq 0} \frac{| Th |}{| h |}.$$

But

$$\begin{aligned}\int_M \| Ty \|^p \mu_t(dy) &= -\int_0^\infty \lambda^p \, dn(\lambda) \\ &= p \int_0^\infty n(\lambda) \lambda^{p-1} \, d\lambda \leqslant p \int_0^\infty m\left(\frac{\lambda}{\| T \|}\right) \lambda^{p-1} \, d\lambda \\ &= \| T \|^p p \int_0^\infty m(\tau) \tau^{p-1} \, d\tau \\ &= \| T \|^p \int_M \| y \|^p \mu_t(dy).\end{aligned}$$

Thus[2]

$$\int_B \| Sy \|^p p_t(dy) \leqslant \| T \|^p \int_B \| y \|^p p_t(dy). \tag{36}$$

In particular, since

$$\int_B \| y \| p_t(dy) = t^{1/2} \int_B \| z \| p_1(dz),$$

$$| \operatorname{trace} [T(D^2 p_t f)(x)] | \leqslant C t^{-1/2} \int_B \| z \| p_1(dz) \| T \| . \tag{37}$$

Since the dual space of the space of completely continuous operators is the space of trace class operators under the pairing $\langle R, S \rangle = \operatorname{trace} [RS]$ and since the restrictions of test operators to H are dense in the space of completely continuous operators, it follows

[2] The inequality $\| Ty \| \leqslant \| T \| \| y \|$ is false in general when $\| T \|$ denotes the norm of T as an operator on H.

from (37) that $(D^2p_tf)(x)$ is a trace class operator with trace class norm

$$\| (D^2p_tf)(x) \|_1 \leqslant Ct^{-1/2} \int_B \| z \| p_1(dz). \tag{38}$$

We shall show that, for $a > 0$, the function $(D^2p_tf)(x)$ is uniformly continuous from $[a, \infty) \times B$ into the Banach space of trace class operators on H. Let $e_1, e_2, \ldots$ be an o.n. basis of H lying in B^* and let P_n be the projection of H onto span $(e_1, \ldots, e_n)$. By Lemma 3.2 of [9] the sequence $P_n(D^2p_tf)(x)$ converges to the operator $(D^2p_tf)(x)$ in trace class norm for each x and t. Now from (10) there follows

$$| \operatorname{trace} [T\{(D^2p_tf)(x) - (D^2p_sf)(x')\}] |$$
$$= \left| \int_B \{t^{-1}f(x + t^{1/2}z) - s^{-1}f(x' + s^{1/2}z)\}\{\langle Tz, z\rangle - \operatorname{trace} T\} p_1(dz) \right|$$
$$\leqslant \left(\int_B \{t^{-1}f(x + t^{1/2}z) - s^{-1}f(x' + s^{1/2}z)\}^2 p_1(dz) \right)^{1/2} \| T \|_2 ,$$

and consequently,

$$\| (D^2p_tf)(x) - (D^2p_sf)(x') \|_2$$
$$\leqslant \left(\int_B \{t^{-1}f(x + t^{1/2}z) - s^{-1}f(x' + s^{1/2}z)\}^2 p_1(dz) \right)^{1/2}. \tag{39}$$

Using the boundedness and Lip 1 character of f and the square-integrability of $\| z \|$ with respect to p_1, one establishes easily that the right side of (39) goes to zero uniformly as $\| x - x' \|$ and $| t - s |$ go to zero for t and s bounded away from zero. Thus the map $(t, x) \to (D^2p_tf)(x)$ is uniformly continuous on $[a, \infty) \times B$ into the Hilbert space of Hilbert-Schmidt operators on H. Hence so also is the map $(t, x) \to P_n(D^2p_tf)(x)$ for each n. Now, on the space of bounded operators on H with ranges contained in span $(e_1, \ldots, e_n)$, the Hilbert-Schmidt norm and trace class norm are equivalent for each n, because if A has rank at most n, one readily verifies by diagonalization of $(A^*A)^{1/2}$ that

$$\| A \|_1 = \operatorname{trace} [(A^*A)^{1/2}] \leqslant n^{1/2}[\operatorname{trace} (A^*A)]^{1/2} = n^{1/2} \| A \|_2$$

while $\| A \|_2 \leqslant \| A \|_1$ always holds. Thus for each n, the map $(t, x) \to P_n(D^2p_tf)(x)$ is uniformly continuous on $[a, \infty) \times B$ into the Banach space of trace class operators on H. It suffices to prove therefore that $P_n(D^2p_tf)(x)$ converges to $(D^2p_tf)(x)$ in trace class norm uniformly on $[a, \infty) \times B$. Let S be a test operator on B and T its restriction to H. Then, if Q_n is the continuous extension of P_n to B, $S(Q_n - Q_m)$ is also a test operator and, replacing S by $S(Q_n - Q_m)$ in (35), one obtains

$$|\operatorname{trace}[T(P_n - P_m)(D^2 p_t f)(x)]| \leqslant Ct^{-1} \int_B \| S(Q_n - Q_m) y \| p_t(dy). \quad (40)$$

Now the same argument which produced (36), when applied to the seminorm $\| y \|_1 = \| (Q_n - Q_m) y \|$ yields

$$|\operatorname{trace}[T(P_n - P_m)(D^2 p_t f)(x)]| \leqslant Ct^{-1} \int_B \| (Q_n - Q_m) y \| p_t(dy) \| T \|$$

and therefore it follows as before that

$$\| (P_n - P_m)(D^2 p_t f)(x) \|_1 \leqslant Ct^{-1/2} \int_B \| (Q_n - Q_m) z \| p_1(dz). \quad (41)$$

Now, by Corollary 5.2 of [9], $\| (Q_n - Q_m) z \|$ goes to zero in probability[3] with respect to p_1. Hence, if

$$F_{n,m}(\lambda) = p_1(\{z : \| (Q_n - Q_m) z \| > \lambda\}),$$

then $F_{n,m}(\lambda) \to 0$ as $n, m \to \infty$ for each $\lambda > 0$. Let

$$F(\lambda) = p_1(\{z : \| z \| > \lambda\}).$$

If $n > m$ then $\| P_n - P_m \| = 1$ and, by Theorem 5 of [9], $F_{n,m}(\lambda) \leqslant F(\lambda)$ for all $n \geqslant 0$. Thus

$$\int_B \| (Q_n - Q_m) z \| p_1(dz) = -\int_0^\infty \lambda \, dF_{n,m}(\lambda) = \int_0^\infty F_{n,m}(\lambda) \, d\lambda$$

and since

$$\int_0^\infty F(\lambda) \, d\lambda = \int_B \| z \| p_1(dz) < \infty,$$

it follows by dominated convergence that

$$\int_0^\infty F_{n,m}(\lambda) \, d\lambda \to 0.$$

[3] Note that $\| (Q_n - Q_m) z \|$ may be identified with $\| (P_n - P_m) z \|^{\sim}$ by Corollary 3 of [11].

Thus the right side of (41) approaches zero uniformly on $[a, \infty) \times B$ and the uniform continuity of $(D^2 p_t f)(x)$ is established.

In order to show that $(p_t f)(x)$ is a differentiable function of t at each t we choose a number $\delta > 0$ and a finite-dimensional projection P on H with range contained in B^* such that, for all x in B, there holds

$$|\operatorname{trace}[(I - P)(D^2 p_t f)(x)]| \leqslant \delta$$

as well as

$$Ct^{-1/2} \int_B \|(I - Q) z\| \, p_1(dz) \leqslant \delta,$$

where Q is the continuous extension of P to B. Such a projection P always exists for any positive number δ by the preceding paragraph. B is the direct sum of QB and $(I - Q) B$ and relative to the cartesian product decomposition $B = (QB) \times (I - Q) B$ the measure p_s is for any $s > 0$ a product $p_s = p_s' \times p_s''$ where p_s' is Gauss measure in QB and p_s'' is Wiener measure in $(I - Q) B$. Let $s > 0$ and put $\epsilon = s - t$. Define $(p_s' p_t'' f)(x)$ to be $\int\int f(x + u + v) \, p_s'(du) \, p_t''(dv)$, where the u integral is over QB and the v integral is over $(I - Q) B$. Then, on the one hand,

$$\begin{aligned}(p_s' p_t'' f)(x) &= \int\int f\left(x + \left(\frac{s}{t}\right)^{1/2} u + v\right) p_t'(du) \, p_t''(dv) \\ &= \int_B f\left(x + \left(\frac{s}{t}\right)^{1/2} Qy + (I - Q) y\right) p_t(dy) \\ &= \int_B f(x + L_\epsilon y) \, p_t(dy), \end{aligned} \tag{42}$$

where

$$L_\epsilon = \left(\frac{s}{t}\right)^{1/2} Q + (I - Q) = \left(\left(\frac{s}{t}\right) Q + (I - Q)\right)^{1/2} = (I + \epsilon t^{-1} Q)^{1/2}.$$

On the other hand,

$$\begin{aligned}(p_s' p_t'' f)(x) &= \int\int f\left(x + u + \left(\frac{t}{s}\right)^{1/2} v\right) p_s'(du) \, p_s''(dv) \\ &= \int_B f\left(x + Qy + \left(\frac{t}{s}\right)^{1/2} (I - Q) y\right) p_s(dy). \end{aligned} \tag{43}$$

Consequently, writing

$$\left| \frac{(p_s f)(x) - (p_t f)(x)}{\epsilon} - \left(\frac{1}{2}\right) \operatorname{trace}(D^2 p_t f)(x) \right| \leqslant I + II + III \tag{44}$$

where

$$I = |\, \epsilon^{-1}[(p_s' p_t'' f)(x) - p_t f(x)] - (\tfrac{1}{2}) \operatorname{trace} [P(D^2 p_t f)(x)] \,|,$$

$$II = |\, \epsilon^{-1}[(p_s f)(x) - (p_s' p_t'' f)(x)] \,|,$$

and

$$III = (\tfrac{1}{2}) \,|\, \operatorname{trace} [(I - P)(D^2 p_t f)(x)] \,|,$$

we have first of all III $\leqslant \delta/2$. Secondly, from (42) there follows

$$I \leqslant \left| \epsilon^{-1} \int_B [f(x + L_\epsilon y) - f(x + y)]\, p_t(dy) - (\tfrac{1}{2}) \operatorname{trace} [P(D^2 p_t f)(x)] \right|,$$

and, by Lemma 3.1, $\lim_{s \to t} I = 0$ uniformly in x.

Moreover, using (43) we have

$$\begin{aligned} II &= \left| \epsilon^{-1} \int_B \left[f(x + y) - f\left(x + \left\{Q + \left(\frac{t}{s}\right)^{1/2} (I - Q)\right\} y\right)\right] p_s(dy) \right| \\ &\leqslant C \,|\, \epsilon \,|^{-1} \int_B \left\| y - \left\{Q + \left(\frac{t}{s}\right)^{1/2} (I - Q)\right\} y \right\| p_s(dy) \\ &= C \,|\, \epsilon \,|^{-1} \left| 1 - \left(\frac{t}{s}\right)^{1/2} \right| \int_B \| (I - Q) y \| \, p_s(dy) \\ &= C \left| \frac{s^{1/2} - t^{1/2}}{s - t} \right| \int_B \| (I - Q) z \| \, p_1(dz). \end{aligned}$$

Hence

$$\limsup_{s \to t} \sup_{x \in B} II \leqslant C 2^{-1} t^{-1/2} \int \| (I - Q) z \| \, p_1(dz) \leqslant \frac{\delta}{2}.$$

Thus the $\limsup_{s \to t}$ of the left side of (44) does not exceed δ and in view of the arbitrariness of δ, Eq. (31) is established as well as the uniform differentiability of $p_t f$.

Finally we assume that, in addition to being uniformly Lip 1 on B, f has bounded support. From Theorem 2, Eq. (14) we have

$$\| (D^2 G f)(x) \|_1 \leqslant \int_0^\infty \| (D^2 p_t f)(x) \|_1 \, dt.$$

From the estimate (38) it follows that

$$\int_0^1 \| (D^2 p_t f)(x) \|_1 \, dt < \infty.$$

Now returning to the equality in (34) we note that, since $L_\epsilon \to I$ in B-operator norm as $\epsilon \to 0$, $\| L_\epsilon^{-1} \| \leqslant 2$ for all sufficiently small ϵ so that if the function $y \to f(x + y)$ vanishes for $\| y \| > N$ then $f(x + L_\epsilon y)$ vanishes for $\| y \| > 2N$ when ϵ is sufficiently small. Thus

$$| \operatorname{trace} [T(D^2 p_t f)(x)] | = 2 \lim_{\epsilon \to 0} \left| \epsilon^{-1} \int_{\|y\| \leqslant 2N} \{ f(x + L_\epsilon y) - f(x + y) \} p_t(dy) \right|$$
$$\leqslant 2C \lim_{\epsilon \to 0} \left| \epsilon^{-1} \int_{\|y\| \leqslant 2N} \| L_\epsilon y - y \| \, p_t(dy) \right| .$$

The discussion following (34) applies without change to yield

$$| \operatorname{trace} [T(D^2 p_t f)(x)] | \leqslant C t^{-1} \int_{\|y\| \leqslant 2N} \| Sy \| \, p_t(dy)$$

instead of (35).

Now

$$\int_{\|y\| \leqslant 2N} \| Sy \| \, p_t(dy) \leqslant [p_t(\| y \| \leqslant 2N)]^{1/2} \left(\int_B \| Sy \|^2 \, p_t(dy) \right)^{1/2},$$

and from (36) this is dominated by

$$[p_t(\| y \| \leqslant 2N)]^{1/2} \left[\int_B \| y \|^2 \, p_t(dy) \right]^{1/2} \| T \| .$$

Thus in view of the degree of arbitrariness of T we have

$$\| (D^2 p_t f)(x) \|_1 \leqslant C t^{-1/2} [p_t(\| y \| \leqslant 2N)]^{1/2} \left[\int_B \| z \|^2 \, p_1(dz) \right]^{1/2}. \tag{45}$$

By (18) the right side of (45) goes to zero faster than $t^{-n/4}$ for all n. Hence $\int_1^\infty \| (D^2 p_t f)(x) \|_1 \, dt < \infty$. Thus $(D^2 G f)(x)$ is a trace class operator. Moreover,

$$\begin{aligned} \operatorname{trace} (D^2 G f)(x) &= \int_0^\infty \operatorname{trace} [(D^2 p_t f)(x)] \, dt \\ &= \lim_{\epsilon \downarrow 0, R \uparrow \infty} \int_\epsilon^R \frac{2 \partial (p_t f)(x)}{\partial t} \, dt \\ &= 2 \lim_{\epsilon \downarrow 0, R \uparrow \infty} [(p_R f)(x) - (p_\epsilon f)(x)] \\ &= -2 f(x). \end{aligned}$$

This concludes the proof of Theorem 3.

We have already pointed out that the semigroup of measures p_t are strongly continuous when acting in the Banach space $\mathfrak{A}$ of bounded uniformly continuous functions on B. The following corollary is merely a restatement of part of the preceding theorem.

COROLLARY 3.1. *Assume $\|\cdot\|$ is in $L^2(p_1)$. If f is a bounded uniformly* Lip 1 *function on B then, for any $s > 0$, the function $p_s f$ is in the domain of the infinitesimal generator A of the semigroup p_t when the semigroup acts in the space $\mathfrak{A}$. Moreover,*

$$(Ap_s f)(x) = (\tfrac{1}{2}) \operatorname{trace} [(D^2 p_s f)(x)]. \tag{46}$$

COROLLARY 3.2. *Assume $\|\cdot\|$ is in $L^2(p_1)$. The set of functions u satisfying the following four conditions is dense in the domain of A in the graph norm.*

(i) *u is in the domain of A,*

(ii) *$D^2u(x)$ is trace class for each x in B,*

(iii) $(Au)(x) = (\frac{1}{2})$ trace $[D^2u(x)]$,

(iv) *the map $x \rightarrow D^2u(x)$ from B into the Banach space of trace class operators is bounded and uniformly continuous.*

The proof relies on the following lemma.

LEMMA 3.2.1. *A bounded uniformly continuous real-valued function f on a metric space (M, ρ) can be uniformly approximated by bounded* Lip 1 *functions with supports contained in the support of f.*

Proof. By considering $\max(f, 0)$ and $-\min(f, 0)$, it suffices to consider only nonnegative functions. Given $\epsilon > 0$ let $\mathfrak{A}_n$ be the collection of all nonnegative uniformly continuous functions f on M such that $\sup\{f(x) : x \in M\} \leqslant n\epsilon$. We shall show by induction on n that for every function f in $\mathfrak{A}_n$, $n = 2, 3, \ldots$ there is a Lip 1 function g such that $\sup_x |f(x) - g(x)| \leqslant 2\epsilon$, and support of g is contained in support of f. The assertion is clearly true if $n = 2$ for then one can take $g = 0$. Assuming the assertion is true for all $n \leqslant k$ we prove it is true for $n = k + 1$ where $k \geqslant 2$. Let f be in $\mathfrak{A}_{k+1}$ but not in $\mathfrak{A}_k$. If $f - (\inf_x f(x))$ is in $\mathfrak{A}_k$ we are done. Thus we may assume $\inf_x f(x) = 0$. Let $A = \{y : f(y) \leqslant (k-1)\epsilon\}$. Then A is not empty. There exists a number $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $\rho(x, y) < \delta$. Thus if $C = \{x : f(x) > k\epsilon\}$ then $\rho(x, A) \geqslant \delta$ for all x in C. Let $v(x) = \epsilon\delta^{-1} \min(\delta, \rho(x, A))$. Then $0 \leqslant v(x) \leqslant \epsilon$ and $v(x) = 0$ on A and $v(x) = \epsilon$ on C. v is a Lip 1 function and $0 \leqslant f - v \leqslant k\epsilon$. Thus $f - v$ is in $\mathfrak{A}_k$ and support $(f - v)$ is contained in support f. By the induction hypothesis there is a Lip 1 function g such that $\sup_x |f(x) - (v(x) + g(x))| \leqslant 2\epsilon$, while support g is contained in support $(f - v)$. Thus $v + g$ is the desired Lip 1 function with support contained in support f.

Proof of Corollary 3.2. Let v be a function in the domain of A. Put $g = Av - v$. Then $v = (A - I)^{-1} g$, i.e.,

$$v(x) = -\int_0^\infty e^{-t}(p_t g)(x)\, dt. \tag{47}$$

g is a bounded uniformly continuous function on B. Given $\epsilon > 0$, let f be a bounded Lip 1 function on B such that $\sup_x |f(x) - g(x)| < \epsilon$. Let $u = (A - I)^{-1} f$, i.e.,

$$u(x) = -\int_0^\infty e^{-t}(p_t f)(x)\, dt. \tag{48}$$

Now $\| (A - I)^{-1} \| \leqslant 1$ so that $\| u - v \|_\infty < \epsilon$. Moreover,

$$\begin{aligned} \| Au - Av \|_\infty &\leqslant \| (A - I)(u - v) \|_\infty + \| u - v \|_\infty \\ &\leqslant \| f - g \|_\infty + \| u - v \|_\infty < 2\epsilon. \end{aligned}$$

Hence functions of the form (48) where f is a bounded Lip 1 function are dense in the domain of A in the graph norm. It remains to show that u satisfies (ii)-(iv). By Lemma 2.1,

$$D^2 u(x) = -\int_0^\infty e^{-t}(D^2 p_t f)(x)\, dt,$$

where the integral converges in Hilbert-Schmidt norm. By Theorem 3 the integrand is a trace class operator for each t and is moreover continuous as a function of t on $(0, \infty)$ into the space of trace class operators. The estimate (38) shows that

$$\| D^2 u(x) \|_1 \leqslant \int_0^\infty e^{-t} \| (D^2 p_t f)(x) \|_1\, dt < \infty.$$

Hence u satisfies (ii). Moreover by Theorem 3 the map $(t, x) \to (D^2 p_t f)(x)$ is uniformly continuous on $[a, \infty) \times B$ into the space of trace class operators for each $a > 0$ and, since $\int_a^\infty e^{-t}(D^2 p_t f)(x)\, dt$ approaches $D^2 u(x)$ in trace class norm uniformly on B by (38), it follows that $D^2 u(x)$ satisfies (iv).

Finally,

$$\begin{aligned}(\tfrac{1}{2})\text{ trace }[D^2u(x)] &= -\int_0^\infty e^{-t}\,(\tfrac{1}{2})\text{ trace }[(D^2p_tf)(x)]\,dt\\ &= -\int_0^\infty \frac{e^{-t}\partial(p_tf)(x)}{\partial t}\,dt\\ &= -[e^{-t}(p_tf)(x)]_0^\infty - \int_0^\infty e^{-t}(p_tf)(x)\,dt\\ &= f(x) + u(x),\end{aligned}$$

and since $f = (A - I)\,u$, (iii) now follows.

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