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Invariance of simultaneous similarity and equivalence of matrices under extension of the ground field

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ABSTRACT

In this work, we give a new and elementary proof that simultaneous similarity and simultaneous equivalence of families of matrices are invariant under extension of the ground field, a result which is non-trivial for finite fields and first appeared in an earlier paper of Klinger and Levy.

1. Introduction

In this article, we let $\mathbb{K}$ denote a field, $\mathbb{L}$ a field extension of $\mathbb{K}$, and n and p two positive integers.

Definition 1. Two families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ of matrices of $M_n(\mathbb{K})$ indexed over the same set I are said to be **simultaneously similar** when there exists $P \in GL_n(\mathbb{K})$ such that

$$\forall i \in I, \quad PA_iP^{-1} = B_i$$

(such a matrix P will then be called a **base change matrix** with respect to the two families).

Two families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ of matrices of $M_{n,p}(\mathbb{K})$ indexed over the same set I are said to be **simultaneously equivalent** when there exists a pair $(P, Q) \in GL_n(\mathbb{K}) \times GL_p(\mathbb{K})$ such that

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$$\forall i \in I, \quad PA_iQ = B_i.$$

Of course, those relations extend the familiar relations of similarity and equivalence respectively on $M_n(\mathbb{K})$ dans $M_{n,p}(\mathbb{K})$, and they are equivalence relations respectively on $M_n(\mathbb{K})^I$ and $M_{n,p}(\mathbb{K})^I$.

The simultaneous similarity of matrices is generally regarded upon as a “wild problem" where finding a useful characterisation by invariants seems out of reach. See [1] for an account of the problem and an algorithmic approach to its solution (for that last matter, also see [3]).

In this respect, our very limited goal here is to establish the following two results:

Theorem 1. *Let $\mathbb{K} - \mathbb{L}$ be a field extension and I be a set.*

Let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of matrices of $M_n(\mathbb{K})$.

Then $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are simultaneously similar in $M_n(\mathbb{K})$ if and only if they are simultaneously similar in $M_n(\mathbb{L})$.

Theorem 2. *Let $\mathbb{K} - \mathbb{L}$ be a field extension and I be a set.*

Let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of matrices of $M_{n,p}(\mathbb{K})$.

Then $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are simultaneously equivalent in $M_{n,p}(\mathbb{K})$ if and only if they are simultaneously equivalent in $M_{n,p}(\mathbb{L})$.

Remarks 1

(i) In both theorems, the “only if" part is trivial.
(ii) It is an easy exercise to derive Theorem 1 from Theorem 2. However, we will do precisely the opposite!

2. A proof for simultaneous similarity

2.1. A reduction to special cases

In order to prove Theorem 2, we will not, *contra* [3], try to give a canonical form for simultaneous similarity. Instead, we will focus on base change matrices and prove directly that if one exists in $M_n(\mathbb{L})$, then another (possibly the same) also exists in $M_n(\mathbb{K})$. To achieve this, we will prove the theorem in the two following special cases:

(i) $\mathbb{K}$ has at least n elements;
(ii) $\mathbb{K} - \mathbb{L}$ is a separable quadratic extension.

Assuming these cases have been solved, let us immediately prove the general case. Case (i) handles the situation where $\mathbb{K}$ is infinite. Assume now that $\mathbb{K}$ is finite, and choose a positive integer N such that $(\#\,\mathbb{K})^{2^N} \geqslant n$.

Since $\mathbb{K}$ is finite, there exists (see Section V.4 of [4]) a tower of N quadratic separable extensions

$$\mathbb{K} \subset K_1 \subset K_2 \subset \cdots \subset K_N.$$

We let $\mathbb{M}$ denote a compositum extension of K_N and $\mathbb{L}$ (as extensions of $\mathbb{K}$):

$$\begin{tikzcd}
\mathbb{K} \arrow[r, dash] \arrow[d, dash] & K_1 \arrow[r, dash] & K_2 \arrow[r, dash] & \cdots \arrow[r, dash] & K_N \arrow[d, dash] \\
\mathbb{L} \arrow[rrrr, dash] & & & & \mathbb{M}.
\end{tikzcd}$$

Assume the families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ of matrices of $M_n(\mathbb{K})$ are simultaneously similar in $M_n(\mathbb{L})$. Then they are also simultaneously similar in $M_n(\mathbb{M})$. However, $\#\,K_N = (\#\,\mathbb{K})^{2^N} \geqslant n$, so this simultaneous similarity also holds in $M_n(K_N)$. Using case (ii) by induction, we then obtain that $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are simultaneously similar in $M_n(\mathbb{K})$.

2.2. The case $\# \mathbb{K} \geqslant n$

The line of reasoning here is folklore, but we reproduce the proof for sake of completeness. Let then $P \in GL_n(\mathbb{L})$ be such that

$$\forall i \in I, \quad PA_iP^{-1} = B_i,$$

so

$$\forall i \in I, \quad PA_i = B_iP.$$

Let V denote the $\mathbb{K}$-linear subspace of $\mathbb{L}$ spanned by the coefficients of P, and choose a basis $(x_1, \dots, x_N)$ of V. Decompose then

$$P = x_1P_1 + \cdots + x_NP_N$$

with $P_1, \dots, P_N$ in $M_n(\mathbb{K})$, and let W be the $\mathbb{K}$-linear subspace of $M_n(\mathbb{K})$ spanned by the N-tuple $(P_1, \dots, P_N)$. Since the A_i's and the B_i's have all their coefficients in $\mathbb{K}$, the previous relations yield:

$$\forall i \in I, \ \forall k \in [\![1, N]\!], \quad P_kA_i = B_iP_k$$

hence

$$\forall i \in I, \ \forall Q \in W, \quad QA_i = B_iQ.$$

It thus suffices to prove that W contains a non-singular matrix.

However, the polynomial $\det(Y_1\,P_1 + \cdots + Y_N\,P_N) \in \mathbb{K}[Y_1, \dots, Y_N]$ is homogeneous of total degree n and is non-zero because

$$\det(x_1 \cdot P_1 + \cdots + x_N \cdot P_N) = \det(P) \neq 0.$$

Since $n \leqslant \# \mathbb{K}$, we conclude that the map $Q \mapsto \det Q$ does not totally vanish on W, which proves that $W \cap GL_n(\mathbb{K})$ is non-empty.

2.3. The case $\mathbb{L}$ is a separable quadratic extension of $\mathbb{K}$

We choose an arbitrary element $\varepsilon \in \mathbb{L} \setminus \mathbb{K}$ and let σ denote the non-identity automorphism of the $\mathbb{K}$-algebra $\mathbb{L}$. Assume $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ are simultaneously similar in $M_n(\mathbb{L})$, and let $P \in GL_n(\mathbb{L})$ be such that

$$\forall i \in I, \quad PA_iP^{-1} = B_i.$$

We first point out that the problem is essentially unchanged should P be replaced with a $\mathbb{K}$-equivalent matrix of $GL_n(\mathbb{L})$.

Indeed, let $(P_1, P_2) \in GL_n(\mathbb{K})^2$, and set $P'' := P_1PP_2^{-1} \in GL_n(\mathbb{L})$, and $A_i'' := P_2A_i(P_2)^{-1}$ and $B_i'' := P_1B_i(P_1)^{-1}$ for all $i \in I$. Then:

$$\forall i \in I, \quad P''A_i''(P'')^{-1} = B_i''.$$

Since it follows directly from the definition that $(A_i)_{i \in I}$ and $(A_i'')_{i \in I}$ are simultaneously similar in $M_n(\mathbb{K})$, and that it is also true of $(B_i)_{i \in I}$ and $(B_i'')_{i \in I}$, it will suffice to show that $(A_i'')_{i \in I}$ and $(B_i'')_{i \in I}$ are simultaneously similar in $M_n(\mathbb{K})$, knowing that they are simultaneously similar in $M_n(\mathbb{L})$.

Returning to P, we split it as

$$P = Q + \varepsilon R \quad \text{with } (Q, R) \in M_n(\mathbb{K})^2.$$

The previous remark then reduces the proof to the case where the pair (Q, R) is canonical in terms of Kronecker reduction (see Chapter XII of [2] and our Section 4). More roughly, we can assume, since P is non-singular, that, for some $q \in [\![0, n]\!]$:

$$Q = \begin{bmatrix} M & 0 \\ 0 & I_{n-q} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix},$$

where $M \in M_q(\mathbb{K})$, N is a nilpotent matrix of $M_{n-q}(\mathbb{K})$, and we have let I_k denote the unit matrix of $M_k(\mathbb{K})$.

Let $i \in I$. Applying σ coefficient-wise to $PA_iP^{-1} = B_i$, we get:

$$\sigma(P)A_i\sigma(P)^{-1} = B_i = PA_iP^{-1},$$

hence A_i commutes with $\sigma(P)^{-1}P$. We now claim the following result:

Lemma 3. *Under the preceding assumptions, any matrix of $M_n(\mathbb{K})$ that commutes with $\sigma(P)^{-1}P$ also commutes with P.*

Assuming this lemma holds, we deduce that $\forall i \in I,\ PA_iP^{-1} = A_i$, hence $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are equal, thus simultaneously similar in $M_n(\mathbb{K})$, which finishes our proof.

Proof of Lemma 3. Let $A \in M_n(\mathbb{K})$ which commutes with $\sigma(P)^{-1}P$. Applying σ, we deduce that A also commutes with $P^{-1}\sigma(P)$, hence with $I_n + (\sigma(\varepsilon) - \varepsilon)P^{-1}R$, hence with $P^{-1}R$ since $\sigma(\varepsilon) \neq \varepsilon$.

Notice then that

$$P^{-1}R = \begin{bmatrix} (M + \varepsilon \cdot I_q)^{-1} & 0 \\ 0 & (I_{n-q} + \varepsilon N)^{-1}N \end{bmatrix}$$

with $(M + \varepsilon \cdot I_q)^{-1}$ non-singular and $(I_{n-q} + \varepsilon N)^{-1}N$ nilpotent, so A, which stabilizes both $\mathrm{Im}(P^{-1}R)^n$ and $\mathrm{Ker}(P^{-1}R)^n$, must be of the form

$$A = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \quad \text{for some } (C, D) \in M_q(\mathbb{K}) \times M_{n-q}(\mathbb{K}).$$

Commutation of A with $P^{-1}R$ ensures that C commutes with $(M + \varepsilon \cdot I_q)^{-1}$, whereas D commutes with $(I_{n-q} + \varepsilon N)^{-1}N = \varepsilon^{-1} \cdot I_{n-q} - \varepsilon^{-1} \cdot (I_{n-q} + \varepsilon N)^{-1}$ hence with $(I_{n-q} + \varepsilon N)^{-1}$. It follows that A commutes with P^{-1}, hence with P. $\square$

3. A proof for simultaneous equivalence

We will now derive Theorem 2 from Theorem 1. Under the assumptions of Theorem 2, we choose an arbitrary object a that does not belong to I, and define

$$C_a = D_a := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{M}_{n+p}(\mathbb{K})$$

and, for $i \in I$,

$$C_i = \begin{bmatrix} 0 & A_i \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D_i = \begin{bmatrix} 0 & B_i \\ 0 & 0 \end{bmatrix} \quad \text{in } \mathrm{M}_{n+p}(\mathbb{K}).$$

The following two conditions are then equivalent:

(i) $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are simultaneously equivalent;
(ii) $(C_i)_{i\in I\cup\{a\}}$ and $(D_i)_{i\in I\cup\{a\}}$ are simultaneously similar.

Indeed, if condition (i) holds, then we choose $(P, Q) \in GL_n(\mathbb{K}) \times GL_p(\mathbb{K})$ such that $\forall i \in I,\ PA_iQ = B_i$, set $R := \begin{bmatrix} P & 0 \\ 0 & Q^{-1} \end{bmatrix}$, and remark that $R \in GL_{n+p}(\mathbb{K})$ and

$$\forall i \in I \cup \{a\}, \quad RC_iR^{-1} = D_i.$$

Conversely, assume condition (ii) holds, and choose $R \in GL_{n+p}(\mathbb{K})$ such that

$$\forall i \in I \cup \{a\}, \quad RC_iR^{-1} = D_i.$$

Equality $RC_aR^{-1} = D_a$ then entails that R has the form

$$R = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \quad \text{for some } (P, Q) \in GL_n(\mathbb{K}) \times GL_p(\mathbb{K}),$$

and the other relations then imply that

$$\forall i \in I, \quad PA_iQ^{-1} = B_i.$$

Using equivalence of (i) and (ii) with both fields $\mathbb{K}$ and $\mathbb{L}$, Theorem 2 follows easily from Theorem 1.

4. Appendix: on the Kronecker reduction of matrix pencils

Attention was brought to us that, in [2], the proof that every pencil of matrix is equivalent to a canonical one fails for finite fields. We will give a correct proof here in the case of a "weak" canonical form (that is all we need here, and reducing further to a true canonical form is not hard from there using the theory of elementary divisors).

Notation 2. For $n \in \mathbb{N}$, set $L_n = \begin{bmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} \in M_{n,n+1}(\mathbb{K})$ and $K_n = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix} \in M_{n,n+1}(\mathbb{K})$; and, for arbitrary objects a and b, define the Jordan matrix:

$$J_n(a, b) = \begin{bmatrix} a & b & 0 & \\ 0 & a & b & \\ & & \ddots & \ddots \\ & & & \end{bmatrix} \in M_n(\{0, a, b\}).$$

Theorem 4 (Kronecker reduction theorem for pencils of matrices)**.** *Let A and B in $M_{n,p}(\mathbb{K})$. We choose an indeterminate X. Then there are non-singular matrices $(P_1, Q_1) \in GL_n(\mathbb{K}) \times GL_p(\mathbb{K})$ such that $P_1(A + XB)Q_1$ is block-diagonal with every non-zero diagonal block having one of the following forms, and only one of the first type*:

- *$P + XI_r$ for some non-singular $P \in GL_r(\mathbb{K})$*;
- *$J_r(1, X)$; $J_r(X, 1)$; $L_r + XK_r$; $(L_r + XK_r)^t$.*

This decomposition is unique up to permutation of blocks and up to similarity on the non-singular matrix P.

We will only prove here that such a decomposition exists. Uniqueness is not needed here so we will leave it as an exercise for the reader.

We will consider A and B as linear maps from $E = \mathbb{K}^p$ to $F = \mathbb{K}^n$. Without loss of generality, we may assume $\operatorname{Ker} A \cap \operatorname{Ker} B = \{0\}$ and $\operatorname{Im} A + \operatorname{Im} B = F$. We define inductively two towers $(E_k)_{k \in \mathbb{N}}$ and $(F_k)_{k \in \mathbb{N}}$ of respective linear subspaces of E and F by:

(a) $E_0 = \{0\}$; $F_0 = A(\{0\}) = \{0\}$;
(b) $\forall k \in \mathbb{N},\ E_{k+1} = B^{-1}(F_k)$ and $F_{k+1} = A(E_{k+1})$.

Notice that $E_1 = \operatorname{Ker} B$. The sequences $(E_k)_{n \geqslant 0}$ and $(F_k)_{n \geqslant 0}$ are clearly non-decreasing so we can find a smallest integer N such that $E_N = E_k$ for every $k \geqslant N$. Hence $F_N = F_k$ for every $k \geqslant N$, and $E_N = B^{-1}(F_N)$. It follows that $A(E_N) = F_N$ and $B(E_N) \subset F_N$. We now let f and g denote the linear maps from E_N to F_N induced by A and B.

From there, the proof has two independent major steps:

Lemma 5. *There are bases* $\mathbf{B}$ *and* $\mathbf{C}$ *respectively of* E_N *and* F_N *such that* $M_{\mathbf{B},\mathbf{C}}(f)+XM_{\mathbf{B},\mathbf{C}}(g)$ *is block-diagonal with all non-zero blocks having one of the forms* $J_r(1,X)$ *or* L_s+XK_s.

Lemma 6. *There are splittings* $E=E_N\oplus E''$ *and* $F=F_N\oplus F''$ *such that* $A(E'')\subset F''$ *and* $B(E'')\subset F''$.

Assuming those lemmas are proven, let us see how we can easily conclude:

- We deduce from the two previous lemmas that $A+XB$ is $\mathbb{K}$-equivalent to some $\begin{bmatrix} A''+X\,B'' & 0 \\ 0 & C(X)\end{bmatrix}$ where $C(X)$ is block-diagonal with all non-zero blocks of the form $J_r(1,X)$ or $L_s+X\,K_s$, and A'' and B'' have coefficients in $\mathbb{K}$, with $\operatorname{Ker} B''=\{0\}$; it will thus suffice to prove the existence of a canonical form for the pair (A'',B'');
- applying the first step of the proof to the matrices $(A'')^t$ and $(B'')^t$, we find that $A''+XB''$ is $\mathbb{K}$-equivalent to some $\begin{bmatrix} A''+X\,B'' & 0 \\ 0 & D(X)\end{bmatrix}$ where $D(X)$ is block-diagonal with all non-zero blocks of the form $J_r(1,X)^t$ (which is $\mathbb{K}$-similar to $J_r(1,X)$) or $(L_s+XK_s)^t$, and A'' and B'' have coefficients in $\mathbb{K}$, with $\operatorname{Ker} B''=\{0\}$ and $\operatorname{coker} B''=\{0\}$. It follows that B'' is non-singular.
- Finally, $(B'')^{-1}(A''+XB'')=(B'')^{-1}A''+X\cdot I_k$ for some integer k, and the pair (A'',B'') can thus be reduced by using the Fitting decomposition of $(B'')^{-1}A''$ combined with a Jordan reduction of its nilpotent part: this yields a block-diagonal matrix $\mathbb{K}$-equivalent to $A''+XB''$ with all diagonal blocks of the form $J_r(X,1)$ or $P+X\cdot I_s$ for some non-singular P. This completes the proof of existence.

Proof of Lemma 6. We proceed by induction.

Assume, for some $k\in[\![1,N]\!]$, that there are splittings $E=E_N\oplus E''$ and $F=F_N\oplus F''$ such that $A(E'')\subset F_k\oplus F''$ and $B(E'')\subset F_k\oplus F''$. Since $B^{-1}(F_N)=E_N$, the subspaces F_N and $B(E'')$ are independent. We can therefore find some F'' such that $F_k\oplus F''=F_k\oplus F''$, $F_N\oplus F''=F$ and $B(E'')\subset F''$. Choose then a basis $(e_1,\dots,e_p)$ of E'', and decompose $A(e_i)=f_i+f_i''$ for all $i\in[\![1,p]\!]$, with $f_i\in F_k$ and $f_i''\in F''$. For $i\in[\![1,p]\!]$, we have $f_i=A(g_i)$ for some $g_i\in E_k$. Then $(e_1-g_1,\dots,e_p-g_p)$ still spans a complementary subspace E'' of E_N in E, and we now have $A(e_i-g_i)\in F''$ and $B(e_i-g_i)\in F''\oplus F_{k-1}$ for all $i\in[\![1,p]\!]$. Hence $E=E_N\oplus E''$ and $F=F_N\oplus F''$, now with $A(E'')\subset F_{k-1}\oplus F''$ and $B(E'')\subset F_{k-1}\oplus F''$. The condition is thus proven at the integer $k-1$. By downward induction, we find that it holds for $k=0$. $\square$

Proof of Lemma 5. The argument is similar to the standard proof of the Jordan reduction theorem.

- Split $F_N=F_{N-1}\oplus W_{N,N}$ and $E_N=E_{N-1}\oplus V_{N,N}\oplus V''_{N,N}$ such that $E_{N-1}\oplus V''_{N,N}=E_{N-1}+(E_N\cap\operatorname{Ker} f)$, $V''_{N,N}\subset\operatorname{Ker} f$ and $f(V_{N,N})=W_{N,N}$ (so f induces an isomorphism from $V_{N,N}$ to $W_{N,N}$). Set $W_{N,N-1}=g(V_{N,N})$ and $W''_{N,N-1}=g(V''_{N,N})$. Remark that $F_{N-2}\oplus W_{N,N-1}\oplus W''_{N,N-1}\subset F_{N-1}$, and split $F_{N-1}=F_{N-2}\oplus W_{N,N-1}\oplus W''_{N,N-1}\oplus W_{N-1,N-1}$.
- We then proceed by downward induction to define four families of linear subspaces $(V_{\ell,k})_{1\leqslant k\leqslant \ell\leqslant N}$, $(V''_{\ell,k})_{1\leqslant k\leqslant \ell\leqslant N}$, $(W_{\ell,k})_{1\leqslant k\leqslant \ell\leqslant N}$ and $(W''_{\ell,k})_{1\leqslant k\leqslant \ell-1\leqslant N-1}$ such that:

  (i) for every $k\in[\![1,N]\!]$,
$$E_k=E_{k-1}\oplus V_{k,k}\oplus V_{k+1,k}\oplus\cdots\oplus V_{N,k}\oplus V''_{k,k}\oplus V''_{k+1,k}\oplus\cdots\oplus V''_{N,k};$$

  (ii) for every $k\in[\![1,N]\!]$,
$$F_k=F_{k-1}\oplus W_{k,k}\oplus W_{k+1,k}\oplus\cdots\oplus W_{N,k}\oplus W''_{k+1,k}\oplus W''_{k+2,k}\oplus\cdots\oplus W''_{N,k};$$

  (iii) for every $k\in[\![1,N]\!]$, $E_{k-1}+(E_k\cap\operatorname{Ker} f)=E_{k-1}\oplus V''_{k,k}$ and $V''_{k,k}\subset\operatorname{Ker} f$;

(vi) for every $\ell \in [\![1, N]\!]$ and $k \in [\![2, \ell]\!]$, g induces an isomorphism $g_{\ell,k} : V_{\ell,k} \xrightarrow{\simeq} W_{\ell,k-1}$ and an isomorphism $g''_{\ell,k} : V''_{\ell,k} \xrightarrow{\simeq} W''_{\ell,k-1}$;

(v) for every $\ell \in [\![1, N]\!]$ and $k \in [\![1, \ell]\!]$, f induces an isomorphism $f_{\ell,k} : V_{\ell,k} \xrightarrow{\simeq} W_{\ell,k}$ and, if $k < \ell$, an isomorphism $f''_{\ell,k} : V''_{\ell,k} \xrightarrow{\simeq} W''_{\ell,k}$.

$$\begin{tikzcd}
 & V_{\ell,1} \arrow[dl, "g"'] \arrow[d, "f"', "\simeq"] & \cdots \arrow[dl, "g"'] & V_{\ell,\ell-1} \arrow[dl, "g"', "\simeq"] \arrow[d, "f"', "\simeq"] & V_{\ell,\ell} \arrow[dl, "g"', "\simeq"] \arrow[d, "f"', "\simeq"] \\
\{0\} & W_{\ell,1} & \cdots & W_{\ell,\ell-1} & W_{\ell,\ell}.
\end{tikzcd}$$

$$\begin{tikzcd}
 & V'_{\ell,1} \arrow[dl, "g"'] \arrow[d, "f"', "\simeq"] & \cdots \arrow[dl, "g"'] & V'_{\ell,\ell-1} \arrow[dl, "g"', "\simeq"] \arrow[d, "f"', "\simeq"] & V'_{\ell,\ell} \arrow[dl, "g"', "\simeq"] \arrow[d, "f"'] \\
\{0\} & W'_{\ell,1} & \cdots & W'_{\ell,\ell-1} & \{0\}.
\end{tikzcd}$$

- Set $\ell \in [\![1, N]\!]$. Define

$$G_\ell = V_{\ell,1} \oplus \cdots \oplus V_{\ell,\ell}, \quad G''_\ell = V''_{\ell,1} \oplus \cdots \oplus V''_{\ell,\ell},$$
$$H_\ell = W_{\ell,1} \oplus \ldots \oplus W_{\ell,\ell} \quad \text{and} \quad H''_\ell = W''_{\ell,1} \oplus \ldots \oplus W''_{\ell,\ell-1}.$$

Notice that:

$$f(G_\ell) = H_\ell, \quad g(G_\ell) \oplus W_{\ell,\ell} = H_\ell, \quad f(G''_\ell) = H''_\ell \quad \text{and} \quad g(G''_\ell) = H''_\ell.$$

From there, it is easy to conclude.
- Let $n_\ell = \dim W_{\ell,\ell}$. Remark that $\dim V_{\ell,k} = \dim W_{\ell,k} = n_\ell$ for every $k \in [\![1, \ell]\!]$ and choose a basis $\mathbf{C}_{\ell,\ell}$ of $W_{\ell,\ell}$. Define $\mathbf{B}_{\ell,\ell} = f_{\ell,\ell}^{-1}(\mathbf{C}_{\ell,\ell})$, $\mathbf{C}_{\ell,\ell-1} := g_{\ell,\ell}(\mathbf{B}_{\ell,\ell})$ and proceed by induction to recover a basis for $V_{\ell,k}$ and $W_{\ell,k}$ for every suitable k: by glueing together those bases, we recover respective bases $(\mathbf{B}_{\ell,1}, \ldots, \mathbf{B}_{\ell,\ell})$ and $(\mathbf{C}_{\ell,1}, \ldots, \mathbf{C}_{\ell,\ell})$ of G_ℓ and H_ℓ and remark that f and g induce linear maps from G_ℓ to H_ℓ with respective matrices $L_{\ell-1} \otimes I_{n_\ell}$ and $K_{\ell-1} \otimes I_{n_\ell}$ in those bases (remember that $E_1 = \operatorname{Ker} g$). A simple permutation of bases shows that those linear maps can be represented by $I_{n_\ell} \otimes L_{\ell-1}$ and $I_{n_\ell} \otimes K_{\ell-1}$ in a suitable common pair of bases.
- Proceeding similarly for G''_ℓ and H''_ℓ, but starting from a basis of $V''_{\ell,\ell}$, we obtain that f and g induce linear maps from G''_ℓ to H''_ℓ and there is a suitable choice of bases so that their matrices are respectively $I_s \otimes I_\ell$ and $I_s \otimes J_\ell(0, 1)$ for some integer s.
- Notice that we have defined splittings

$$E_N = G_1 \oplus G''_1 \oplus G_2 \oplus G''_2 \oplus \cdots \oplus G_N \oplus G''_N$$

and

$$F_N = H_1 \oplus H''_1 \oplus H_2 \oplus H''_2 \oplus \cdots \oplus H_N \oplus H''_N,$$

therefore Lemma 5 is proven by glueing together the various bases built here. $\square$

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