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# The cone condition and $t$-designs

Peter Dukes[a], Richard M. Wilson[b]

[a] Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3P4, Canada
[b] Department of Mathematics, 253-37, California Institute of Technology, Pasadena, CA 91125, USA

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**Abstract**

The existence of a $t$-$(v, k, \lambda)$ design implies that certain 'almost constant' vectors belong to the convex cone generated by the columns of the incidence matrix of $t$-subsets versus $k$-subsets of a $v$-set. We prove that some vectors are not in, or in a few cases are in, this cone—whether a design exists or not. When certain vectors are shown not to be in this cone, the implication is an inequality on the parameters or a condition on the structure of a $t$-design. We unify a number of known inequalities for $t$-designs, and derive some new ones concerning $t$-wise balanced designs, with this approach.


## 1. Introduction and summary

By a $t$-vector on a set $V$, we mean a function $f$ assigning a real number $f(T)$ to each $t$-element subset $T$ of $V$. Let $\mathcal{F}$ be a family (multiset) of subsets of $V$ with $|A| \in \{k_1, \ldots, k_\ell\}$ for every $A \in \mathcal{F}$. We use the term *block* for a member of $\mathcal{F}$ (i.e. with multiplicity $\geq 1$). Given a nonnegative integer valued $t$-vector $f$ on $V$, we say that $\mathcal{F}$ *realizes* $f$ when the number of members of $\mathcal{F}$, counting multiplicities, that contain $T$ is exactly $f(T)$ for every $t$-subset $T$ of $V$. We also say $f$ is *realizable* with 'block sizes in $\{k_1, \ldots, k_\ell\}$'.

When $\mathcal{F}$ realizes a constant vector $f_\lambda$ with $f_\lambda(T) = \lambda$ for all $T$, then the pair $(V, \mathcal{F})$ is called a $t$-*wise balanced design* ($t$BD) of index $\lambda$ on $v$ points. If all blocks of such a $t$BD have (a single) size $k$, it is also known as a $t$-$(v, k, \lambda)$ *design*.

We will give necessary conditions for the realizability of certain nonnegative integral $t$-vectors $f$. These imply many inequalities on the parameters of $t$BDs containing subconfigurations of various kinds.

*E-mail addresses:* dukes@math.uvic.ca (P. Dukes), rmw@caltech.edu (R.M. Wilson).

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We often find useful a 'functional notation' when dealing with vectors and matrices: $h(i)$ will denote the $i$th coordinate of a vector $h$ and $M(i, j)$ the entry in row $i$ and column $j$ of the matrix $M$. We will usually not distinguish between row vectors and column vectors. For integers $t, k, v$ with $t \le k \le v$, let $W_{tk}^v$ denote the $\binom{v}{t}$ by $\binom{v}{k}$ matrix whose rows are indexed by the $t$-subsets of a $v$-set $X$, whose columns are indexed by the $k$-subsets of $X$, and where the entry in row $T$ and column $K$ is 1 if $T \subseteq K$ and 0 otherwise. $C_{tk}^v$ will denote the convex cone generated by the columns of $W_{tk}^v$.

A family $\mathcal{F}$ of subsets of $V$ may be identified with a nonnegative integer valued vector $h$ indexed by the subsets of $V$, where $h(K)$ is the multiplicity of $K$ in $\mathcal{F}$. It is clear that $f$ is realized by a family $\mathcal{F}$ with block sizes in $\{k_1, \ldots, k_\ell\}$ if and only if $Wh = f$, where $W$ is the $t$-incidence matrix for the set of subsets of the allowed sizes, i.e.

$$W = [W_{tk_1}^v, W_{tk_2}^v, \ldots, W_{tk_\ell}^v]. \tag{1.1}$$

In summary, $f$ is realizable (with the restriction on block sizes) when and only when $f$ is a *nonnegative integral* linear combination of the columns of $W$. A necessary condition for $f$ to be realizable is thus that $f$ is a nonnegative real (or rational) linear combination of the columns of $W$, i.e. that $f$ is in the convex cone generated by the columns of $W$. We call this the *cone condition* for the realizability of $f$. In general, deciding whether the cone condition holds is a linear programming problem (a feasibility question).

**Remark.** That $f$ is an integral linear combination of the columns of $W$ may be called the *integrality condition* for the realizability of $f$. This has been discussed elsewhere; see [8].

We point out in Section 2 that the cone generated by the columns of $W$ is in fact equal to the cone $C_{tk}^v$ when $k$ is the *minimum* of $k_1, k_2, \ldots, k_\ell$. Section 2 also covers some preliminaries on convex geometry.

A nonnegative constant vector $f_\lambda$ is *always* in the cone $C_{tk}^v$ (cf. Section 2), so it may at first appear that the cone condition provides no information relevant to $t$-designs. But a number of interesting inequalities arise when we consider the $t$-vector realized by a family $\mathcal{F}$ obtained by *deleting* a configuration consisting of as few as one or two blocks from the family of blocks of a $t$-design.

For example, if a single block is deleted from the family of blocks of a $t$-$(v, k, \lambda)$ design, $k < v$, we obtain a family $\mathcal{F}$ of $k$-subsets of a $v$-set $V$ that realizes the $t$-vector $f'$ with

$$f'(T) = \begin{cases} \lambda - 1 & \text{if } T \subseteq K, \\ \lambda & \text{otherwise} \end{cases}$$

for $t$-subsets $T$ of $V$, where $K$ is the set of points of the deleted block. A particular instance of Theorem 1.2 below shows that if $t \ge 2s$, $v \ge k + s$ and $f' \in C_{tk}^v$, then $b \ge \binom{v}{s}$, where $b = \lambda \binom{v}{t} / \binom{k}{t}$ is the number of blocks of the design. The case $s = 1$ is the well known Fisher Inequality for 2-designs: $b \ge v$.

The cone condition is more-or-less naturally adapted to give necessary conditions for designs with 'holes'. An *incomplete* $t$BD on a set $V$ with a *hole* $U \subseteq V$ is a family $\mathcal{F}$ of subsets of $V$ that realizes, for some $\lambda$, the $t$-vector $f$ given by

$$f(T) = \begin{cases} 0 & \text{if } T \subseteq U, \\ \lambda & \text{otherwise}. \end{cases} \tag{1.2}$$

Note that if $U$ is a block of a $t$BD with index $\lambda = 1$, then removing $U$ from the family of blocks produces an incomplete $t$BD with a hole $U$. Kreher and Rees [4] confirmed a conjecture of Kramer [3] (the case $\lambda = 1$) by proving that if $t \le u < v$ and there exists an incomplete $t$BD on $v$ points with a hole $U$ of size $u$ in which all blocks have sizes $\ge t+1$, then

$$u \le \begin{cases} (v-1)/2 & \text{if } t \ge 2 \text{ is even,} \\ v/2 & \text{if } t \ge 3 \text{ is odd.} \end{cases}$$

A short proof of the Kreher–Rees result was given in [9] as an illustration of the cone condition. In Section 3, we prove a somewhat more general result:

**Theorem 1.1.** *Let $U$ be a $u$-subset of a $v$-set $V$ with $t < u \le v$. Assume $t - s$ is even for some $s \le t$, and let $f$ be the $t$-vector on $V$ given by*

$$f(T) = \begin{cases} 0 & \text{if } |T \cap U| \ge t - s, \\ \lambda & \text{otherwise.} \end{cases} \tag{1.3}$$

(1) *$f \in C^v_{t,t+1}$ if and only if $u \le (v-s-1)/2$.*
(2) *If $f$ is realized by a family $\mathcal{F}$ with block sizes $\ge t+1$, then $u = (v-s-1)/2$ if and only if every block has at most $t-s$ points outside $U$.*

The following theorem will be proved in Section 4.

**Theorem 1.2.** *Let $s, t, k, w, v$ be positive integers with $t \ge 2s$ and $t \le k, w \le v - s$. Let $U$ be a $w$-subset of a $v$-set $V$. Given real numbers $\lambda$, $\nu$, $0 \le \nu \le \lambda$, consider the $t$-vector $f$ on $V$ defined by*

$$f(T) = \begin{cases} \lambda - \nu & \text{if } T \subseteq U, \\ \lambda & \text{otherwise.} \end{cases} \tag{1.4}$$

If $f$ is in the cone $C^v_{tk}$, then

$$\lambda \binom{v}{t} \ge \nu \binom{w}{t} \sum_{i=0}^{s} \left( \binom{v}{i} - \binom{v}{i-1} \right) \frac{\binom{k}{i}\binom{v-w}{i}}{\binom{w}{i}\binom{v-k}{i}}. \tag{1.5}$$

More generally, the same inequality holds if we relax the condition $f(T) = \lambda - \nu$ for $T \subseteq U$ to the condition that the *average* value of $f$ on $t$-subsets $T \subseteq U$ is $\lambda - \nu$.

We comment on the case of equality in (1.5) in Section 4.

The inequality (1.5) holds when there exists an $\nu$-fold block (a block of multiplicity $\nu$) of size $w$, $t \le w \le v - s$, in a $t$BD with minimum block size $\ge k$ on $v$ points with index $\lambda$, because deleting $\nu$ copies of a block with point set $U$ produces a family that realizes the $t$-vector $f$ in (1.4).

In the case $w = k$, (1.5) simplifies to

$$\lambda \frac{\binom{v}{t}}{\binom{k}{t}} \ge \nu \binom{v}{s}.$$

The left-hand side is the number $b$ of blocks of a $t$-$(v, k, \lambda)$ design. So a corollary of Theorem 1.2 is that $b \ge m\binom{v}{s}$ when there is a $m$-fold block in a nontrivial $t$-$(v, k, \lambda)$ design and $t \ge 2s$. This was first proved in [7] and includes as special cases the inequality due to Mann for $t = 2$, and Ray-Chaudhuri and Wilson's generalization of Fisher's Inequality for $m = 1$.

A $t$-$(w, k, \nu)$ design on a point set $U$ is *enclosed* in a $t$-$(v, k, \lambda)$ design on point set $V$ when $U \subseteq V$ and the block family of the first design is contained in the block family of the second. By considering blocks in the second design but not the first, we obtain a family $\mathcal{F}$ of $k$-subsets which realizes the $t$-vector $f$ in (1.4). So again (1.5) is a necessary condition. If $\nu = \lambda$, the first design is a *subdesign* of the second. If additionally we take $t = 2$ (i.e. $s = 1$), (1.5) reduces after some simplification to the well known inequality

$$w \leq \frac{v-1}{k-1} \tag{1.6}$$

on the size of a proper subdesign of a 2-design. For general $t$, taking $\nu = \lambda$ gives a constraint on the size $w$ of a hole in an incomplete $t$BD with minimum block size $k$. While equality can occur in many cases, it should be noted that we do not obtain the result $w \leq (v-1)/2$ when $k = t+1$ that was proved in [4] and is contained in Theorem 1.1.

Another corollary of Theorem 1.2 is the upper bound derived in [6] for the number $n$ of blocks contained in a $w$-subset $U$ of the $v$ points of a $t$-$(v, k, \lambda)$ design. If the $n$ blocks are deleted from the block set, we obtain a family realizing a $t$-vector $f$ whose average value on $t$-subsets $T \subseteq U$ is $\lambda - \nu$, where $\nu = n\binom{k}{t}/\binom{w}{t}$.

A dual inequality, also from [6], concerning $n$ blocks which contain a $w$-set $U$ in their intersection follows by an easy modification. Given a collection $\{A_1, \ldots, A_b\}$ of blocks of a set $V$, its *supplement* is the family $\{V \setminus A_1, \ldots, V \setminus A_b\}$. The supplement of a $t$-$(v, k, \lambda)$ design is a $t$-$(v, v-k, \lambda')$ design with $\lambda' = \lambda\binom{v-k}{t}/\binom{k}{t}$. If there are $n$ blocks containing a $w$-subset in their intersection in some design, then there are $n$ blocks contained in a $(v-w)$-subset in the supplementary design. Alternatively, the $t$-vector

$$f(T) = \begin{cases} \lambda' - \nu & \text{if } T \subseteq V \setminus U, \\ \lambda' & \text{otherwise} \end{cases}$$

can be tested for containment in $C^v_{t,v-k}$ to obtain

$$\lambda'\binom{v}{t} \geq \nu\binom{v-w}{t}\sum_{i=0}^{s}\left(\binom{v}{i} - \binom{v}{i-1}\right)\frac{\binom{w}{i}\binom{v-k}{i}}{\binom{k}{i}\binom{v-w}{i}}.$$

Perhaps surprisingly, it will be shown that the following cone condition for certain 2-vectors $f$ is a consequence of the same general result as Theorem 1.2.

**Theorem 1.3.** *Let $A_1$ and $A_2$ be $k$-subsets of a $v$-set that meet in $\mu$ points, and let $f$ be the 2-vector so that*

$$f(T) = \begin{cases} \lambda - 2 & \textit{if } T \subseteq A_1 \cap A_2, \\ \lambda - 1 & \textit{if } T \subseteq A_1 \textit{ or } T \subseteq A_2, \textit{but not both}, \\ \lambda & \textit{otherwise}. \end{cases} \tag{1.7}$$

*If $f \in C^v_{2k}$, then*

$$k + \lambda - r \leq \mu \leq r - k - \lambda + \frac{\lambda k}{r}, \tag{1.8}$$

*where $r = \lambda(v-1)/(k-1)$.*

This result also has a natural interpretation for designs. Deleting two blocks $A_1$ and $A_2$ with $|A_1 \cap A_2| = \mu$ from the blocks of a 2-$(v, k, \lambda)$ design yields a family realizing the 2-vector $f$ in (1.7). The reader may recognize the necessary conditions (1.8) as Connor's inequalities on block intersection size in a 2-design.

The idea of the cone condition is simple. However, there are often difficult computations involved in simplifying the resulting expressions. Moreover, it is perhaps unfortunate that after such simplification many of the same classical inequalities appear. On the other hand, it is our hope that the cone condition will be seen as a new general approach which unifies several earlier results. For example, the main theorem in Section 4 includes both the extension [7] of Connor's inequalities to $t$-designs with $t \geq 2$ and the various extensions of Fisher's inequality discussed earlier. Originally, these were obtained with different methods. But the cone condition (after some calculations) gives both.

## 2. Preliminaries on cones and inequalities

We begin with an easy observation.

**Lemma 2.1.** *A $t$-vector $f$ is in the convex cone $C$ generated by columns of*

$$W = [W^v_{tk_1}, W^v_{tk_2}, \ldots, W^v_{tk_\ell}]$$

*if and only if $f$ is in the cone $C^v_{tk}$, where $k = \min\{k_1, \ldots, k_\ell\}$.*

**Proof.** Clearly, $C^v_{tk} \subseteq C$. Now for $H \subseteq V$, Let $e_H$ be the 0-1 $t$-vector with $e_H(T) = 1$ if and only if $T \subseteq H$. For $t \leq k \leq h$, and $|H| = h$, we have

$$e_H = \frac{1}{\binom{h-t}{k-t}} \sum_{K \subset H, |K|=k} e_K.$$

So the columns of $W^v_{tk_j}$ are nonnegative linear combinations of the columns of $W^v_{tk}$. It follows that $C = \cup_j C^v_{tk_j} \subseteq C^v_{tk}$. □

The Minkowski–Farkas lemma asserts that either a system of linear equations $Wh = f$ has a nonnegative solution $h$, or there exists a vector $g$ so that $gW$ has all nonnegative coordinates but the scalar product $gf$ is negative, but not both.

In particular, we can prove that a $t$-vector $f$ is not in the cone $C^v_{tk}$ by exhibiting a $t$-vector $g$ such that $gW \geq \mathbf{0}$ but $gf < 0$.

We first restate the Minkowski–Farkas lemma in our notation:

**Lemma 2.2.** *Let $V$ be a $v$-set and $f$ a $t$-vector on $V$. Then $f \in C^v_{tk}$ if and only if, for every $t$-vector $g$ such that*

$$\sum_{T \subseteq K} g(T) \geq 0$$

*for every $k$-subset $K$ of $V$, we have*

$$\sum_{T \subseteq V} g(T) f(T) \geq 0.$$

(*The summations are over all $t$-subsets $T$ contained in $K$ and $V$, respectively.*)

In what follows, we will primarily call upon the 'only if' part of Lemma 2.2. This is the easier implication of the two. We rephrase this in terms of 'realizability' and make a standard observation about equality.

**Lemma 2.3.** *If a $t$-vector $f$ is realized by a family $\mathcal{F}$ of $k$-sets and $g$ is a $t$-vector so that*

$$\sum_{T\subseteq K} g(T) \geq 0 \tag{2.1}$$

*for every $k$-subset $K$ of $V$, then*

$$\sum_{T\subseteq V} g(T)f(T) \geq 0.$$

*Equality holds if and only if*

$$\sum_{T\subseteq A} g(T) = 0$$

*for every block $A$ of $\mathcal{F}$.*

**Proof.** Suppose $g$ has the property above. Let $h(K)$ be the multiplicity of a $k$-set $K$ in $\mathcal{F}$. Then, in matrix notation, $gf = g(Wh) = (gW)h \geq 0$. In other notation,

$$\begin{aligned}\sum_T g(T)f(T) &= \sum_T g(T)\left(\sum_{K:T\subseteq K} h(K)\right)\\ &= \sum_K h(K)\left(\sum_{T:T\subseteq K} g(T)\right) \geq 0,\end{aligned}$$

with equality if and only if at least one of $h(K)$ or $\sum_{T\subseteq K} g(T)$ vanishes for every $k$-set $K$. □

In general, a $g$ satisfying (2.1) is normal to some hyperplane 'supporting' the cone. The convex cones in question here are all polyhedral. So there are a finite number of *facets*, or faces of codimension 1. A facet can be described by a dual $t$-vector $g$. The direction of $g$ is toward the interior of the cone. The hard part of Lemma 2.2, i.e. the sufficient conditions for $f \in C^v_{tk}$, can therefore be reduced to checking

$$\sum_{T\subseteq V} g(T)f(T) \geq 0$$

for the finitely many $g$ corresponding to facets of $C^v_{tk}$. If we could efficiently list all facets, the cone condition on $f$ would become simple. But there seem to be very many facets, even when $t = 2$ and $k = 3$.

On the other hand, it is possible to describe some facets of the cone $C^v_{tk}$. Pick an arbitrary subset $U$ of $V$, $|U| \geq t$. Let $p(X)$ be a polynomial of degree $t$ which is nonnegative on $\{0, 1, \ldots, k\}$ and vanishes at exactly $t$ of these integers. (Thus, the zeros appear in pairs of consecutive integers.) Write

$$p(X) = \sum_{j=0}^{t} a_j \binom{X}{j}\binom{k-X}{t-j}$$

and set $g(T) = a_j$ if and only if $|T \cap U| = j$. The details showing such a $g$ is a facet appear in [1]. When $t = 2$ and $p(X) = (X - c)(X - c - 1)$, $1 \le c \le k - 2$, the entries of $g$ are (up to scalar multiple)

$$a_0 = \frac{c+1}{k-c-1}, \qquad a_1 = -1, \qquad a_2 = \frac{k-c}{c}. \tag{2.2}$$

Together with three 'degenerate' facets corresponding to $(a_0, a_1, a_2) = (1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, these are all facets of this form.

In general, suppose we have a partition of the $v$-set $V$ into sets $U_1, U_2, \dots, U_d$. We say a $t$-vector $h$ is *invariant* under this partition if $h(T)$ depends only on how $T$ intersects the partition; that is, on the sizes $|T \cap U_j|$, for $j = 1, \dots, d$. It is an easy and convenient fact that the necessary and sufficient cone conditions in Lemma 2.2 are simplified when $f$ is invariant under a coarse partition. In more detail, let $H$ denote the subgroup $\mathrm{Sym}(U_1) \times \cdots \times \mathrm{Sym}(U_d)$ of $\mathrm{Sym}(V)$. Suppose $h$ is any $t$-vector and consider 'averaging' over the partition to obtain $h^*$, where

$$h^*(T) = \frac{1}{|H|} \sum_{\sigma \in H} h(\sigma(T)).$$

If $f$ is an invariant $t$-vector, we have $gf = g^* f$ for any $g$.

**Lemma 2.4.** . *Suppose $f$ is a $t$-vector invariant under some partition $U_1, \dots, U_d$ of a $v$-set $V$. Then $f \in C_{tk}^v$ if and only if the conclusion of Lemma 2.2 holds for all $t$-vectors $g$ invariant under the same partition.*

Given nonnegative integers $d$ and $n$, let $\Sigma_d(n)$ denote the set of nonnegative integral vectors $(a_1, a_2, \dots, a_d)$ so that $a_1 + a_2 + \cdots + a_d = n$.

Suppose we have a partition of the $v$-set $V$ into sets $U_1, U_2, \dots, U_d$, of respective sizes $u_1, u_2, \dots, u_d$, and we are given a $t$-vector $f$ invariant under this partition, i.e. there are constants $f_{i_1 i_2 \dots i_d}$ for $(i_1, i_2, \dots, i_d) \in \Sigma_d(t)$ so that

$$f(T) = f_{i_1 i_2 \dots i_d} \quad \text{whenever } |T \cap U_\alpha| = i_\alpha,\ \alpha = 1, 2, \dots, d.$$

**Lemma 2.5.** *For $f$ as above, $f$ is in the cone $C_{tk}^v$ if and only if there is a nonnegative solution*

$$(x_{j_1 j_2 \dots j_d}), \quad (j_1, j_2, \dots, j_d) \in \Sigma_d(k),$$

*to the system of linear equations*

$$\sum_{(j_1, j_2, \dots, j_d) \in \Sigma_d(k)} \binom{j_1}{i_1}\binom{j_2}{i_2}\cdots\binom{j_d}{i_d} x_{j_1 j_2 \dots j_d} = \binom{u_1}{i_1}\binom{u_2}{i_2}\cdots\binom{u_d}{i_d} f_{i_1 i_2 \dots i_d},$$
$$(i_1, i_2, \dots, i_d) \in \Sigma_d(t).$$

**Remark.** If $f$ is realized by a family $\mathcal{F}$, we obtain a solution of the above system by taking $x_{j_1 j_2 \cdots j_d}$ to be the number of blocks $K$ in $\mathcal{F}$ that satisfy $|K \cap U_j| = j_i$, $i = 1, 2, \dots, d$.

The following theorem explains in part why polynomials arise in our later proofs.

**Theorem 2.6.** *For $f$ as above, $f$ is in the cone $C_{tk}^v$ if and only if*

$$\sum_{(i_1, i_2, \dots, i_d) \in \Sigma_d(t)} g_{i_1 i_2 \dots i_d} f_{i_1 i_2 \dots i_d} \binom{u_1}{i_1}\binom{u_2}{i_2}\cdots\binom{u_d}{i_d} \ge 0 \tag{2.3}$$

*for every polynomial*

$$p(X_1, \ldots, X_d) = \sum_{(i_1,\ldots,i_d)\in\Sigma_d(t)} g_{i_1 i_2 \ldots i_d} \binom{X_1}{i_1}\binom{X_2}{i_2}\cdots\binom{X_d}{i_d}$$

*that is nonnegative on all vectors* $(X_1, \ldots, X_d) \in \Sigma_d(k)$, $(X_1, \ldots, X_d) \leq (u_i, \ldots, u_d)$.

**Example 2.1.** Consider a partition $U_1, U_2, U_3$ of an 8-set with $|U_1| = |U_2| = 2$ and $|U_3| = 4$. The (degree two) polynomial

$$p(X_1, X_2, X_3) = 2\binom{X_1}{2} + 2\binom{X_2}{2} - X_1X_2 + X_2X_3$$

is nonnegative on nonnegative integer triples which sum to 3; thus it corresponds to a supporting $t$-vector $g$ of $C^8_{23}$ (with entries 2, 1, −1, 0) invariant under the partition. In fact, it can be shown that $p$ corresponds to a facet.

## 3. Some examples for $t$-designs

In this section, various applications of the cone condition are given with interpretations for $t$-designs.

Say that a block family $\mathcal{F}$ realizing the $t$-vector in (3.1) below is an incomplete $t$BD with a hole $U$ of *strength* $s$. The special case $s = 0$ was discussed in Section 1.

**Theorem 3.1.** *Let $U$ be a $u$-subset of a $v$-set $V$ with $t < u \leq v$. Assume $t - s$ is even for some $s \leq t$, and let $f$ be the $t$-vector on $V$ given by*

$$f(T) = \begin{cases} 0 & \text{if } |T \cap U| \geq t - s, \\ \lambda & \text{otherwise}. \end{cases} \tag{3.1}$$

(1) $f \in C^v_{t,t+1}$ *if and only if* $v \geq 2u + s + 1$.
(2) *If $f$ is realized by a family $\mathcal{F}$ with block sizes $\geq t + 1$, then $u = (v - s - 1)/2$ if and only if every block has at most $t$ points outside $U$.*

**Proof.** Define the polynomial $p(X) = (-1)^t \binom{X-1}{t}$. Clearly, $p(x) \geq 0$ for $x = 0, 1, \ldots, t$; in fact, $p(x)$ vanishes for all of these integers except $x = 0$. We will make use of the identity

$$\sum_{i=0}^{t-r} \frac{(-1)^i}{\binom{t}{i}} \binom{X}{i}\binom{Y}{t-i} = \frac{t+1}{X+Y-t}\left[\frac{(-1)^{t-r}}{\binom{t+1}{r}}\binom{X}{t-r+1}\binom{Y}{r} + \binom{Y}{t+1}\right] \tag{3.2}$$

for $0 \leq r \leq t$. This holds as a polynomial identity in indeterminants $X$ and $Y$ for any nonnegative integer $t$, and for nonnegative integers $x, y, t$ provided $x + y \neq t$. After some easy calculations, (3.2) is seen to be an instance of Vandermonde's identity. In particular, setting $r = 0$ gives

$$p(X) = \sum_{i=0}^{t} \frac{(-1)^i}{\binom{t}{i}} \binom{X}{i}\binom{t+1-X}{t-i}. \tag{3.3}$$

Assume $t - s \geq 2$ is even and let $U$ be a $u$-subset of the $v$-set $V$. Define for $t$-subsets $T \subseteq V$,

$$g(T) = \frac{(-1)^i}{\binom{t}{i}} \quad \text{whenever } |T \cap U| = i.$$

For any $K \subseteq V$ with $|K| = t+1$,

$$\sum_{T \subseteq K} g(T) = p(|K \cap U|) \geq 0 \tag{3.4}$$

by (3.3). Now by Lemma 2.3, if $f \in C^v_{t,t+1}$ then

$$0 \leq \sum_T g(T) f(T) = \lambda \sum_{i=0}^{t-s-1} \frac{(-1)^i}{\binom{t}{i}} \binom{u}{i} \binom{v-u}{t-i}.$$

Applying (3.2) again with $r = s+1$ together with the identity $\binom{\alpha}{\beta}\binom{\beta}{\gamma} = \binom{\alpha}{\gamma}\binom{\alpha-\gamma}{\beta-\gamma}$ gives

$$\binom{v-u-s-1}{t-s} \geq \binom{u}{t-s},$$

which is equivalent to $v \geq 2u+s+1$. Conversely, suppose $v \geq 2u+s+1$. It must be shown that there is a nonnegative $(t+1)$-vector $h$ such that $Wh = f$. Since $f$ is invariant under the partition $U, V \setminus U$, this is equivalent to the existence of solutions $c_i \geq 0$ to

$$M(c_0, \ldots, c_{t+1})^\top = (\overbrace{1, \ldots, 1}^{t-s}, 0, \ldots, 0)^\top, \tag{3.5}$$

where $M_{ij} = \binom{u-i}{j-i}\binom{v-u-t+i}{i-j+1}$ for $i = 0, \ldots, t$, $j = 0, \ldots, t+1$. Indeed, there are $M_{ij}$ $(t+1)$-subsets of $V$ which meet $U$ in $j$ points and contain a given $t$-subset meeting $U$ in $i$ points; so we can take $c_j = \lambda h(K)\binom{u}{j}\binom{v-u}{t+1-j}$ if $|K \cap U| = j$. Setting $c_j = 0$ for $j \geq t-s$, the system (3.5) reduces to

$$\begin{pmatrix} a & b & & & \\ & a+1 & b-1 & & \\ & & a+2 & & \\ & & & \ddots & \\ & & & & b-r+1 \\ & & & & a+r \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

for $a = v-u-t$, $b = u$, and $r = t-s-1$. By an identity similar to (3.2), we find

$$c_j = \frac{1 + (-1)^{r-j}\binom{b-j}{r+1-j}\binom{a+r}{r+1-j}^{-1}}{a+b},$$

$j = 0, \ldots, r$ is a solution. So we have $c_j \geq 0$ if and only if $b - j \geq a + r$ whenever $r - j$ is odd, or $v \geq 2u+s+1$. This proves (1).

For (2), equality holds if and only if we have equality in (3.4) for every $(t+1)$-set $K$ contained in a block $A$ of a realization $\mathcal{F}$ of $f$. This occurs if and only if $s < |A \setminus U| \leq t$ for every block $A$. □

**Remarks.** If $t-s$ is odd, we have the inequality $u \leq (v-s)/2$. It is curious that the inequality in Theorem 3.1 is the same as that on an incomplete $t$BD with $v+s$ points and a strength 0 hole of size $u+s$, yet there appears to be no easy combinatorial reduction to strength 0.

It is very difficult in general to determine simple and effective necessary and sufficient conditions for certain $t$-vectors $f$ to belong to $C^v_{tk}$. However, for $t = 2$ and $f$ invariant under a 'bipartition' $(U, V \setminus U)$, a characterization is possible.

**Theorem 3.2.** *Let $U$ be a $w$-subset of a $v$-set $V$ with $2 \le w < v-1$. Let $0 \le \nu \le \lambda$ be real numbers and $f$ the 2-vector given by*

$$f(T) = \begin{cases} \lambda - \nu & \text{if } T \subseteq U, \\ \lambda & \text{otherwise.} \end{cases}$$

*Then $f \in C^v_{2k}$ if and only if*

$$\frac{\nu}{\lambda} \le \frac{c(c+1)(v-w)(v-w-1)}{(k-c)(k-c-1)w(w-1)} - \frac{2c(v-w)}{(k-c)(w-1)} + 1, \tag{3.6}$$

*where $c = \lfloor \frac{w(k-1)}{v-1} \rfloor$.*

**Proof.** The given vector $f$ is invariant under the partition $(U, V \setminus U)$. So $f \in C^v_{2k}$ if and only if the condition in Lemma 2.2 holds for all 2-vectors $g$ which also respect this partition. From the discussion in Section 2, we know all of these facets. For a given choice of $c$ in (2.2), the condition $\sum_{T \subset V} g(T)f(T) \ge 0$ is equivalent to (3.6) above. Some calculus shows that the (integral) value of $c$ which minimizes the right side of (3.6) is $c = \lfloor \frac{w(k-1)}{v-1} \rfloor$, as required. □

It can be shown that (3.6) is always at least as strong as the case $t = 2$ of (1.5). Similarly, improvements on (1.5) for $t > 2$ are possible from the cone condition. Unfortunately, the necessary and sufficient conditions for larger $t$ are not so simple as (3.6). We merely present an example here. A more detailed discussion is found in [1].

**Example 3.1.** Let $|V| = 17$ and $K \subset V$ with $|K| = 8$. Consider the question of realizing the 4-vector

$$f(T) = \begin{cases} 4 & \text{if } T \subset K, \\ 5 & \text{otherwise.} \end{cases}$$

Let $(a_0, \ldots, a_4) = (5, -3, 3, -5, 15)$ and define $g(T) = a_i$ whenever $|T \cap B| = i$. It is simple to verify that $\sum_{T \subset K} g(T) \ge 0$ for every 8-subset $K$. Now

$$\sum_{T \subset V} g(T)f(T) = 25\binom{8}{0}\binom{9}{4} - 15\binom{8}{1}\binom{9}{3} + 15\binom{8}{2}\binom{9}{2} - 25\binom{8}{3}\binom{9}{1} + 60\binom{8}{4}\binom{9}{0} < 0,$$

so $f \notin C^{17}_{4,8}$ by Lemma 2.2. Thus, $f$ is not realizable by 8-subsets and there can be no 4-(17,8,5) design. The nonexistence of a design with these parameters was first shown by Delsarte.

Often, while $f$ may not be invariant under a bipartition, it is nonetheless nice to choose $g$ in this way. The following result offers an illustration.

**Theorem 3.3.** *Let $w \ge k \ge 2$ and $v = w(k-1)+1$. Suppose $U_1, U_2$ are distinct $w$-subsets of a $v$-set $V$. Let $\lambda > 0$ and $f$ the 2-vector given by*

$$f(T) = \begin{cases} 0 & \text{if } T \subseteq U_1 \text{ or } T \subseteq U_2, \\ \lambda & \text{otherwise.} \end{cases} \tag{3.7}$$

*If $f \in C^v_{2k}$ then $|U_1 \cap U_2| \ge \frac{w-1}{k-1}$.*

**Proof.** Let $w' = |U_1 \cap U_2|$. We (partially) define the 2-vector $g$ by

$$g(T) = \begin{cases} k-1 & \text{if } T \cap U_1 = \emptyset, \\ -\binom{k-1}{2} & \text{if } |T \cap U_1| = 1. \end{cases}$$

It is easy to check that on $k$-subsets $K$ with $|K \cap U_1| \le 1$ we have $\sum_{T \subseteq K} g(T) \ge 0$. Therefore by Lemma 2.3,

$$0 \le \sum_{T \subset V} g(T) f(T) = \frac{1}{2}(k-1)(w-w')((k-1)w' - (w-1)),$$

where the last equality requires some simplification. This proves $w' \ge (w-1)/(k-1)$. □

From Eq. (1.6), the existence of a 2-$(w, k, \lambda)$ subdesign in a 2-$(v, k, \lambda)$ design implies $w \le (v-1)/(k-1)$. Applying (1.6) again to bound the intersection size $w'$ of two (different) such subdesigns gives $w' \le (w-1)/(k-1)$. The existence of a pair of 2-$(w, k, \lambda)$ subdesigns on point sets $U_1, U_2$ implies the realizability of the 2-vector $f$ in (3.7). So, by Theorem 3.3, two distinct maximum proper subdesigns (on $w$ points) in a 2-$(w(k-1)+1, k, \lambda)$ design must intersect in exactly $(w-1)/(k-1)$ points. For $\lambda = 1$, this reduces to a special case of Proposition 4.2 in [5] on the intersection of two subdesigns. It is noteworthy that the original proof uses $\lambda = 1$, but Theorem 3.3 works for any $\lambda$. However, the result in [5] permits $v > w(k-1)+1$ and $|U_1| \ne |U_2|$ and it seems that we do not recover the general result from the cone condition.

Consider now the 2-vector $f$ in (1.7) obtained from 'deleting' a pair of blocks $A_1, A_2$ meeting in $\mu$ points. This $f$, and all relevant supporting 2-vectors $g$, are invariant under the partition $((A_1 \cup A_2)^c, A_1 \setminus A_2, A_2 \setminus A_1, A_1 \cap A_2)$. There are seven corresponding orbits of 2-subsets $T = \{x, y\}$.

| $x \in$ | $y \in$ | Number of edges | $f(\{x,y\})$ | $g(\{x,y\})$ |
|---|---|---|---|---|
| $(A_1 \cup A_2)^c$ | $(A_1 \cup A_2)^c$ | $\binom{v-2k+\mu}{2}$ | $\lambda$ | $a_1$ |
| $A_1 \triangle A_2$ | $(A_1 \cup A_2)^c$ | $2(k-\mu)(v-2k+\mu)$ | $\lambda$ | $a_2$ |
| $A_1 \cap A_2$ | $(A_1 \cup A_2)^c$ | $\mu(v-2k+\mu)$ | $\lambda$ | $a_3$ |
| $A_1 \setminus A_2$ | $A_2 \setminus A_1$ | $(k-\mu)^2$ | $\lambda$ | $a_4$ |
| $A_i \setminus A_j$ | $A_i \setminus A_j$ | $2\binom{k-\mu}{2}$ | $\lambda - 1$ | $a_5$ |
| $A_1 \cap A_2$ | $A_1 \triangle A_2$ | $2\mu(k-\mu)$ | $\lambda - 1$ | $a_6$ |
| $A_1 \cap A_2$ | $A_1 \cap A_2$ | $\binom{\mu}{2}$ | $\lambda - 2$ | $a_7$ |

It is a linear programming question to find $(a_1, \ldots, a_7)$ given $v, k, \lambda, \mu$ so that the quantity $\sum_T g(T) f(T)$ is minimized subject to $\sum_{T \subset K} g(T) \ge 0$ for all $k$-subsets $K$. (Without loss, it can be assumed that $a_1 \in \{\pm 1, 0\}$.) When this minimum is negative, we can conclude by Lemma 2.3 that $f$ is not in the cone $C^v_{2k}$, hence not realizable. Several possible values of $\mu$ allowed by (1.8) can be ruled out with this method; see [1] for a table. Similar restrictions on block intersection size in 2-designs have been noted by Greig [2].

**Example 3.2.** Let $v = 21$, $k = 6$, $t = 2$, and $\lambda = 2$. The inequalities (1.8) give $0 \le \mu \le 3$ as a necessary condition on realizability of $f$. However, $\mu = 0, 3$ can be ruled out using the argument above with

$$\mu = 0: \quad (a_1, \ldots, a_7) = (1, -1/2, 0, 0, 11/2, 1, 0)$$
$$\mu = 3: \quad (a_1, \ldots, a_7) = (1, -1/2, -2, -2, 11/2, 10, 0).$$

In each case, it is simple to check that $\sum_T g(T)f(T) = -15$ and that $\sum_{T\subset K} g(T) \geq 0$ for $|K| = 6$. So in any 2-(21,6,2) design, two blocks must meet in either $\mu = 1$ or 2 points.

The cone condition can be used to derive necessary conditions on the intersection pattern of three or more blocks, but we do not have elegant general results at this time. An example is given below.

**Example 3.3.** Let $|V| = 22$ and suppose $A_1, A_2, A_3$ are 8-subsets of $V$ with $|A_1 \cap A_2 \cap A_3| = |A_i \cap A_j| = 1$ for $i \neq j$. (Note that $(A_1 \cup A_2 \cup A_3)^c = \emptyset$ in this case.) Let $f$ be the 2-vector defined by

$$f(T) = \begin{cases} 3 & \text{if } T \subset A_i \text{ for some } i, \\ 4 & \text{otherwise.} \end{cases}$$

Let $\{z\} = A_1 \cap A_2 \cap A_3$. Define the 2-vector $g$ by

$$g(\{x, y\}) = \begin{cases} -1 & \text{if } x \in A_i, y \in A_j, i \neq j, z \notin \{x, y\} \\ 3 & \text{if } \{x, y\} \subset A_i \text{ for some } i, z \notin \{x, y\} \\ 1/7 & \text{if } \{x, y\} \subset A_i, \text{ for some } i, z \in \{x, y\} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $\sum_{T\subset K} g(T) \geq 0$ for all 8-subsets $K$. But the dot product $gf$ is $-12$, and so $f \notin C^{22}_{2,8}$. Consequently, in any 2-(22,8,4) design, there do not exist three blocks which meet both pairwise and threewise in a single point. This has also been observed by Greig [2].

## 4. A general result and proofs

Consider $n$ subsets $W_i$, $i = 1, \ldots, n$, of a $v$-set $V$ and work with the partition of $V$ into $2^n$ regions defined by the $W_i$. To each subset $S$ of indices in $\{1, \ldots, n\}$, we associate the region $W_S = (\cap_{i\in S} W_i) \cap (\cap_{j\notin S} W_j^c)$. In full generality, applying Theorem 2.6 to a $t$-vector $f$ invariant under $\{W_S : S \subset \{1, \ldots, n\}\}$ requires the consideration of nonnegative polynomials in $2^n$ variables $(X_S)$, one per region. However, our present application actually involves polynomials in far fewer variables.

Let $t \leq k$ be fixed positive integers. A polynomial $p(X_1, \ldots, X_{d-1})$ of degree at most $t$ can be expressed in the basis

$$\mathcal{B}_{tk} = \left\{ \binom{X_1}{i_1}\binom{X_2}{i_2}\cdots\binom{X_d}{i_d} : i_1 + \cdots + i_d = t \right\},$$

where $X_d = k - X_1 - \cdots - X_{d-1}$. Suppose the corresponding coefficients of $p$ in $\mathcal{B}_{tk}$ are $a_{i_1 i_2 \ldots i_d}$. Observe that if $a_{i_1 i_2 \ldots i_d}$ depends only on, say, $i_1 + \cdots + i_r, i_{r+1}, \ldots, i_d$, then by the Vandermonde convolution identity $p$ is in fact a linear combination of

$$\left\{ \binom{X_1 + \cdots + X_r}{j}\binom{X_r}{i_{r+1}}\cdots\binom{X_d}{i_d} : j + i_{r+1} + \cdots + i_d = t \right\}.$$

In any case, for a fixed $t$ and $k$, we define a 'homogeneous' extension of $p$ with respect to $\mathcal{B}_{tk}$ by

$$\widehat{p}(X_1, \ldots, X_{d-1}; X_d) = \sum_{i_1+\cdots+i_d=t} a_{i_1 i_2 \ldots i_d} \binom{X_1}{i_1}\binom{X_2}{i_2}\cdots\binom{X_d}{i_d}.$$

The variables $X_1, \ldots, X_d$ are independent in $\widehat{p}$. Note that the operation $p \mapsto \widehat{p}$ is linear. We may refer to $X_d$ as the 'homogenizing variable' in $\hat{p}$.

For $U \subset V$, let $f$ be the $t$-vector defined by

$$f(T) = \begin{cases} \lambda - m & \text{if } T \subseteq U, \\ \lambda & \text{otherwise.} \end{cases}$$

Let $U_1, \ldots, U_d$ be an invariant partition for $f$ as in Section 2. With the notation above, the inequality (2.3) can be restated as

$$\lambda \widehat{p}(|U_1|, \ldots, |U_d|) - m\widehat{p}(|U_1 \cap U|, \ldots, |U_d \cap U|) \geq 0, \tag{4.1}$$

provided that $p(X_1, \ldots, X_{d-1}) \geq 0$ on nonnegative integers $X_1, \ldots, X_{d-1}$ with sum at most $k$. We will use this formulation of (2.3) in what follows.

Now fix $v$ and $k$. For nonnegative integers $s$, define

$$\Psi_{s,w}(X) = \sum_{i=0}^{s} (-1)^{s-i} \frac{\binom{v-s}{i}\binom{w-i}{s-i}\binom{k-1-i}{s-i}}{\binom{s}{i}} \binom{X}{i}.$$

These polynomials arise in the theory of association schemes in connection with the idempotents of the Johnson scheme. We now use these polynomials to specialize (4.1).

**Lemma 4.1.** *Let $s, t, k, w_i, v$ be positive integers with $t \geq 2s$ and $t \leq k, w_i \leq v - s$ for $i = 1, \ldots, n$. Let $V$ be a $v$-set and $W_i \subset V$ with $|W_i| = w_i$, $|W_i \cap W_j| = w_{ij}$, etc. For each $i$, let $m_i$ be a nonnegative real number. Let $f$ be a $t$-vector with*

$$f(T) = \lambda - \sum_{i : W_i \supseteq T} m_i.$$

*If $f \in C^v_{tk}$, then the $n \times n$ matrix with $(i, j)$-entry*

$$\lambda \widehat{p}_{ij}(w_{ij}, w_i - w_{ij}, w_j - w_{ij}; v - w_i - w_j + w_{ij}) - \sum_{h=1}^{n} m_h \widehat{p}_{ij}(w_{hij}, w_{hi} - w_{hij}, w_{hj} - w_{hij}; w_h - w_{hi} - w_{hj} + w_{hij})$$

*is positive semidefinite, where $p_{ij}(X_{ij}, X_i, X_j) = \Psi_{s,w_i}(X_i + X_{ij})\Psi_{s,w_j}(X_j + X_{ij})$.*

**Proof.** Let $R$ be the matrix defined in the theorem and $\mathbf{z} = (z_1, \ldots, z_n)$ be a vector of real numbers. Consider the polynomial with $2^n - 1$ variables $\{X_S : S \neq \emptyset\}$ given by

$$q((X_S)) = \left( \sum_i z_i \Psi_{s,w_i} \left( \sum_{S \ni i} X_S \right) \right)^2.$$

In Theorem 2.6, take $g$ to be the $t$-vector whose entries are determined by the coefficients of $q$ in $\mathcal{B}_{tk}$. Since $q$ is nonnegative and of degree $2s \leq t$, (4.1) asserts that

$$0 \leq gf = \lambda \widehat{q}(\mathbf{u}) - \sum_h m_h \widehat{q}(\mathbf{u}_h), \tag{4.2}$$

where $\mathbf{u}(S) = |W_S|$ and $\mathbf{u}_h(S) = |W_S \cap W_h|$. We have taken $S = \emptyset$ to index the homogenizing variable. Now the coefficients in $\mathcal{B}_{tk}$ of the polynomial

$$q_{ij}((X_S : S \neq \emptyset)) = \Psi_{s,w_i}\left( \sum_{S \ni i} X_S \right) \Psi_{s,w_j}\left( \sum_{S \ni j} X_S \right)$$

depend only on the four sums of indices over positions $S$ which (i) contain both $i$ and $j$, (ii) contain $i$ but not $j$, (iii) contain $j$ but not $i$, and (iv) contain neither $i$ nor $j$. So after a substitution,

$$\widehat{q}_{ij}(\mathbf{u}) = \widehat{p}_{ij}(w_{ij}, w_i - w_{ij}, w_j - w_{ij}; v - w_i - w_j + w_{ij})$$

and

$$\widehat{q}_{ij}(\mathbf{u}_h) = \widehat{p}_{ij}(w_{hij}, w_{hi} - w_{hij}, w_{hj} - w_{hij}; w_h - w_{hi} - w_{hj} + w_{hij}).$$

From (4.2) and linearity, it follows that $\mathbf{z}R\mathbf{z}^\top \geq 0$. $\square$

Although we have not attempted to simplify $\widehat{p}_{ij}$ in general, it should be noted that the polynomials $\Psi_{s,k}$ enjoy useful properties with respect to a change of basis from $\mathcal{B}_{tk}$ to $\mathcal{B}_{tv}$. We omit the tedious calculations, but details can be found in [1].

**Lemma 4.2.** *Suppose $t \geq 2s$.*

(a) *Let $p(X) = (\Psi_{s,w}(X))^2$. Then*

$$\widehat{p}(w; v - w) = \gamma \Psi_{s,w}(k),$$

*where* $\gamma = \binom{v}{t}\binom{v}{s}^{-1}\binom{w}{s}\binom{k}{t}^{-1}\binom{v-k}{s}$.

(b) *Let $p(X_{12}, X_1, X_2) = (\Psi_{s,k}(X_1 + X_{12}))(\Psi_{s,k}(X_2 + X_{12}))$. Then*

$$\widehat{p}(\mu, k - \mu, k - \mu; v - 2k + \mu) = \gamma \Psi_{s,k}(\mu),$$

*where* $\gamma = \binom{v}{t}\binom{v}{s}^{-1}\binom{k}{s}\binom{k}{t}^{-1}\binom{v-k}{s}$.

With (a) above, the proof of Theorem 1.2 reduces to an easy application of Lemma 4.1. We restate Theorem 1.2 below and sketch the proof.

**Theorem 4.3.** *Let $s, t, k, w, v$ be positive integers with $t \geq 2s$ and $t \leq k, w \leq v - s$. Let $U$ be a $w$-subset of a $v$-set $V$. Given real numbers $\lambda, m$, $0 \leq m \leq \lambda$, consider the $t$-vector $f$ on $V$ defined by*

$$f(T) = \begin{cases} \lambda - m & \text{if } T \subseteq U, \\ \lambda & \text{otherwise}. \end{cases}$$

*If $f$ is in the cone $C^v_{tk}$, then*

$$\lambda \binom{v}{t} \geq m \binom{w}{t} \sum_{i=0}^{s} \left( \binom{v}{i} - \binom{v}{i-1} \right) \frac{\binom{k}{i}\binom{v-w}{i}}{\binom{w}{i}\binom{v-k}{i}}. \tag{4.3}$$

*More generally, the same inequality holds if we relax the condition $f(T) = \lambda - m$ for $T \subseteq U$ to the condition that the average value of $f$ on $t$-subsets $T \subseteq U$ is $\lambda - m$.*

**Proof.** This is the case $n = 1$ with $m_1 = m$ and $w_1 = w$. Take $p(x) = (\Psi_{s,w}(x))^2$. By Lemma 4.1,

$$\lambda \widehat{p}(w; v - w) \geq m \binom{w}{t} \widehat{p}(w; 0) = m \binom{w}{t} \binom{k}{t}^{-1} p(k).$$

This can be rewritten, by Lemma 4.2(a), as

$$\lambda \binom{v}{t}\binom{v}{s}^{-1}\binom{v-k}{s}\binom{w}{s} \Psi_{s,w}(k) \geq m \binom{w}{t} (\Psi_{s,w}(k))^2. \tag{4.4}$$

The following calculation of $\Psi_{s,w}(k)$ can be found in [6]:

$$\Psi_{s,w}(k) = \frac{\binom{w}{s}\binom{v-k}{s}}{\binom{v}{s}} \sum_{i=0}^{s} \left( \binom{v}{i} - \binom{v}{i-1} \right) \frac{\binom{k}{i}\binom{v-w}{i}}{\binom{w}{i}\binom{v-k}{i}}. \tag{4.5}$$

In particular, $\Psi_{s,w}(k) > 0$ as $s \leq k$, $w \leq v - s$. The inequality (4.3) now follows from (4.4) and (4.5). □

We now state a condition for equality resulting from Lemma 2.3. The reader may recognize this as a generalization of the 'integer roots criterion' for tight $t$-designs, [7].

**Theorem 4.4.** *If $\mathcal{F}$ is a family of $k$-sets realizing the $t$-vector $f$ as above, then equality occurs in Theorem 4.3 if and only if $|A \cap U|$ is a root of $\Psi_{s,w}(X)$ for every block $A$ of $\mathcal{F}$.*

Finally, we consider another specialization of Lemma 4.1.

**Theorem 4.5.** *Let $s, t, k, v$ be positive integers with $t \geq 2s$ and $t \leq k \leq v - s$. Let $V$ be a $v$-set and $A_i \subset V$ be $k$-subsets of $V$ for $i = 1, \ldots, n$. Let $f$ be a $t$-vector with*

$$f(T) = \lambda - |\{i : A_i \supseteq T\}|.$$

*Let $\mu_{ij} = |A_i \cap A_j|$ and define the $n$ by $n$ matrix $P = [\Psi_{s,k}(\mu_{ij})]$. If $f \in C^v_{tk}$, then $\lambda\gamma I_n - P$ is positive semidefinite, where $\gamma = \binom{v}{t}\binom{v}{s}^{-1}\binom{k}{s}\binom{k}{t}^{-1}\binom{v-k}{s}$.*

**Proof.** Take $m_i = 1$, $w_i = k$ and $W_i = A_i$ for all $i$ in Lemma 4.1. Let $p_{ij}(X_{ij}, X_i, X_j) = \Psi_{s,k}(X_i + X_{ij})\Psi_{s,k}(X_j + X_{ij})$. By Lemma 4.2(b), we obtain

$$\widehat{p}_{ij}(\mu_{ij}, k - \mu_{ij}, k - \mu_{ij}; v - 2k + \mu_{ij}) = \gamma\Psi_{s,k}(\mu_{ij}),$$

the $(i, j)$-entry of $\gamma P$. And since $\widehat{p}(X_1, \ldots, X_{d-1}, k - X_1 - \cdots - X_{d-1}) = p(X_1, \ldots, X_{d-1})$, we have

$$\widehat{p}(\mu_{hij}, \mu_{hi} - \mu_{hij}, \mu_{hj} - \mu_{hij}; k - \mu_{hi} - \mu_{hj} + \mu_{hij}) = \Psi_{s,k}(\mu_{hi})\Psi_{s,k}(\mu_{hj}).$$

Summing over $h$, this is the $(i, j)$-entry of $P^2$. So by Lemma 4.1, $\lambda\gamma P - P^2$ is positive semidefinite. From this it follows that $P$ and $\lambda\gamma I - P$ are both positive semidefinite. □

As a consequence of Theorem 4.3, for $t \geq 2s$ and $k \leq v - s$, if a $t$-$(v, k, \lambda)$ design has $n$ different blocks with pairwise intersection sizes $\mu_{ij}$, then the same matrix is positive semidefinite. This was first proved by Wilson in [7].

Observe the identity (4.5) with $w = k$ becomes $\Psi_{s,k}(k) = \binom{k}{s}\binom{v-k}{s} > 0$. So the case $n = 2$ with $\mu = \mu_{12} = |A_1 \cap A_2|$ reduces to

$$\det \begin{pmatrix} \lambda\gamma - \Psi_{s,k}(k) & \Psi_{s,k}(\mu) \\ \Psi_{s,k}(\mu) & \lambda\gamma - \Psi_{s,k}(k) \end{pmatrix} \geq 0,$$

or

$$\binom{k}{s}\binom{v-k}{s} \pm \Psi_{s,k}(\mu) \leq \lambda \binom{v}{t}\binom{v}{s}^{-1}\binom{k}{s}\binom{k}{t}^{-1}\binom{v-k}{s}.$$

We leave it to the reader to verify that the case $s = 1$, $t = 2$ gives the condition stated in Theorem 1.3.

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