A 2n-Point interpolation formula with its applications to q-identities

Sandy H.L. Chen *, Amy M. Fu

Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, PR China

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A B S T R A C T

Based on Krattenthaler’s determinantal formula and divided difference operators, we give a 2n-point interpolation formula for a polynomial of degree \( \leq n \) in one variable. Several applications of this formula, such as q-identities related to divisor functions, finite forms of the quintuple product identity and a bibasic hypergeometric identity, are discussed. We also give an expansion formula for \( \prod_{i=1}^{n}(y-uq^{i-1}) \) by using the supersymmetric complete functions and determinant evaluation.

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1. Introduction

Let \( A = \{a_1, a_2, \ldots \} \) be a set of indeterminates. The \( n \)-point Newton interpolation formula for a function \( f(x) \) gives the unique polynomial \( P(x) \) of degree \( n-1 \):

\[
P(x) = f(x) - f(a_1) \partial_1 \partial_2 \cdots \partial_n \prod_{i=1}^{n}(x - a_i) = f(a_1) + \sum_{i=1}^{n-1} f(a_1) \partial_1 \partial_2 \cdots \partial_{i} \prod_{j=1}^{i}(x - a_j),
\]

where we take \( a_{n+1} = x \) and the divided difference \( \partial_i (i \geq 1) \), acting on its left, is defined by

\[
f(a_1, \ldots, a_i, a_{i+1}, \ldots) \partial_i = \frac{f(a_1, \ldots, a_i, a_{i+1}, \ldots) - f(a_1, \ldots, a_{i+1}, \ldots)}{a_i - a_{i+1}}.
\]

Unlike Newton’s formula (1.1), Lagrange’s interpolation formula does not need the knowledge of the difference of a function:

\[
P(x) = \sum_{i=1}^{n} \frac{\prod_{j \neq i}(x - a_j)}{\prod_{j \neq i}(a_i - a_j)} f(a_i).
\]

Based on the following determinantal formula due to Krattenthaler [7], we shall introduce a 2n-point interpolation formula which gives a unique polynomial of degree \( \leq n \). Given three sets of variables \( X = \{x_1, \ldots, x_{n+1}\} \), \( A = \{a_1, \ldots, a_n\} \), and \( B = \{b_1, \ldots, b_n\} \), Krattenthaler has shown that

\[
\det((x_i - a_j) \cdots (x_i - a_n)(x_i - b_1) \cdots (x_i - b_{n-1}))_{i,j=1}^{n+1} = \prod_{1 \leq i < j \leq n+1} (x_i - x_j) \prod_{1 \leq i \leq n} (a_i - b_i).
\]

* Corresponding author.
E-mail addresses: chenhuanlin@mail.nankai.edu.cn (S.H.L. Chen), fu@nankai.edu.cn (A.M. Fu).

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Taking $x_{n+1} = y$ and using Laplace’s expansion to expand the determinant along the last row, (1.3) can be reduced to the following formula:

$$f(y) = \sum_{k=0}^{n} c_k (y - a_{k+1}) \cdots (y - a_n) (y - b_1) \cdots (y - b_k),$$

(1.4)

where $f(y) = (y-x_1) \cdots (y-x_n)$ and $c_k$ is a quotient whose numerator is the minor of the above determinant with respect to the entry $(y-a_{k+1}) \cdots (y-a_n) (y-b_1) \cdots (y-b_k)$ and denominator is the product $(-1)^k \prod_{1 \leq i < j \leq n} (a_j - b_i) \prod_{1 \leq i < j \leq n} (x_i - x_j)$. Applying the techniques of the divided differences to determine the coefficients $c_k, 0 \leq k \leq n$, we obtain the main result of this paper and we will prove it in the next section.

**Theorem 1.1.** Suppose $f(y)$ is a polynomial in $y$ with degree $\leq n$. Given two sets of points $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ with $a_i \neq b_j, 1 \leq i, j \leq n$, we have the following 2$n$-point interpolation formula:

$$f(y) = f(b_1) \prod_{i=1}^{n} \frac{(y - a_i)}{(b_1 - a_i)} + f(a_1) \prod_{i=1}^{n} \frac{(y - b_i)}{(a_1 - b_i)} + \sum_{k=1}^{n-1} \frac{f(b_1)}{\prod_{i=1}^{n-k+1} (b_i - a_i)} \partial_1 \cdots \partial_k (b_{k+1} - a_{n-k+1}) \prod_{i=1}^{n-k} (y - a_i) \prod_{i=1}^{k} (y - b_i).$$

(1.5)

**Theorem 1.1** can be regarded as a terminating case of the Newton type rational interpolation formula for a formal power series $f(y)$ given in [4]:

$$f(y) = f(b_1) + \sum_{n=1}^{\infty} f(b_1) \prod_{i=1}^{n-1} (1 - b_1 c_i) \partial_1 \cdots \partial_n (1 - b_{n+1} c_n) \prod_{i=1}^{n} \frac{y - b_i}{1 - yc_i},$$

where $b_1, b_2, \ldots, c_1, c_2, \ldots$ are complex numbers and the series is convergent when $|y| < 1$, \(\lim_{n \to \infty} b_1 \cdots b_n = 0\), and $\lim_{n \to \infty} y^n b_1 \cdots b_n = 0$.

Consider the case $f(y) = 1$ in **Theorem 1.1**. Comparing with the reminders of Newton’s and Lagrange’s interpolation formulas for the function $1/(x - y)$, we are led to the following identity.

**Theorem 1.2.** We have

$$\frac{1}{x - b_1} \prod_{i=1}^{n} \frac{(y - a_i)}{(b_1 - a_i)} + \frac{1}{x - a_1} \left( \prod_{i=1}^{n} \frac{(y - b_i)}{(a_1 - b_i)} - \prod_{i=1}^{n} \frac{(y - b_i)}{(x - b_i)} \right) + \sum_{k=1}^{n-1} \frac{1/(x - b_1)}{\prod_{i=1}^{n-k+1} (b_i - a_i)} \partial_1 \cdots \partial_k (b_{k+1} - a_{n-k+1}) \prod_{i=1}^{n-k} (y - a_i) \prod_{i=1}^{k} (y - b_i)$$

$$= \frac{1}{x - b_1} + \sum_{k=1}^{n-1} \frac{1}{x - b_1} \prod_{j=1}^{k} \frac{(y - b_j)}{(x - b_{j+1})} = \sum_{k=1}^{n-1} \frac{1}{x - b_k} \prod_{j \neq k} \frac{(y - b_j)}{(b_k - b_j)}.$$

The last equality has already appeared in [3] in the proofs of several q-identities related to divisor functions. A short proof of **Theorem 1.2** and some applications of **Theorem 1.1**, such as q-identities related to divisor functions, finite forms of the quintuple product identity and a bigaussian hypergeometric identity, will be discussed in Section 3.

**Theorem 1.3.** We have

$$\prod_{i=1}^{n} (y - uq^{i-1}) = \prod_{k=0}^{n} \left[ \frac{n}{k} \right] \prod_{i=1}^{k} (uq^{n-i} - uq^{1-k}) \prod_{i=1}^{n-k} \frac{(wq - uq^{i-1})}{(uq^{k+1} - uq^{1-k})} \prod_{i=1}^{k} \frac{(y - wq^{i})}{(y - uq^{i-n})},$$

(1.6)

where $\left[ \frac{n}{k} \right]$ is the q-Gauss coefficient defined by

$$\left[ \frac{n}{k} \right] = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}},$$

where $(a; q)_n$ is the q-shifted factorial defined by

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n = 0, 1, \ldots \infty.$$
Theorem 1.3 can be deduced from Theorem 1.1 through the following specializations:

\[ f(y) = \prod_{i=1}^{n} (y - uq^{j-1}), \quad A = \{vq^{-1}, \ldots, vq^{-n}, v\}, \quad B = \{wq, wq^2, \ldots, wq^n\}. \]

Instead of using Theorem 1.1, we shall give an alternative proof of Eq. (1.6) in Section 4 by evaluating the numerator of \( c_k \), \( 0 \leq k \leq n \).

Note that Theorem 1.3 can be considered as a variation of the terminating \( \phi q_5 \) summation formula [5]. On the other hand, one can view the terminating \( \phi q_5 \) summation formula as an interpolation formula. Writing \( wv^{-1} \) as \( a, wv^{-1} \) and \( uv^{-1} \) as \( c \), we find

\[
\frac{(aq; q)_n (aq/bc; q)_n}{(aq/b; q)_n (aq/c; q)_n} = \sum_{k=0}^{n} \frac{(1 - aq^{2k})(a; q)_k (b; q)_k (c; q)_k (q^{-n}; q)_k}{(1 - a)(q; q)_k (aq/b; q)_k (aq/c; q)_k (aq^{n+1}; q)_k} \left( \frac{aq^{n+1}}{bc} \right)^k.
\]

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, it suffices to verify the following lemma. The main technique we use in our proof is the following Leibniz formula [8]:

\[
f(x_1)g(x_1) \partial_1 \partial_2 \cdots \partial_n = \sum_{k=0}^{n} \left( f(x_1) \partial_1 \cdots \partial_k \right) \left( g(x_{k+1}) \partial_{k+1} \cdots \partial_n \right). \tag{2.1}
\]

Lemma 2.1. We have

\[
c_k = \begin{cases} \frac{f(b_1)}{\prod_{i=1}^{n} (b_1 - a_i)}, & \text{if } k = 0, \\ \frac{f(b_1)}{\prod_{i=1}^{n} (b_1 - a_i)} \partial_1 \cdots \partial_k (b_{k+1} - a_k), & \text{if } 1 \leq k \leq n - 1, \\ \frac{f(a_n)}{\prod_{i=1}^{n} (a_n - b_i)}, & \text{if } k = n, \end{cases}
\]

where \( c_k \) and \( f(y) \) are given as in (1.4).

Proof. According to Eq. (1.3), we have

\[
det((x_1 - a_{j+1}) \cdots (x_1 - a_n)(x_i - b_1) \cdots (x_i - b_j))_{i,j=1}^{n} = \prod_{i=1}^{n} (x_i - b_1) det((x_i - a_{j+1}) \cdots (x_i - a_n)(x_i - b_2) \cdots (x_i - b_j))_{i,j=1}^{n} = (-1)^n f(b_1) \prod_{1 \leq i < j \leq n} (x_i - x_j) \prod_{2 \leq i \leq n} (a_j - b_i).
\]

Therefore,

\[
c_0 = \frac{f(b_1)}{\prod_{i=1}^{n} (b_1 - a_i)}.
\]

Similarly, one has

\[
c_n = \frac{f(a_n)}{\prod_{i=1}^{n} (a_n - b_i)}.
\]

Specializing \( y \) to \( b_2 \) in (1.4) to get

\[
f(b_2) = \frac{f(b_1) \prod_{i=1}^{n} (b_2 - a_i)}{\prod_{i=1}^{n} (b_1 - a_i)} + c_1(b_2 - b_1) \prod_{i=2}^{n} (b_2 - a_i),
\]
which implies
\[
c_1 = \left( f(b_1) / \prod_{i=1}^{n} (b_1 - a_i) - f(b_2) / \prod_{i=1}^{n} (b_2 - a_i) \right) (b_2 - a_1) / (b_1 - b_2)
\]
\[
= \frac{f(b_1)}{\prod_{i=1}^{n} (b_1 - a_i)} \partial_1 (b_2 - a_1).
\]

Let \( g(y) = f(y) / (y - a_1) \cdots (y - a_n) \). Rewrite (1.4) as
\[
g(y) = g(b_1) + \sum_{k=1}^{n-1} c_k (y - b_1) \cdots (y - b_k) / (y - a_1) \cdots (y - a_k) + \frac{f(a_n)}{\prod_{i=1}^{n} (a_n - b_i)} \prod_{i=1}^{n} (y - b_i).
\]
(2.3)

Multiplying both sides by \( (y - a_1) \cdots (y - a_{i-1}) \), then applying the operator \( \partial_1 \cdots \partial_i \), we have
\[
g(y_1)(y_1 - a_1) \cdots (y_1 - a_{i-1}) \partial_1 \cdots \partial_i \bigg|_{y_1=b_1, 1 \leq i \leq n}
\]
\[
= \sum_{k=1}^{n-1} c_k (y_1 - b_1) \cdots (y_1 - b_k) / (y_1 - a_1) \cdots (y_1 - a_k) \partial_1 \cdots \partial_i \bigg|_{y_1=b_1, 1 \leq i \leq n}
\]
\[
+ \sum_{k=1}^{n-1} c_k (y_1 - b_1) \cdots (y_1 - b_k) \partial_1 \cdots \partial_i \bigg|_{y_1=b_1, 1 \leq i \leq n}.
\]
(2.4)

Since \( \partial_1 \cdots \partial_i \) decreases degree by \( i \) and \( (y_1 - a_{k+1}) \cdots (y_1 - a_{i-1}) (y_1 - b_1) \cdots (y_1 - b_k) \) is a polynomial of degree \( i - 1 \), so the first sum on the right side vanishes.

Consider the case \( i \leq k \). By the Leibniz type formula (2.1), we find
\[
\frac{(y_1 - b_1) \cdots (y_1 - b_k)}{(y_1 - a_1) \cdots (y_1 - a_k)} \partial_1 \cdots \partial_i \bigg|_{y_1=b_1, 1 \leq i \leq n}
\]
\[
= \left( \frac{(y_1 - b_1) \cdots (y_1 - b_i)}{(y_1 - a_1) \cdots (y_1 - a_i)} \right) \partial_1 \cdots \partial_i \bigg|_{y_1=b_1, 1 \leq i \leq n} + \left( \frac{(y_2 - b_2) \cdots (y_2 - b_k)}{(y_2 - a_1) \cdots (y_2 - a_k)} \partial_1 \cdots \partial_i \bigg|_{y_2=b_1, 1 \leq i \leq n} + \cdots + \right.
\]
\[
= \left( \frac{(y_1 - b_1) \cdots (y_1 - b_k)}{(y_1 - a_1) \cdots (y_1 - a_k)} \partial_1 \cdots \partial_i \bigg|_{y_1=b_1, 1 \leq i \leq n} = \begin{cases} 0, & \text{if } i < k, \\ 1 / (b_{k+1} - a_k), & i = k. \end{cases}
\]

Now (2.4) becomes
\[
g(b_1)(b_1 - a_1) \cdots (b_1 - a_{k-1}) \partial_1 \cdots \partial_k = \frac{f(b_1)}{(b_1 - a_k) \cdots (b_1 - a_n)} \partial_1 \cdots \partial_k = \frac{c_k}{(b_{k+1} - a_k)}.
\]

as desired. \( \square \)

It is easy to see that Theorem 1.1 can be deduced from (1.4) and Lemma 2.1 by replacing \( a_i \) by \( a_{n-i+1} \) for \( 1 \leq i \leq n \). There is an obvious symmetry between \( a_i \)'s and \( b_i \)'s in Theorem 1.1:
\[
\sum_{k=1}^{n-1} \frac{f(b_1)}{\prod_{i=1}^{n-k} (b_1 - a_i)} \partial_1 \cdots \partial_k (b_{k+1} - a_{n-k+1}) \prod_{i=1}^{n-k} (y - a_i) \prod_{i=1}^{k} (y - b_i)
\]
\[
= \sum_{k=1}^{n-1} \frac{f(a_1)}{\prod_{i=1}^{n-k} (a_1 - b_i)} \partial_1 \cdots \partial_k (a_{k+1} - b_{n-k+1}) \prod_{i=1}^{n-k} (y - b_i) \prod_{i=1}^{k} (y - a_i),
\]

which implies the following identity.
Corollary 2.2. For $1 \leq k \leq n - 1$, we have
\[
\frac{f(b_1)}{\prod_{i=1}^{n-k+1} (b_1 - a_i)} \partial_1 \cdots \partial_k = - \frac{f(a_1)}{\prod_{i=1}^k (a_1 - b_i)} \partial_1 \cdots \partial_{n-k},
\]  
where $f$ is a polynomial with degree $\leq n$.

3. Interpolation formulas for $f(y) = 1$

As an immediate consequence of Theorem 1.1, we have
\[
1 = \prod_{i=1}^n \frac{y - a_i}{b_1 - a_i} + \prod_{i=1}^n \frac{y - b_i}{a_1 - b_i} + \sum_{k=1}^{n-1} \frac{1}{\prod_{i=1}^{k-1} (b_{k+1} - a_i)} \partial_1 \cdots \partial_{n-k}(b_{k+1} - a_{n-k+1}) \prod_{i=1}^{n-k} (y - a_i) \prod_{j=1}^k (y - b_j),
\]  
which is a $2n$-point interpolation formula for $f(y) = 1$. In this section, we shall apply Eq. (3.1) to derive several $q$-identities and a biaf hypergeometric identity.

3.1. $q$-identities related to divisor functions

Multiply both sides of (3.1) by $1/(y - a_1)$ and then set $a_1 = x$, $b_i = y$. Since
\[
\frac{1}{y - b_1} \partial_1 \cdots \partial_{n-1} = \frac{1}{(y - b_1) \cdots (y - b_n)},
\]  
the last term of the summation in (3.1) becomes
\[
\frac{1}{(b_1 - a_2)(b_1 - x)} \partial_1 \cdots \partial_{n-1}(y - a_2) \prod_{i=1}^{n-1} (y - b_i)
\]  
\[
= \left( \frac{1}{(x - a_2)(a_2 - b_1)} + \frac{1}{(a_2 - x)(x - b_1)} \right) \partial_1 \cdots \partial_{n-1}(y - a_2) \prod_{i=1}^{n-1} (y - b_i)
\]  
\[
= \frac{1}{a_2 - x} \left( \prod_{i=1}^{n-1} \frac{y - b_i}{a_2 - b_i} - \prod_{i=1}^{n-1} \frac{y - b_i}{x - b_i} \right) + \frac{1}{y - x} \prod_{i=1}^{n-1} (y - b_i).
\]  
For $1 \leq i \leq n - 1$, replacing $a_{i+1}$ by $a_i$, and multiplying both sides by $-1$, we find
\[
\frac{1}{x - y} \left( 1 - \prod_{i=1}^{n-1} \frac{y - b_i}{y - a_i} \right) = \frac{1}{x - b_1} \prod_{i=1}^{n-1} \frac{y - a_i}{(y - b_i)} + \frac{1}{x - a_1} \left( \prod_{i=1}^{n-1} \frac{y - b_i}{a_1 - b_i} - \prod_{i=1}^{n-1} \frac{y - b_i}{a_1 - x} \right)
\]  
\[
+ \sum_{k=1}^{n-2} \frac{1}{x - b_1} \partial_1 \cdots \partial_k(b_{k+1} - a_{n-k}) \prod_{j=1}^{n-k-1} (y - a_j) \prod_{j=1}^k (y - b_j).
\]  
Since the $n$-point Newton’s formula and the $n$-point Lagrange’s formula for $1/(x - y)$ have the same remainder $\prod_{i=1}^n (y - b_i)((x - y) \prod_{i=1}^n (x - b_i))^{-1}$, Theorem 1.2 holds.

Let $A = \{q^{M-1}, \ldots, q^{M-n}\}, \quad B = \{q^{-1}, q^{-2}, \ldots, q^{-n}\}, \quad x = q^M, \quad y = 1$. Theorem 1.2 implies
\[
\sum_{k=0}^n \frac{M!}{k!(M+n-1-k)!} \frac{q^{k+1}}{1 - q^{k+1}} = \sum_{k=1}^{n+1} \frac{q^k}{1 - q^k} \frac{M!}{k!(M+n-1-k)!} = \sum_{k=1}^{n+1} \frac{n+1}{k} \frac{(-1)^{k-1} q^{k+1/2}}{1 - q^{M+k}}.
\]

Note that the second equality is given by Uchimura [9], see also [3].

3.2. Finite forms of the quintuple product identity

By setting $a_i = vq^{i-n}$ and $b_i = wq^i$ for $1 \leq i \leq n$ in (3.1), one can verify that
\[
\frac{1}{\prod_{i=1}^{n-k+1} (b_1 - vq^{i-n})} \partial_1 \cdots \partial_k = \sum_{k=1}^{n+1} \frac{q^k}{1 - q^k} \frac{M!}{k!(M+n-1-k)!} \frac{(-1)^{k-1} q^{k+1/2}}{1 - q^{M+k}}.
\]
Substituting the above relation into (3.1), we obtain
\[
1 = \sum_{k=0}^{n} \binom{n}{k} \prod_{j=1}^{n-k} (y - vq^{j-n}) \prod_{j=1}^{k} (y - wq^{j}) \prod_{j=1}^{n-k} (wq^{k+1} - vq^{j-n}) \prod_{j=1}^{k} (vq^{1-k} - wq^{j}) .
\] (3.2)

Setting \( v = w^{-1} \) and \( y = q \) in (3.2), we reach the following \( q \)-identity [2]:
\[
1 = \sum_{k=0}^{n} (1 + wq^{k}) \binom{n}{k} \frac{(w; q)_{n+1}}{(w^2q^k; q)_{n+1}} w^k q^{k^2}.
\]

Note that above identity is a finite form of Watson’s quintuple product identity [5]:
\[
\sum_{k=-\infty}^{\infty} (1 - wq^k) w^{3k} q^{\frac{3k^2-k}{2}} = (q; q)_\infty (w; q)_\infty (q/w; q)_\infty (w^2 q; q^2)_\infty (q/w^2; q^2)_\infty.
\]

Consider the case \( v = w^{-1} q^{-1} \) and \( y = 1 \). Then (3.2) implies another finite form of the quintuple product identity [6]:
\[
1 = \sum_{k=0}^{n} (1 - w^2 q^{2k+1}) \binom{n}{k} \frac{(wq; q)_{n+1}}{(w^2q^{k+1}; q)_{n+1}} w^k q^{k^2}.
\]

### 3.3. A bibasic hypergeometric identity

Let \( f(y) = 1 \) and \( a_i = p^i, b_i = q^{-i} \) for \( 1 \leq i \leq n \). Then (2.5) implies the following \( p, q \)-identity:
\[
\sum_{j=0}^{n-k} (-1)^{j} p^{\left(\frac{j+1}{2}\right)-(n-k)(j+1)} (p; q)_{n-k-j}(pq^{j+1}; q)_{k+1} = \sum_{j=0}^{k} (-1)^{j} q^{\left(\frac{j+1}{2}\right)+(j+1)(n-k)} (q; q)(q; q)_{k-j}(pq^{j+1}; p)_{n-k+1}.
\] (3.3)

**Proof.** By the partial fraction expansion, we find
\[
\frac{q^2}{\prod (z - q^j)} = \sum_{j=0}^{n-k} (-1)^j q^{\left(\frac{n-j}{2}\right)} (q; q)(q; q)_{n-j}(z - q^{j+1}).
\]

Thus,
\[
\frac{1}{(b_1 - p) \cdots (b_1 - p^{n-k+1}) \partial_1 \cdots \partial_k} = \sum_{j=0}^{n-k} (-1)^{j} p^{\left(\frac{n-j}{2}\right)-(n-k)(j+1)} (p; q)_{n-k-j}(b_1 - p^{j+1}) \partial_1 \cdots \partial_k
\]
\[
= \sum_{j=0}^{n-k} (-1)^{j+1} q^{\left(\frac{j+1}{2}\right)-(n-k)(j+1)} (q; q)(q; q)_{k-j}(pq^{j+1}; q)_{k+1}.
\]

In the same manner, we deduce that
\[
\frac{1}{(a_1 - q^{-1}) \cdots (a_1 - q^{-k-1}) \partial_1 \cdots \partial_{n-k}} = \sum_{j=0}^{k} (-1)^{k+j+1} q^{\left(\frac{k+1}{2}\right)+(j+1)(n-k)} (q; q)(q; q)_{k-j}(pq^{j+1}; p)_{n-k+1}.
\]

Now (3.3) immediately follows from (2.5). This completes the proof. \( \square \)

### 4. An expansion of the \( q \)-shifted factorials

In this section, we shall give a different proof of **Theorem 1.3** by computing certain minors of Krattenthaler’s determinant. Expanding the determinant in (1.3) with respect to the last row, we obtain
\[
\prod_{1 \leq s < j \leq n} (x_j - x_s) \prod_{i=1}^{n} (x_i - y) \prod_{1 \leq s < j \leq n} (a_j - b_i)
\]
\[
= \sum_{k=1}^{n+1} (-1)^{n+k+1} C_{n,k}(X, A, B)(y - a_k) \cdots (y - a_n)(y - b_1) \cdots (y - b_{k-1}).
\] (4.1)

where \( C_{n,k}(X, A, B) \) denotes the co-factor of the determinant with respect to the entry \( (y - a_k) \cdots (y - a_n)(y - b_1) \cdots (y - b_{k-1}) \).
To present our proof, let us give a quick review of some basic properties of supersymmetric complete functions. Given two sets of indeterminates $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$, the supersymmetric complete function $h_n(X - Y)$ is defined by

$$h_n(X - Y) = \prod_{x \in X} (1 - x t) \prod_{y \in Y} (1 - x t) = \sum_{k=0}^{n} (-1)^k e_k(Y) h_{n-k}(X),$$

where $[t^n] f(t)$ stands for the coefficient of $t^n$ in $f(t)$, $e_k(X)$ denotes the $k$-th elementary symmetric function and $h_k(X)$ denotes the $k$-th complete symmetric function in $X$. Clearly, $h_0(X - Y) = 1$.

**Lemma 4.1 ([8]).** Let $\{j_1, j_2, \ldots, j_n\}$ be a sequence of integers, and let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be sets of indeterminates. The following relation holds

$$\det\left( h_{j+k-j}(X_k - Y_k) \right)_{k,l=1}^{n} = \det\left( h_{j+k-l}(X_k - Y_k - D_{k-l}) \right)_{k,l=1}^{n},$$

where $D_0, D_1, \ldots, D_{n-1}$ are sets of indeterminates such that the cardinality of $D_i$ is equal to or less than $i$.

**Lemma 4.2.** For $1 \leq k \leq n + 1$, set

$$Y_j = \begin{cases} \{a_j, a_{n}, a_{n+1}, \ldots, a_{n+1}\}, & 1 \leq j < k; \\ \{a_{n+1}, a_{n+1}, a_{n+1}, \ldots, a_{n+1}\}, & k \leq j \leq n, \\ \{x_1, x_2, \ldots, x_n\} & j = n + 1. \end{cases}$$

Then we have

$$C_{n,k}(X, A, B) = \prod_{1 \leq i < j \leq n} (x_i - x_j) \det(h_i(X - Y_j))_{i,j=1}^{n} = (-1)^{\binom{n}{2}} \prod_{1 \leq i < j \leq n} (x_i - x_j) \det(e_{i-1}(Y_j))_{i,j=1}^{n+1},$$

**Proof.** The identity (4.2) easily follows from the definition of the supersymmetric complete function and Lemma 4.1. Since $h_0(X - Y) = 1$, $h_n(X - Y) = \sum_{k=0}^{n} (-1)^k e_k(X) h_{n-k}(X) = 0$, $n \neq 0$, we deduce that

$$\det(h_i(X - Y_j))_{i,j=1}^{n} = (-1)^n \det(h_{i-1}(X - Y_j))_{i,j=1}^{n+1} = (-1)^n \det(h_{i-j}(X))_{i,j=1}^{n+1} \det((-1)^{i-1} e_{i-1}(Y_j))_{i,j=1}^{n+1} = (-1)^{\binom{n}{2}} \det(e_{i-1}(Y_j))_{i,j=1}^{n+1}.$$}

This completes the proof. \(\square\)

Setting $X = \{u, uq^{-1}, \ldots, uq^{1-n}\}$, $A = \{v, vq^{-1}, \ldots, vq^{1-n}\}$ and $B = \{wq, wq^2, \ldots, wq^n\}$ in (4.1), we find

$$\prod_{i=1}^{n} (y - uq^{i-1}) \prod_{1 \leq i < j \leq n} (vq^{j-i} - wq) = \sum_{k=1}^{n+1} (-1)^{k+1} D_{n,k}(X, A, B) \prod_{i=1}^{n} (y - uq^{i-1}) \prod_{i=1}^{n} (vq^{j-i} - wq),$$

where

$$D_{n,k}(X, A, B) = C_{n,k}(X, A, B) \left/ \prod_{1 \leq i < j \leq n} (uq^{j-i} - uq^{i-1}) \right..$$

To prove (1.6), it suffices to establish the following theorem.

**Theorem 4.3.** For $1 \leq k \leq n + 1$, we have

$$D_{n,k}(X, A, B) = (-1)^{n+1-k} \left[ \prod_{i=0}^{n-k} (wq - uq^i) \prod_{i=1}^{k-1} (uq^{n-i} - vq^{2-k}) \prod_{j=k}^{n} (vq^{j-i} - wq). \right]$$

It is convenient for us to present the proof of Theorem 4.3 via two steps. In the first step (Lemma 4.4) we shall evaluate a special case of $D_{n,k}(X, A, B)$, where $u = wq^2$. The second step (Lemma 4.5) is to show that $D_{n,k}(X, A, B)$ is a polynomial in $u$ with $n$ roots $wq, w, \ldots, wq^{1+k-n}, vq^{1-n}, \ldots, vq^{2-k-n}$. 


Lemma 4.4. Let $A' = \{v, vq^{-1}, \ldots, vq^{-n}\}$ and $B' = \{wq, wq^2, wq^3, \ldots, wq^{m+1}\}$. For $1 \leq k \leq n + 1$, we have

$$D'_{n,k}(A', B') = (-1)^{\binom{n+1}{2}} w^{n-k+1} q^{n-k+1} (q; q)_{n-k+1} \prod_{i=1}^{k-1} \left(\prod_{j=1}^{n} (vq^{-j} - wq^j) \prod_{i=1}^{k-1} (wq^{2-k} - wq^{n+2})\right). \quad (4.5)$$

where

$$D'_{n,k}(A', B') = (-1)^{\binom{n}{2}} D_{n,k}(X, A, B) \bigg|_{u = wq^2}.$$

Proof. From Lemma 4.2 it follows that

$$D'_{n,k}(A', B') = \det(e_{i-1}(Y_j))_{i,j=1}^{n+1}, \quad (4.6)$$

where

$$Y_j = \begin{cases} \{vq^{1-j}, \ldots, vq^{1-n}, wq, \ldots, wq^{k-1}\}, & \text{if } 1 \leq j < k; \\ \{vq^{2}, \ldots, vq^{1-n}, wq, \ldots, wq^{k}\}, & \text{if } k \leq j \leq n; \\ \{wq^2, \ldots, wq^{n+1}\}, & \text{if } j = n + 1. \end{cases}$$

We shall proceed by induction on $n$. When $n = 1$ and $k = 1, 2$, we have $D'_{1,1}(\{v\}, \{wq, wq^2\}) = -wq(1 - q)$, $D'_{1,2}(\{v\}, \{wq, wq^2\}) = -(v - wq^2)$, which are in accordance with the right hand side of (4.5). We now assume that (4.5) holds for $1 \leq n \leq m - 1$, where $m \geq 2$. Since

$$e_j(Y_1) - e_j(Y_{k+1}) = (vq^{1-k} - wq^k) e_{j-1}(vq^{-k}, \ldots, wq^{k-1})$$

and

$$e_j(Y_n) - e_j(Y_{m+1}) = (wq - wq^{m+1}) e_{j-1}(wq^2, \ldots, wq^m),$$

it is easy to verify that

$$D'_{m,1}(A', B') = (-1)^{\binom{m+1}{2}} w^m q^m (q; q)_m \prod_{i=1}^{m} \prod_{j=1}^{m} (vq^{1-j} - wq^j),$$

which is equal to the right side of (4.5).

We now consider that case $1 \leq k \leq m - 1$. Since

$$e_j(Y_1) - e_j(Y_{k+2}) = (vq^{1-k} - wq^k) e_{j-1}(vq^{-k}, \ldots, wq^{k-1}) + (vq^{-k} - wq^{k+1}) e_{j-1}(vq^{1-k}, \ldots, wq^k),$$

it follows that

$$D'_{m,k+1}(A', B') = (-1)^m \prod_{i=1}^{m} (vq^{1-i} - wq^i) (wq - wq^{m+1})$$

$$\times \left( \frac{D'_{m-1,k}(A' \setminus \{v\}, B' \setminus \{wq^{m+1}\})}{vq^{k-1} - wq^k} + \frac{D'_{m-1,k+1}(A' \setminus \{v\}, B' \setminus \{wq^{m+1}\})}{vq^{-k} - wq^{k+1}} \right) .$$

where $A \setminus B$ denotes the set difference of $A$ and $B$.

By the inductive hypothesis, we get

$$D'_{m-1,k}(A' \setminus \{v\}, B' \setminus \{wq^{m+1}\}) = (-1)^{\binom{m}{2}} w^{m-k} q^{m-k} (q; q)_{m-k} \prod_{i=1}^{m-1} \prod_{j=1}^{m-1} (vq^{-j} - wq^j) \prod_{i=1}^{k-1} (vq^{1-k} - wq^{m+1})$$

and

$$D'_{m-1,k+1}(A' \setminus \{v\}, B' \setminus \{wq^{m+1}\}) = (-1)^{\binom{m}{2}} w^{m-k-1} q^{m-k-1} (q; q)_{m-k-1}$$

$$\times \prod_{i=1}^{m} \prod_{j=1}^{m} (vq^{-j} - wq^j) \prod_{i=1}^{k-1} (vq^{-k} - wq^{m+1}).$$
Therefore,
\[
D_{m,k+1}(A', B') = (-1)^{\frac{m+1}{2}} \prod_{i=1}^{k} (vq^{1-k} - wq^{j+2}) \left( \frac{w(1-q^k)}{vq^{-k} - wq^k} + \frac{(v - wq^k)q^{-k}}{vq^{-k} - wq^k} \right),
\]
as desired.

Finally, we are left with the case \( k = m + 1 \). Since, by (1.3),
\[
D_{m,m+1}(X, A, B) \prod_{1 \leq i \leq j \leq m} (uq^{j-1} - uq^{i-1}) = \det \left( (uq^{j-1} - vq^{i-1}) \cdots (uq^{j-1} - wq^{i-1}) \right)_{i,j=1}^m
\]
we have
\[
D_{m,k}^r(A', B') = (-1)^{\frac{m+1}{2}} D_{m,k}(X, A, B) \bigg|_{u = wq^2}
\]
\[
= (-1)^{\frac{m+1}{2}} \prod_{i=1}^{m} (uq^{1-m} - vq^{i-1}) \prod_{j=1}^{m-1} (vq^{j-1} - wq^j)
\]
\[
= (-1)^{\frac{m+1}{2}} \prod_{i=1}^{m} (vq^{1-m} - wq^{i+1}) \prod_{j=1}^{m-1} (vq^{j-1} - wq^j),
\]
which is equal to the right hand side of (4.5). This completes the proof. \( \square \)

**Lemma 4.5.** We have
\[
D_n(X, A, B) = C \prod_{i=0}^{n-k} (uq^i - wq) \prod_{i=1}^{k-1} (uq^n - vq^2 - k),
\]
where \( C \) is independent of \( u \).

**Proof.** We view \( D_n(X, A, B) \) as a polynomial in \( u \) of degree \( n \) with coefficients depending on \( v \) and \( w \). The essence of the proof is to show that \( wq, w, \ldots, wq^{1+k-n}, vq^{1-n}, \ldots, vq^{3-k-n} \) are the roots of the polynomial.

For \( u = wq^2 \), where \( 1 \leq i \leq n - k + 1 \), let
\[
D_0 = \emptyset, \quad D_1 = \{wq^{2-i}\}, \ldots, D_i = \{wq^{2-i}, \ldots, w\}, \\
D_i = \{wq^{2-i}, \ldots, w, wq^{1+i}\}, \ldots, D_{n-1} = \{wq^{2-i}, \ldots, w, wq^{1+n}\}, \ldots, D_{n} = \{wq^{2-i}, \ldots, wq^{3-n}\}.
\]

Clearly, \( e_k(X) = 0 \) if the cardinality of \( X \) is less than \( k \). In view of **Lemma 4.1**, \( D_n(X, A, B) \) can be transformed into a determinant whose \((i,j)\)-th entry is equal to 0 if
\[(i,j) \in \{(i,j) : 2 \leq k \leq n - 1 \text{ and } n - j + 2 \leq i \leq n, \text{ or } k \leq j \leq n \text{ and } n - k + 1 \leq i \leq n\}.
\]
Thus \( D_n(X, A, B) \big|_{u = wq^2} = 0 \).

Similarly, for \( u = vq^{2-n-i} \), \( 1 \leq k \leq 1 \), we take
\[
D_0 = \emptyset, \quad D_1 = \{vq^{2-n-i}\}, \ldots, D_i = \{vq^{2-n-i}, \ldots, vq^{-n}\}, \\
D_i = \{vq^{2-n-i}, \ldots, vq^{-n}, vq^{1-i}\}, \quad D_{n-i} = \{vq^{2-n-i}, \ldots, vq^{2-n}, vq^{1-i}, \ldots, vq^{-n}\}.
\]
Now \( D_n(X, A, B) \) becomes a determinant with the \((i,j)\)-th entry being 0 if
\[(i,j) \in \{(i,j) : k \leq j \leq n - 1 \text{ and } j + 1 \leq i \leq n, \text{ or } 1 \leq j \leq k - 1 \text{ and } k - 1 \leq i \leq n\}.
\]
So we deduce that \( D_n(X, A, B) \big|_{u = vq^{2-n-i}} = 0 \). This completes the proof. \( \square \)

We are now ready to complete the proof of **Theorem 4.3**.

**Proof of Theorem 4.3.** In **Lemma 4.5**, we have established that
\[
D_n(X, A, B) = C \prod_{i=0}^{n-k} (uq^i - wq) \prod_{i=1}^{k-1} (uq^n - vq^2 - k),
\]
where \( C \) is independent of \( u \).
To determine $C$, we set $u = wq^2$. Applying Lemmas 4.2 and 4.4, we obtain
\begin{align*}
C \sum_{i=0}^{n-k} (wq^{2+i} - wq) \prod_{i=1}^{k-1} (uq^{n-i} - vq^{2-k}) = &\, (-1)^n u^{n-k+1} q^{n-k+1} (q; q)_{n-k+1} \\
\times &\left[ \sum_{k=1}^{n-1} \prod_{i=1}^{k-1} (vq^{1-j} - wq^j) \prod_{j=1}^{k-1} (vq^{2-k} - wq^{n-i+2}) \right].
\end{align*}
This implies that
\begin{align*}
C = &\left[ \sum_{k=1}^{n-1} \prod_{i=1}^{k-1} (vq^{1-j} - wq^j) \right] \\
\times &\prod_{j=1}^{k-1} (vq^{2-k} - wq^{n-i+2}).
\end{align*}
Therefore,
\begin{equation}
D_{n,k}(X, A, B) = (-1)^{n-k+1} \left[ \sum_{k=1}^{n-1} \prod_{i=1}^{k-1} (wq - uq^i) \prod_{j=1}^{k-1} (uq^{n-i} - vq^{2-k}) \right] \prod_{j=1}^{k-1} \prod_{i=j}^{k-1} (vq^{1-j} - wq^j),
\end{equation}
as desired. This completes the proof. □

To conclude this paper, we give two special cases of (1.6). Bailey [1] found the following two identities as the $q$-analogues of Dixon’s theorem for the cubic sums of binomial coefficients:
\begin{equation}
\sum_{k=-n}^{n} (-1)^k \left[ \binom{2n}{n+k} \right]^3 q^{3(k+1)/2} = \frac{(q; q)^{3n}}{q(q; q)_n^3},
\end{equation}
and
\begin{equation}
\sum_{k=-n}^{n} (-1)^k \left[ \binom{2n+1}{n+k+1} \right]^3 q^{3(k+1)/2} = \frac{(q; q)^{3n+1}}{q(q; q)_n^3}.
\end{equation}
Replacing $y$ by $uq^{3n}$, $v$ and $w$ by $uq^{2n-1}$ in (1.6), we obtain
\begin{equation}
\left( q^{2n+1}; q \right)_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} q^{3k(k-1)/2} \frac{(q^{n-k+1}; q)_k^2}{(q^k; q)_k} \frac{(q^{n-k+1}; q^2)_n^2}{(q^{2k+1}; q^2)_n^2}.
\end{equation}
Multiplying both sides by $(q; q)_{2n+1}/(q; q)^3_n$ gives (4.8).

Taking $y = uq^{2n+1}$, $v = uq^{2n-1}$ and $w = uq^{2n}$ in (1.6), we find
\begin{equation}
\left( q^{2n+2}; q \right)_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} q^{3k(k+1)/2} \frac{(q^{n-k+1}; q)_k^2}{(q^k; q)_k} \frac{(q^{n-k+1}; q^2)_n^2}{(q^{2k+2}; q^2)_n^2}.
\end{equation}
Multiplying both sides by $(q; q)_{2n+1}/(q; q)^3_n$, we arrive at (4.9).

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References