\((\mathbb{Z}_2)^k\)-actions and the minimal data of the normal bundle\(\star\)

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1. Introduction

Let \(\text{MO}_n\) denote the unoriented cobordism group and \(\text{MO}_* = \sum_{n \geq 0} \text{MO}_n\) the unoriented cobordism ring \([1]\), and let \(J'_{n,k}\) denote the group of the \(n\)-dimensional unoriented cobordism classes containing a representative that admits a \((\mathbb{Z}_2)^k\)-action with the fixed point set of constant dimension \(n - r\). Then \(J'_{n,k} = \sum_{n \geq r} J_{n,k}\) forms an ideal in \(\text{MO}_*\).

We write \((\mathbb{Z}_2)^k\) for the group being generated by the elements \(t_1, t_2, \ldots, t_k\) subject to the relation \(t_i^2 = 1\) and \(t_it_j = t_jt_i\) \((i, j = 1, 2, \ldots, k)\), and \(\text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)\) for the set of homomorphisms \((\mathbb{Z}_2)^k \to \mathbb{Z}_2 = \{1, -1\}\), consisting of \(2^k\) distinct homomorphisms which we label \(f_i\), \(i = 0, 1, \ldots, 2^k - 1\), where \(f_0\) is the trivial homomorphism. Thus if \(1 \leq i \leq 2^k - 1\), there is a generator \(t_i\) in \((\mathbb{Z}_2)^k\) such that \(f_i(t_i) = -1\). We note that for \(1 \leq i \leq 2^k - 1\), the kernel of the homomorphism \(\ker f_i \cong (\mathbb{Z}_2)^{k-1}\).

Every irreducible representation of \((\mathbb{Z}_2)^k\) is one-dimensional and has the form \(\lambda : (\mathbb{Z}_2)^k \times \mathbb{R} \to \mathbb{R}\) with \(\lambda(t, y) = f_i(t)y\) for some \(f_i\). Hence if \(e\) is a real vector bundle with a \((\mathbb{Z}_2)^k\)-action covering a compact Hausdorff space with a \((\mathbb{Z}_2)^k\)-action, there are \(2^k\) subbundles \(e_i\) of \(e\) with \(e = e_0 \oplus e_1 \oplus \cdots \oplus e_{2^k-1}\) such that the action of \(t_j\) on \(e\) corresponds to \((f_0(t_j)y_0, f_1(t_j)y_1, \ldots, f_{2^k-1}(t_j)y_{2^k-1})\), where \(y_i \in E(e_i)\), the total space of \(e_i\), \(i = 0, 1, \ldots, 2^k - 1\).

Let \(T : ((\mathbb{Z}_2)^k \times M^n) \to M^n\) be a smooth \((\mathbb{Z}_2)^k\)-action with the fixed point set \(F\) of constant dimension \(n - r\). Then \(F\) is a disjoint union of closed submanifolds of \(M^n\). Let \(N_i\) represent the set of fixed points of \(k\)\(f_i\) and \(\bar{T}_j = T(t_j, -)\). If \(t_j \notin \ker f_i\), from \([3]\) we know that \(F\) is the fixed point set of the involution \((N_i, \bar{T}_j|_{N_i})\). The normal bundle \(e\) to the fixed point set \(F\) of \((M^n, T)\) can be decomposed as \(e = e_1 \oplus e_2 \oplus \cdots \oplus e_{2^k-1}\), and \(e_1 \to F\) is justly the normal bundle to the fixed point set \(F\) of the involution \((N_i, \bar{T}_j|_{N_i})\). Since \(f_0 = 1, e_0\) is the trivial zero-dimensional bundle and may be ignored. In this way, \(F\) and the ordered set of the \(2^k - 1\) vector bundles \(\{e_i\}_{i=1}^{2^k-1}\) constitute the fixed point data of the action on \(M^n\).
Let $I = \{(a_1, a_2, \ldots, a_{2^k-1}) \mid \sum_{i=1}^{2^k-1} a_i = r, \ a_i \text{ is a non-negative integer}\}$. Then $I$ is said to be the set of total $2^k - 1$ partitions of $r$.

For $A \subseteq I$, let $J'_{n,k}(A)$ denote the set of the classes represented by a manifold $M^n$ with a $(\mathbb{Z}_2)^k$-action having the fixed point data $(F^{n-r}, [\varepsilon_1])_{i=1}^{2^k-1}$ such that for each component $F_j$ of $F^{n-r}$,

$$\left(\dim(\varepsilon_1|F_j), \dim(\varepsilon_2|F_j), \ldots, \dim(\varepsilon_{2^k-1}|F_j)\right) \in A.$$ Then $J'_{n,k}(A) = \sum_{i \geq r} J'_{n,k}(A)$ forms an ideal in $MO_*$.

Let $r_1 = (2, 1, 0)$, $r_2 = (1, 0, 2)$, $r_3 = (0, 2, 1)$, $r_4 = (1, 2, 1)$, $r_5 = (0, 1, 2)$ and $r_6 = (2, 0, 1)$. For $A = \{r_1, r_2, r_3, r_4, r_5, r_6\}$, $\{r_1, r_2, r_3, r_4\}$ and $\{r_2, r_3, r_5\}$. Pergher showed $J'_{3,2}(A) = 0$ in [2]. For $A = \{(2, 1, 0), (2, 0, 1), (1, 1, 1)\}$, Wu and Guo computed the ideal $J'_{3,2}(A)$ in [11]. For $A = \{(2, 2, 0), (2, 2, 2), (0, 2, 2)\}$, Wu and Guo determined the ideal $J'_{2,2}(A)$ in [12].

In another way, one may consider to study $J'_{n,k}(A)$. Let $\|A\|$ represent the number of the elements of $A$. The minimal data of the normal bundle for $J'_{n,k}$ is defined by

$$\|J'_{n,k}\| = \min\{\|A\| \mid J'_{n,k}(A) = J'_{n,k}, \ A \subseteq I\},$$

which reflects the complexity of the $(\mathbb{Z}_2)^k$-actions. It is interesting to determine $\|J'_{n,k}\|$ or a smaller upper bound of $\|J'_{n,k}\|$.

The main results of this paper are

**Theorem 1.1.** $\|J'_{n,k}\| = 3, k \geq 2$.

**Theorem 1.2.**

1. $\|J'_{n,k}\| = 3, k = 2$.
2. $\|J'_{n,k}\| \leq 3, k \geq 3$.

**Theorem 1.3.**

$$\|J'_{n,k}\| \leq \begin{cases} 8, & r = 4, k \geq 2, \\ 9, & r = 5, k = 2. \end{cases}$$

2. Preliminary

If an $n$-dimensional cobordism class $[M^n]$ can be expressed as a sum of products of lower-dimensional cobordism classes in $MO_*$, then the class is called decomposable; otherwise it is indecomposable. Let $RP(n)$ be the real projective space of dimension $n$ and $\chi: MO_* = \sum MO_{2^n} \rightarrow \mathbb{Z}_2$ the mod 2 Euler characteristic. Throughout this paper $w$ denotes the total Stiefel–Whitney class and $w_i$ the $i$-th Stiefel–Whitney class, $\equiv$ denotes congruence mod 2. Binomial coefficient are $\binom{m}{n} = m!/(n!(m-n)!)$.

**Lemma 2.1.** ([4]) Let $RP(n_1, n_2, \ldots, n_l)$ be the projective space bundle of $\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l$ of $RP(n_1) \times RP(n_2) \times \cdots \times RP(n_l)$, where $\lambda_i$ is the pullback of the canonical line bundle over the $i$-th factor. Then for $l > 1$, $[RP(n_1, n_2, \ldots, n_l)]$ is indecomposable in $MO_*$ if and only if

$$s = \binom{m}{n_1} + \binom{m}{n_2} + \cdots + \binom{m}{n_l} \equiv 1 \pmod{2},$$

where $m = l + \sum_{i=1}^{l} n_i - 2$.

The manifold $RP(n_1, n_2, \ldots, n_l)$ has dimension $m + 1$. If $n_{i+1} = n_{i+2} = \cdots = n_l = 0$, $RP(n_1, n_2, \ldots, n_l)$ will sometimes be written as $RP(n_1, \ldots, n_l; l)$.

**Lemma 2.2.** ([5]) If $m = \sum_{i=1}^{l} m_i 2^i$, and $n = \sum_{i=1}^{l} n_i 2^i$ with $0 \leq m_i, n_i \leq 1$, then

$$\binom{m}{n} \equiv 1 \pmod{2} \iff \forall i, n_i \leq m_i.$$

**Lemma 2.3.** ([6]) If $0 < r < 2^k$, then
Lemma 2.4. \[J_{s,k}^r = \begin{cases} (0), & r = 1, \\
\sum_{n=0}^{\infty} MO_n, & r \text{ even}, \\
\sum_{n=0}^{\infty} MO_n \cap \ker \chi, & r > 1, \text{ odd}. \end{cases}\]

Let \(I, I'\) be the set of the total \(2^k - 1\) partitions of \(r\) and \(r'\) respectively.

\[P = \{ (a_1,i, a_2,i, \ldots, a_{2^k-1,i}) \mid i = 1, \ldots, m \} \subseteq I, \]
\[Q = \{ (b_1,i, b_2,i, \ldots, b_{2^k-1,i}) \mid j = 1, \ldots, n \} \subseteq I'.\]

Let \(\sigma = \left(\begin{array}{c} 1 \ 2 \ \ldots \ 2^k - 1 \\ n_1 \ n_2 \ \ldots \ n_{2^k-1} \end{array}\right) \in S_{2^k-1}\), where \(S_{2^k-1}\) is the symmetric group on \(2^k - 1\) symbols, \(\sigma P = \{(a_1,i, a_2,i, \ldots, a_{n_{2^k-1},i}) \mid i = 1, \ldots, m\}\), and \(P \oplus \theta = \{(a_1,i, a_2,i, \ldots, a_{2^k-1,i}, 0, 0, \ldots, 0) \mid i = 1, \ldots, m\}\). Then we have \(2^k + 1\).

**Lemma 2.4.**

(1) \(J_{s,k}^r(P) = J_{s,k}^r(\sigma P), \forall \sigma \in S_{2^k-1}\).

(2) \(J_{s,k}^r(P) \cdot J_{s,k}^r(Q) \subseteq J_{s,k}^r(P \oplus Q)\).

(3) \(J_{s,k}^r(P) \subseteq J_{s,k+1}^r(P \oplus \theta)\).

**Proof.** (1) Let \(\sigma = \left(\begin{array}{c} 1 \ 2 \ \ldots \ 2^k - 1 \\ n_1 \ n_2 \ \ldots \ n_{2^k-1} \end{array}\right)\). If \(M^n\) admits a \((\mathbb{Z}_2)^k\)-action with the fixed point data \((F^{n-r}, \{e_i\}_{i=1}^{2^k-1})\) such that for each component \(F_j\) of \(F^{n-r}\),

\[(\dim(e_1|F_j), \dim(e_2|F_j), \ldots, \dim(e_{2^k-1}|F_j)) \in P,\]

then

\[(\dim(e_1|F_j), \dim(e_2|F_j), \ldots, \dim(e_{2^k-1}|F_j)) \in \sigma P.\]

So \(J_{s,k}^r(P) \subseteq J_{s,k}^r(\sigma P)\). Similarly \(J_{s,k}^r(\sigma P) \subseteq J_{s,k}^r(\sigma^{-1}(\sigma P)) = J_{s,k}^r(P)\), therefore \(J_{s,k}^r(P) = J_{s,k}^r(\sigma P)\).

(2) For \([M] \in J_{s,k}^r(P)\) and \([M'] \in J_{s,k}^r(Q)\), by using the diagonal rule we can define a \((\mathbb{Z}_2)^k\)-action on \(M \times M'\) such that \([M \times M'] \in J_{s,k+1}^r(P \oplus Q)\).

(3) Let \(M^n\) admit a \((\mathbb{Z}_2)^k\)-action with the fixed point data \((F^{n-r}, \{e_i\}_{i=1}^{2^k-1})\) such that for each component \(F_j\) of \(F^{n-r}\),

\[(\dim(e_1|F_j), \dim(e_2|F_j), \ldots, \dim(e_{2^k-1}|F_j)) \in P.\]

By adding a new generator \(t_{k+1}\) acting as the identity on \(M\), a \((\mathbb{Z}_2)^k\)-action on \(M\) can be extended to a \((\mathbb{Z}_2)^{k+1}\)-action with the same fixed point set. Hom\((\mathbb{Z}_2)^k, \mathbb{Z}_2\) consists of \(2^k\) distinct homomorphisms which we label \(f_i, i = 0, 1, \ldots, 2^k - 1\). Hom\((\mathbb{Z}_2)^{k+1}, \mathbb{Z}_2\) consists of \(2^{k+1}\) distinct homomorphisms which we label \(f'_j, j = 0, 1, \ldots, 2^{k+1} - 1\). Let \(f'_j(t_{k+1}) = 1(j = 0, 1, \ldots, 2^k - 1), f'_j(t_{k+1}) = -1(j = 2^k, 2^k + 1, \ldots, 2^{k+1} - 1)\). Then the group ker \(f_i = \ker f'_j\) for \(i = 0, 1, \ldots, 2^k - 1\).

Since \(t_{k+1}\) acts as the identity on \(M\), \(f'_j(t_{k+1}) = -1\), then \(N(\ker f'_j) = F\). If \(f'_j(t_{k+1}) = 1\), then \(N(\ker f'_j) = N(\ker f_j)\), where \(N(\ker f'_j)\) represents the set of fixed points of \(\ker f'_j\).

Let \(e'_i\) be the normal bundle of \(F\) in \(N(\ker f'_j)\). Then for each component \(F_i\) of \(F\),

\[(\dim(e'_1|F_i), \dim(e'_2|F_i), \ldots, \dim(e'_{2^k-1}|F_i), 0, 0, \ldots, 0) \]
\[= (\dim(e_1|F_i), \dim(e_2|F_i), \ldots, \dim(e_{2^k-1}|F_i), 0, 0, \ldots, 0) \in P \oplus \theta.\]

So \(J_{s,k}^r(P) \subseteq J_{s,k+1}^r(P \oplus \theta)\).

**Lemma 2.5.** If \(0 < r < 2^k\), then \(\|J_{s,k+1}^r\| \leq \|J_{s,k}^r\|\).

**Proof.** Let \(J_{s,k}^r(A) = J_{s,k}^r(A \oplus \theta)\). By Lemma 2.4, \(J_{s,k}^r(A) \subseteq J_{s,k+1}^r(A \oplus \theta)\). According to Lemma 2.3, \(J_{s,k}^r = J_{s,k+1}^r\), so \(J_{s,k+1}^r(A \oplus \theta) \subseteq J_{s,k+1}^r = J_{s,k}^r(A) \subseteq J_{s,k+1}^r(A \oplus \theta)\). \(J_{s,k+1}^r(A \oplus \theta) = J_{s,k+1}^r(A)\). Hence \(\|J_{s,k+1}^r\| \leq \|A\| = \|J_{s,k}^r\|\).
Lemma 3.1. \([RP(n_1, n_2, n_3)] \in J_{s,2}^2(A)\) for \(A = \{(1, 0, 1), (1.1, 0), (0, 1, 1)\}\).

Proof. Let
\[
T_1[y_1, y_2, y_3] = [-y_1, y_2, y_3],
T_2[y_1, y_2, y_3] = [-y_1, -y_2, y_3].
\]
Then \(T_1, T_2\) define a \((\mathbb{Z}_2)^2\)-action on \(RP(n_1, n_2, n_3)\) with the fixed point set
\[
F = RP(\lambda_1) \cup RP(\lambda_2) \cup RP(\lambda_3)
= RP(n_1) \times RP(n_2) \times RP(n_3) \times \{e_1, e_2, e_3\},
\]
and
\[
F_{T_1} = RP(\lambda_1) \cup RP(\lambda_2 + \lambda_3),
F_{T_2} = RP(\lambda_1 + \lambda_2) \cup RP(\lambda_3),
F_{T_1T_2} = RP(\lambda_1 + \lambda_3) \cup RP(\lambda_2),
\]
where \(F_{T_1}, F_{T_2}\), and \(F_{T_1T_2}\) denotes the fixed point set of \(T_1, T_2,\) and \(T_1T_2\) respectively. The components of \(F\) correspond to \(A = \{(1, 0, 1), (1.1, 0), (0, 1, 1)\}\). Hence \([RP(n_1, n_2, n_3)] \in J_{s,2}^2(A)\). □

Proof of Theorem 1.1. From Theorem 8.1 in [4], there are indecomposable manifolds \(RP(n_1, n_2, n_3)\) in every dimension. According to Theorem 3.2 in [2] and Lemma 3.1, \(J_{s,2}^2(A) = \sum_{n \geq 2} \text{MO}_n\), so \(\|J_{s,2}^2\| \leq 3\). From Lemma 2.5, we obtain \(\|J_{s,2}^2\| \leq 3, k \geq 2\).

Next we show \(\|J_{s,2}^2\| \geq 3, k \geq 2\).

By Lemma 2.3, we know \([RP(2)] \in J_{s,2}^2\). Let \(T : (\mathbb{Z}_2)^k \times \text{RP}(2) \rightarrow \text{RP}(2)\) be a smooth \((\mathbb{Z}_2)^k\)-action on \(\text{RP}(2)\) with isolated fixed points. \((\mathbb{Z}_2)^k\) is generated by \(k\) commuting involutions \(t_1, t_2, \ldots, t_k\). \(\text{Hom}( (\mathbb{Z}_2)^k, \mathbb{Z}_2)\) consists of \(2^k\) distinct homomorphisms which we label \(f_1, f_2, \ldots, f_k\). Each \((\mathbb{Z}_2)^k\)-dimensional irreducible representations which we label \(Y_i : t_i(s) = f_i(t_i)s, 0 \leq i \leq 2^k - 1, 1 \leq j \leq k\).

Following [1], by \(R_+((\mathbb{Z}_2)^k)\) we denote the unoriented representation algebra of the group \((\mathbb{Z}_2)^k\). Then \(R_+((\mathbb{Z}_2)^k)\) is the polynomial algebra \(\mathbb{Z}_2[Y_0, Y_1, \ldots, Y_{2^k - 1}]\) (see [1]). If \(y\) is an isolated fixed point of the action \((\text{RP}(2), T)\), then the local representation class \(X(y)\) at \(y\) can be decomposed by
\[
X(y) = Y_0^{p_0} Y_1^{p_1} \cdots Y_{2^k - 1}^{p_{2^k - 1}}, \quad p_0 + p_1 + \cdots + p_{2^k - 1} = 2,
\]
where \(p_i\) is the dimension of the component of \(N_i\) containing \(y\). If \(y \notin N_i\), then \(p_i = 0\). By Theorem 32.6 in [1], if \((\mathbb{Z}_2)^k\) acts smoothly on \(\text{RP}(2)\) with isolated fixed points, then there are at least two isolated fixed points. Since \(X(\text{RP}(2)) = X(F) = 1\), the number of isolated fixed points is odd and at least three.

(1) If there are three of fixed points, \(y_1, y_2,\) and \(y_3\) whose local representation classes are distinct, then by decomposition of local representation class, we know that the fixed point data at \(y_1, y_2,\) and \(y_3\) are distinct.

(2) If there are only two local representation classes, then for \(X(y_1) = X(y_2)\), we can cancel off these two fixed points without changing equivariant class \([(\mathbb{Z}_2)^k, \text{RP}(2)]\) in \(\mathbb{Z}_2((\mathbb{Z}_2)^k)\) (see [1]), and so we use this method to cancel off these points whose local representation classes are the same. Finally we obtain \([(\mathbb{Z}_2)^k, V^2]\) with only one isolated fixed point and the equivariant class \([(\mathbb{Z}_2)^k, V^2] = [(\mathbb{Z}_2)^k, \text{RP}(2)]\). This contradicts to Theorem 32.6 in [1].

(3) If there are only one local representation classes, then similarly we also get a contradiction. Hence \(\|J_{s,2}^2\| \geq 3, k \geq 2\), and the result follows. □

4. \(\|J_{s,2}^2\| (k \geq 2)\)

Before proving Theorem 1.2, let us introduce some lemmas.

Lemma 4.1. ([8]) For a \((\mathbb{Z}_2)^2\)-action \((M^n, T)\), the manifold \(M^n\) is bordant to the union of \(\text{RP}(\lambda \oplus \varepsilon_2) \oplus \varepsilon_1 \oplus R\) and \(\text{RP}(\lambda \oplus \varepsilon_2) \oplus \lambda \oplus \varepsilon_1 \oplus R\), where the first fibers over \(\text{RP}(\varepsilon_2 \oplus R)\) and then over \(F\), and the second fibers over \(\text{RP}(\varepsilon_2 \oplus R)\) and then over \(F\).

Lemma 4.2. ([1]) Let \(k^h \rightarrow V^n\) be a smooth \(k\)-plane bundle over a closed manifold. If either \(k = 2\) or \(n = 1\), then \([\text{RP}(k^h)] = 0\).
Let \( I = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, a, b, c, v\} \), where \( \tau_1 = (2, 1, 0), \tau_2 = (1, 0, 2), \tau_3 = (0, 2, 1), \tau_4 = (1, 2, 0), \tau_5 = (0, 1, 2), \tau_6 = (2, 0, 1), a = (3, 0, 0), b = (0, 3, 0), c = (0, 0, 3) \) and \( v = (1, 1, 1) \).

In the following we take \( x_4 = [RP(1, 1, 0)] \), then the Stiefel–Whitney number \( w^4_1(x_4) = 1 \). By Lemma 2.3, \( x_4 \in \mathcal{J}^3_{s,2} \).

**Lemma 4.3.** If \( A \subseteq I \) and \( \|A\| = 1 \), then \( x_4 \notin \mathcal{J}^2_{s,2}(A) \) and \( \mathcal{J}^3_{s,2}(A) \neq \mathcal{J}^2_{s,2} \).

**Proof.** (1) If \( A = \{\tau_1\} \), then \( \|A\| = 1 \), by Theorem 4.2 in [2], \( \mathcal{J}^3_{s,2}(A) = 0 \).

(2) If \( A = \{a, b\} \) or \( \{c\} \), by [2] and [9], \( \mathcal{J}^3_{s,2}(A) = \mathcal{J}^2_{s,1}(A) = \{\alpha \in MO_n | w_1^i w_{n-i}(\alpha) = w_1^i w_{n-i}s_5(\alpha) = 0, 5 \leq i \leq n, 0 \leq j \leq n, n \geq 3\} \).

(3) If \( A = \{v\} \), by Theorem 4.1 in [2],

\[ \mathcal{J}^3_{s,2}(A) = \{\alpha \in MO_n | w_1^i w_{n-i}(\alpha) = 0, i = 0, 1, 2, \ldots, n, n \geq 3\} \]

From \( w^4_1(x_4) = 1 \), we know that if \( \|A\| = 1 \), \( x_4 \notin \mathcal{J}^2_{s,2}(A) \). Hence \( \mathcal{J}^3_{s,2}(A) \neq \mathcal{J}^2_{s,2} \). \( \square \)

**Lemma 4.4.** If \( A \subseteq \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\} \) and \( \|A\| = 2 \), then \( x_4 \notin \mathcal{J}^3_{s,2}(A) \) and \( \mathcal{J}^3_{s,2}(A) \neq \mathcal{J}^2_{s,2} \).

**Proof.** According to Theorem 4.1 in [2], if \( A = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7\} \) or \( \{\tau_3, \tau_5\} \), then \( \mathcal{J}^3_{s,2}(A) = 0 \).

Let \( A = \{\tau_3, \tau_6\} \). If \( x_4 \in \mathcal{J}^3_{s,2}(A) \), by Lemma 4.1 \( x_4 \) is represented by the union of the following fiberings:

1. \( RP(\lambda \oplus \varepsilon_3) \rightarrow RP(e_2 \oplus R) \rightarrow F_{e_1} \);
2. \( RP(\lambda \oplus e_2 \oplus \lambda \oplus R) \rightarrow RP(\varepsilon_3) \rightarrow F_{\varepsilon_1} \);
3. \( RP(\lambda \oplus e_3 \oplus e_1 \oplus R) \rightarrow F_{e_1} \);
4. \( RP(\lambda \oplus \varepsilon_1 \oplus R) \rightarrow RP(\varepsilon_3) \rightarrow F_{\varepsilon_1} \).

From \( \dim(\lambda \oplus \varepsilon_3) = 2 \) and \( \dim(RP(\varepsilon_3)) = \dim(F_{\varepsilon_1}) = 1 \), by Lemma 4.2 we know that the classes of (1)–(4) are zero, i.e. \([RP(\lambda \oplus \varepsilon_3) \rightarrow RP(\lambda \oplus e_2 \oplus \lambda \oplus R)] = [RP(\lambda \oplus e_3 \oplus e_1 \oplus R)] = [RP(\lambda \oplus \varepsilon_3 \oplus e_1 \oplus R)] = 0 \). On the other hand, \( x_4 \neq 0 \), this is a contradiction. So \( x_4 \notin \mathcal{J}^3_{s,2}(A) \) for \( A = \{\tau_3, \tau_6\} \). Let \( \sigma_1 = (132) \) and \( \sigma_2 = (123) \). Then \( \sigma_1 A = \{\tau_2, \tau_4\}, \sigma_2 A = \{\tau_1, \tau_5\} \).

By Lemma 2.4, \( x_4 \notin \mathcal{J}^3_{s,2}(\sigma_1 A) = \mathcal{J}^3_{s,2}(\sigma_2 A) = \mathcal{J}^3_{s,2} \).

Hence if \( A \subseteq \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\} \) and \( \|A\| = 2 \), then \( x_4 \notin \mathcal{J}^3_{s,2}(A) \) and \( \mathcal{J}^3_{s,2}(A) \neq \mathcal{J}^2_{s,2} \). \( \square \)

**Lemma 4.5.** If \( A = \{a, \tau_1\} \), \( \{b, \tau_1\}, \{c, \tau_1\}, i = 1, 2, \ldots, 6 \), then \( x_4 \notin \mathcal{J}^3_{s,2}(A) \) and \( \mathcal{J}^3_{s,2}(A) \neq \mathcal{J}^2_{s,2} \).

**Proof.** Let \( A = \{a, \tau_1\} \). If \( x_4 \in \mathcal{J}^3_{s,2}(A) \), by Lemma 4.1 \( x_4 \) is represented by the union of the following fiberings:

1. \( RP(\varepsilon_1 \oplus R) \rightarrow F_{\varepsilon_1} \);
2. \( RP(\lambda \oplus \varepsilon_1 \oplus R) \rightarrow F_{\varepsilon_1} \);
3. \( RP(\varepsilon_1 \oplus e_2 \oplus R) \rightarrow F_{\varepsilon_1} \).

According to Lemma 4.2, the classes of (1) and (2) are zero. For (3), let Stiefel–Whitney class \( w(\varepsilon_1 \oplus R) = w(\varepsilon_1) = 1 + A_1 + A_2, w(e_2 \oplus R) = w(e_2) = 1 + v, \) and \( c \) (respectively \( d \)) the characteristic class of \( \lambda \) (respectively of the line bundle over \( RP(\varepsilon_1 \oplus R) \)). Then

\[
w(RP(\varepsilon_1 \oplus R)) = w(RF_1)(1 + c)^2 + (1 + c)v[1 + (1 + d)a_1 + (1 + d)dA_2] = (1 + c^2 + cv + v)[1 + d^2 + d^2 + A_1 + d^2A_1 + A_2 + dA_2]
= (1 + v)(1 + d + d^2 + A_1 + A_2)
= 1 + d + A_1 + v + d^2 + A_2 + dv + A_1v + d^2v + A_2v.
\]

Because \( \varepsilon_1 \) is the pullback from \( F, A^2_1 = 0 \) and \( A_2 = 0 \),

\[
w_f^4(RP(\varepsilon_1 \oplus R)) = (d + A_1 + v)^4 = d^4 = d \cdot d^3 = d(dA_1) = d^2A_1 = d^2A^2_1 = 0.
\]

So \( w_f^4(x_4) = 0 \), this is a contradiction. Thus \( x_4 \notin \mathcal{J}^3_{s,2}(A) \).

Let \( A = \{a, \tau_2\} \). If \( x_4 \in \mathcal{J}^3_{s,2}(A) \), then by Lemma 4.1 \( x_4 \) is represented by the union of the following fiberings:

1. \( RP(\varepsilon_1 \oplus R) \rightarrow F_{\varepsilon_1} \);
2. \( RP(\lambda \oplus \varepsilon_1 \oplus R) \rightarrow F_{\varepsilon_1} \).
(3) $w(RP(\lambda \oplus \varepsilon) \oplus \varepsilon \oplus R) \to F_2$.
(4) $w(RP(\lambda \oplus \varepsilon) \oplus \varepsilon) \to w(F_3) \to F_2$.

By Lemma 4.2, the classes of (1)–(3) are zero. The total Stiefel–Whitney class of (4) is

$$w(RP(\lambda \oplus \varepsilon) \oplus \varepsilon) = w(F_2)[(1 + c)^2 + (1 + cv)(1 + d)A_1 + (1 + d)A_2]$$

$$= (1 + c^2 + cv + 1)(1 + d^2 + d^2 + A_1 + A_2 + 2dA_2)$$

$$= (1 + v)(1 + d + d^2 + A_1 + A_2)$$

$$= 1 + d + A_1 + v + d^2 + 2A + dv + A_1 + d^2 + 2A_2v,$$

where $c$ (respectively $d$) is the characteristic class of $\lambda$ (respectively of the line bundle over $RP(\lambda \oplus \varepsilon \oplus R)$).

Let $w_1(\varepsilon_1) = u$. Then $A_1 = c + u$ and $A_2 = cu$, $w_4^4(RP(\lambda \oplus \varepsilon_1 \oplus R)) = (d + A_1 + v)^4 = d^4 - (d^2 + A_1 + A_2 - d^2)A_1 = d^4 - d^3A_1 + d^2A_2 = d^2(A_1 + dA_2)$. By [10] and [14], $w_4^4(RP(\lambda \oplus \varepsilon_1 \oplus R)) = (A_1^2 + A_2)(RP(\varepsilon)) = (c^2 + cu)(RP(\varepsilon)) = (c^2 + cu)(RP(\varepsilon)) = 0$. So $w_4^4(x_4) = 0$, this is a contradiction. Thus $x_4 \notin J_{3}^2(A)$.

For $A = [a, \tau_1], [a, \tau_2], [a, \tau_3]$ and $[a, \tau_6]$, similarly we can obtain $x_4 \notin J_{3}^2(A)$.

Hence if $A = (a, \tau_i), i = 1, 2, \ldots, 6$, then $x_4 \notin J_{3}^3(A)$ and $J_{3}^3(A) \neq J_{3}^3$. According to Lemma 2.4, if either $A = [b, \tau_1]$ or $[c, \tau_6], i = 1, 2, \ldots, 6$, then $x_4 \notin J_{3}^3(A)$ and $J_{3}^3(A) \neq J_{3}^3$. □

**Lemma 4.6.** If $A = [v, \tau_1], [v, a], [v, b]$ or $[v, c], i = 1, 2, \ldots, 6$, then $x_4 \notin J_{3}^3(A)$ and $J_{3}^3(A) \neq J_{3}^3$.

**Proof.** Let $A = [v, \tau_6]$. If $x_4 \in J_{3}^3(A)$, by Lemma 4.1 $x_4$ is represented by the union of the following fiberings:

(1) $RP(\lambda \oplus \varepsilon_3 \oplus \varepsilon_1 \oplus R) \to RP(\varepsilon_2) \to F_v$;
(2) $RP(\lambda \oplus \varepsilon_3 \oplus \lambda \oplus \varepsilon_1 \oplus R) \to RP(\varepsilon_3) \to F_v$;
(3) $RP(\lambda \oplus \varepsilon_3 \oplus \varepsilon_1 \oplus R) \to F_{2}$;
(4) $RP(\lambda \oplus \varepsilon_1 \oplus R) \to RP(\varepsilon_3) \to F_{6}$.

By Lemma 4.2, the classes of (2)–(4) are zero. For (1), similar arguments show that $w_4^4(RP(\lambda \oplus \varepsilon_3 \oplus \varepsilon_1 \oplus R)) = 0$. So $w_4^4(x_4) = 0$, this is a contradiction. Thus $x_4 \notin J_{3}^3(A)$. By Lemma 2.4, if $A = (v, \tau_1), i = 1, 2, \ldots, 6$, then $x_4 \notin J_{3}^3(A)$. Similarly, if $A = [a, v], [b, v]$ or $[v, c], i = 1, 2, \ldots, 6$, then $x_4 \notin J_{3}^3(A)$ and $J_{3}^3(A) \neq J_{3}^3$. □

**Lemma 4.7.** If $A \subseteq [(3, 0, 0), (0, 3, 0), (0, 0, 3)]$ and $\|A\| = 2$, then $x_4 \notin J_{3}^3(A)$ and $J_{3}^3(A) \neq J_{3}^3$.

**Proof.** If $A \subseteq [(3, 0, 0), (0, 3, 0), (0, 0, 3)]$, from [2] we know that $J_{3}^3(A) = J_{3}^3[\alpha \in MO_n | w_4^4(w_n - j)(\alpha) = w_4^4(w_n - s_5(\alpha)) = 0, 5 \leq i \leq n, 0 \leq j \leq n, n \geq 3]$. So $x_4 \notin J_{3}^3(A)$ and $J_{3}^3(A) \neq J_{3}^3$. □

**Proof of Theorem 1.2.** (1) Let $A = [(2, 1, 0), (2, 0, 1), (1, 1, 1)]$. By Theorem 2 in [11], $J_{3}^3(A) = J_{3}^3$, hence $\|J_{3}^3\| \leq 3$.

According to Lemmas 4.3–4.7, $\|J_{3}^3\| \geq 3$, hence $\|J_{3}^3\| = 3$.

(2) By Lemma 2.5 and (1), $\|J_{3}^3\| \leq \|J_{3}^3\| = 3 (k \geq 3)$. □

5. $\|J_{3}^4(k)| k \geq 2 \|$ and $\|J_{3}^5(k)\|

Let $B = [(3, 1, 1), (3, 2, 0), (3, 0, 2), (2, 1, 2), (2, 2, 1), (1, 2, 2), (4, 0, 1), (4, 1, 0), (5, 0, 0)]$. Then we have

**Lemma 5.1.** $RP(5, n_2, n_3) \in J_{3}^5(B)$.

**Proof.** Let $T_{1,1} \oplus$ be an involution on $RP(5)$ with the fixed point set $RP(2) \cup RP(2)$ and $T_{1,1} \oplus$ an involution on $\lambda_1$ covering $T_{1,1}$.

Let $RP(n_2)$ and $RP(n_3)$ admit the trivial action of $(Z_2)^0$. Define $T_1, T_2$ to be the involution on $RP(5, n_2, n_3)$ induced by $-1 \times 1 \times 1$ and $T_{1,1} \times 1 \times 1$ on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$. Then $T_1, T_2$ defines a $(Z_2)^3$-action on $RP(5, n_2, n_3)$ with the fixed point set $F = [RP(2) \cup RP(2)] \times [RP(n_2) \times [n_2, e_2, e_3]]$. The components of $F$ correspond to [(3, 1, 1), (4, 1, 0), (4, 0, 1), (4, 1, 0), (5, 0, 0)]. Hence $RP(5, n_2, n_3) \in J_{3}^5(B)$. □

**Lemma 5.2.** $[RP(n_1, n_2, \ldots, n_{10})] \in J_{3}^5(B)$. 

Proof. Let

\[ T_1[y_1, y_2, \ldots, y_5, y_6, \ldots, y_{10}] = [-y_1, -y_2, \ldots, -y_5, y_6, \ldots, y_{10}] \]
\[ T_2[y_1, y_2, \ldots, y_5, y_6, \ldots, y_{10}] = [y_1, y_2, \ldots, y_5, y_6, \ldots, y_{10}] \]

Then \((T_1, T_2)\) defines a \((\mathbb{Z}_2)^2\)-action on \(RP(n_1, n_2, \ldots, n_{10})\) with the fixed point set

\[ F = RP(n_1, n_2, \ldots, n_5) \cup RP(n_6, n_7, \ldots, n_{10}) \]

and

\[ F_{T_1} = RP(n_1, n_2, \ldots, n_5) \cup RP(n_6, n_7, \ldots, n_{10}) \]
\[ F_{T_2} = RP(n_1, n_2, \ldots, n_5) \cup RP(n_6, n_7, \ldots, n_{10}) \]
\[ F_{T_1} \circ T_2 = RP(n_1, n_2, \ldots, n_5) \cup RP(n_6, n_7, \ldots, n_{10}) \]

The components of \(F\) correspond to \((5, 0, 0)\), therefore \([RP(n_1, n_2, \ldots, n_{10})] \in J_{n,2}^B(\mathbb{B}). \]

\[ \text{Lemma 5.3. There exist indecomposable classes } x_n \in J_{n,2}^B(\mathbb{B}) \text{ for } n \geq 8. \]

Proof. By [13] and Lemmas 5.1–5.2, the lemma is established. \(\square\)

\[ \text{Proof of Theorem 1.3. Case 1. } r = 4 \text{ and } k = 2. \]

Let \(C = \{(2, 2, 0), (2, 0, 2), (0, 2, 2), (2, 1, 1), (1, 2, 1), (2, 1, 1), (1, 2, 1), (1, 3, 0), (0, 3, 1)\}\). By exhibiting special generators of \(MO_*\), we show \(J_{2,2}^4 \subset \mathcal{J}_{2,2}^4\), hence \(\|J_{2,2}^4\| \leq 8\).

Take a system of generators of \(MO_*\) as follows:

(i) Let \(x_2 = [RP(2)], x_4 = [RP(1, 0, 1)], x_6 = [RP(3, 0, 0, 0)]\). According to Lemma 2.7 and Lemma 2.8 in [12], \(x_5 \in J_{2,2}^4\).

(ii) For \(n = 2^m - 4 \geq 4\), take \(x_n = [RP(3, 2^m - 5, 0)]\). In order to show \(x_n \in J_{2,2}^4\), we construct a \((\mathbb{Z}_2)^2\)-action on \(RP(3, 2^m - 5, 0)\) Let \(T\) be an involution on \(RP(3, 2^m - 5, 0)\), let

\[ T_1[y_1, y_2, y_3] = [-\tilde{T}_1 y_1, y_2, y_3] \]
\[ T_2[y_1, y_2, y_3] = [\tilde{T}_1 y_1, -y_2, y_3] \]

Then \((T_1, T_2)\) defines a \((\mathbb{Z}_2)^2\)-action on \((3, 2^m - 5, 0)\) with the fixed point set

\[ F = \{RP(1) \cup RP(3)\} \times RP(2^m - 5) \times RP(0) \times \{e_1, e_2, e_3\} \]

The components of \(F\) correspond to \((1, 2, 1)(3, 0, 0), (0, 3, 1)\). So \(x_n \in J_{2,2}^4(\mathbb{C}). \)

(iv) For \(n \geq 8\), \(n \neq 2^m - 1\) and \(2^m\), by Lemma 2.4 in [12], there are indecomposable classes \(x_n \in J_{2,2}^4(\mathbb{C}). \)

From Lemma 2.4, we know that \(x_2^3, x_4^2, x_8 \in J_{2,2}^4(\mathbb{C})\). By [7], \(x_4 \notin J_{2,2}^4\). Since \(J_{2,2}^4(\mathbb{C})\) is an ideal in \(MO_*\), \(J_{2,2}^4(\mathbb{C})\) consists of \(4\)-dimensional decomposable classes and all \(n\)-dimensional classes with \(n > 4\). By [7], \(J_{2,2}^4(\mathbb{C}) = J_{2,2}^4\).

Case 2. \(r = 4\) and \(k = 3\). Let

\[ D = \{(1, 1, 1, 0, 1, 0, 0), (0, 1, 0, 1, 0, 1), (1, 0, 1, 1, 0, 1), (0, 1, 0, 1, 1, 0), (1, 0, 1, 1, 0, 1), (2, 0, 0, 0, 0, 0), (0, 0, 2, 0, 0, 0, 0), (0, 2, 0, 0, 0, 0, 0)\}. \]

By exhibiting special generators of \(MO_*\), we show \(J_{3,3}^4(D) = \sum_{n \geq 4} MO_n = J_{3,3}^4\), hence \(\|J_{3,3}^4\| = 8\).

Let \(x_2 = [RP(2)], x_5 = [RP(2, 0, 0, 0)]. By Lemma 2.5 and Lemma 2.7 in [12], \(x_2^2, x_5 \in J_{3,2}^4(P) \subseteq J_{3,2}^4\) where \(P = \{(2, 2, 0), (2, 0, 2), (0, 2, 2)\}\). According to Lemma 2.4, \(x_2^2, x_5 \in J_{3,3}^4(D)\). By [7], \(J_{3,3}^4(D) = J_{3,3}^4\), hence \(\|J_{3,3}^4\| = 8\).

Thus, for \(n \geq 4\), \(n \neq 5\), and \(n \neq 2^m - 1\), by Corollary 3.2 in [6], there exist indecomposable classes \(x_n = [RP(n_1, n_2, n_3, n_4, n_5)]\). For \(RP(n_1, n_2, n_3, n_4, n_5)\), let

\[ T_1[y_1, y_2, y_3, y_4, y_5] = [y_1, y_2, -y_3, y_4, -y_5] \]
\[ T_2[y_1, y_2, y_3, y_4, y_5] = [y_1, -y_2, y_3, y_4, y_5] \]
\[ T_3[y_1, y_2, y_3, y_4, y_5] = [y_1, y_2, -y_3, -y_4, y_5] \]
Then \((T_1, T_2, T_3)\) defines a \((\mathbb{Z}_2)^3\)-action on \(RP(n_1, n_2, \ldots, n_5)\) with the fixed point set

\[
F = \text{RP}(n_1) \times \text{RP}(n_2) \times \text{RP}(n_3) \times \text{RP}(n_4) \times \text{RP}(n_5) \times \{e_1, e_2, e_3, e_4, e_5\}
\]

and

\[
F_{T_1, T_2} = \text{RP}(\lambda_1 \oplus \lambda_2) \cup \text{RP}(\lambda_3 \oplus \lambda_5) \cup \text{RP}(\lambda_2),
\]

\[
F_{T_1, T_3} = \text{RP}(\lambda_1 \oplus \lambda_2) \cup \text{RP}(\lambda_3) \cup \text{RP}(\lambda_4) \cup \text{RP}(\lambda_5),
\]

\[
F_{T_2, T_3} = \text{RP}(\lambda_1 \oplus \lambda_3) \cup \text{RP}(\lambda_3 \oplus \lambda_4) \cup \text{RP}(\lambda_2),
\]

\[
F_{T_1, T_2, T_1} = \text{RP}(\lambda_1) \cup \text{RP}(\lambda_2 \oplus \lambda_3) \cup \text{RP}(\lambda_3) \cup \text{RP}(\lambda_4),
\]

\[
F_{T_1, T_2, T_2, T_3} = \text{RP}(\lambda_1) \cup \text{RP}(\lambda_2 \oplus \lambda_3) \cup \text{RP}(\lambda_4) \cup \text{RP}(\lambda_5).
\]

The components of \(F\) correspond to \(((1, 1, 1, 0, 0, 0), (0, 1, 0, 1, 0, 1), (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0), (0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)).\) Thus \(x_9 \in J_{a,3}^4(D)\) for \(n \geq 4, n \neq 5,\) and \(n \neq 2^k - 1.\)

By Lemma 2.5, \(\|J_{a,3}^4\| \leq \|J_{a,3}^4\| \leq 8 (k \geq 4).\)

Case 4, \(r = 4\) and \(k = 2.\)

Take \(x_2 = [\text{RP}(2)], x_4 = [\text{RP}(1, 1, 0)], x_5 = [\text{RP}(2, 0, 0, 0)]\) and \(x_6 = [\text{RP}(3, 0, 0, 0)].\) For \(n \geq 8,\) take \(x_9\) as in Lemma 5.3.

By [6], \(x_6^1 \notin J_{a,2}^5(B),\) where \(B = [(3, 1, 1), (3, 2, 0), (3, 0, 2), (2, 1, 2), (2, 2, 1), (1, 2, 1), (4, 0, 0), (4, 1, 0), (4, 1, 0), (5, 0, 0)].\)

By [13], \(J_{a,2}^5(B) = J_{a,2}^5(B),\) where \(Q = \{(2, 0, 1), (2, 1, 0), (1, 1, 1),\}\), and \(x_4, x_5, x_6 \in J_{a,2}^5(Q),\) where \(Q = \{(2, 0, 1), (2, 1, 0), (1, 1, 1),\}\), by Lemma 2.4, \(x_2x_4, x_2x_5, x_2x_6, x_4x_5, x_5x_6, x_4^2, x_5^2, x_6^2 \in J_{a,2}^5(B).\) By [13],

\[
J_{n,2}^5(B) = J_{n,2}^5(B, M_{0}, \ker \chi, D_{n, \ker \chi}, n \geq 7, n = 5, 6,
\]

where \(\chi : \operatorname{MO}_n \to \mathbb{Z}_2\) denotes the mod 2 Euler characteristic. Hence \(J_{a,2}^5(B) = J_{a,2}^5(B, M_{0}, \ker \chi, D_{n, \ker \chi}, n \geq 7, \|J_{a,2}^5\| \leq 9. \)

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References

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