Optimality Conditions and Duality Models for a Class of Nonsmooth Constrained Fractional Optimal Control Problems

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Both parametric and nonparametric necessary and sufficient optimality conditions are established for a class of nonsmooth constrained optimal control problems with fractional objective functions and linear dynamics. Moreover, using the forms and contents of these optimality principles, four parametric and eight parameter-free duality models are constructed and weak, strong, and strict converse duality theorems are proved. These optimality and duality results contain, as special cases, similar results for fractional optimal control problems containing square roots of positive semidefinite quadratic forms in their objective and constraint functions. The optimality and duality criteria presented in this paper generalize a number of existing results for optimal control problems and subsume a fairly large number of cognate results obtained previously in the areas of finite-dimensional linear, fractional, and nonlinear programming.

1. INTRODUCTION

In this paper, we shall establish parametric and nonparametric necessary and sufficient optimality conditions and construct several duality models for the following nonstandard optimal control problem containing arbitrary norms:

(P) Minimize

$$\varphi(x, u) = \frac{\int_a^b [f(x(t), u(t), t) + \|K(t)x(t)\|_K + \|L(t)u(t)\|_L] dt}{\int_a^b [g(x(t), u(t), t) - \|M(t)x(t)\|_M - \|N(t)u(t)\|_N] dt}$$
subject to
\[ x(a) = 0, \quad x(b) = 0, \tag{1.1} \]
\[ Dx(t) = A(t)x(t) + B(t)u(t), \quad t \in [a, b], \tag{1.2} \]
\[ h_i(x(t), u(t), t) + \| P_i(t)x(t) \|_{P_i} + \| Q_i(t)u(t) \|_{Q_i} \leq 0, \quad t \in [a, b], i \in k, \tag{1.3} \]
\[ x \in C^0[a, b], \quad u \in \text{PWS}^m[a, b], \tag{1.4} \]

where \( C^0[a, b] \) is the space of all continuous \( n \)-dimensional vector functions \( x:[a, b] \rightarrow \mathbb{R}^n \) (\( n \)-dimensional Euclidean space) defined on the compact interval \([a, b]\) of the real line \( \mathbb{R} \), with the norm \( \| x \|_\infty + \| Dx \|_\infty \), where the differentiation operator \( D \) is defined by the relation

\[ y = Dx \iff x(t) = \int_a^t y(\tau) \, d\tau; \]

thus \( D = d/dt \) except at discontinuities of the piecewise smooth function \( y:[a, b] \rightarrow \mathbb{R}^n \); \( \text{PWS}^m[a, b] \) is the space of all piecewise smooth \( m \)-dimensional vector functions defined on \([a, b]\), with the norm \( \| x \|_\infty; f, g, \) and \( h_i, \) \( i \in k = \{1, 2, \ldots, k\} \), are continuously differentiable real-valued functions defined on \( \mathbb{R}^n \times \mathbb{R}^m \times [a, b] \); \( f(\cdot, \cdot, t), -g(\cdot, \cdot, t), \) and \( h_i(\cdot, \cdot, t), i \in k \), are convex on \( \mathbb{R}^n \times \mathbb{R}^m \) throughout \([a, b] \); \( A(t), B(t), K(t), L(t), M(t), N(t), P_i(t), \) and \( Q_i(t), i \in k \), are, respectively, \( n \times n, n \times m, p \times n, q \times m, r \times n, s \times m, p_i \times n, \) and \( q_i \times m \) matrices whose entries are continuous real-valued functions defined on \([a, b] \); \( \| \cdot \|_K, \| \cdot \|_L, \| \cdot \|_M, \| \cdot \|_N, \| \cdot \|_{P_i}, \) and \( \| \cdot \|_{Q_i}, i \in k \), are arbitrary norms, and the numerator of the objective function is nonnegative and its denominator is positive for all state-control pairs \((x, u)\) satisfying the constraints of \((P)\).

Despite the fact that the functions \( f(\cdot, \cdot, t), -g(\cdot, \cdot, t), \) and \( h_i(\cdot, \cdot, t), i \in k \), are assumed to be convex, \((P)\) is not a convex control problem because of the fractional form of its objective function. In fact, under the hypotheses specified above, this function is quasiconvex. However, this particular feature of \((P)\) will not enter explicitly into our analysis inasmuch as our main results for \((P)\) will be derived with the help of an auxiliary equivalent convex problem.

Finite-dimensional analogues of \((P)\) are known as fractional programming problems in the discipline of mathematical programming, and have been the subject of numerous investigations in the past three decades, resulting in a fairly large literature. One of the primary reasons for such an immense interest in these problems appears to be their capability in providing realistic models for some important classes of problems in the areas of operations research, management science, and economics. This ability is due to the fact that in many areas, including resource allocation,
transportation, production planning, inventory control, financial management, maintenance and replacement scheduling, and reliability assessment, ratios such as profit/capital, profit/revenue, return/cost, return/risk, cost/time, profit/time, etc., can serve as useful measures of system performance. Proper characterization of the efficiency of these measures often requires optimization of certain ratios which, in turn, gives rise to the formulation of fractional programming problems. In noneconomic situations, problems of this type have arisen in information theory, stochastic programming, numerical analysis, approximation theory, multifacility location theory, decomposition of large-scale mathematical programming problems, and goal programming, among others. For comprehensive surveys and extensive lists of references dealing with several aspects of fractional programming, including modeling properties, actual and potential areas of applications, optimality conditions, duality formulations, sensitivity and stability analysis, and computational algorithms, the reader is referred to [1, 7, 8, 10, 11, 13].

In contrast to the status of finite-dimensional fractional programming, it appears that similar infinite-dimensional optimization problems, and especially fractional variational and optimal control problems, have not yet received much attention in the related literature. Some limited results for fractional optimal control problems are given in [2, 3, 14, 20], and some applications of optimality conditions for these problems are discussed in [15–18].

In the present study we shall introduce two sets of optimality principles and twelve duality models for (P). The first set of optimality results will be derived in Section 3 by resorting to a Dinkelbach-type parametric approach whereby the nonconvex problem (P) is replaced by an equivalent convex problem depending on a certain parameter. Consequently, the optimality conditions obtained via this auxiliary problem are of a parametric nature. Subsequently, we shall obtain our second set of optimality conditions by essentially eliminating this parameter. Utilizing the forms and contents of these optimality results as a basis, we shall formulate in Section 4 four parametric duality models for (P) and prove appropriate duality theorems. We shall continue our discussion of duality for (P) in Sections 5 and 6 by constructing eight additional parameter-free duality models and proving weak, strong, and strict converse duality theorems. In Section 7 we shall briefly discuss the relevance and applicability of these optimality and duality results to a related class of fractional optimal control problems containing square roots of positive semidefinite quadratic forms, obtained as a special case of (P) by taking all the norms to be the $l_2$-norm. Finally, in Section 8 we shall briefly point out the possibility of extending the results of this paper to some more general and new optimal control models.
Evidently, a salient feature of (P) is the presence of arbitrary norms. Optimization problems containing norms arise naturally in many areas of the decision sciences, applied mathematics, and engineering. These problems occur most frequently in location theory, approximation theory, and engineering design. A number of references dealing with optimization problems involving norms are given in [21] (see also [4–6, 19]).

Because of the fractional form of the objective function and the existence of the terms involving arbitrary norms, the optimality and duality criteria developed here for (P) and its special cases, generalize and improve a number of existing results in the area of optimal control theory, including those of [6], and furthermore, subsume the optimal control analogues of a fairly large number of cognate results obtained previously for several classes of finite-dimensional linear, nonlinear, and fractional programming problems. In particular, (P) embodies some interesting extensions of the linear-quadratic optimal control problem which is one of the most useful and frequently utilized nonlinear optimal control models.

In view of the prevalence of optimal control models in all disciplines of engineering, operations, management science, and economics, and because of the utility of fractional objective functions in these and other related areas, it is reasonable to expect that in the future many classes of dynamic optimization problems with single fractional, multiple fractional, and generalized fractional objective functions will be utilized with greater frequency which will, in turn, necessitate more systematic investigations of various theoretical and computational aspects of fractional optimal control problems. The results presented in this work may prove useful in stimulating further interest in this vastly unexplored area of optimal control theory.

2. PRELIMINARIES

As pointed out earlier, this paper consists of essentially two interrelated parts. In the first part, we shall employ an indirect approach and consequently obtain parameter-dependent optimality and duality results via an auxiliary problem. Subsequently, combining these results with a direct method, we shall develop in the second part a number of parameter-free optimality and duality principles for (P). In our parametric approach, we shall need a set of optimality conditions given in [6] for the following special case of (P):

\[
\text{(P1) } \min_{(x, u) \in \mathbb{R}} \int_a^b [f(x(t), u(t), t) + \|K(t)x(t)\|_K + \|L(t)u(t)\|_L] \, dt,
\]

where \( \mathcal{N} \) (assumed to be nonempty) is the feasible set of (P), that is,

\[
\mathcal{N} = \{ (x, u) \in C[a, b] \times \text{PWS}^n[a, b] : (1.1)-(1.3) \text{ hold} \}.
\]
Although this problem is not considered explicitly in [6] in the exact form stated above, the method used in [4–6] can easily be modified and adapted for (P1). This would lead to the following optimality result for (P1).

**Theorem 2.1** [6]. Assume that the linear map \((x, u) \rightarrow Dx(t) - A(t)x(t) - B(t)u(t)\) is surjective at a feasible solution \((x^*, u^*)\) of (P1), and that the constraints of (P1) satisfy Slater’s constraint qualification (SCQ), that is, assume that there exists \((\tilde{x}, \tilde{u}) \in C^n[a, b] \times \text{PWS}^m[a, b]\) such that \(\tilde{x}(a) = \tilde{x}(b) = 0\) and

\[
D\tilde{x}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t), \quad t \in [a, b],
\]

\[
h_i(\tilde{x}(t), \tilde{u}(t), t) + \|P_i(t)\tilde{x}\|_{P(i)} + \|Q_i(t)\tilde{u}\|_{Q(i)} < 0,
\]

\(t \in [a, b], i \in \mathbb{k}.

Then \((x^*, u^*)\) is an optimal solution of (P1) if and only if there exist \(v^* \in \text{PWS}^n[a, b], w^* \in \text{PWS}^k[a, b], \alpha^* \in \text{PWS}^n[a, b], \beta^* \in \text{PWS}^k[a, b], \xi^* \in \text{PWS}^k[a, b], \) and \(\eta^* \in \text{PWS}^k[a, b], i \in \mathbb{k}\), such that the following relations hold for all \(t \in [a, b]:\)

\[
\nabla_1 f(x^*, u^*, t) + K(t)^T \alpha^*(t) + A(t)^T v^*(t)
+ \sum_{i=1}^{k} w_i^*(t) \left[ \nabla h_i(x^*, u^*, t) + P_i(t)^T \xi^*(t) \right] + C_{dx}(t) = 0,
\]

\[
\nabla_2 f(x^*, u^*, t) + L(t)^T \beta^*(t) + B(t)^T v^*(t)
+ \sum_{i=1}^{k} w_i^*(t) \left[ \nabla h_i(x^*, u^*, t) + Q_i(t)^T \eta^*(t) \right] = 0,
\]

\[
\sum_{i=1}^{k} w_i^*(t) \left[ h_i(x^*, u^*, t) + \|P_i(t)x^*\|_{P(i)} + \|Q_i(t)u^*\|_{Q(i)} \right] = 0,
\]

\[
\|\alpha^*(t)\|_{L} \leq 1, \quad \|\beta^*(t)\|_{L} \leq 1,
\]

\[
\|\xi^*(t)\|_{P(i)} \leq 1, \quad \|\eta^*(t)\|_{Q(i)} \leq 1, i \in \mathbb{k},
\]

\[
\alpha^*(t)^T K(t)x^* = \|K(t)x^*\|_{L}, \quad \beta^*(t)^T L(t)u^* = \|L(t)u^*\|_{L},
\]

\[
\xi^*(t)^T P_i(t)x^* = \|P_i(t)x^*\|_{P(i)}, \quad \eta^*(t)^T Q_i(t)u^* = \|Q_i(t)u^*\|_{Q(i)}, i \in \mathbb{k},
\]

where \(\text{PWS}^k[a, b] = \{w \in \text{PWS}^k[a, b]; w(t) \geq 0 \text{ for all } t \in [a, b]\}, \nabla F\) and \(\nabla^2 F\) denote the partial gradients of the function \(F: \mathbb{R}^n \times \mathbb{R}^m \times [a, b] \rightarrow \mathbb{R}, (x(t), u(t), t) \rightarrow F(x(t), u(t), t),\) with respect to its first and second argu-
ments; that is, \( \nabla_1 F = (\partial F / \partial x_1(t), \ldots, \partial F / \partial x_n(t))^T \) and \( \nabla_2 F = (\partial F / \partial u_1(t), \ldots, \partial F / \partial u_m(t))^T \); \( E^T \) is the transpose of the matrix \( E \), and \( \| \cdot \|^s \) denotes the dual norm to \( \| \cdot \|_s \).

In the statement of the above theorem, the argument \( t \) of the functions \( x^* \) and \( u^* \) was omitted for the sake of notational simplicity. This practice will be continued throughout the sequel.

Because of our assumptions concerning the underlying function spaces and data of \( P \), the surjectivity property required in Theorem 2.1 is always ensured by the well-known existence theory for linear systems of ordinary differential equations.

3. OPTIMALITY CONDITIONS

In this section we shall combine Theorem 2.1 with a certain auxiliary problem to obtain a set of necessary and sufficient optimality conditions for \( P \). The auxiliary equivalent nonfractional problem making our indirect approach possible has the form

\[
\min_{(x, u) \in \mathbb{R}} \int_a^b \left[ f(x, u, t) + \| K(t) x \|_K + \| L(t) u \|_L \right.
\]

\[
- \lambda [g(x, u, t) - \| M(t) x \|_M - \| N(t) u \|_N] \right] dt,
\]

where \( \lambda \in \mathbb{R} \) is a parameter.

It is well known in the area of fractional programming [8, 11] that certain aspects of \( P \) can be investigated indirectly via \( (E\lambda) \). The relationships between \( P \) and \( (E\lambda) \) that are needed for our present purposes are stated in the following lemma.

**Lemma 3.1 [9].** Let \( \lambda^* \) be the optimal value of \( P \) and let \( e(\lambda) \) be the optimal value of \( (E\lambda) \) for any fixed \( \lambda \in \mathbb{R} \) such that \( (E\lambda) \) has an optimal solution. Then the following assertions are valid:

(a) If \( (x^*, u^*) \) is an optimal solution of \( P \), then \( (x^*, u^*) \) is an optimal solution of \( (E\lambda^*) \) and \( e(\lambda^*) = 0 \).

(b) If \( (E\lambda^*) \) has an optimal solution \( (\tilde{x}, \tilde{u}) \) for some \( \tilde{\lambda} \in \mathbb{R} \) with \( e(\tilde{\lambda}) = 0 \), then \( (\tilde{x}, \tilde{u}) \) is an optimal solution of \( P \) and \( \tilde{\lambda} = \lambda^* \).

Now making use of this equivalence result in conjunction with Theorem 2.1, we can establish the main result of this section.

**Theorem 3.1.** Assume that the constraints of \( P \) satisfy (SCQ) (see Theorem 2.1). Then \( (x^*, u^*) \in \mathbb{R} \) is an optimal solution of \( P \) if and only if there exist \( \lambda^* \in \mathbb{R}, \nu^* \in \text{PWS}^\alpha[a, b], w^* \in \text{PWS}^\alpha[a, b], \alpha^* \in \)
\[ \nabla_1 f(x^*, u^*, t) + K(t)^\top \alpha^*(t) - \lambda^* \left[ \nabla_2 g(x^*, u^*, t) - M(t)^\top \gamma^*(t) \right] \\
+ A(t)^\top v^*(t) + \sum_{i=1}^{k} w_i^*(t) \left[ \nabla_i h_i(x^*, u^*, t) + P_i(t)^\top \xi^i(t) \right] \\
+ Dv^*(t) = 0, \quad (3.1) \]

\[ \nabla_2 f(x^*, u^*, t) + L(t)^\top \beta^*(t) - \lambda^* \left[ \nabla_2 g(x^*, u^*, t) - N(t)^\top \delta^*(t) \right] \\
+ B(t)^\top v^*(t) + \sum_{i=1}^{k} w_i^*(t) \left[ \nabla_i h_i(x^*, u^*, t) + Q_i(t)^\top \eta^i(t) \right] = 0, \quad (3.2) \]

\[ \sum_{i=1}^{k} w_i^*(t) \left[ h_i(x^*, u^*, t) + \|P_i(t) x^*\|_{P(i)} + \|Q_i(t) u^*\|_{Q(i)} \right] = 0, \quad (3.3) \]

\[ \int_a^b \left\{ f(x^*, u^*, t) + \|K(t) x^*\|_K + \|L(t) u^*\|_L \\
- \lambda^* \left[ g(x^*, u^*, t) - \|M(t) x^*\|_M - \|N(t) u^*\|_N \right] \right\} dt = 0, \quad (3.4) \]

\[ \|\alpha^*(t)\|_K \leq 1, \quad \|\beta^*(t)\|_L \leq 1, \quad \|\gamma^*(t)\|_M \leq 1, \quad \|\delta^*(t)\|_N \leq 1, \]

\[ \|\xi^i(t)\|_{P(i)} \leq 1, \quad \|\eta^i(t)\|_{Q(i)} \leq 1, \quad i \in \mathbb{K}, \quad (3.5) \]

\[ \alpha^*(t)^\top K(t) x^* = \|K(t) x^*\|_K, \quad \beta^*(t)^\top L(t) u^* = \|L(t) u^*\|_L, \]

\[ \gamma^*(t)^\top M(t) x^* = \|M(t) x^*\|_M, \quad \delta^*(t)^\top N(t) u^* = \|N(t) u^*\|_N, \]

\[ \xi^i(t)^\top P_i(t) x^* = \|P_i(t) x^*\|_{P(i)}, \quad \eta^i(t)^\top Q_i(t) u^* = \|Q_i(t) u^*\|_{Q(i)}, \quad i \in \mathbb{K}. \quad (3.6) \]

**Proof.** Since \((x^*, u^*)\) is an optimal solution of \((P)\), by part (a) of Lemma 3.1, \((x^*, u^*)\) is an optimal solution of \((EP \lambda^*)\) where \(\lambda^* = \varphi(x^*, u^*)\) (optimal value of \((P)\)). It is clear from our assumptions set forth in the description of \((P)\) that \(\lambda^* > 0\) and that \((EP \lambda^*)\) is a convex problem. Hence by Theorem 2.1, there exist \(\psi^*, \omega^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \xi^i, \) and \(\eta^i, i \in \mathbb{K},\) as specified above, such that (3.1)–(3.6) hold. Similarly, the sufficiency part of the theorem follows from Theorem 2.1 and part (b) of Lemma 3.1.
We shall next show that the sufficiency part of Theorem 3.1 remains valid under slightly different conditions. For this we need the following generalized Cauchy inequality.

**Lemma 3.2 [19]**. For any \( y, z \in \mathbb{R}^l \), one has \( y^T z \leq \|y\| \|z\| \).

**Theorem 3.2**. Let \( (x^*, u^*) \in \mathbb{R}^m \) and \( \lambda^* = \varphi(x^*, u^*) \), and assume that there exist \( v^*, w^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \zeta^*, \iota \in \mathbb{R}^k \), as specified in Theorem 3.1, such that (3.3)–(3.6) and the following inequalities hold for all \( t \in [a, b] \):

\[
\begin{align*}
\left\{ \nabla_1 f(x^*, u^*, t)^T + \alpha^*(t)^T K(t) - \lambda^* \left[ \nabla_1 g(x^*, u^*, t)^T - \gamma^*(t)^T M(t) \right] \\
+ v^*(t)^T A(t) + \sum_{i=1}^k w^*_i(t) \left[ \nabla_2 h_i(x^*, u^*, t)^T + \zeta^*_i(t)^T P_i(t) \right] \\
+ Dw^*_i(t)^T \right\}(y - x^*) \geq 0
\end{align*}
\]
for all \( y \in C^m[a, b] \) such that \( (y, u) \in \mathbb{R}^m \) for some \( u \in \mathbb{R}^m[a, b] \),

(3.7)

\[
\begin{align*}
\left\{ \nabla_2 f(x^*, u^*, t)^T + \beta^*(t)^T L(t) - \lambda^* \left[ \nabla_2 g(x^*, u^*, t)^T - \delta^*(t)^T N(t) \right] \\
+ v^*(t)^T B(t) + \sum_{i=1}^k w^*_i(t) \left[ \nabla_2 h_i(x^*, u^*, t)^T + \eta^*_i(t)^T Q_i(t) \right] \right\}(z - u^*) \\
\geq 0
\end{align*}
\]
for all \( z \in \mathbb{R}^m[a, b] \) such that \( (x, z) \in \mathbb{R}^m \) for some \( x \in C^m[a, b] \).

(3.8)

Then \((x^*, u^*) \) is an optimal solution of \((P)\).

**Proof.** Let \((x, u)\) be an arbitrary feasible solution of \((P)\). Then

\[
\int_a^b \left\{ f(x, u, t) + \|K(t)x\|_K + \|L(t)u\|_L \right. \\
- \lambda^* \left[ g(x, u, t) - \|M(t)x\|_M - \|N(t)u\|_N \right] \} \, dt
\]
\[
\begin{align*}
&= \int_a^b \left\{ f(x,u,t) + \|K(t)x\|_M + \|L(t)u\|_L \\
&\quad - \lambda^* \left[ g(x,u,t) - \|M(t)x\|_M - \|N(t)u\|_N \right] \\
&\quad - f(x^*,u^*,t) - \alpha^*(t)^T K(t)x^* - \beta^*(t)^T L(t)u^* \\
&\quad + \lambda^* \left[ g(x^*,u^*,t) - \gamma^*(t)^T M(t)x^* - \delta^*(t)^T N(t)u^* \right] \right\} \, dt \\
&\quad (\text{by } (3.4) \text{ and } (3.6)) \\
&\geq \int_a^b \left\{ \nabla_1 f(x^*,u^*,t)^T (x - x^*) + \nabla_2 f(x^*,u^*,t)^T (u - u^*) \\
&\quad - \lambda^* \left[ \nabla_1 g(x^*,u^*,t)^T (x - x^*) + \nabla_2 g(x^*,u^*,t)^T (u - u^*) \right] \\
&\quad + \|K(t)x\|_M + \|L(t)u\|_L + \lambda^* \|M(t)x\|_M + \lambda^* \|N(t)u\|_N \\
&\quad - \alpha^*(t)^T K(t)x^* - \beta^*(t)^T L(t)u^* \\
&\quad - \lambda^* \gamma^*(t)^T M(t)x^* - \lambda^* \delta^*(t)^T N(t)u^* \right\} \, dt \\
&\quad (\text{by the convexity of } f(\cdot,\cdot, t) \text{ and } -g(\cdot,\cdot, t), \text{ and nonnegativity of } \lambda^*) \\
&\geq \int_a^b \left\{ -\alpha^*(t)^T K(t) - \lambda^* \gamma^*(t)^T M(t) \right\} (x - x^*) \\
&\quad - \left[ \beta^*(t)^T L(t) + \lambda^* \delta^*(t)^T N(t) \right] (u - u^*) \\
&\quad + \|K(t)x\|_M + \|L(t)u\|_L + \lambda^* \|M(t)x\|_M + \lambda^* \|N(t)u\|_N \\
&\quad - \alpha^*(t)^T K(t)x^* - \beta^*(t)^T L(t)u^* \\
&\quad - \lambda^* \gamma^*(t)^T M(t)x^* - \lambda^* \delta^*(t)^T N(t)u^* \\
&\quad - \left\{ v^*(t)^T A(t) + \sum_{i=1}^k w^*_i(t) \right. \\
&\quad \times \left[ \nabla_1 h_i(x^*,u^*,t)^T + \xi^*(t)^T P_i(t) \right] + Dv^*(t) \right\} (x - x^*) \\
&\quad - \left\{ v^*(t)^T B(t) + \sum_{i=1}^k w^*_i(t) \right. \\
&\quad \times \left[ \nabla_2 h_i(x^*,u^*,t)^T + \eta^*(t)^T Q_i(t) \right] \right\} (u - u^*) \, dt \\
&\quad (\text{by } (3.7) \text{ and } (3.8))
\end{align*}
\]
\[ \geq \int_a^b \left\{ -\|K(t)x\|_K \|\alpha^*(t)\|_K^2 - \lambda^* \|M(t)x\|_M \|y^*(t)\|_M^2 \\
- \|L(t)u\|_L \|\beta^*(t)\|_L^2 - \lambda^* \|N(t)u\|_N \|\delta^*(t)\|_N^2 \\
+ \|K(t)x\|_K + \|L(t)u\|_L + \lambda^* \|M(t)x\|_M + \lambda^* \|N(t)u\|_N \\
+ v^*(t)^T [Dx - Dx^* - A(t)(x - x^*) - B(t)(u - u^*)] \\
- \sum_{i=1}^k w^*_i(t) \left[ \|P_i(t)x\|_{P(i)} \|\xi^*(t)\|_{P(i)}^2 \\
+ \|Q_i(t)u\|_{Q(i)} \|\eta^*(t)\|_{Q(i)}^2 \\
- \xi^*(t)^T P_i(t)x^* - \eta^*(t)^T Q_i(t)u^* \\
+ \sum_{i=1}^k w^*_i(t) \left[ h_i(x^*, u^*, t) - h_i(x, u, t) \right] \right] dt \]

(by Lemma 3.2, integration by parts, convexity of \( h_i(\cdot, \cdot, t), i \in k, \) and nonnegativity of \( \lambda^* \) and \( w^*(t) \))

\[ \geq \int_a^b \left\{ v^*(t)^T [Dx - A(t)x - B(t)u] \\
- v^*(t)^T [Dx^* - A(t)x^* - B(t)u^*] \\
+ \sum_{i=1}^k w^*_i(t) \left[ h_i(x^*, u^*, t) + \xi^*(t)^T P_i(t)x^* + \eta^*(t)^T Q_i(t)u^* \right] \\
- \sum_{i=1}^k w^*_i(t) \left[ h_i(x, u, t) + \|P_i(t)x\|_{P(i)} + \|Q_i(t)u\|_{Q(i)} \right] dt \]

(by (3.5) and nonnegativity of \( \lambda^* \) and \( w^*(t) \))

\[ \geq 0 \]

(by (3.3), (3.6), feasibility of \( (x, u) \) and \( (x^*, u^*) \), and nonnegativity of \( w^*(t) \)). Therefore, it follows that \( \varphi(x, u) \geq \lambda^* = \varphi(x^*, u^*) \). Since \( (x, u) \) is an arbitrary feasible solution of \( (\mathcal{P}) \), we conclude that \( (x^*, u^*) \) is an optimal solution of \( (\mathcal{P}) \). \]

Although the proofs of Theorem 3.2 and the sufficiency part of Theorem 3.1 are almost identical, their requirements are essentially different. We shall point out some of the differences between these theorems in the
next section when we propose and discuss two duality models for (P) which
are motivated by the form of the sufficiency conditions specified in these
two theorems.

Parameter-free versions of Theorems 3.1 and 3.2 are obtained by replac-
ing the parameter $\lambda^*$ by $\varphi(x^*, u^*)$ and redefining the multiplier functions
associated with the dynamical equations and inequality constraints. Thus
eliminating $\lambda^*$, Theorems 3.1 and 3.2 can be restated as follows.

**Theorem 3.3.** Assume that the constraints of (P) satisfy (SCQ). Then
$(x^*, u^*) \in \bar{N}$ is an optimal solution of (P) if and only if there exist $v^*$, $w^*$,
$\alpha^*$, $\beta^*$, $\gamma^*$, $\delta^*$, $\xi^*$, and $\eta^*$, $i \in k$, as specified in Theorem 3.1, such that
(3.3), (3.5), (3.6), and the following inequalities hold for all $t \in [a, b]:$

$$
\begin{align*}
\Psi(x^*, u^*) \left[ \nabla_1 f(x^*, u^*, t) + K(t)^T \alpha^*(t) \right] \\
- \Phi(x^*, u^*) \left[ \nabla_1 g(x^*, u^*, t) - M(t)^T \gamma^*(t) \right] + A(t)^T v^*(t) \\
+ \sum_{i=1}^k w_i^*(t) \left[ \nabla_i h_i(x^*, u^*, t) + P_i(t)^T \xi^*(t) \right] + Dv^*(t) = 0,
\end{align*}
$$

$$
\begin{align*}
\Psi(x^*, u^*) \left[ \nabla_2 f(x^*, u^*, t) + L(t)^T \beta^*(t) \right] \\
- \Phi(x^*, u^*) \left[ \nabla_2 g(x^*, u^*, t) - N(t)^T \delta^*(t) \right] + B(t)^T v^*(t) \\
+ \sum_{i=1}^k w_i^*(t) \left[ \nabla_i h_i(x^*, u^*, t) + Q_i(t)^T \eta^*(t) \right] = 0,
\end{align*}
$$

where $\Phi(x^*, u^*)$ is equal to the numerator of the objective function evaluated
at $(x^*, u^*)$, and $\Psi(x^*, u^*)$ to its denominator evaluated at $(x^*, u^*)$.

**Theorem 3.4.** Assume that the constraints of (P) satisfy (SCQ). Then
$(x^*, u^*) \in \bar{N}$ is an optimal solution of (P) if and only if there exist $v^*$, $w^*$,
$\alpha^*$, $\beta^*$, $\gamma^*$, $\delta^*$, $\xi^*$, and $\eta^*$, $i \in k$, as specified in Theorem 3.1, such that
(3.3), (3.5), (3.6), and the following inequalities hold for all $t \in [a, b]:$

$$
\begin{align*}
\left\{ \Psi(x^*, u^*) \left[ \nabla_1 f(x^*, u^*, t)^T + \alpha^*(t)^T K(t) \right] \\
- \Phi(x^*, u^*) \left[ \nabla_1 g(x^*, u^*, t)^T - \gamma^*(t)^T M(t) \right] + v^*(t)^T A(t) \\
+ \sum_{i=1}^k w_i^*(t) \left[ \nabla_i h_i(x^*, u^*, t)^T + \xi^*(t)^T P_i(t) \right] \\
+ Dv^*(t)^T \right\} (x - x^*) \geq 0
\end{align*}
$$
for all $x \in C^a[a, b]$ such that $(x, u) \in \tilde{X}$ for some $u \in \text{PWS}^m[a, b]$, 
\[
\begin{aligned}
\Psi(x^*, u^*) \left[ \nabla_2 f(x^*, u^*, t)^T + \beta^*(t)^T L(t) \right] \\
- \Phi(x^*, u^*) \left[ \nabla_2 g(x^*, u^*, t)^T - \delta^*(t)^T N(t) \right] + v^*(t)^T B(t) \\
+ \sum_{i=1}^k w_i^*(t) \left[ \nabla_2 h_i(x^*, u^*, t)^T + \eta^*(t)^T Q_i(t) \right] (u - u^*) \geq 0
\end{aligned}
\]
for all $u \in \text{PWS}^m[a, b]$ such that $(x, u) \in \tilde{X}$ for some $x \in C^a[a, b]$.

Making use of Theorems 3.3 and 3.4, we shall formulate and discuss eight parameter-free duality models for (P) in Sections 5 and 6.

4. DUALITY MODEL I

Utilizing Theorems 3.1 and 3.2, we next show that the following are dual problems for (P):

\textbf{(D1)} \quad \text{Maximize } \lambda \\
subject to (1.1), (1.4), and 
\[
\begin{aligned}
\nabla_1 f(x, u, t) + K(t)^T \alpha(t) - \lambda \left[ \nabla_1 g(x, u, t) - M(t)^T \gamma(t) \right] \\
+ A(t)^T v(t) + \sum_{i=1}^k w_i(t) \left[ \nabla_1 h_i(x, u, t) + P_i(t)^T \xi_i(t) \right] \\
+ Dv(t) = 0, \quad t \in [a, b], \\
\nabla_2 f(x, u, t) + L(t)^T \beta(t) - \lambda \left[ \nabla_2 g(x, u, t) - N(t)^T \delta(t) \right] \\
+ B(t)^T v(t) + \sum_{i=1}^k w_i(t) \left[ \nabla_2 h_i(x, u, t) + Q_i(t)^T \eta_i(t) \right] = 0,
\end{aligned}
\]
$\quad t \in [a, b]$. \hspace{1cm} (4.1)

\[
\begin{aligned}
\int_a^b \left[ f(x, u, t) + \|K(t)x\|_K + \|L(t)u\|_L \\
- \lambda \left[ g(x, u, t) - \|M(t)x\|_M - \|N(t)u\|_N \right] \\
+ v(t)^T \left[ -Dx + A(t)x + B(t)u \right] \\
+ \sum_{i=1}^k w_i(t) \left[ h_i(x, u, t) + \|P_i(t)x\|_{\|P_i\|} + \|Q_i(t)u\|_{\|Q_i\|} \right] \right) dt \geq 0,
\end{aligned}
\]
$\quad t \in [a, b]$. \hspace{1cm} (4.2)
\[ \| \alpha(t) \|_K^* \leq 1, \quad \| \beta(t) \|_K^* \leq 1, \quad \| \gamma(t) \|_M^* \leq 1, \]
\[ \| \delta(t) \|_N^* \leq 1, \quad t \in [a, b], \]  
\[ \| \zeta(t) \|_{P(i)}^* \leq 1, \quad \| \eta(t) \|_{Q(i)}^* \leq 1, \quad t \in [a, b], \ i \in \mathbb{K}, \]  
\[ \alpha(t)^T K(t)x = \| K(t)x \|_K, \quad \beta(t)^T L(t)u = \| L(t)u \|_L, \]
\[ \gamma(t)^T M(t)x = \| M(t)x \|_M, \quad \delta(t)^T N(t)u = \| N(t)u \|_N, \quad t \in [a, b], \]  
\[ \zeta(t)^T P_i(t)x = \| P_i(t)x \|_{P(i)}, \]
\[ \eta(t)^T Q_i(t)u = \| Q_i(t)u \|_{Q(i)}, \quad t \in [a, b], \ i \in \mathbb{K}, \]  
\[ \lambda \in \mathbb{R}_+, \quad \nu \in \text{PWS}_a[a, b], \quad w \in \text{PWS}_k[a, b], \quad \alpha \in \text{PWS}_a[0, b], \]
\[ \beta \in \text{PWS}_a[a, b], \quad \gamma \in \text{PWS}'[a, b], \quad \delta \in \text{PWS}'[a, b], \]
\[ \zeta \in \text{PWS}_a[a, b], \quad \eta \in \text{PWS}_a[a, b], \ i \in \mathbb{K}; \]  

(D1) Maximize \( \lambda \)  
subject to (1.1), (1.4), (4.3)–(4.8), and

\[ \begin{align*}
\left( \nabla_1 f(x, u, t) + \alpha(t)^T K(t) - \lambda \left[ \nabla_1 g(x, u, t)^T - \gamma(t)^T M(t) \right] + v(t)^T A(t) + \sum_{i=1}^{k} w_i(t) \left[ \nabla_1 h_i(x, u, t)^T + \zeta_i(t)^T P_i(t) \right] + Dv(t)^T \right) (y - x) \geq 0, \quad t \in [a, b], \text{ for all } y \in C^n[a, b] \\
\text{such that } (y, u) \in \mathcal{N} \text{ for some } u \in \text{PWS}_m[a, b].
\end{align*} \]

\[ \begin{align*}
\left( \nabla_2 f(x, u, t) + \beta(t)^T L(t) - \lambda \left[ \nabla_2 g(x, u, t)^T - \delta(t)^T N(t) \right] + v(t)^T B(t) + \sum_{i=1}^{k} w_i(t) \left[ \nabla_2 h_i(x, u, t)^T + \eta_i(t)^T Q_i(t) \right] \right) (z - u) \geq 0, \quad t \in [a, b], \text{ for all } z \in \text{PWS}_m[a, b] \\
\text{such that } (x, z) \in \mathcal{N} \text{ for some } x \in C^n[a, b].
\end{align*} \]
Comparing (D1) and (D1), we observe that (D1) is relatively more general than (D1) in the sense that any feasible solution of (D1) is also feasible for (D1), but the converse is not necessarily true. Furthermore, we see that (4.1) and (4.2) together form a system of \( n + m \) equations, whereas (4.9) and (4.10) are two inequalities which in general cannot be expressed as equivalent systems of equations. Therefore, (D1) and (D1) are essentially different problems. However, despite these apparent differences, the statements and proofs of all the duality theorems for (P)–(D1) and (P)–(D1) are almost identical and, therefore, we consider only the pair (P)–(D1).

The next two theorems show that (D1) is a dual problem for (P).

**Theorem 4.1 (Weak Duality).** Let \((\bar{x}, \bar{u})\) and

\[
(x, u, \lambda, v, w, \alpha, \beta, \gamma, \delta, \xi^1, \ldots, \xi^k, \eta^1, \ldots, \eta^k)
\]

be arbitrary feasible solutions of (P) and (D1), respectively. Then \(\varphi(\bar{x}, \bar{u}) \geq \lambda\).

**Proof.** Since

\[
\int_b^a \{f(\bar{x}, \bar{u}, t) + \|K(t)\bar{x}\|_k + \|L(t)\bar{u}\|_l - \lambda[g(\bar{x}, \bar{u}, t) - \|M(t)\bar{x}\|_M - \|N(t)\bar{u}\|_N]\} \, dt
\]

\[
- \int_b^a \{f(x, u, t) + \|K(t)x\|_k + \|L(t)u\|_l - \lambda[g(x, u, t) - \|M(t)x\|_M - \|N(t)u\|_N]\} \, dt
\]

\[
= \int_a^b \{f(\bar{x}, \bar{u}, y) + \|K(t)\bar{x}\|_k + \|L(t)\bar{u}\|_l - \lambda[g(\bar{x}, \bar{u}, y) - \|M(t)\bar{x}\|_M - \|N(t)\bar{u}\|_N]\} \, dt
\]

\[
- \int_a^b \{f(x, u, t) + \alpha(t)^T K(t)x + \beta(t)^T L(t)u - \lambda[g(x, u, t) - \gamma(t)^T M(t)x - \delta(t)^T N(t)u]\} \, dt \quad \text{(by (4.6))}
\]

\[
\geq \int_a^b \left\{\nabla_x f(x, u, t)^T (\bar{x} - x) + \nabla_u f(x, u, t)^T (\bar{u} - u)
\right.
\]

\[
- \lambda\left[\nabla_x g(x, u, t)^T (\bar{x} - x) + \nabla_u g(x, u, t)^T (\bar{u} - u)\right]
\]

\[
+ \|K(t)\bar{x}\|_k + \|L(t)\bar{u}\|_l + \lambda \|M(t)\bar{x}\|_M + \lambda \|N(t)\bar{u}\|_N
\]

\[
- \alpha(t)^T K(t)x - \beta(t)^T L(t)u - \lambda \gamma(t)^T M(t)x - \lambda \delta(t)^T N(t)u\right\} \, dt
\]
(by the convexity of $f(.,t)$ and $-g(.,t)$, and nonnegativity of $\lambda$)

$$\begin{align*}
&= \int_a^b \left\{ - \left[ (t)^TK(t) + \lambda g(t)^TM(t) \right](\bar{x} - x) \\
&\quad - \left[ (t)^TL(t) + \lambda (t)N(t) \right](\bar{u} - u) + \|K(t)\bar{x}\|_K \\
&\quad + L(t)\bar{u} + \lambda \|M(t)\bar{x}\|_M + \lambda \|N(t)\bar{u}\|_N - (t)^TK(t)x \\
&\quad - \beta(t)^TL(t)u - \lambda (t)^TM(t)x - \lambda (t)^TN(t)u \right\}
\end{align*}$$

$$\begin{align*}
&\quad - \left\{ \langle t, A(t) + \sum_{i=1}^k w_i(t) \left[ \nabla h_i(x,u,t)^T \\
&\quad + \xi i(t)P_i(t) \right] + Du(t)^T \right\} \bar{x} - x \\
&\quad - \left\{ \langle t, B(t) + \sum_{i=1}^k w_i(t) \left[ \nabla h_i(x,u,t)^T \\
&\quad + \eta i(t)Q_i(t) \right] \rangle \bar{u} - u \right\} dt \quad (by (4.1) and (4.2))
\end{align*}$$

$$\begin{align*}
&\geq \int_a^b \left\{ - \|K(t)\bar{x}\|_K \|\alpha(t)\|_K^* - \lambda \|M(t)\bar{x}\|_M \|\gamma(t)\|_M^* \\
&\quad - \|L(t)\bar{u}\|_L \|\beta(t)\|_L^* - \lambda \|N(t)\bar{u}\|_N \|\delta(t)\|_N^* + \|K(t)\bar{x}\|_K \\
&\quad + L(t)\bar{u} + \lambda \|M(t)\bar{x}\|_M + \lambda \|N(t)\bar{u}\|_N \\
&\quad + \langle t, [D\bar{x} - Dx - A(t)(\bar{x} - x) - B(t)(\bar{u} - u)] \rangle \\
&\quad - \sum_{i=1}^k w_i(t) \left[ \|P_i(t)\bar{x}\|_{P_i} \|\xi i(t)\|_{P_i}^* + \|Q_i(t)\bar{u}\|_{Q_i} \|\eta i(t)\|_{Q_i}^* \\
&\quad - \xi i(t)P_i(t)x - \eta i(t)Q_i(t)u \right] \\
&\quad + \sum_{i=1}^k w_i(t) \left[ h_i(x,u,t) - h_i(\bar{x},\bar{u},t) \right] \right\} dt
\end{align*}$$
(by Lemma 3.2, integration by parts, convexity of $h_i(\cdot, t)$, $i \in k$, and nonnegativity of $\lambda$ and $w(t)$)

$$
\geq \int_a^b \left\{ v(t)^T \left[ D\bar{x} - A(t)\bar{x} - B(t)\bar{u} \right] + v(t)^T \left[ -Dx + A(t)x + B(t)u \right]
+ \sum_{i=1}^k w_i(t) \left[ h_i(x, u, t) + \zeta'(t)^T P_i(t)x + \eta'(t)^T Q_i(t)u \right]
- \sum_{i=1}^k w_i(t) \left[ h_i(\bar{x}, \bar{u}, t) + \|P_i(t)\bar{x}\|_{P(t)} + \|Q_i(t)\bar{u}\|_{Q(t)} \right] \right\} dt
$$

(by (4.4), (4.5), and nonnegativity of $\lambda$ and $w(t)$)

$$
\geq \int_a^b \left\{ v(t)^T \left[ -Dx + A(t)x + B(t)u \right]
+ \sum_{i=1}^k w_i(t) \left[ h_i(x, u, t) + \|P_i(t)\bar{x}\|_{P(t)} + \|Q_i(t)\bar{u}\|_{Q(t)} \right] \right\} dt
$$

(by the primal feasibility of $(\bar{x}, \bar{u})$, nonnegativity of $w(t)$, and (4.7) we have the inequality

$$
\int_a^b \left\{ f(\bar{x}, \bar{u}, t) + \|K(t)\bar{x}\|_k + \|L(t)\bar{u}\|_k
- \lambda \left[ g(\bar{x}, \bar{u}, t) - \|M(t)\bar{x}\|_m - \|N(t)\bar{u}\|_n \right] \right\} dt
\geq \int_a^b \left\{ f(x, u, t) + \|K(t)x\|_k + \|L(t)u\|_k
- \lambda \left[ g(x, u, t) - \|M(t)x\|_m - \|N(t)u\|_n \right] \right\} dt
+ v(t)^T \left[ -Dx + A(t)x + B(t)u \right]
+ \sum_{i=1}^k w_i(t) \left[ h_i(x, u, t) + \|P_i(t)x\|_{P(t)} + \|Q_i(t)u\|_{Q(t)} \right] \right\} dt
\geq 0 \quad \text{(by (4.3))},
$$

which yields $\varphi(\bar{x}, \bar{u}) \geq \lambda$.

**Theorem 4.2 (Strong Duality).** Let $(x^*, u^*)$ be an optimal solution of (P) and assume that the constraints of (P) satisfy (SCQ). Then there exist $\lambda^* \in \mathbb{R}_+$, $\nu^* \in \text{PWS}^*[a, b]$, $w^* \in \text{PWS}^t[a, b]$, $\alpha^* \in \text{PWS}^t[a, b]$, $\beta^* \in \mathbb{R}_+$.
Theorem 4.3 (Strict Converse Duality). Let \((x^*, u^*)\) and \((\tilde{x}, \tilde{u}, \lambda, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\xi}^1, ..., \tilde{\xi}^k, \tilde{\eta}^1, ..., \tilde{\eta}^k)\) be optimal solutions of \((P)\) and \((D_1)\), respectively, and assume that the constraints of \((P)\) satisfy (SCQ). Assume furthermore that \(f(., t)\) or \(-g(., t)\) is strictly convex throughout \([a, b]\) for at least one \(i \in \mathbb{k}\) with the corresponding component \(w_i(t)\) of \(w(t)\) positive on \([a, b]\). Then \((\tilde{x}(t), \tilde{u}(t)) = (x^*(t), u^*(t))\) for all \(t \in [a, b]\), that is, \((\tilde{x}, \tilde{u})\) is an optimal solution of \((P)\), and \(\varphi(x^*, u^*) = \lambda^*\).

Proof. Suppose, on the contrary, that \((\tilde{x}(t), \tilde{u}(t)) \neq (x^*(t), u^*(t))\) on a subset of \([a, b]\) with positive length. From Theorem 4.2 we know that there exist \(x^*, u^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \xi^i, \eta^i, \), and \(\eta^*\), \(i \in \mathbb{k}\), as specified in Theorem 4.2, such that

\[
(x^*, u^*, \lambda^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \xi^1, ..., \xi^k, \eta^1, ..., \eta^k)
\]

is an optimal solution of \((D_1)\) and \(\varphi(x^*, u^*) = \lambda^*\). Now proceeding as in the proof of Theorem 4.1 with \((\tilde{x}, \tilde{u})\) replaced by \((x^*, u^*)\) and

\[
(x, u, \lambda, v, w, \alpha, \beta, \gamma, \delta, \xi^1, ..., \xi^k, \eta^1, ..., \eta^k)
\]

by

\[
(\tilde{x}, \tilde{u}, \tilde{\lambda}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\xi}^1, ..., \tilde{\xi}^k, \tilde{\eta}^1, ..., \tilde{\eta}^k)
\]

we arrive at the strict inequality \(\varphi(x^*, u^*) > \tilde{\lambda}\), in contradiction to the fact that \(\varphi(x^*, u^*) = \lambda^*\). Therefore, we must have \((\tilde{x}(t), \tilde{u}(t)) = (x^*(t), u^*(t))\) for all \(t \in [a, b]\), and \(\varphi(x^*, u^*) = \lambda\).
The structure of the duality models (D1) and (D1) is based directly on the form and features of the optimality conditions of Theorems 3.1 and 3.2. As it can easily be seen from the proof of Theorem 4.1, there is some redundancy in the statements of (D1) and (D1) which can be eliminated without invalidating the duality relations between (P) and (D1) and (D1). More precisely, if (4.6) and (4.7) are deleted and (4.3) is modified accordingly, then the following reduced versions of (D1) and (D1) are also dual problems for (P):

\[(E1) \quad \text{Maximize } \lambda \text{ subject to (1.1), (1.4), (4.1), (4.2), (4.4), (4.5), (4.8), and} \]

\[
\int_a^b \left\{ f(x,u,t) + \alpha(t)^T K(t)x + \beta(t)^T L(t)u \\
- \lambda \left[ g(x,u,t) - \gamma(t)^T M(t)x - \delta(t)^T N(t)u \right] \\
+ \nu(t)^T \left[ -Dx + A(t)x + B(t)u \right] \\
+ \sum_{i=1}^k w_i(t) \left[ h_i(x,u,t) + \xi(t)^T P_i(t)x + \eta(t)^T Q_i(t)u \right] \right\} dt \geq 0;
\]

\[(4.11)\]

\[(E\bar{E}1) \quad \text{Maximize } \lambda \text{ subject to (1.1), (1.4), (4.4), (4.5), and (4.8)--(4.11).} \]

It can readily be verified that with only minor modifications, Theorems 4.1–4.3 are also valid for (P)–(E1) (and (P)–(\(E\bar{E}1\)).

5. DUALITY MODEL II

The duality models presented in the preceding section were motivated by Theorems 3.1 and 3.2 and consequently are of a parametric nature. In this section we make use of Theorems 3.3 and 3.4 to formulate two parameter-free duality models for (P) and prove weak, strong, and strict converse duality theorems. Specifically, we show that the following are dual problems for (P):

\[(D\bar{I}1) \quad \text{Maximize} \]

\[
\int_a^b \left[ f(x,u,t) + \|K(t)x\|_K + \|L(t)u\|_L \right] dt \\
\int_a^b \left[ g(x,u,t) - \|M(t)x\|_M - \|N(t)u\|_N \right] dt
\]
subject to
\[ x(a) = 0, \quad x(b) = 0 \quad (5.1) \]
\[
\Psi(x, u) \left[ \nabla_t f(x, u, t) + K(t)^T \alpha(t) \right]
- \Phi(x, u) \left[ \nabla_t g(x, u, t) - M(t)^T \gamma(t) \right] + A(t)^T v(t)
+ \sum_{i=1}^{k} w_i(t) \left[ \nabla_t h_i(x, u, t) + P_i(t)^T \zeta^i(t) \right] + Dv(t) = 0,
\quad t \in [a, b], \quad (5.2)\]
\[
\Psi(x, u) \left[ \nabla_t f(x, u, t) + L(t)^T \beta(t) \right]
- \Phi(x, u) \left[ \nabla_t g(x, u, t) - N(t)^T \delta(t) \right]
+ B(t)^T v(t) + \sum_{i=1}^{k} w_i(t) \left[ \nabla_t h_i(x, u, t) + Q_i(t)^T \eta^i(t) \right] = 0,
\quad t \in [a, b], \quad (5.3)\]
\[
v(t)^T [-Dx + A(t)x + B(t)u] + \sum_{i=1}^{k} w_i(t) [h_i(x, u, t)
\quad + \|P_i(t)x\|_{P_i(t)} + \|Q_i(t)u\|_{Q_i(t)}] \geq 0, \quad t \in [a, b], \quad (5.4)\]
\[
\|\alpha(t)\|_K^* \leq 1, \quad \|\beta(t)\|_N^* \leq 1, \quad \|\gamma(t)\|_M^* \leq 1,
\quad \|\delta(t)\|_N^* \leq 1, \quad \|\zeta(t)^i\|_{P_i(t)}^* \leq 1, \quad \|\eta(t)^i\|_{Q_i(t)}^* \leq 1, \quad t \in [a, b], \quad i \in \mathbb{K}, \quad (5.5)\]
\[
\alpha(t)^TK(t)x = \|K(t)x\|_K, \quad \beta(t)^TL(t)u = \|L(t)u\|_L,
\quad \gamma(t)^TM(t)x = \|M(t)x\|_M,
\quad \delta(t)^TN(t)u = \|N(t)u\|_N, \quad t \in [a, b], \quad (5.7)\]
\[
\zeta(t)^iP_i(t)x = \|P_i(t)x\|_{P_i(t)}, \quad \eta(t)^iQ_i(t)u = \|Q_i(t)u\|_{Q_i(t)}, \quad t \in [a, b], \quad i \in \mathbb{K}, \quad (5.8)\]
\[
x \in C^\alpha[a, b], \quad u \in \text{PWS}^\alpha[a, b], \quad v \in \text{PWS}^\alpha[a, b], \quad w \in \text{PWS}^\ell[a, b],
\quad \alpha \in \text{PWS}^\alpha[a, b], \quad \beta \in \text{PWS}^\alpha[a, b], \quad \gamma \in \text{PWS}^\alpha[a, b],
\quad \delta \in \text{PWS}^\alpha[a, b], \quad \zeta^i \in \text{PWS}^\alpha[a, b], \quad \eta^i \in \text{PWS}^\alpha[a, b], \quad i \in \mathbb{K}, \quad (5.9)\]
where \(\Psi\) and \(\Phi\) are as defined in Theorem 3.3;
Maximize
\[
\int_a^b \left[ f(x, u, t) + \|K(t)x\|_K + \|L(t)u\|_L \right] dt
\]
subject to (5.1), (5.4)–(5.9), and
\[
\begin{cases}
\Psi(x, u) \left[ \nabla f(x, u, t)^T + \alpha(t)^T K(t) \right] \\
- \Phi(x, u) \left[ \nabla g(x, u, t)^T - \gamma(t)^T M(t) \right] + \nu(t)^T A(t) \\
+ \sum_{i=1}^k w_i(t) \left[ \nabla h_i(x, u, t)^T + \xi_i(t)^T P_i(t) + D\nu(t)^T \right] (y - x) \geq 0,
\end{cases}
\]
for all \( y \in C^n[a, b] \) such that \((y, u) \in \bar{\Phi} \) for some \( u \in PWS^m[a, b] \), \( \xi \) (5.10)

\[
\begin{cases}
\Psi(x, u) \left[ \nabla g(x, u, t)^T + \beta(t)^T L(t) \right] \\
- \Phi(x, u) \left[ \nabla g(x, u, t)^T - \delta(t)^T N(t) \right] \\
+ \nu(t)^T B(t) + \sum_{i=1}^k w_i(t) \left[ \nabla h_i(x, u, t)^T + \eta_i(t)^T Q_i(t) \right] (z - u) \geq 0,
\end{cases}
\]
for all \( z \in PWS^m[a, b] \) such that \((x, z) \in \bar{\Psi} \) for some \( x \in C^n[a, b] \). \( \xi \) (5.11)

The remarks made earlier concerning the relationships between (D1) and (D1) are, of course, also applicable to (DII) and (DII).

Throughout this section and the next, it will be assumed that \( \Phi(x, u) \geq 0 \) and \( \Psi(x, u) > 0 \) for all \( x \) and \( u \) such that \((x, u, v, w, \alpha, \beta, \gamma, \delta, \xi^1, \ldots, \xi^k, \eta^1, \ldots, \eta^k) \) is a feasible solution of the dual problem under consideration.

Just as in the case of (P)–(D1), we now proceed to prove weak, strong, and strict converse duality theorems for (P)–(DII).

**Theorem 5.1 (Weak Duality).** Let \((\bar{x}, \bar{u})\) and
\[
(x, u, v, w, \alpha, \beta, \gamma, \delta, \xi^1, \ldots, \xi^k, \eta^1, \ldots, \eta^k)
\]
be arbitrary feasible solutions of \((P)\) and \((D_{II})\), respectively. Then

\[
\varphi(\bar{x}, \bar{u}) \geq \psi(x, u, v, w, \alpha, \beta, \gamma, \delta, \xi^1, \ldots, \xi^k, \eta^1, \ldots, \eta^h),
\]

where \(\psi\) is the objective function of \((D_{II})\).

**Proof.** Since

\[
\begin{align*}
\int_a^b & \left[ g(x, u, t) - M(t)x \right] dt \\
\times & \int_a^b \left[ f(\bar{x}, \bar{u}, t) + K(t)\bar{x} + L(t)\bar{u} \right] dt \\
- & \int_a^b \left[ f(\bar{x}, \bar{u}, t) + K(t)x + L(t)u \right] dt \\
\times & \int_a^b \left[ g(\bar{x}, \bar{u}, t) - M(t)\bar{x} - N(t)\bar{u} \right] dt \\
= & \Psi(x, u) \left\{ \int_a^b \left[ f(\bar{x}, \bar{u}, t) + K(t)\bar{x} + L(t)\bar{u} \right] dt \\
- \int_a^b \left[ f(\bar{x}, \bar{u}, t) + K(t)x + L(t)u \right] dt \right\} \\
- & \Phi(x, u) \left\{ \int_a^b \left[ g(\bar{x}, \bar{u}, t) - M(t)\bar{x} - N(t)\bar{u} \right] dt \\
- \int_a^b \left[ g(x, u, t) - M(t)x - N(t)u \right] dt \right\} \\
\geq & \Psi(x, u) \int_a^b \left[ \nabla f(x, u, t)^T(\bar{x} - x) + \nabla g(x, u, t)^T(\bar{u} - u) \\
+ K(t)\bar{x} + L(t)\bar{u} \right] dt \\
- & \Phi(x, u) \left\{ \int_a^b \left[ \nabla f(x, u, t)^T(\bar{x} - x) + \nabla g(x, u, t)^T(\bar{u} - u) \\
- M(t)\bar{x} - N(t)\bar{u} \right] dt \right\} \\
+ & \Psi(x, u) \left\{ \int_a^b \left[ \nabla f(x, u, t)^T(\bar{x} - x) + \nabla g(x, u, t)^T(\bar{u} - u) \\
- M(t)x - N(t)u \right] dt \right\} \\
- & \Phi(x, u) \left\{ \int_a^b \left[ \nabla f(x, u, t)^T(\bar{x} - x) + \nabla g(x, u, t)^T(\bar{u} - u) \\
- M(t)\bar{x} - N(t)\bar{u} \right] dt \right\} \\
\end{align*}
\]


(by the convexity of $f(\cdot,\cdot,t)$ and $-g(\cdot,\cdot,t)$, and nonnegativity of $\Psi(x,u)$ and $\Phi(x,u)$)

\[
\begin{aligned}
&= \int_a^b \left\{ \Psi(x,u) \left[ -\alpha(t)^T K(t)(\bar{x} - x) - \beta(t)^T L(t)(\bar{u} - u) \\
&\quad + \|K(t)\bar{x}\|_K + \|L(t)\bar{u}\|_L - \|K(t)x\|_K - \|L(t)u\|_L \right] \\
&\quad - \left( v(t)^TA(t) + \sum_{i=1}^k w_i(t) \left[ \nabla \phi_i(x,u,t)^T + \xi^i(t)^T P_i(t) \right] \\
&\quad + Dv(t)^T \right)(\bar{x} - x) \\
&\quad + \Phi(x,u) \left[ -\gamma(t)^T M(t)(\bar{x} - x) - \delta(t)^T N(t)(\bar{u} - u) \\
&\quad + \|M(t)\bar{x}\|_M + \|N(t)\bar{u}\|_N - \|M(t)x\|_M - \|N(t)u\|_N \right] \\
&\quad - \left( v(t)^TB(t) + \sum_{i=1}^k w_i(t) \left[ \nabla \phi_i(x,u,t)^T \\
&\quad + \eta^i(t)^T Q_i(t) \right] \right)(\bar{u} - u) \right\} \, dt \quad (\text{by } (5.2) \text{ and } (5.3))
\end{aligned}
\]

\[
\begin{aligned}
&\geq \int_a^b \left\{ \Psi(x,u) \left[ -\|K(t)\bar{x}\|_K \|\alpha(t)\|^2_K + \alpha(t)^T K(t)x \\
&\quad - \|L(t)\bar{u}\|_L \|\beta(t)\|^2_L + \beta(t)^T L(t)u + \|K(t)\bar{x}\|_K \\
&\quad + \|L(t)\bar{u}\|_L - \|K(t)x\|_K - \|L(t)u\|_L \right] \\
&\quad - v(t)^TA(t)(\bar{x} - x) - \sum_{i=1}^k w_i(t) \left[ \nabla \phi_i(x,u,t)^T (\bar{x} - x) \\
&\quad + \|P_i(t)\bar{x}\|_{\mu_{(i)}} \|\xi^i(t)\|^2_{\mu_{(i)}} - \xi^i(t)^T P_i(t)x \right] + v(t)^TD(\bar{x} - x) \\
&\quad + \Phi(x,u) \left[ -\|M(t)\bar{x}\|_M \|\gamma(t)\|^2_M + \gamma(t)^T M(t)x \\
&\quad - \|N(t)\bar{u}\|_N \|\delta(t)\|^2_N + \delta(t)^T N(t)u + \|M(t)\bar{x}\|_M \\
&\quad + \|N(t)\bar{u}\|_N - \|M(t)x\|_M - \|N(t)u\|_N \right] \right\}
\end{aligned}
\]
\[-v(t)^T B(t)(\overline{u} - u) - \sum_{i=1}^k w_i(t) \left[ \nabla_2 h_i(x, u, t)^T (\overline{u} - u) \right. \\
\quad + \left. \left\| Q_i(t) \overline{u} \right\|_{Q(i)} \left\| \eta'(t) \right\|^*_{Q(i)} - \eta'(t)^T Q_i(t) u \right] \right\} dt \]

(by the nonnegativity of $\Phi(x, u)$, $\Psi(x, u)$, and $w(t)$, Lemma 3.2, and integration by parts)

\[\geq \int_a^b \left\{ v(t)^T \left[ D\tilde{x} - A(t)\tilde{x} - B(t)\overline{u} \right] \\
\quad - v(t)^T \left[ Dx - A(t)x - B(t)u \right] \\
\quad - \sum_{i=1}^k w_i(t) \left[ \nabla_1 h_i(x, u, t)^T (\tilde{x} - x) + \nabla_2 h_i(x, u, t)^T (\overline{u} - u) \right. \\
\quad \left. - \zeta'(t)^T P_i(t)x - \eta'(t)^T Q_i(t)u + \left\| P_i(t)\tilde{x} \right\|_{P(i)} + \left\| Q_i(t)\overline{u} \right\|_{Q(i)} \right] \right\} dt \]

(by (5.5)–(5.7) and nonnegativity of $\Phi(x, u)$, $\Psi(x, u)$, and $w(t)$)

\[\geq \int_a^b \left\{ v(t)^T \left[ -Dx + A(t)x + B(t)u \right] \\
\quad + \sum_{i=1}^k w_i(t) \left[ h_i(x, u, t) + \zeta'(t)^T P_i(t)x + \eta'(t)^T Q_i(t)u \right] \\
\quad - \sum_{i=1}^k w_i(t) \left[ h_i(\tilde{x}, \overline{u}, t) + \left\| P_i(t)\tilde{x} \right\|_{P(i)} + \left\| Q_i(t)\overline{u} \right\|_{Q(i)} \right] \right\} dt \]

(by the primal feasibility of $(\tilde{x}, \overline{u})$, convexity of $h_i(\cdot, \cdot, t)$, $i \in K$, and nonnegativity of $w(t)$)

\[\geq 0\]

(by the primal feasibility of $(\tilde{x}, \overline{u})$, nonnegativity of $w(t)$, (5.4), and (5.8)), it
follows that

\[
\varphi(\bar{x}, \bar{u}) = \frac{\int_a^b [f(x, u, t) + \|K(t)x\|_K + \|L(t)u\|_L] \, dt}{\int_a^b [g(x, u, t) - \|M(t)x\|_M - \|N(t)u\|_N] \, dt}
\]

\[
\geq \frac{\int_a^b [f(x, u, t) + \|K(t)x\|_K + \|L(t)u\|_L] \, dt}{\int_a^b [g(x, u, t) - \|M(t)x\|_M - \|N(t)u\|_N] \, dt}
\]

\[= \psi(x, u, v, w, \alpha, \beta, \gamma, \delta, \xi, \eta, \ldots)\]

**Theorem 5.2 (Strong Duality).** Let \((x^*, u^*)\) be an optimal solution of (P) and assume that the constraints of (P) satisfy (SCQ). Then there exist \(v^* \in \text{PWS}[a, b], w^* \in \text{PWS}[a, b], \alpha^* \in \text{PWS}[a, b], \beta^* \in \text{PWS}[a, b], \gamma^* \in \text{PWS}[a, b], \delta^* \in \text{PWS}[a, b], \xi^i \in \text{PWS}[a, b], \) and \(\eta^i \in \text{PWS}[a, b], i \in k, \) such that \((x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \xi^i, \ldots, \eta^i)\) is an optimal solution of (DII).

**Proof.** By Theorem 3.3, there exist \(v^*, w^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \xi^i, \) and \(\eta^i, i \in k, \) as specified above, such that \((x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \xi^i, \ldots, \eta^i)\) is a feasible solution of (DII). Since (P) and (DII) have the same objective function, optimality of this feasible solution for (DII) follows from Theorem 5.1.

**Theorem 5.3 (Strict Converse Duality).** Let \((x^*, u^*)\) and \((\tilde{x}, \tilde{u}, v, w, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\xi}, \tilde{\eta}, \ldots)\) be optimal solutions of (P) and (DII), respectively, and assume that the constraints of (P) satisfy (SCQ). Assume furthermore that \(f(\cdot, \cdot, t)\) or \(-g(\cdot, \cdot, t)\) is strictly convex throughout \([a, b]\) and \(\Phi(\tilde{x}, \tilde{u}) > 0\), or \(h(\cdot, \cdot, t)\) is strictly convex throughout \([a, b]\) for at least one \(i \in k\) with the corresponding component \(\tilde{w}(t)\) of \(\tilde{w}(t)\) positive on \([a, b]\). Then \((\tilde{x}(t), \tilde{u}(t)) = (x^*(t), u^*(t))\) for all \(t \in [a, b], \) that is, \((\tilde{x}, \tilde{u})\) is an optimal solution of (P).

**Proof.** This is similar to the proof of Theorem 4.3.

Evidently, a prominent feature of (DII) is the fact that it has the same objective function as the primal problem (P). A dual problem of this type was originally formulated in [12] for a linear fractional programming problem. Subsequently, similar dual problems for fractional and other kinds of nonlinear programming problems were proposed in the related literature.
As in the case of (D1) and (D1), we observe that the constraints (5.7) and (5.8) are superfluous and hence can be dropped if appropriate changes are made in (5.4) and in the objective functions of (DII) and (DIII). Indeed, implementing these alterations will yield the following reduced versions of (DII) and (DIII):

(EII) \textbf{M}aximize

\[ \int_a^b \left[ f(x,u,t) + \alpha(t)^T K(t)x + \beta(t)^T L(t)u \right] dt \]

\[ \int_a^b \left[ g(x,u,t) - \gamma(t)^T M(t)x - \delta(t)^T N(t)u \right] dt \]

subject to (5.1), (5.2), and (5.3) with $\Phi(x,u)$ replaced by $\Gamma(x,u,\alpha,\beta)$ and $\Psi(x,u)$ by $\Delta(x,u,\gamma,\delta)$, (5.5), (5.6), (5.9), and

\[ v(t)^T \left[ -Dx + A(t)x + B(t)u \right] \]

\[ + \sum_{i=1}^k w_i(t) \left[ h_i(x,u,t) + \zeta^i(t)^T P_i(t)x + \eta^i(t)^T Q_i(t)u \right] \geq 0, \quad t \in [a,b], \quad (5.12) \]

where $\Gamma(x,u,\alpha,\beta)$ is the numerator of the objective function of (EII) and $\Delta(x,u,\gamma,\delta)$ its denominator;

(ŒII) \textbf{M}aximize

\[ \int_a^b \left[ f(x,u,t) + \alpha(t)^T K(t)x + \beta(t)^T L(t)u \right] dt \]

\[ \int_a^b \left[ g(x,u,t) - \gamma(t)^T M(t)x - \delta(t)^T N(t)u \right] dt \]

subject to (5.1), (5.5), (5.6), (5.9), (5.10), and (5.11) with $\Phi(x,u)$ replaced by $\Gamma(x,u,\alpha,\beta)$ and $\Psi(x,u)$ by $\Delta(x,u,\gamma,\delta)$, and (5.12).

The proofs of the weak, strong, and strict converse duality theorems for (P)–(EII) and (P)–(ŒII) are almost identical to those of Theorems 5.1–5.3, and hence omitted.

6. DUALITY MODEL III

In this section we discuss two other pairs of parameter-free duality models for (P). We begin by showing that the following variants of (DII)
and (DIII) are dual problems for (P):

\[(\text{DIII}) \quad \text{Maximize} \]

\[
\frac{\Phi(x, u) + \Omega(x, u, v, w)}{\Psi(x, u)}
\]

subject to (5.1), (5.4)–(5.9), and

\[
\Psi(x, u)\left\{ \nabla f(x, u, t) + K(t)^T \alpha(t) + A(t)^Tv(t) \\
+ \sum_{i=1}^{k} w_i(t) \left[ \nabla h_i(x, u, t) + P_i(t)^T \xi_i(t) \right] + Dv(t) \right\} \\
- [\Phi(x, u) + \Omega(x, u, v, w)] \left[ \nabla g(x, u, t) - M(t)^T \gamma(t) \right] = 0, \\
t \in [a, b], \quad (6.1)
\]

\[
\Psi(x, u)\left\{ \nabla f(x, u, t) + L(t)^T \beta(t) + B(t)^Tv(t) \\
+ \sum_{i=1}^{k} w_i(t) \left[ \nabla h_i(x, u, t) + Q_i(t)^T \eta_i(t) \right] \right\} \\
- [\Phi(x, u) + \Omega(x, u, v, w)] \left[ \nabla g(x, u, t) - N(t)^T \delta(t) \right] = 0, \\
t \in [a, b], \quad (6.2)
\]

where \( \Phi \) and \( \Psi \) are as defined in Theorem 3.3 and

\[
\Omega(x, u, v, w) = \int_a^b \left\{ v(t)^T \left[ -Dx + A(t)x + B(t)u \right] \\
+ \sum_{i=1}^{k} w_i(t) \left[ h_i(x, u, t) + \|P_i(t)x\|_{\rho(t)} + \|Q_i(t)u\|_{\varrho(t)} \right] \right\} dt;
\]

\[(\text{DIII}) \quad \text{Maximize} \]

\[
\frac{\Phi(x, u) + \Omega(x, u, v, w)}{\Psi(x, u)}
\]
subject to (5.1), (5.4)–(5.9), and
\[
\begin{align*}
\left\{ \Psi(x,u) & \cdots + \alpha(t)^T K(t) + v(t)^T A(t) \\
& + \sum_{i=1}^k w_i(t) \left[ \nabla h_i(x,u,t)^T + \xi_i(t)^T P_i(t) \right] + Dv(t)^T \right) \\
& - \left[ \Phi(x,u) + \Omega(x,u,v,w) \right] \left[ \nabla g(x,u,t)^T - \gamma(t)^T M(t) \right] \right) (y-x) \\
& \geq 0,
\end{align*}
\]
t \in [a,b], for all y \in C^n[a,b]
such that (y,u) ∈ \hat{\mathcal{H}} for some u ∈ PWS^m[a,b], (6.3)

\[
\begin{align*}
\left\{ \Psi(x,u) & \cdots + \alpha(t) = \beta(t)^T L(t) + v(t)^T B(t) \\
& + \sum_{i=1}^k w_i(t) \left[ \nabla h_i(x,u,t)^T + \eta_i(t)^T Q_i(t) \right] \right) \\
& - \left[ \Phi(x,u) + \Omega(x,u,v,w) \right] \left[ \nabla g(x,u,t)^T - \delta(t)^T N(t) \right] \right) (z-u) \\
& \geq 0,
\end{align*}
\]
t \in [a,b], for all z \in PWS^m[a,b]
such that (x,z) ∈ \hat{\mathcal{H}} for some x ∈ C^n[a,b]. (6.4)

The remarks made earlier concerning the relationships between (D1) and (DII) are, of course, also applicable to (DIII) and (DIII).

We next state and prove weak, strong, and strict converse duality theorems for (P)–(DIII).

**Theorem 6.1 (Weak Duality).** Let (\tilde{x}, \tilde{u}) and (x, u, v, w, \alpha, \beta, \gamma, \delta, \xi^1, \ldots, \xi^k, \eta^1, \ldots, \eta^k) be arbitrary feasible solutions of (P) and (DIII), respectively. Then \( \varphi(\tilde{x}, \tilde{u}) \geq \varphi(x, u, v, w, \alpha, \beta, \gamma, \delta, \xi^1, \ldots, \xi^k, \eta^1, \ldots, \eta^k) \), where \( \omega \) is the objective function of (DIII).

**Proof.** Since
\[
\begin{align*}
\Psi(x,u) \Phi(\tilde{x}, \tilde{u}) & - \left[ \Phi(x,u) + \Omega(x,u,v,w) \right] \Psi(\tilde{x}, \tilde{u}) \\
& = \Psi(x,u) \int_a^b \left[ f(\tilde{x}, \tilde{u}, t) + \|K(t)\tilde{x}\|_{\mathcal{L}} + \|L(t)\tilde{u}\|_{\mathcal{L}} \\
& - f(x,u,t) - \|K(t)x\|_{\mathcal{L}} - \|L(t)u\|_{\mathcal{L}} \right] dt
\end{align*}
\]
\[
- \Phi(x,u) \int_a^b \left[ g(\bar{x},\bar{u},t) - \|M(t)\bar{x}\|_M - \|N(t)\bar{u}\|_N \right. \\
- g(x,u,t) + \|M(t)x\|_M + \|N(t)u\|_N \right] dt \\
- \Psi(\bar{x},\bar{u}) \Omega(x,u,v,w) \\
\geq \Psi(x,u) \int_a^b \left[ \nabla_1 f(x,u,t)^T (\bar{x} - x) + \nabla_2 f(x,u,t)^T (\bar{u} - u) \\
+ \|K(t)\bar{x}\|_K + \|L(t)\bar{u}\|_L - \|K(t)x\|_K - \|L(t)u\|_L \right] dt \\
- \Phi(x,u) \int_a^b \left[ \nabla_1 g(x,u,t)^T (\bar{x} - x) + \nabla_2 g(x,u,t)^T (\bar{u} - u) \\
- \|M(t)\bar{x}\|_M - \|N(t)\bar{u}\|_N + \|M(t)x\|_M + \|N(t)u\|_N \right] dt \\
- \Psi(\bar{x},\bar{u}) \Omega(x,u,v,w)
\]

(by the convexity of \(f(\ldots,t)\) and \(-g(\ldots,t)\), and nonnegativity of \(\Phi(x,u)\) and \(\Psi(x,u)\))

\[
= \int_a^b \left\{ -\Psi(x,u) \left\{ \alpha(t)^T K(t)(\bar{x} - x) + \beta(t)^T L(t)(\bar{u} - u) \\
- \|K(t)\bar{x}\|_K - \|L(t)\bar{u}\|_L + \|K(t)x\|_K + \|L(t)u\|_L \\
+ \sum_{i=1}^k \omega_i(t) \left\{ \left[ \nabla_1 h_i(x,u,t)^T + \zeta_i(t)^T P_i(t) \right](\bar{x} - x) \\
+ \left[ \nabla_2 h_i(x,u,t)^T + \eta_i(t)^T Q_i(t) \right](\bar{u} - u) \right\} \\
+ \left[ v(t)^T A(t) + Dw(t)^T \right](\bar{x} - x) + v(t)^T B(t)(\bar{u} - u) \right\} \\
+ \Phi(x,u) \left[ -\gamma(t)^T M(t)(\bar{x} - x) - \delta(t)^T N(t)(\bar{u} - u) \\
- \|M(t)x\|_M - \|N(t)u\|_N + \|M(t)\bar{x}\|_M + \|N(t)\bar{u}\|_N \right] \\
+ \Omega(x,u,v,w) \left[ \nabla_1 g(x,u,t)^T (\bar{x} - x) + \nabla_2 g(x,u,t)^T (\bar{u} - u) \right] \\
- \gamma(t)^T M(t)(\bar{x} - x) - \delta(t)^T N(t)(\bar{u} - u) \right\} dt \\
- \Phi(\bar{x},\bar{u}) \Omega(x,u,v,w) \quad (\text{by (6.1) and (6.2)})
\]
\[ \geq \int_{a}^{b} \left\{ \Psi(x, u) \left( -\|K(t)\bar{x}\|_K \|\alpha(t)\|^a_t + \alpha(t)^T K(t) x \\
- \|L(t)\bar{u}\|_L \|\beta(t)\|^b_t + \beta(t)^T L(t) u \\
+ \|K(t)\bar{x}\|_K + \|L(t)\bar{u}\|_L \\
- \sum_{i=1}^{k} w_i(t) \left[ \nabla h_i(x, u, t)^T (\bar{x} - x) \\
+ \|P_i(t)\bar{x}\|_{P_i(t)} \|\xi^i(t)\|_{P_i(t)} \\
- \xi^i(t)^T P_i(t)x + \nabla h_i(x, u, t)^T (\bar{u} - u) \\
+ \|Q_i(t)\bar{u}\|_{Q_i(t)} \|\eta^i(t)\|_{Q_i(t)} - \eta^i(t)^T Q_i(t) u \right] \\
+ v(t)^T \left[ D\bar{x} - A(t)\bar{x} - B(t)\bar{u} \right] \\
+ v(t)^T \left[ -Dx + A(t)x + B(t)u \right] \right) \right\} \]

+ \Phi(x, u) \left[ -\|M(t)\bar{x}\|_M \|\gamma(t)\|^a_{\gamma} + \gamma(t)^T M(t) x \\
- \|N(t)\bar{u}\|_N \|\delta(t)\|^b_{\delta} + \delta(t)^T N(t) u - \|M(t)x\|_M \\
- \|N(t)u\|_N + \|M(t)\bar{x}\|_M + \|N(t)\bar{u}\|_N \right] \\
+ \Omega(x, u, v, w) \left[ -g(x, u, t) + g(\bar{x}, \bar{u}, t) \\
- \|M(t)\bar{x}\|_M \|\gamma(t)\|^a_{\gamma} + \gamma(t)^T M(t) x \\
- \|N(t)\bar{u}\|_N \|\delta(t)\|^b_{\delta} + \delta(t)^T N(t) u \\
- g(\bar{x}, \bar{u}, t) + \|M(t)\bar{x}\|_M + \|N(t)\bar{u}\|_N \right) \right] \right\} d\tau \\
(by \ the \ nonnegativity \ of \ w(t), \ \Phi(x, u), \ \Psi(x, u), \ \text{and} \ \Omega(x, u, v, w); \ \text{Lemma} \ 3.2, \ \text{integration \ by \ parts, \ convexity \ of} \ -g(., ., t), \ \text{and \ definition \ of} \ \Psi(\bar{x}, \bar{u})) \]

\[ \geq \int_{a}^{b} \left\{ \Psi(x, u) \left( v(t)^T \left[ -Dx + A(t)x + B(t)u \right] \right) \right\} \right\} \]

+ \sum_{i=1}^{k} w_i(t) \left[ h_i(x, u, t) - h_i(\bar{x}, \bar{u}, t) - \|P_i(t)\bar{x}\|_{P_i(t)} \]
\[
- \left( \| Q_i(t) x \|_{Q_i} + \xi_i(t)^T P_i(t) x + \eta_i(t) Q_i(t) u \right)
- \Omega(x, u, v, w) \left[ g(x, u, t) - \gamma(t)^T M(t) x - \delta(t)^T N(t) u \right] dt
\]

(by (5.5)-(5.7), nonnegativity of \( w(t) \), \( \Phi(x, u) \), \( \Psi(x, u) \), and \( \Omega(x, u, v, w) \); primal feasibility of \( (x, \bar{u}) \), and convexity of \( h_i(\cdot, \cdot, t), i \in \mathcal{I} \)

\[
\geq 0
\]

(by the nonnegativity of \( w(t) \) and \( \Psi(x, u) \), primal feasibility of \( (x, \bar{u}) \), (5.4), (5.7), (5.8), and definitions of \( \Psi(x, u) \) and \( \Omega(x, u, v, w) \)), we conclude that

\[
\varphi(x, \bar{u}) = \frac{\Phi(x, u) + \Omega(x, u, v, w)}{\Psi(x, u)} = \omega(x, u, v, w, \alpha, \beta, \gamma, \delta, \zeta^1, \ldots, \zeta^k, \eta^1, \ldots, \eta^k) \tag*{\blacksquare}
\]

**Theorem 6.2 (Strong Duality).** Let \((x^*, u^*)\) be an optimal solution of \((P)\) and assume that the constraints of \((P)\) satisfy \((SCQ)\). Then there exist \( v^* \in PWS^*([a, b], \mathbb{R}^n), w^* \in PWS^*([a, b], \mathbb{R}^m), \alpha^* \in PWS^*([a, b], \mathbb{R}^p), \beta^* \in PWS^*([a, b], \mathbb{R}^q), \gamma^* \in PWS^*([a, b], \mathbb{R}^r), \delta^* \in PWS^*([a, b], \mathbb{R}^s), \zeta^* \in PWS^*([a, b], \mathbb{R}^t), \text{ and } \eta^* \in PWS^*([a, b], \mathbb{R}^u), i \in \mathcal{I}, \) such that \((x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \zeta^1, \ldots, \zeta^k, \eta^1, \ldots, \eta^k)\) is an optimal solution of \((DIII)\) and \( \varphi(x^*, u^*) = \omega(x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \zeta^1, \ldots, \zeta^k, \eta^1, \ldots, \eta^k)\).

**Proof.** By Theorem 3.3, there exist \( \bar{v} \in PWS^*([a, b], \mathbb{R}^m), \bar{w} \in PWS^*([a, b], \mathbb{R}^n), \) and \( \alpha^*, \beta^*, \gamma^*, \delta^*, \zeta^*, \text{ and } \eta^*, i \in \mathcal{I}, \) as specified above, such that the following relations hold for all \( t \in [a, b] \):

\[
\Psi(x^*, u^*) \left[ \nabla_1 f(x^*, u^*, t) + K(t)^T \alpha^*(t) \right]
- \Phi(x^*, u^*) \left[ \nabla_1 g(x^*, u^*, t) - M(t)^T \gamma^*(t) \right] + A(t)^T \bar{v}(t)
+ \sum_{i=1}^{k} \bar{w}_i(t) \left[ \nabla_1 h_i(x^*, u^*, t) + P_i(t)^T \zeta^i(t) \right] + D \bar{v}(t) = 0, \tag*{(6.5)}
\]

\[
\Psi(x^*, u^*) \left[ \nabla_2 f(x^*, u^*, t) + L(t)^T \beta^*(t) \right]
- \Phi(x^*, u^*) \left[ \nabla_2 g(x^*, u^*, t) - N(t)^T \delta^*(t) \right] + B(t)^T \bar{v}(t)
+ \sum_{i=1}^{k} \bar{w}_i(t) \left[ \nabla_2 h_i(x^*, u^*, t) + Q_i(t)^T \eta^i(t) \right] = 0, \tag*{(6.6)}
\]
\[
\sum_{i=1}^{k} \bar{w}_i(t)\left[h_i(x^*, u^*, t) + \|P_i(t)x^*\|_{P(i)} + \|Q_i(t)u^*\|_{Q(i)} \right] = 0, \tag{6.7}
\]

\[
\|\alpha^*(t)\|_k^* \leq 1, \quad \|\beta^*(t)\|_L^* \leq 1, \quad \|\gamma^*(t)\|_M^* \leq 1, \quad \|\delta^*(t)\|_N^* \leq 1, \quad \|\xi^*(t)\|_{P(i)}^* \leq 1, \quad \|\eta^*(t)\|_{Q(i)}^* \leq 1, \quad i \in k, \tag{6.8}
\]

\[
\alpha^*(t)^T K(t)x^* = \|K(t)x^*\|_K^*, \quad \beta^*(t)^T L(t)u^* = \|L(t)u^*\|_L^*, \tag{6.9}
\]

\[
\gamma^*(t)^T M(t)x^* = \|M(t)x^*\|_M^*, \quad \delta^*(t)^T N(t)u^* = \|N(t)u^*\|_N^*, \tag{6.10}
\]

Now, if we let \(v^* = \overline{v}/\Psi(x^*, u^*), \ w^* = \overline{w}/\Psi(x^*, u^*), \) and notice that \(\Omega(x^*, u^*, v^*, w^*) = 0, \) then (6.5)–(6.7) can be rewritten as

\[
\Psi(x^*, u^*) \left\{ \nabla_1 f(x^*, u^*, t) + K(t)^T\alpha^*(t) + A(t)^Tv^*(t) \right. \\
+ \sum_{i=1}^{k} w_i^*(t) \left[ \nabla_2 h_i(x^*, u^*, t) + P_i(t)^T\xi^*(t) \right] + D\nu^*(t) \left. \right\} \\
- \left[ \Phi(x^*, u^*) + \Omega(x^*, u^*, v^*, w^*) \right] \\
\times \left[ \nabla_1 g(x^*, u^*, t) - M(t)^T\gamma^*(t) \right] = 0, \quad t \in [a, b], \tag{6.11}
\]

\[
\Psi(x^*, u^*) \left\{ \nabla_2 f(x^*, u^*, t) + L(t)^T\beta^*(t) + B(t)^Tv^*(t) \right. \\
+ \sum_{i=1}^{k} w_i^*(t) \left[ \nabla_2 h_i(x^*, u^*, t) + Q_i(t)^T\eta^*(t) \right] \left. \right\} \\
- \left[ \Phi(x^*, u^*) + \Omega(x^*, u^*, v^*, w^*) \right] \\
\times \left[ \nabla_2 g(x^*, u^*, t) - N(t)^T\delta^*(t) \right] = 0, \quad t \in [a, b], \tag{6.12}
\]

\[
\sum_{i=1}^{k} w_i^*(t) \left[ h_i(x^*, u^*, t) + \|P_i(t)x^*\|_{P(i)} + \|Q_i(t)u^*\|_{Q(i)} \right] = 0, \\
t \in [a, b]. \tag{6.13}
\]

From (6.8)–(6.13) and primal feasibility of \((x^*, u^*)\) it is clear that \(y^* = (x^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*, \delta^*, \xi^*, \eta^*)\) is a feasible solution of (DIII). Since \(\Omega(x^*, u^*, v^*, w^*) = 0, \) we have \(\varphi(x^*, u^*) = \omega(y^*). \) Now optimality of \(y^*\) for (DIII) follows from Theorem 6.1.
**Theorem 6.3 (Strict Converse Duality).** Let $(x^*, u^*)$ and $(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\xi}^1, ..., \tilde{\xi}^k, \tilde{\eta}^1, ..., \tilde{\eta}^k)$ be optimal solutions of $(P)$ and $(D_{III})$, respectively, and assume that the constraints of $(P)$ satisfy (SCQ). Assume furthermore that $f(.,.,t)$ or $-g(.,.,t)$ is strictly convex throughout $[a, b]$ (and $\Phi(\tilde{x}, \tilde{u}) > 0$), or $h(.,.,t)$ is strictly convex throughout $[a, b]$ for at least one $i \in k$ with the corresponding component $\tilde{w}_i(t)$ of $\tilde{w}(t)$ positive on $[a, b]$. Then $(\tilde{x}(t), \tilde{u}(t)) = (x^*(t), u^*(t))$ for all $t \in [a, b]$; that is, $(\tilde{x}, \tilde{u})$ is an optimal solution of $(P)$, and $\varphi(x^*, u^*) = \omega(x^*, u^*, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\xi}^1, ..., \tilde{\xi}^k, \tilde{\eta}^1, ..., \tilde{\eta}^k)$.

**Proof.** This is similar to the proof of Theorem 4.3.

We close this section with the determination of the reduced forms of $(D_{III})$ and $(D_{III'})$. These dual problems are obtained by deleting (5.7) and (5.8), and then altering the objective functions and the remaining constraints of $(D_{III})$ and $(D_{III'})$ accordingly. These modifications lead to the following relatively more streamlined versions of $(D_{III})$ and $(D_{III'})$:

(EIII) \[ \text{Maximize} \quad \frac{\Gamma(x, u, \alpha, \beta) + \Pi(x, u, v, w, \xi, \eta)}{\Delta(x, u, \gamma, \delta)} \]

subject to (5.1), (5.5), (5.6), (5.9), (5.12), and (6.1) and (6.2) with $\Phi(x, u)$ replaced by $\Gamma(x, u, \alpha, \beta)$, $\Psi(x, u)$ by $\Delta(x, u, \gamma, \delta)$, and $\Omega(x, u, v, w)$ by $\Pi(x, u, v, w, \xi, \eta)$, where

\[
\Pi(x, u, v, w, \xi, \eta) = \int_a^b \left[ v(t)^T [ -Dx + A(t)x + B(t)u ] \right.
+ \sum_{i=1}^k w_i(t) \left[ h_i(x, u, t) + \xi^i(t)^T P_i(t)x + \eta^i(t)^T Q_i(t)u \right] dt;
\]

(ÊIII) \[ \text{Maximize} \quad \frac{\Gamma(x, u, \alpha, \beta) + \Pi(x, u, v, w, \xi, \eta)}{\Delta(x, u, \gamma, \delta)} \]

subject to (5.1), (5.5), (5.6), (5.9), and (6.3) and (6.4) with $\Phi(x, u)$ replaced by $\Gamma(x, u, \alpha, \beta)$, $\Psi(x, u)$ by $\Delta(x, u, \gamma, \delta)$, and $\Omega(x, u, v, w)$ by $\Pi(x, u, v, w, \xi, \eta)$.

Following the pattern of Theorems 6.1–6.3, one can easily state and prove similar duality results for $(P)$–(EIII) and $(P)$–(ÊIII).
7. PROBLEMS CONTAINING SQUARE ROOTS OF POSITIVE SEMIDEFINITE QUADRATIC FORMS

In this section we briefly discuss a special case of \( P \) obtained by choosing all the norms to be the \( l_2 \)-norm \( \| \cdot \|_2 \).

Let \( \| \cdot \|_K, \| \cdot \|_L, \| \cdot \|_M, \| \cdot \|_N, \| \cdot \|_{P(i)}, \text{ and } \| \cdot \|_{Q(i)}, \) \( i \in k, \) be the \( l_2 \)-norm, and define \( E(t) = K(t)^T K(t), \) \( F(t) = L(t)^T L(t), \) \( G(t) = M(t)^T M(t), \) \( H(t) = N(t)^T N(t), \) \( R_i(t) = P_i(t)^T P_i(t), \) and \( S_i(t) = Q_i(t)^T Q_i(t), \) \( i \in k. \) Then it is clear that \( E(t), G(t), \) and \( R_i(t), \) \( i \in k, \) are \( n \times n \) symmetric positive semidefinite matrices; \( F(t), H(t), \) and \( S_i(t), \) \( i \in k, \) are \( m \times m \) symmetric positive semidefinite matrices, and, therefore, the functions \( x(t) \to [ \langle x(t)^T E(t) x(t) \rangle ]^{1/2}, \) \( x(t) \to [ \langle x(t)^T G(t) x(t) \rangle ]^{1/2}, \) and \( x(t) \to [ \langle x(t)^T R_i(t) x(t) \rangle ]^{1/2}, \) \( i \in k, \) are convex on \( \mathbb{R}^n, \) and the functions \( u(t) \to [ \langle u(t)^T F(t) u(t) \rangle ]^{1/2}, \) \( u(t) \to [ \langle u(t)^T H(t) u(t) \rangle ]^{1/2}, \) and \( u(t) \to [ \langle u(t)^T S_i(t) u(t) \rangle ]^{1/2}, \) \( i \in k, \) are convex on \( \mathbb{R}^m. \)

With these choices of the norms and matrices, \( P \) and \( P^1 \) become

\[
(P^*) \quad \text{Minimize} \quad \int_a^b \left\{ f(x, u, t) + \left[ x^T E(t) x \right]^{1/2} + \left[ u^T F(t) u \right]^{1/2} \right\} dt
\]

subject to (1.1), (1.2), (1.4), and

\[
h_i(x, u, t) + \left[ x^T R_i(t) x \right]^{1/2} + \left[ u^T S_i(t) u \right]^{1/2} \leq 0, \quad t \in [a, b], \ i \in k; \tag{7.1}
\]

\[
(P^1) \quad \text{Minimize} \quad \int_a^b \left\{ f(x, u, t) + \left[ x^T E(t) x \right]^{1/2} + \left[ u^T F(t) u \right]^{1/2} \right\} dt,
\]

where

\[
\mathcal{H}^* = \{ (x, u): (1.1), (1.2), (1.4), \text{ and } (7.1) \text{ hold} \}.
\]

Evidently, Theorems 3.1–3.4, the twelve duality models \((D_1), (\bar{D}_1), (E_1), (\bar{E}_1), (D_1), (\bar{D}_1), (E_1), (\bar{E}_1), (D_1), (\bar{D}_1), (E_1), \) and \((\bar{E}_1)), \) and the corresponding duality theorems can be specialized and restated for \((P^*)\) and \((P^1)\) in a straightforward manner. We shall not explicitly state these results.
Various classes of finite-dimensional mathematical programming problems containing square roots of positive semidefinite quadratic forms have been investigated and a number of optimality and duality results for these problems have been published in the related literature. In practical situations, these problems have arisen in stochastic programming, multifacility location problems, and portfolio planning, among others. A fairly extensive list of references pertaining to several aspects of these problems is given in [21].

8. CONCLUDING REMARKS

In this paper we have introduced and discussed parametric and nonparametric necessary and sufficient optimality criteria and several parametric and nonparametric duality models for a class of nonsmooth constrained optimal control problems with fractional objective functions and linear dynamics. Although the results developed here subsume several existing results in the area of optimal control theory and contain the optimal control analogues of a great variety of kindred results obtained previously for various classes of finite-dimensional mathematical programming problems, they are essentially restricted to a particular class of optimal control problems, namely, problems with convex-concave fractional objective functions, convex constraints, and linear dynamics.

Research efforts aimed at extending the results of this paper to more general types of fractional and conventional optimal control problems with nonlinear dynamics and generalized convex functions, and possibly to more general optimal control models such as continuous minmax, multiobjective fractional, and generalized fractional optimal control problems, will constitute worthwhile undertakings.

It appears that at least some of the optimality and duality results of the present study can be generalized for the following two classes of unorthodox control problems which contain (P) and several other problems as special cases:

(P2)

\[
\begin{align*}
\text{Minimize} & \quad \max_{(x,u) \in \mathbb{R}} \frac{\int_a^b [f_i(x,u,t) + \|K_i(t)x\|_{k(i)} + \|L_i(t)u\|_{k(i)}] \, dt}{\int_a^b [g_i(x,u,t) - \|M_i(t)x\|_{m(i)} - \|N_i(t)u\|_{m(i)}] \, dt;}
\end{align*}
\]
Finite-dimensional counterparts of (P2) and (P3) are known as general fractional and multiobjective fractional programming problems, respectively, and have been the subject of numerous investigations in the past few years. In contrast, their infinite-dimensional analogues, especially optimal control and variational problems, have not yet received much attention in the literature of optimization theory.

In subsequent papers, we shall explore the possibility of developing an optimality-duality theory for (P2) and a proper efficiency-duality theory for (P3).

REFERENCES