Asymptotic behavior of solutions of second order quasilinear differential equations with delay depending on the unknown function

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Received 5 August 2003; received in revised form 26 August 2004

Abstract

The asymptotic behavior of the nonoscillatory solutions of quasilinear differential equations of second order with delay depending on the unknown function is considered. The main results given by [Bainov et al. (J. Comput. Appl. Math. 91 (1998) 87–96) and Wong (Funkcial. Ekvac. 11 (1968) 207–234)] are improved and generalized.

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MSC: 34C10; 34K15

Keywords: Asymptotic behavior of solutions; Differential equations of second order with delay

1. Introduction

This paper is concerned with the asymptotic behavior of the nonoscillatory solutions of a class of quasilinear differential equations with delay depending on the unknown function of the form

\[(r(t)|x'(t)|^{2-1}x'(t))' + f(t, x(t), x(\Delta(t, x(t)))) = 0, \quad t \geq 0.\] (1)
By a solution of (1) in the interval $[T, \infty)$, we mean a function $x : (T_1, \infty) \to R$ where $T_1 = \inf\{\Delta(t, x) : t \geq T, x \in R\}$, which is continuously differential on $[T, \infty)$ together with $r(t)|x'(t)|^{2-1}x'(t)$ and satisfies the equation at every point of $[T, \infty)$. Our attention will be restricted to those solutions $x(t)$ of (1) which satisfy $\sup\{|x(t)| : t \geq T_1\} > 0$ for every $T_1 \geq T$. A nontrivial solution is called to be oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. Thus a nonoscillatory solution is either eventually positive or eventually negative.

Eq. (1) has been the object of intensive studies in recent years because it can be considered as a natural generalization of the important equations:

$$x''(t) + f(t, x(t)) = 0, \quad (2)$$

$$|x'(t)|^{2-1}x'(t)' + f(t, x(t)) = 0, \quad (3)$$

$$(r(t)x'(t))' + f(t, x(t), x(\Delta(t, x(t)))) = 0. \quad (4)$$

On the one hand, it becomes in some case a one-dimensional version (polar form) of important partial differential equations of the form

$$\text{div}(|Du|^{m-2}Du) + f(x, u) = 0.$$ 

Many authors have studied the Eqs. (2) and (3). We refer to [4–6,8,9,7,10,11]. However, the differential equations of form (4) with delay depending on the unknown function have been investigated only in the papers [1–3] up to now.

Our purpose here is to develop the nonoscillation theory for such a general case of (1). This work was motivated by the paper of Bainov et al. [3] in which a detailed analysis of nonoscillatory properties was given for the Eq. (4). We will follow closely the presentation of Bainov et al. [3], and show that all of their results not only can be generalized to (1), but also can be improved, i.e., the condition that $f(t, u, v)$ of (1) is nondecreasing in $u$ and $v$ for each fixed $t \geq T$ can be weakened. Our main results are stated and proved in Section 3. In Section 2, we give preliminary notes and some lemmas and in Section 4 we give some remarks and an example illustrating the results.

2. Preliminary notes and some lemmas

Let $T \in R_+ = [0, \infty)$. Now introduce the following conditions:

H1. $\alpha > 0$ is constant.
H2. $r \in C(R_+, R_+)$ and $r(t) > 0, t \in R_+$.
H3. $\int_0^\infty (1/r(s))^{1/\alpha} ds = +\infty$.
H4. $f \in C(R_+ \times R^2, R)$.
H5. There exists $T \in R_+$ such that $uf(t, u, v) > 0$ for $t \geq T, u \cdot v > 0$.
H6. $\Delta \in C(R_+ \times R, R)$.
H7. There exists a function $A_*(t) \in C(R_+, R)$ and $T \in R_+$ such that $\lim_{t \to +\infty} A_*(t) = +\infty$ and $A_*(t) \leq \Delta(t, x)$ for $t \geq T, x \in R$.
H8. There exists a function $A^*(t) \in C(R_+, R)$ and $T \in R_+$ such that $A^*(t)$ is a nondecreasing function for $t \geq T$ and $A(t, x) \leq A^*(t) \leq t$ for $t \geq T, x \in R$.
H9. For any positive constants \( l \) and \( L \), with \( l < L \) there exist positive constant \( \theta \) and \( \Theta \), depending possibly on \( l \) and \( L \) such that \( l \leq |u| < L \) implies
\[
\theta f(t, l, v) \leq |f(t, u, v)| \leq \Theta f(t, L, v)
\]
and \( l \leq |v| \leq L \) implies
\[
\theta f(t, u, l) \leq |f(t, u, v)| \leq \Theta f(t, u, L).
\]

H10. For any positive constant \( l \) and \( L \) with \( l < L \) there exist positive constants \( \theta f(t, l, v) \) and \( \Theta f(t, u, L) \), depending possibly on \( l \) and \( L \) such that \( l \leq |u| \leq L \) implies
\[
\theta f(t, l, v) \leq |f(t, u, v)| \leq \Theta f(t, u, L)
\]
and \( l \leq |v| \leq L \) implies
\[
\theta f(t, u, l) \leq |f(t, u, v)| \leq \Theta f(t, u, L),
\]
where \( R(t) = \int_0^t (1/r(s))^{1/2} \, ds \).

Introduce the functions
\[
R(t) = \int_0^t (1/r(s))^{1/2} \, ds, \quad R(t, T) = \int_T^t (1/r(s))^{1/2} \, ds.
\]

Now we give two lemmas, which are useful in the proof of our main results.

**Lemma 1.** Let conditions H2 and H3 hold and \( x(t) \) be continuously differentiable on \([T, \infty)\) together with \( r(t)|x'(t)|^{p-1}x'(t) \). Suppose that
\[
x(t)(r(t)|x'(t)|^{p-1}x'(t))' < 0, \quad t \geq T.
\]
Then we have \( x(t)x'(t) > 0 \) for \( t \geq T \).

**Proof.** Let \( x(t) > 0 \) and \( x(t)(r(t)|x'(t)|^{p-1}x'(t))' < 0 \) for \( t \geq T \). Then \( r(t)|x'(t)|^{p-1}x'(t) \) is a decreasing function for \( t \geq T \). The case \( x'(t) \leq 0 \) is not possible. If we suppose that there exist \( k > 0 \) and \( T_1 \geq T \) such that
\[
r(t)|x'(t)|^{p-1}x'(t) \leq -k, \quad t \geq T_1.
\]
Then by H3 we can obtain that \( x(t) \to -\infty \) as \( t \to \infty \), which is a contradiction. \( \square \)

**Lemma 2.** Let conditions H1–H8 hold and \( x(t) \) be a nonoscillatory solution of (1). Then \( x(t) \) possesses one of the following properties:

(P1) \( \lim_{t \to +\infty} \frac{x(t)}{R(t)} = \text{const} \neq 0, \quad \lim_{t \to +\infty}|x(t)| = +\infty. \)

(P2) \( \lim_{t \to +\infty} \frac{x(t)}{R(t)} = 0, \quad \lim_{t \to +\infty}|x(t)| = +\infty. \)

(P3) \( \lim_{t \to +\infty} \frac{x(t)}{R(t)} = 0, \quad \lim_{t \to +\infty}|x(t)| = \text{const} \neq 0. \)

**Proof.** Let \( x(t) > 0 \) for \( t \geq T_0 \geq T \geq 0 \), it follows from condition H7 that there exists \( T_1 \geq T_0 \) such that \( x(\Delta(t, x(t))) > 0 \) for \( t \geq T_1 \) and from H5 and (1), we conclude that there exists \( T_2 \geq T_1 \) such that
(r(t)|x'(t)|^{z-1}x'(t))' < 0 \text{ for } t \geq T_2 \geq T_1. \text{ Therefore } r(t)|x'(t)|^{z-1}x'(t) \text{ is decreasing function for } t \geq T_2. \text{ By Lemma 1, } r(t)|x'(t)|^{z-1}x'(t) > 0 \text{ for } t \geq T_2. \text{ Then, there exists the limit}

\lim_{t \to +\infty} r(t)(x'(t))^2 = L \in [0, \infty). \tag{5}

It is easy to prove that

\lim_{t \to +\infty} \frac{x(t)}{R(t)} = L^{1/z}. \tag{6}

In fact, from (5) we can get

\lim_{t \to +\infty} \frac{x'(t)}{(r(t))^{1/z}} = L^{1/z},

\lim_{t \to +\infty} \frac{x'(t)}{R'(t)} = L^{1/z}.

From H3, \( R(t) \to \infty \) as \( t \to \infty \), by the above equality, (6) is true. From (6), if \( L > 0 \), then \( x(t) \) possesses property (P1). Let \( L = 0 \). Since \( x'(t) > 0 \) for \( t \geq T_2 \) and \( x(t) \) is an increasing function then either \( \lim_{t \to +\infty} x(t) = \text{const} \neq 0 \) (and \( x(t) \) possesses property (P3)), or \( \lim_{t \to +\infty} x(t) = +\infty \) (and \( x(t) \) possesses property (P2)). \( \Box \)

3. Main results

**Theorem 1.** Let conditions H1–H8 and H10 hold. Then:

1. If Eq. (1) has a nonoscillatory solution with property (P1), then there exists a constant \( c \neq 0 \) such that

\[
\int_{\infty}^{\infty} |f(s, cR(s), cR^*(s))| < +\infty.
\]

2. If for some \( c \neq 0 \) we have

\[
\int_{\infty}^{\infty} |f(s, cR(s), cR^*(s))| < +\infty,
\]

then Eq. (1) has a solution with the property (P1).

**Proof.** 1. Let Eq. (1) have a solution \( x(t) \) for which

\[
\lim_{t \to +\infty} \frac{x(t)}{R(t)} = L \neq 0.
\]

Without loss of generality, we suppose that \( L > 0 \). Then there exist \( c > 0 \) and \( T_0 \geq T \geq 0 \) such that

\[ x(t) \geq cR(t) \]
and
\[ cR(A^*(t)) \geq x(A(t, x(t))) \geq cR(A_+(t)), \quad t \geq T_0. \] (9)

The integration of Eq. (1) shows that
\[
\int_{T_0}^{\infty} f(s, x(s), x(A(s, x(s)))) \, ds \leq r(T_0)[x'(T_0)]^2. \] (10)

Inequalities (9) and (10) and H10 imply that
\[
r(T_0)[x'(T_0)]^2 \geq \int_{T_0}^{\infty} f(s, x(s), x(A(s, x(s)))) \, ds \geq \theta \int_{T_0}^{\infty} f(s, cR(s), x(A(s, x(s)))) \, ds \geq \theta^2 \int_{T_0}^{\infty} f(s, cR(s), cR(A_+(s))) \, ds,
\]
i.e.,
\[
\int_{T_0}^{\infty} f(s, cR(s), cR(A_+(s))) \, ds < +\infty. \] (11)

2. Let inequality (8) be fulfilled with a constant \( c \neq 0 \). Without loss of generality, we suppose that \( c > 0 \). We choose \( m \) and \( T \) such that \( 0 < m \leq c/2 \) and \( T > 0 \) such that
\[
\theta^2 \int_{T}^{\infty} f(s, 2mR(s), 2mR(A_+(s))) \, ds \leq (2^x - 1)m^2.
\]

Consider now the nonempty, closed, bounded, convex subset \( D \) of \( C([T_{-1}, +\infty), R) \) given by
\[
D = \left\{ x \in C([T_{-1}, +\infty), R) \left| \begin{array}{l}
x(t) = 0, \\
mR(t, T) \leq x(t) \leq 2mR(t, T),
\end{array} \right\} \right. \quad T_{-1} \leq t < T
\]
and for every \( x \in D \) define the operator \( Sx \):
\[
(Sx)(t) = \begin{cases} 
0, & T_{-1} \leq t < T, \\
\int_{T}^{t} \left( \frac{1}{r(s)} \right)^{1/x} \left[ m^2 + \int_s^{+\infty} f(u, x(u), x(A(u, x(u)))) \, du \right]^{1/2} \, ds, & t \geq T.
\end{cases}
\]

We can easily show that \( S(D) \subseteq D \). In order to apply the operator \( S \), the Tychonov fixed point theorem, it is sufficient now to prove that \( S \) is continuous in \( D \) and that \( S(D) \) is relatively compact in \( C([T_{-1}, +\infty), R) \).

Let \( \{x_n\} \subseteq D \), \( x_n \to x, x \in D \), we need to prove that \( Sx_n \to Sx \), e.g., the sequences \( \{Sx_n\} \) tends, uniformly on every compact set of \([T_{-1}, +\infty), to \ Sx.\)
For \( t > T \) we have
\[
| (Sx_n(t) - (Sx)(t)) |
\]
\[
= \left| \int_T^t \left( \frac{1}{r(s)} \right)^{1/z} \left[ m^2 + \int_s^{+\infty} f(u, x_n(u), x_n(A(u, x_n(u)))) \, du \right]^{1/z} \, ds \right|
\]
\[
- \left| \int_T^t \left( \frac{1}{r(s)} \right)^{1/z} \left[ m^2 + \int_s^{+\infty} f(u, x(u), x(A(u, x(u)))) \, du \right]^{1/z} \, ds \right|
\]
\[
\leq \left| \int_T^t \left( \frac{1}{r(s)} \right)^{1/z} \left[ \left\{ m^2 + \int_s^{+\infty} f(u, x_n(u), x_n(A(u, x_n(u)))) \, du \right\}^{1/z} - \left\{ m^2 + \int_s^{+\infty} f(u, x(u), x(A(u, x(u)))) \, du \right\}^{1/z} \right] \, ds \right|.
\]
Hence from the continuity of the functions \( f, A \), we get that the sequence \( \{Sx_n\} \) tends, uniformly on every compact set of \([T_{-1}, +\infty)\), to \( Sx \).

For \( t \in [T_{-1}, T] \), the above assertions follow from the definition of the operator \( S \).

Concerning the compactness of \( S(D) \) in \( C([T_{-1}, +\infty), R) \) it is sufficient to show that if \( \{x_n\} \subset D \), then the sequence \( \{Sx_n\} \) is quasi-bounded and quasi-continuous on every compact set of \([T_{-1}, +\infty)\).

The quasi-bounded easily follows taking into account that \( S(D) \subset D \) and \( D \) is a bounded subset of \( C([T_{-1}, +\infty), R) \). Let us prove the quasi-continuity of the sequence \( \{Sx_n\} \).

For \( t_1, t_2 \in [T, +\infty) \) we have
\[
| (Sx_n)(t_2) - (Sx_n)(t_1) |
\]
\[
= \left| \int_{t_1}^{t_2} \left( \frac{1}{r(s)} \right)^{1/z} \left[ m^2 + \int_s^{+\infty} f(u, x_n(u), x_n(A(u, x_n(u)))) \, du \right]^{1/z} \, ds \right|
\]
\[
- \left| \int_{t_1}^{t_2} \left( \frac{1}{r(s)} \right)^{1/z} \left[ m^2 + \int_s^{+\infty} f(u, x(u), x(A(u, x(u)))) \, du \right]^{1/z} \, ds \right|
\]
\[
\leq \left| \int_{t_1}^{t_2} \left( \frac{1}{r(s)} \right)^{1/z} \left[ \left\{ m^2 + \int_s^{+\infty} f(u, x_n(u), x_n(A(u, x_n(u)))) \, du \right\}^{1/z} - \left\{ m^2 + \int_s^{+\infty} f(u, x(u), x(A(u, x(u)))) \, du \right\}^{1/z} \right] \, ds \right|.
\]
Since \( x_n \in D \), we get \( x_n(t) \leq cR(t) \), \( x_n(A(t, x_n(t))) \leq cR(A^*(t)) \). Taking into account H10, hence we obtain
\[
| (Sx_n)(t_2) - (Sx_n)(t_1) | \leq \int_{t_1}^{t_2} \left( \frac{1}{r(s)} \right)^{1/z} \left\{ m^2 + \Theta^2 \int_s^{+\infty} f(u, cR(u), cR(A^*(u))) \, du \right\}^{1/z} \, ds.
\]

So that the quasi-continuity of the sequence \( \{Sx_n\} \) in \([T, +\infty)\) is proved. For \( t \in [T_{-1}, T] \), the assertion follows from the definition of the operator \( S \). For \( t_1 \in [T_{-1}, T] \), \( t_2 \in [T, +\infty) \), or vice versa, in view of
\[
(Sx_n)(t_2) - (Sx_n)(t_1) = (Sx_n)(t_2) - (Sx_n)(T) + (Sx_n)(T) - (Sx_n)(t_1),
\]
the assertion easily follows reasoning as in the above cases. Hence the Schauder–Tychonov fixed point theorem ensures the existence of a function \( x \in D \) such that \( x = Sx \), i.e.,

\[
x(t) = \int_T^t \left( \frac{1}{r(s)} \right)^{1/2} \left[ m^2 + \int_s^{+\infty} f(u, x(u), x(A(u, x(u)))) \right]^{1/2} ds, \quad t \geq T.
\]

It is easy to see that \( x(t) \) is a positive solution of (1) on \([T, \infty)\) with the desired property \( \lim_{t \to +\infty} (x(t)/R(t)) = m \neq 0 \). This finishes the proof. □

**Theorem 2.** Suppose that conditions H1–H9 hold. Then Eq. (1) has a nonoscillatory solution having the property (P3) if and only if

\[
\int_T^\infty \left( \frac{1}{r(t)} \int_t^\infty |f(s, c, c)| ds \right)^{1/2} dt < +\infty \quad \text{for some } c \neq 0. \tag{12}
\]

**Proof.** Let Eq. (1) have a nonoscillatory solution (P3): \( \lim_{t \to \infty} x(t) = \text{const} \neq 0 \). There is no loss of generality in assuming that \( \lim_{t \to \infty} x(t) > 0 \), so that there exist positive constant \( l, L \) and \( T_1 \) such that \( l \leq x(t) \leq L, \ l \leq x(A(t, x(t))) \leq L, \ t \geq T \). Condition (H9) then implies that

\[
f(t, x(t), x(A(t, x(t)))) \geq \theta^2 f(t, l, l), \quad t \geq T_1
\]

for some constant \( \theta > 0 \). Integrating (1) from \( t \) to \( +\infty \), and noting that \( x'(t) > 0 \) for \( t \geq T_1 \), we have

\[
\frac{1}{r(t)} \int_t^\infty f(s, x(s), x(A(s, x(s)))) ds = [x'(t)]^2 - [x'(+\infty)]^2 < [x'(t)]^2, \quad t \geq T_1.
\]

It follows that

\[
\int_{T_1}^t \left[ \frac{1}{r(s)} \int_s^\infty f(s, x(s), x(A(s, x(s)))) ds \right]^{1/2} ds \leq x(t) - x(T_1) \leq x(t),
\]

which, combined with (13), yields

\[
\theta^{2/\alpha} \int_{T_1}^t \left[ \frac{1}{r(s)} \int_s^\infty f(s, l, l) ds \right]^{1/2} ds \leq L.
\]

Suppose that (12) holds for some \( c \neq 0 \). We may assume that \( c > 0 \). By the condition H9 there is a constant \( \Theta \) such that \( c/2 \leq x(t) \leq c \) implies \( f(t, x(t), x(A(t, x(t)))) \leq \Theta^2 f(t, c, c) \) for \( t \geq T_1 \). Choose \( T > 0 \) so large that

\[
\Theta^{2/\alpha} \int_T^\infty \left[ \frac{1}{r(t)} \int_t^\infty f(s, c, c) ds \right]^{1/2} dt \leq \frac{c}{2},
\]

and define the set

\[
D = \left\{ x \in C([T_1, +\infty), R) \mid \begin{array}{ll}
    x(t) = \frac{c}{2}, & T_1 \leq t < T \\
    c & \leq x(t) \leq c, \quad t \geq T
\end{array} \right\}
\]
and the operator \( S: D \to C([T_{-1}, +\infty), R) \) by the formula

\[
Sx(t) = \begin{cases} 
    \frac{c}{2}, & T_{-1} \leq t < T \\
    c - \int_t^\infty \left( \frac{1}{r(s)} \int_s^\infty f(u, x(u), x(\Lambda(u, x(u)))) \, du \right)^{1/\alpha} \, ds, & t \geq T.
\end{cases}
\]

As the proof of Theorem 1, it is routinely verified that:

(i) \( S \) maps \( D \) into itself;
(ii) \( S \) is a continuous mapping;
(iii) \( S(D) \) is relatively compact in \( C([T_{-1}, \infty)) \).

Therefore, by the Schauder–Tychonoff fixed point theorem, there is a function \( x \in D \) such that \( x = Sx \), i.e.,

\[
x(t) = c - \int_t^\infty \left( \frac{1}{r(s)} \int_s^\infty f(u, x(u), x(\Lambda(u, x(u)))) \, du \right)^{1/\alpha} \, ds, \quad t \geq T.
\]

This shows that \( x(t) \) is a solution of (1) which is positive on \([T_{-1}, \infty)\) and satisfies \( \lim_{t \to +\infty} x(t) = c \).

This completes the proof. \( \square \)

**Theorem 3.** Let conditions H1–H8 and H10 hold. Then Eq. (1) has a nonoscillatory solution having the property (P2) if

\[
\int_0^\infty |f(s, cR(s), cR(\Lambda^*(s)))| \, ds < +\infty \tag{14}
\]

for some nonzero constant \( c \) and

\[
\int_0^\infty \left( \frac{1}{r(t)} \int_t^\infty |f(s, d, d)| \, ds \right)^{1/\alpha} \, dt = +\infty \tag{15}
\]

for each nonzero constant \( d \) for which \( cd > 0 \).

**Proof.** Without loss of generality, we suppose that \( c > 0 \) in (14). Let \( m \in (0, c) \) and \( T \geq 0 \) by such that

\[
\Theta^2 \int_T^\infty f(s, cR(s), cR(\Lambda^*(s))) \, ds \leq m^2
\]

and \( R(T) \geq 1 \). Then

\[
m + mR(t, T) \leq cR(t), \quad t \geq T.
\]

Define the set

\[
D = \left\{ x \in C([T_{-1}, +\infty), R) \mid \begin{array}{l}
    x(t) = m, \\
    m \leq x(t) \leq m + mR(t, T), \quad T_{-1} \leq t < T
\end{array} \right\}
\]

\[
m \leq x(t) \leq \frac{c}{2}, \quad T \geq T
\]

\[
\int_T^\infty f(s, cR(s), cR(\Lambda^*(s))) \, ds \leq m^2
\]

and \( R(T) \geq 1 \). Then

\[
m + mR(t, T) \leq cR(t), \quad t \geq T.
\]

Define the set

\[
D = \left\{ x \in C([T_{-1}, +\infty), R) \mid \begin{array}{l}
    x(t) = m, \\
    m \leq x(t) \leq m + mR(t, T), \quad T_{-1} \leq t < T
\end{array} \right\}
\]

\[
m \leq x(t) \leq \frac{c}{2}, \quad T \geq T
\]

\[
\int_T^\infty f(s, cR(s), cR(\Lambda^*(s))) \, ds \leq m^2
\]

and \( R(T) \geq 1 \). Then

\[
m + mR(t, T) \leq cR(t), \quad t \geq T.
\]
and the operator $S : D \rightarrow C([-1, +\infty), R)$ by the formula

$$Sx(t) = \begin{cases} m, & T_{-1} \leq t < T \\ m + \int_{T}^{t} \left( \frac{1}{r(s)} \int_{s}^{\infty} f(u, x(u), x(A(u, x(u)))) \, du \right)^{1/2} \, ds, & t \geq T. \end{cases}$$

The Schauder–Tychonoff fixed point theorem ensures the existence of a function $x \in D$ such that $x = Sx$, $t \geq T_{-1}$. It can be verified immediately that $x(t)$ is a solution of Eq. (1) and $\lim_{t \to +\infty} x(t)/R(t) = 0$. On the other hand, $x(t) \geq m + \int_{T}^{t} \left( \frac{1}{r(s)} \int_{s}^{\infty} f(u, m, m) \, du \right)^{1/2} \, ds$ implies in view of condition (15) that $\lim_{t \to +\infty} x(t) = +\infty$. Therefore $x(t)$ has property (P2). The proof is completed. □

4. Some remarks and examples

First we give some remarks.

**Remark 1.** If $f(t, u, v)$ satisfies conditions H4 and H5 and is nondecreasing in $u$ and $v$ for each fixed $t \geq 0$, then it satisfies both conditions H9 and H10.

**Remark 2.** Suppose that $f(t, u, v) = q(t)g(u)h(v)$ where $q : [0, \infty) \to (0, +\infty)$ and $g, h : R \to R$ are continuous and $\text{sgn } g(u) = \text{sgn } u$, $\text{sgn } h(v) = \text{sgn } v$. Then $f(t, u, v)$ clearly satisfies condition H9. If in addition $g$ and $h$ have the properties that

$$k_{1}|g(u)g(v)| \leq |g(uv)| \leq K_{1}|g(u)g(v)|, \quad uv \geq 0$$

and

$$k_{2}|h(u)h(v)| \leq |h(uv)| \leq K_{2}|h(u)h(v)|, \quad uv \geq 0$$

for some positive constants $k_{i}$ and $K_{i}$, $i = 1, 2$, it is not difficult to show that $f(t, u, v)$ satisfies condition H10.

**Remark 3.** In the case when $x = 1$ and $f(t, u, v)$ is nondecreasing in $u$ and $v$ for each fixed $t \geq 0$. Theorems 1–3 are reduced to Theorems 5–7 given in paper [3]. Hence, our results improve and generalize the main results of [3] (see the next example).

**Remark 4.** In the case when $r(t) \equiv 1$, $x = 1$ and $f(t, u, v) = f(t, u)$, our results which can be applied to Eq. (2) are the same ones given in [11].

Next we give an example to illustrate the results. For convenience, we only consider the ordinary differential equation.
Example 1. Consider the equation:

\[
(|x'|^{x-1} x')' + \frac{t^u|x|^{n-1}x}{1 + t^u|x|^m} = 0, \quad t \geq 1.
\]

where \(m > 0, n > 0, u > 0, \) and \(v > 0\) are constants. The function

\[ f(t, x) = \frac{t^u|x|^{n-1}x}{1 + t^u|x|^m} \]

satisfies both H9 and H10, since \(0 < l \leq x \leq L\) implies

\[
\begin{align*}
  f(t, l) &\leq f(t, x) \leq f(t, L) \quad \text{for } n \geq m, \\
  f(t, lt) &\leq f(t, xt) \leq f(t, Lt) \quad \text{for } n \geq m, \\
  (\frac{l}{L})^m &f(t, l) \leq f(t, x) \leq (\frac{L}{l})^m f(l, L) \quad \text{for } m > n, \\
  (\frac{l}{L})^m &f(t, lt) \leq f(t, xt) \leq (\frac{L}{l})^m f(l, Lt) \quad \text{for } m > n.
\end{align*}
\]

It is easy to prove that if \(u > z + v + 1\) then (12) holds and if \(u + m > v + n + 1\) then (7) and (8) hold. Therefore, Theorems 1 and 2 show that necessary and sufficient conditions for (16) to have nonoscillatory solutions \(x(t)\) satisfying \(\lim_{t \to +\infty} x(t) = \text{const} \neq 0\) and \(\lim_{t \to +\infty} |\frac{x(t)}{t}| = \text{const} \neq 0\) are, respectively,

\[ u > z + v + 1 \quad \text{and} \quad u + m > v + n + 1. \]

If we take \(z = 1, u = 4, v = 1, m = 2, n = 1\), then from the above assertion we know that the following equation:

\[
x''(t) + \frac{tx}{1 + t^4x^2} = 0, \quad t \geq 1
\]

has nonoscillatory solutions \(x(t)\) satisfying \(\lim_{t \to +\infty} x(t) = \text{const} \neq 0\) and \(\lim_{t \to +\infty} |\frac{x(t)}{t}| = \text{const} \neq 0.\) But the paper [3] fails to Eq. (17), because the function \(f(t, x) = tx/(1 + t^4x^2)\) does not satisfy the condition that \(f(t, x)\) is nondecreasing in \(x\).

References

[4] M. Del Pino, M. Elgueta, R. Manasevich, Generalizing Hartman’s oscillation result for \((|x'|^{p-2}x')' + c(t)|x|^{p-2}x = 0, \quad p > 1, \) Houston J. Math. 17 (1991) 63–70.


