Duality and saddle-point type optimality for generalized nonlinear fractional programming

X.M. Yang, X.Q. Yang, and K.L. Teo

Abstract

In this paper, we establish two theorems of alternative with generalized subconvexlikeness. We introduce two dual models for a generalized fractional programming problem. Theorems of alternative are then applied to establish duality theorems and a saddle-point type optimality condition.

Keywords: Generalized fractional programming; Generalized convexity; Theorems of alternative; Duality models; Duality theorems; Saddle-point optimality conditions

1. Introduction

Consider the following generalized fractional programming problem:

\[ \bar{\theta} = \inf_{x \in K} \left\{ \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)} \mid h_j(x) \leq 0, \ j = 1, 2, \ldots, m \right\}, \]

where \( K_0 = \{ x \mid x \in K, \ h_j(x) \leq 0, \ j = 1, 2, \ldots, m \} \), \( K \) is a subset of \( \mathbb{R}^n \), \( f_i, g_i \ (1 \leq i \leq p) \) and \( h_j, j = 1, 2, \ldots, m \), are real valued functions defined on \( K \), and the functions \( g_i \) are positive on \( K \). Furthermore, the feasible set of (P) is assumed to be nonempty, so we have \( \bar{\theta} < \infty \).

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* Corresponding author.

E-mail address: maxmyang@polyu.edu.hk (X.M. Yang).

1 Current address: Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong.
In [1], Crouzeix et al. obtained duality results for the linear case of problem (P), that is, $f_i$, $g_i$, and $h_j$ are linear, and $K$ is the nonnegative orthant, with the aid of an associated parametric problem. Almost at the same time, Jagannathan and Schaible [2] developed a duality result for (P) using a Farkas’ lemma, in both linear and nonlinear cases and under different assumptions. Later, Xu [3] presented two duality models for a generalized fractional programming problem and established duality theorems for (P), where $f_i$, $g_i$, and $h_j$ are convex functions. For the same problem as in [3], Xu [4] discussed also saddle point type optimality criteria for (P) where the convexity of functions involved is assumed. Recently, Chandra and Kumar [5] considered different Lagrangian functions and established their saddle point type optimality criteria.

In 1992, Yang presented generalized subconvexlike functions and established the first basic theorem of alternative on generalized subconvexlike functions (see [6]). Recently, people think that generalized subconvexlike function is a class of important generalized convexity and basic theorem of alternative on generalized subconvexlike function are very useful in optimization. Therefore, many papers have appeared on generalized subconvexlike functions and their applications to optimization (see [10–14]).

In this paper, first we will present two new theorems of alternative under generalized convexity. Next we introduce two duality models which are the modification of models for a generalized fractional programming problem in [3], and prove duality theorems under generalized convexity assumptions using alternative theorem. Finally, we also obtain a saddle point type optimality condition using generalized convexity.

2. Theorems of alternative

It is well known that theorems of alternative are very important results in optimization problems and that many results can be derived from theorems of alternative. In this section, we present two theorems of alternative with generalized subconvexlikeness.

In what follows, $X$ and $Y$ are real normed spaces, $Y^*$ is the dual space of $Y$, $\Gamma$ is an arbitrary nonempty set in $X$, $S \subset Y$ is a convex cone with nonempty interior and $\Gamma \subset X$ be a nonempty set. If the function $f : \Gamma \to Y$ is generalized subconvexlike, then exactly one of the following systems is consistent:

**Definition 2.1.** The function $f$ is said to be generalized subconvexlike with respect to $S$ (see [6]) if there exists $\rho \in \text{int} S$ such that for any $x, y \in \Gamma$, any $\alpha \in (0, 1)$, any $\epsilon > 0$, there exist $z \in \Gamma$ and $k > 0$ satisfying

$$
\epsilon \rho + \alpha f(x) + (1 - \alpha) f(y) - kf(z) \in S.
$$

The function $f$ is said to be subconvexlike with respect to $S$ (see [7]) if there exists $\rho \in \text{int} S$, such that for any $x, y \in \Gamma$, any $\alpha \in (0, 1)$, any $\epsilon > 0$, there exist $z \in \Gamma$ satisfying

$$
\epsilon \rho + \alpha f(x) + (1 - \alpha) f(y) + f(z) \in S.
$$

**Theorem 2.1** (Basic theorem of alternative [6]). Let $X$ and $Y$ be real normed spaces, and $S \subset Y$ be a convex cone with nonempty interior and $\Gamma \subset X$ be a nonempty set. If the function $f : \Gamma \to Y$ is generalized subconvexlike, then exactly one of the following systems is consistent:
Theorem 2.2. Let $X$ be a real normed space, $\Gamma \subset X$ be an arbitrary nonempty compact set and $f = (f_1, f_2, \ldots, f_n): \Gamma \to \mathbb{R}^n$. Assume that, for any $\epsilon \in \mathbb{R}_{++}^n$, $f - \epsilon$ is generalized subconvexlike on $\Gamma$ (with respect to $\mathbb{R}_{++}^n$) and $f_1, f_2, \ldots, f_n$ are real valued lower semicontinuous functions defined on $\Gamma$. Then $f(x) \leq 0$ is inconsistent on $\Gamma$ if and only if there exists $\lambda \in \mathbb{R}_{++}^n$ such that
\[
\lambda^T f(x) > 0, \quad \forall x \in \Gamma. \tag{2.1}
\]

Proof. We need only prove necessity. Assume that $f(x) \leq 0$ is inconsistent on $\Gamma$. Then there exists $\epsilon \in \mathbb{R}_{++}^n$ such that
\[
f(x) < \epsilon \tag{2.2}
\]
is inconsistent on $\Gamma$. By contradiction, for any $k \in \mathbb{N}$, $\epsilon = (1/k, 1/k, \ldots, 1/k) \in \mathbb{R}_{++}^n$, $f(x) < \epsilon$ is consistent on $\Gamma$. That is, there exist $\{x_k\} \subset \Gamma$ such that
\[
f_i(x_k) < 1/k, \quad i = 1, 2, \ldots, n, \quad \forall k \in \mathbb{N}.
\]
As $\Gamma$ is a compact set, $\{x_k\}$ has a convergent subsequence $\{x_{kj}\}$. Assume $x_{kj} \to \bar{x} \in \Gamma$. It follows from the lower semicontinuity of $f$ that
\[
f_i(\bar{x}) \leq \lim_{j \to \infty} \inf_{i \in \mathbb{N}} f_i(x_{kj}) \leq 0, \quad i = 1, 2, \ldots, n,
\]
which contradicts the fact that $f(x) \leq 0$ is inconsistent on $\Gamma$. Since $f - \epsilon$ is generalized subconvexlike, from Theorem 2.1 and (2.2), there exists $p \in \mathbb{R}_{++}^n \setminus \{0\}$ such that
\[
p(f(x) - \epsilon) \geq 0, \quad \forall x \in \Gamma,
\]
i.e.,
\[
p(f(x)) \geq p^T \epsilon > 0, \quad \forall x \in \Gamma. \tag{2.3}
\]
If $p \in \mathbb{R}_{++}^n$, then by letting $\lambda = p$, inequality (2.3) implies inequality (2.1) holds.
If $p \in \mathbb{R}_{++}^n \setminus \{0\}$ and $p \notin \mathbb{R}_{++}^n$, then we assume without loss of generality that $p_1 > 0$, $\ldots$, $p_k > 0$, $p_{k+1} = \cdots = p_n = 0$. From inequality (2.3), we have
\[
\sum_{i=1}^{k} p_i f_i(x) > 0, \quad \forall x \in \Gamma.
\]
Since $f_i$ are lower semicontinuous functions on $\Gamma$ ($1 \leq i \leq n$), it follows that $g(x) = \sum_{i=1}^{k} p_i f_i(x)$ and $h(x) = \sum_{j=k+1}^{n} p_j f_j(x)$ are lower semicontinuous functions on $\Gamma$. As $\Gamma$ is a nonempty compact set, we know that $g(x)$ and $h(x)$ have extreme minimum on $\Gamma$. Let $\alpha$ and $\beta$ be the minimum values of $g$ and $h$, respectively. Let
\[
\tilde{\alpha} = \frac{\alpha}{\beta}, \quad \tilde{\beta} = |\beta| + 1,
\]
Then,
\[ \sum_{i=1}^{k} p_i f_i(x) + \bar{\alpha} \sum_{i=k+1}^{n} f_i(x) \geq \alpha + \bar{\alpha} \beta > \alpha - \bar{\alpha} \bar{\beta} = 0, \quad \forall x \in \Gamma. \]  

(2.4)

Let

\[ \lambda = (p_1, p_2, \ldots, p_k, \bar{\alpha}, \ldots, \bar{\alpha}) \in \mathbb{R}^n_{++}. \]

Because of (2.4), this implies that

\[ \lambda^T f(x) > 0, \quad \forall x \in \Gamma. \]  

Corollary 2.1. Let \( X \) be a real normed space, \( \Gamma \subset X \) be an arbitrary nonempty compact set and \( f = (f_1, f_2, \ldots, f_n) : \Gamma \rightarrow \mathbb{R}^n \). Assume that \( f \) is a subconvexlike function on \( \Gamma \) (with respect to \( \mathbb{R}^n_+ \)) and \( f_1, f_2, \ldots, f_n \) are real valued lower semicontinuous functions defined on \( \Gamma \). Then \( f(x) \leq 0 \) is inconsistent on \( \Gamma \) if and only if there exists \( \lambda \in \mathbb{R}^n_{++} \) such that

\[ \lambda^T f(x) > 0, \quad \forall x \in \Gamma. \]

Proof. Since the subconvexlikeness of \( f \) on \( \Gamma \) (with respect to \( \mathbb{R}^n_+ \)) implies that for any \( \epsilon \in \mathbb{R}^n_{++} \), \( f - \epsilon \) is subconvexlike on \( \Gamma \) (with respect to \( \mathbb{R}^n_+ \)) and subconvexlikeness implies generalized subconvexlikeness, so Corollary 2.1 holds from Theorem 2.2.

Remark 2.1. Theorem 2.2 and Corollary 2.1 improve and extend Theorem 3.1 in [8]. It is worth observing that the set \( \Gamma \) does not require any convexity, \( \lambda \in \mathbb{R}^n_{++} \) in Fan’s theorem and convexity is generalized to generalized subconvexlikeness.

Remark 2.2. In [9], Fan et al. put forward the following conjecture: If \( f_i \) \((i = 1, 2, \ldots, m)\) are convex functions defined on a convex set \( \Gamma \), then that the system \( f_i(x) \leq 0 \) \((i = 1, 2, \ldots, m)\) is inconsistent on \( \Gamma \) implies that there are nonnegative numbers \( \lambda_i \) \((i = 1, 2, \ldots, m)\) such that \( \sum_{i=1}^{m} \lambda_i f_i(x) > 0 \) \forall \( x \in \Gamma \). He also showed that the conjecture is false by the following example.

Let \( \mathbb{R}^2_+ \) be the Euclidean plane, where points are denoted by \( x = (\xi_1, \xi_2) \). Let \( \Gamma \) be the convex set which is the union of the open half-plane \( \xi_2 > 0 \) and the half-line \( \xi_1 > 0 \) and \( \xi_2 = 0 \). Let \( f_1, f_2 \) be defined on \( \Gamma \) by \( f_1(x) = \xi_1 \) and \( f_2(x) = \xi_2 \). Then the system \( f_i(x) \leq 0 \) \((i = 1, 2)\) is inconsistent on \( \Gamma \), but no pair of nonnegative numbers \( \lambda_1, \lambda_2 \) can satisfy \( \lambda_1 \xi_1 + \lambda_2 \xi_2 > 0 \) for all \( (\xi_1, \xi_2) \in \Gamma \).

By adding the lower semicontinuity of \( f_i \), Theorem 2.1 and Corollary 2.1 show that if the system \( f_i(x) \leq 0 \) \((i = 1, 2, \ldots, m)\) is inconsistent on \( \Gamma \), then, under generalized subconvexlikeness, there exists positive numbers \( \lambda_i \) \((i = 1, 2, \ldots, m)\) such that \( \sum_{i=1}^{m} \lambda_i f_i(x) > 0 \) for all \( x \in \Gamma \).

3. Duality of generalized fractional programming

In [3], Xu present two duality models for a generalized fractional programming problem \( P \) and discussed duality theorems. Now we introduce two new duality models which are modification of Xu’s models.
We define

\[ F(x) = \left( f_1(x), \ldots, f_p(x) \right)^T, \quad G(x) = \left( g_1(x), \ldots, g_p(x) \right)^T, \]

\[ h(x) = \left( h_1(x), \ldots, h_m(x) \right)^T. \]

For \( x \in K \), \( u \in \mathbb{R}^p \) with \( u > 0 \), and \( v \in \mathbb{R}^m \) with \( v > 0 \), we denote

\[ GL(x, u, v) = \frac{u^T F(x) + v^T h(x)}{u^T G(x)}, \]

\[ GK(x, v) = \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)} + \sum_{j=1}^{m} \max_{1 \leq i \leq p} \frac{v_j h_j(x)}{g_j(x)}. \]

Then we define two duals of problem (P),

\[ \sup_{u > 0, v > 0} \inf_{x \in K} GL(x, u, v), \quad (D_1) \]

\[ \sup_{v > 0} \inf_{x \in K} GK(x, v), \quad (D_2) \]

Let \( v(D_i) \) denote the optimal value of \((D_i)\), \( i = 1, 2 \). Now we can prove duality theorems between (P) and \((D_1)\) or (P) and \((D_2)\).

We can easily prove that following weak duality results between (P) and \((D_1)\) or (P) and \((D_2)\).

**Theorem 3.1 (Weak duality).** Let \( x \) be a feasible point of (P). Then, for any \( u \in \mathbb{R}^p \) with \( u > 0 \) and \( v \in \mathbb{R}^m \) with \( v > 0 \), we have

\[ v(D_1) = \sup_{u > 0, v > 0} \inf_{x \in K} GL(x, u, v) \leq \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)}, \]

\[ v(D_2) = \sup_{v > 0} \inf_{x \in K} GK(x, v) \leq \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)}. \]

**Theorem 3.2.** Suppose that \( K \) is compact, and that for any \( \epsilon \in \mathbb{R}^p \) with \( \epsilon > 0 \), \( (F(x) - \tilde{\theta} G(x) + \epsilon e, h(x)) \) are lower semicontinuous and generalized subconvexlike functions on \( K \) with respect to \( \mathbb{R}_+^{p+1} \), \( G(x) \) is lower semicontinuous function on \( K \). Then \( v(D_1) = \delta \).

**Proof.** If \( \delta = -\infty \), then \((D_1) = -\infty \) because of Theorem 3.1. So we focus on the case when \( \delta > -\infty \).

From the definition of \( \delta \), we have

\[ \delta \leq \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)} \quad \text{for all } x \in K_0. \]

That is, the system

\[ (F(x) - \tilde{\delta} G(x)) < 0, \quad h(x) \leq 0, \quad x \in K, \]

has no solution.

Thus, for any \( \epsilon \in \mathbb{R}^p \) with \( \epsilon > 0 \), it follows that

\[ (F(x) - \tilde{\delta} G(x)) + \epsilon e \leq 0, \quad h(x) \leq 0, \quad x \in K, \]

has no solution,
where $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^p$. From the conditions of Theorem 2.2, we see that, for any fixed $\epsilon > 0$, there exist $u \in \mathbb{R}^p$ with $u > 0$, and $v \in \mathbb{R}^m$ with $v > 0$, such that

$$u^T (F(x) - \bar{\theta} G(x) + \epsilon e) + v^T h(x) > 0 \quad \text{for all } x \in K.$$  

Without loss of generality, we assume $u^T e = 1$. By the lower semicontinuity of $G(x)$, we know that for above $u \in \mathbb{R}^p$ with $u > 0$, $u^T G(x)$ is a lower semicontinuous function on $K$. Again, by the facts that $K$ is compact set and the functions $g_i(x)$ ($1 \leq i \leq p$) are positive on $K$, we see that, there exists $\bar{\alpha} > 0$ such that

$$u^T F(x) + v^T h(x) > \bar{\theta} - \epsilon \bar{\alpha} \quad \text{for all } x \in K.$$  

Hence,

$$v(D_1) = \sup_{u > 0, u^T e = 1} \inf_{v > 0} \frac{u^T F(x) + v^T h(x)}{u^T G(x)} > \bar{\theta} - \epsilon \bar{\alpha}.$$  

By $\epsilon > 0$ may any sufficient small, we have

$$v(D_1) \geq \bar{\theta}. \quad (3.1)$$

The combination of (3.1) and the weak duality Theorem 3.1 completes our proof. □

**Theorem 3.3.** Suppose that $K$ is compact, and that for any $\epsilon > 0$ ($\max_{1 \leq i \leq p} (f_i(x) - \bar{\theta} g_i(x)) + \epsilon, h(x)$) are lower semicontinuous generalized subconvexlike functions on $K$, $G(x)$ is lower semicontinuous function on $K$. Then $v(D_2) = \bar{\theta}$.

**Proof.** If $\bar{\theta} = -\infty$, then $\min(D_2) = -\infty$ because of Theorem 3.1. So we focus on the case when $\bar{\theta} > -\infty$.

From definition of $\bar{\theta}$, we have

$$\bar{\theta} \leq \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)} \quad \text{for all } x \in X.$$  

That is, the system

$$\max_{1 \leq i \leq p} (f_i(x) - \bar{\theta} g_i(x)) < 0, \quad h(x) \leq 0, \quad x \in K, \quad \text{has no solution},$$  

or the system

$$\max_{1 \leq i \leq p} (f_i(x) - \bar{\theta} g_i(x)) + \epsilon \leq 0, \quad h(x) \leq 0, \quad x \in K, \quad \text{has no solution}. \quad (3.2)$$

From assumption conditions of the theorem and Theorem 2.2, we see that, for any fixed $\epsilon > 0$, there exist $u \in \mathbb{R}^1$ with $u > 0$, and $v \in \mathbb{R}^m$ with $v > 0$, such that

$$u \left( \max_{1 \leq i \leq p} (f_i(x) - \bar{\theta} g_i(x)) + \epsilon \right) + v^T h(x) > 0 \quad \text{for all } x \in K.$$  

Without loss of generality, we assume $u = 1$,

$$\max_{1 \leq i \leq p} (f_i(x) - \bar{\theta} g_i(x)) + \epsilon + v^T h(x) > 0 \quad \text{for all } x \in K. \quad (3.3)$$
Now, for a fixed $x \in K$, there exists $s \in \{1, 2, \ldots, p\}$ such that
\[
\max_{1 \leq i \leq p} \left( f_i(x) - \bar{\theta} g_i(x) \right) = f_s(x) - \bar{\theta} g_s(x).
\] (3.4)

By lower semicontinuity of $G(x)$ and $K$ is compact set, and the functions $g_i(x)$ ($1 \leq i \leq p$) are positive on $K$, we see that
\[
\max_{1 \leq i \leq p} \frac{1}{g_i(x)} > 0 \quad \text{for all } x \in X.
\]

Then for this $x \in K$, by (3.3), (3.4) and above inequality, we have
\[
0 < \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)} - \bar{\theta} + \frac{\epsilon}{g_i(x)} + \frac{v^T h(x)}{g_i(x)} = \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)} - \bar{\theta} + \frac{\epsilon}{g_i(x)} + \sum_{j=1}^{m} v_j \max_{1 \leq i \leq p} \frac{h_j(x)}{g_i(x)} = GK(x, v) - \bar{\theta} + \frac{\epsilon}{g_i(x)}.
\]

By $\epsilon > 0$ may any sufficient small, we have
\[
v(D_2) \geq \bar{\theta}. \quad (3.5)
\]

The combination of (3.5) and the weak duality Theorem 3.1 completes our proof. \qed

**Remark 3.1.** In [3], Xu defined $(D_1)$ and $(D_2)$ for $u \in \mathbb{R}^p$ with $u \geq 0$, $\|u\| = 1$, and $v \in \mathbb{R}^m$ with $v \geq 0$. In this paper, we define $(D_1)$ and $(D_2)$ for $u \in \mathbb{R}^p$ with $u > 0$, and $v \in \mathbb{R}^m$ with $v > 0$. And we prove weak and strong duality theorems under generalized subconvexlikeness conditions. Therefore, under weaker conditions, we give stronger results than Xu’s.

4. Saddle-point type optimality criteria of generalized fractional programming

Xu gave a saddle-point type optimality criterion for (P) in [4]. Recently, Chandra and Kumar in [5] considered the Lagrangian function
\[
GL(x, u, y) = \frac{u^T F(x) + v^T h(x)}{u^T G(x)}, \quad \forall x \in X, \; u \in \Lambda, \; v \in \mathbb{R}^m_+,
\]
where $\Lambda = \{u \in \mathbb{R}^n: u_i \geq 0, \; \sum_{i=1}^n u_i = 1\}$. And they also obtained another type of saddle-point optimality criteria under the convexity. In this section, we will relax the convexity conditions in [5] to the generalized subconvexlikeness.

**Definition 4.1.** If there exist $\bar{x} \in X$, $\bar{u} \in \Lambda$, and $\bar{v} \in \mathbb{R}^m_+$ such that
\[
GL(\bar{x}, \bar{u}, \bar{v}) \leq GL(\bar{x}, u, v) \leq GL(x, \bar{u}, \bar{v}), \quad \forall x \in X, \; u \in \Lambda, \; v \in \mathbb{R}^m_+,
\]
then the point $(\bar{x}, \bar{u}, \bar{v})$ is said to be a $GL$-saddle point of the problem (P).
Lemma 4.1 [3]. Let $\alpha_i, \beta_i, i = 1, 2, \ldots, p$, be real numbers and $\alpha_i > 0, i = 1, 2, \ldots, p$. Then
\[
\sum_{i=1}^{p} \beta_i \leq \max_{1 \leq i \leq p} \frac{\beta_i}{\alpha_i}.
\]

For a given $\bar{x} \in S$, we denote
\[
\bar{\theta} = \max_{1 \leq i \leq p} \frac{f_i(\bar{x})}{g_i(\bar{x})}.
\]

Theorem 4.1. Suppose that $\bar{x}$ is an optimal solution of problem (P), and suppose that $(F(x) - \bar{\theta}G(x), h(x))$ is a generalized subconvexlike function on $K$ (with respect to $\mathbb{R}^{p+m}_+$), also suppose that $h(x)$ satisfies a constraint qualification. Then, there exist $\bar{u} \in \Lambda$, $\bar{v} \in \mathbb{R}^m_+$ such that $(\bar{x}, \bar{u}, \bar{v})$ is a GL-saddle point of (P) and
\[
\sum_{j=1}^{m} \bar{v}_j h_j(\bar{x}) = 0.
\]

Proof. Since $\bar{x}$ is an optimal solution of problem (P),
\[
\bar{\theta} = \max_{1 \leq i \leq p} \frac{f_i(\bar{x})}{g_i(\bar{x})} \leq \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)}, \quad x \in X.
\]

Thus, the system
\[
(F(x) - \bar{\theta}G(x)) < 0, \quad h(x) \leq 0, \quad x \in K,
\]
has no solution.

From generalized subconvexlikeness of $(F(x) - \bar{\theta}G(x), h(x))$, by Theorem 2.1, there exist $\bar{\alpha} \in \mathbb{R}^p_+$, $\bar{r} \in \mathbb{R}^m_+$ such that
\[
\sum_{i=1}^{p} \bar{\alpha}_i (f_i(x) - \bar{\theta} g_i(x)) + \sum_{j=1}^{m} \bar{r}_j h_j(x) \geq 0, \quad \forall x \in K. \tag{4.1}
\]

Since $h(x)$ satisfies constraint qualification, it is easy to prove that $\bar{\alpha} \in \mathbb{R}^p_+ \setminus \{0\}$, i.e., $\sum_{i=1}^{p} \bar{\alpha}_i > 0$.

Let
\[
\bar{u}_i = \frac{\bar{\alpha}_i}{\sum_{i=1}^{p} \bar{\alpha}_i}, \quad \bar{v}_j = \frac{\bar{r}_j}{\sum_{i=1}^{p} \bar{\alpha}_i}.
\]

Equation (4.1) yields
\[
\sum_{i=1}^{p} \bar{u}_i f_i(x) + \sum_{j=1}^{m} \bar{v}_j h_j(x) \geq \bar{\theta} \sum_{i=1}^{p} \bar{u}_i g_i(x), \quad \forall x \in K.
\]

That is,
\[
\frac{\bar{u}^T F(x) + \bar{v}^T h(x)}{\bar{u}^T G(x)} \geq \bar{\theta}, \quad \forall x \in K. \tag{4.2}
\]
By Lemma 4.1, 
\[
\bar{\theta} = \max_{1 \leq i \leq p} \frac{f_i(\bar{x})}{g_i(\bar{x})} \geq \frac{\bar{u}^T F(\bar{x})}{\bar{u}^T G(\bar{x})}.
\]
Thus, (4.2) yields
\[
\bar{v}^T h(\bar{x}) \geq 0. \tag{4.3}
\]
Since \(h_j(\bar{x}) \leq 0, j = 1, 2, \ldots, m\), we have
\[
\bar{v}^T h(\bar{x}) \leq 0. \tag{4.4}
\]
It follows from (4.3) and (4.4) that
\[
\bar{v}^T h(\bar{x}) = 0. \tag{4.5}
\]
Letting \(x = \bar{x}\) in (4.2) and using (4.5), we have
\[
\bar{u}^T F(\bar{x}) \bar{u}^T G(\bar{x}) \geq \bar{\theta}. \tag{4.6}
\]
By Lemma 4.1 and (4.6), we obtain
\[
\frac{\bar{u}^T F(\bar{x})}{\bar{u}^T G(\bar{x})} \geq \max_{1 \leq i \leq p} \frac{f_i(\bar{x})}{g_i(\bar{x})} = \bar{\theta} \geq \frac{u^T F(\bar{x})}{u^T G(\bar{x})}. \tag{4.7}
\]
From \(h_j(\bar{x}) \leq 0, j = 1, 2, \ldots, m\), and \(v_j \geq 0, j = 1, 2, \ldots, m\), we have
\[
v^T h(\bar{x}) \leq 0. \tag{4.8}
\]
Equations (4.7), (4.8) and (4.5) yield
\[
\frac{u^T F(x) + v^T h(x)}{u^T G(x)} \leq \frac{\bar{u}^T F(\bar{x}) + \bar{v}^T h(\bar{x})}{\bar{u}^T G(\bar{x})}. \tag{4.9}
\]
That is,
\[
GL(\bar{x}, u, v) \leq GL(\bar{x}, \bar{u}, \bar{v}), \quad \forall v \in \mathbb{R}^m_+. \tag{4.10}
\]
On the other hand, from Lemma 4.1 and (4.5), we have
\[
\frac{\bar{u}^T F(\bar{x}) + \bar{v}^T h(\bar{x})}{\bar{u}^T G(\bar{x})} \leq \bar{\theta}, \quad \forall x \in K. \tag{4.11}
\]
From (4.2) and (4.10), we get
\[
\frac{\bar{u}^T F(\bar{x}) + \bar{v}^T h(\bar{x})}{\bar{u}^T G(\bar{x})} \leq \frac{u^T F(x) + v^T h(x)}{u^T G(x)}, \quad \forall x \in K.
\]
That is,
\[
GL(\bar{x}, \bar{u}, \bar{v}) \leq GL(x, u, v), \quad \forall x \in K. \tag{4.12}
\]
Combining (4.9) and (4.11), it follows that \((\bar{x}, \bar{u}, \bar{v})\) is a GL-saddle point of problem (P).
References