Asymptotic method for Dougall’s bilateral hypergeometric sums

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Abstract

By means of the modified Abel lemma on summation by parts, a recurrence relation for Dougall’s bilateral $2\,H_2$-series is established with an extra natural number parameter $m$. Then the steepest descent method allows us to compute the limit for $m \to \infty$, which leads us surprisingly to a completely new proof of the celebrated bilateral $2\,H_2$-series identity due to Dougall (1907). The same approach applies also to the bilateral very well-poised $5\,H_5$-series identity [J. Dougall, On Vandermonde’s theorem and some more general expansions, Proc. Edinburgh Math. Soc. 25 (1907) 114–132].

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For a complex number $x$ and an integer $n$, the shifted factorial is defined through $\Gamma$-function quotient:

$$(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)} \quad \text{where} \quad \Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \, du \quad \text{with} \quad \Re(x) > 0.$$ 

It reads explicitly as

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = \begin{cases} \prod_{k=0}^{n-1} (x + k), & n = 1, 2, \ldots; \\ \frac{(-1)^n}{\prod_{k=1}^{n} (k - x)}, & n = -1, -2, \ldots; \end{cases}$$

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where we suppose that \( x \) is not an integer for the last line. The product and fractional forms of the shifted factorial and the \( \Gamma \)-function are abbreviated respectively as

\[
[A, B, \ldots, C]_n = (A)_n(B)_n \ldots (C)_n, \quad \Gamma[A, B, \ldots, C] = \Gamma(A) \Gamma(B) \ldots \Gamma(C);
\]

\[
\left[ \frac{\alpha, \beta, \ldots, \gamma}{A, B, \ldots, C} \right]_n = \left( \frac{\alpha}_n \frac{\beta}_n \ldots \frac{\gamma}_n \right)_{\frac{A}{n}} \frac{(A)_{\frac{1}{n}}}{\Gamma_{\frac{1}{n}}(A)} \frac{(B)_{\frac{1}{n}}}{\Gamma_{\frac{1}{n}}(B)} \ldots \frac{(C)_{\frac{1}{n}}}{\Gamma_{\frac{1}{n}}(C)}.
\]

Then the celebrated bilateral series identity due to Dougall (1907, [5]) can be reproduced as follows.

**Theorem 1.** (Dougall, 1907, [5]) Let \( a, b, c, d \) be four complex numbers with none of them being integer. Under condition \( \Re (c + d - a - b) > 1 \), there holds the summation formula:

\[
2 \mathcal{H}_2^{2} \left[ \begin{array}{c} a, b \\ c, d \end{array} \right] = \sum_{k=-\infty}^{+\infty} \frac{(a)_k(b)_k}{(c)_k(d)_k} = \Gamma \left[ \begin{array}{c} 1 - a, 1 - b, c, d \\ c - a, d - a, c - b, d - b \end{array} \right].
\]

The original proof due to Dougall has been standard. It consists of computing the contour integral of the complex function

\[ f(z) = \cot(\pi z) \Gamma \left[ \begin{array}{c} a + z, b + z \\ c + z, d + z \end{array} \right] \]

through the Cauchy residue theorem (see Slater [7, §6.1] and [6, §5.3] also). Another proof applies Carlson’s theorem on regular functions (cf. [1, §2.8]) to the Gauss summation formula (cf. Bailey [2, §1.3]):

\[
\mathcal{F}^{2} \left[ \begin{array}{c} a, b \\ c \end{array} \right] = \sum_{k=0}^{+\infty} \frac{(a)_k(b)_k}{k!(c)_k} = \Gamma \left[ \begin{array}{c} c, c - a - b \\ c - a, c - b \end{array} \right]
\]

where \( \Re (c - a - b) > 0 \)

which is obviously the special case \( d = 1 \) of Theorem 1. There is also a proof through transformation from bilateral series to unilateral series, which can be found in Chu [3].

By means of the modified Abel lemma on summation by parts, a recurrence relation for Dougall’s bilateral \( 2 \mathcal{H}_2 \)-series is established with an extra natural number parameter \( m \). Then the steepest descent method allows us to compute the limit for \( m \to \infty \), which leads us surprisingly to a completely new proof of the celebrated bilateral \( 2 \mathcal{H}_2 \)-series identity due to Dougall (1907). Following the same approach, we shall also prove the very well-poised bilateral \( 5 \mathcal{H}_5 \)-series identity (Dougall, 1907, [5]). Both examples illustrate new connections between hypergeometric series and classical asymptotic analysis.

1. **The Abel method on summation by parts and recurrence relation**

For an arbitrary complex sequence \( \{ \tau_k \} \), define the backward and forward difference operators \( \Delta \) and \( \nabla \), respectively, by

\[
\nabla \tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \Delta \tau_k = \tau_k - \tau_{k+1}.
\]

It should be pointed out that \( \Delta \) is adopted for convenience in the present paper, which differs from the usual operator \( \Delta \) only in the minus sign.
Then Abel’s lemma on summation by parts may be reformulated as
\[ +\infty \sum_{k=\infty} A_k \nabla B_k = +\infty \sum_{k=\infty} B_k \triangle A_k \] (1)
provided that both series are convergent and \( A_k B_{k-1} \to 0 \) as \( k \to \pm \infty \).

**Proof.** Let \( m \) and \( n \) be two integers. According to the definition of the backward difference, we have
\[ \sum_{k=m}^{n} A_k \nabla B_k = \sum_{k=m}^{n} A_k (B_k - B_{k-1}) = \sum_{k=m}^{n} A_k B_k - \sum_{k=m}^{n} A_k B_{k-1}. \]
Replacing \( k \) by \( k + 1 \) for the last sum, we get the following expression:
\[ \sum_{k=m}^{n} A_k \nabla B_k = A_{n+1} B_n - A_m B_{m-1} + \sum_{k=m}^{n} B_k (A_k - A_{k+1}) = A_{n+1} B_n - A_m B_{m-1} + \sum_{k=m}^{n} B_k \triangle A_k. \]
Letting \( m \to -\infty \) and \( n \to +\infty \), we get the identity stated in the lemma. \( \square \)

For two sequences \( A_k \) and \( B_k \) defined by
\[ A_k = \left[ \begin{array}{c} a, b \\ c, a + b - c \end{array} \right] \quad \text{and} \quad B_k = \frac{(1 + a + b - c)_k}{(d)_k} \]
it is almost trivial to compute the differences
\[ \triangle A_k = \frac{(c-a)(c-b)}{c(c-a-b)} \left[ \begin{array}{c} a, b \\ 1+c, 1+a+b-c \end{array} \right]_k, \]
\[ \nabla B_k = \frac{c+d-a-b-1}{c-a-b} \frac{(a+b-c)_k}{(d)_k}. \]
By means of the modified Abel lemma on summation by parts, we can manipulate the bilateral Dougall sum as follows:
\[ 2H_2 \left[ \begin{array}{c} a, b \\ c, d \end{array} \right] = \sum_{k=-\infty}^{+\infty} \left[ \begin{array}{c} a, b \\ c, a + b - c \end{array} \right]_k \frac{(a+b-c)_k}{(d)_k} \]
\[ = \frac{c-a-b}{c+d-a-b-1} \sum_{k=-\infty}^{+\infty} A_k \nabla B_k = \frac{c-a-b}{c+d-a-b-1} \sum_{k=-\infty}^{+\infty} B_k \triangle A_k \]
\[ = \frac{(c-a)(c-b)}{c(c+d-a-b-1)} \sum_{k=-\infty}^{+\infty} \frac{(1+a+b-c)_k}{(d)_k} \left[ \begin{array}{c} a, b \\ 1+c, 1+a+b-c \end{array} \right]_k. \]
Canceling the common factors, we have established the following relation:
\[ 2H_2 \left[ \begin{array}{c} a, b \\ c, d \end{array} \right] = 2H_2 \left[ \begin{array}{c} a, b \\ 1+c, d \end{array} \right] \frac{(c-a)(c-b)}{c(c+d-a-b-1)}. \] (2)
For the left member of the last equation is symmetric with respect to \(c\) and \(d\), we have another relation
\[
2H_2 \left[ \frac{a, b}{c, d} \right] = 2H_2 \left[ \frac{a, b}{c, 1 + d} \right] \frac{(d - a)(d - b)}{d(c + d - a - b - 1)}.
\]
(3)
The composition of the last relations, i.e., the application of (3) to the right-hand side of (2), yields the third relation
\[
2H_2 \left[ \frac{a, b}{c, d} \right] = 2H_2 \left[ \frac{a, b}{c + 1, d + m} \right] \frac{(c - a)(c - b)(d - a)(d - b)}{cd(c + d - a - b - 1)^2}.
\]
(4)
Iterating the last recursion for \(m\)-times, we find the following elegant relation.

**Lemma 2** (Recurrence relation for Dougall’s bilateral \(2H_2\)-series).
\[
2H_2 \left[ \frac{a, b}{c, d} \right] = 2H_2 \left[ \frac{a, b}{c + m, d + m} \right] \frac{[c - a, c - b, d - a, d - b]_m}{[c + d - a - b - 1]_2m}.
\]
(5)
We remark that all the bilateral \(2H_2\)-series involved in the recurrence relations displayed in this section are convergent, which is guaranteed by \(\Re(c + d - a - b) > 1\).

2. The saddle point method for bilateral \(2H_2\)-series

Shifting the summation index by \(k \rightarrow k - n\), we have
\[
2H_2 \left[ \frac{a, b}{c + 2n, d + 2n} \right] = \sum_{k=-\infty}^{+\infty} \left[ \frac{a, b}{c + 2n, d + 2n} \right]_{k-n}
\]
\[
= \left[ \frac{a, b}{c + 2n, d + 2n} \right]_{-n} \times \sum_{k=-\infty}^{+\infty} \left[ \frac{a - n, b - n}{c + n, d + n} \right]_{k}
\]
\[
= 2H_2 \left[ \frac{a - n, b - n}{c + n, d + n} \right] \times \left[ \frac{c + n, d + n}{1 - a, 1 - b} \right]_{n}.
\]

Letting \(m = 2n\) in Lemma 2 and then substituting the above displayed relation into the right-hand side of (5), we get the following expression
\[
2H_2 \left[ \frac{a, b}{c, d} \right] = 2H_2 \left[ \frac{a - n, b - n}{c + n, d + n} \right] \times \frac{[c - a, c - b, d - a, d - b]_{2n}}{[1 - a, 1 - b, c, d]_n[c + d - a - b - 1]_{4n}}.
\]
(6)
Recall the asymptotic relation of \(\Gamma\)-function (cf. [1, §1.1])
\[
\Gamma(x + n) \approx (n - 1)!n^x \quad \text{when} \ n \to \infty.
\]
(7)
When \(n \to \infty\), we can evaluate without difficulty the shifted factorial fraction:
\[
\frac{[c - a, c - b, d - a, d - b]_{2n}}{[1 - a, 1 - b, c, d]_n[c + d - a - b - 1]_{4n}} \approx \frac{\sqrt{2\pi n}}{2^n} \frac{[1 - a, 1 - b, c, d]_n[c + d - a - b - 1]}{[c - a, d - a, c - b, d - b]_{2n}}.
\]

We deduce from Theorem 1 that the asymptotic relation stated in the following proposition is valid. Conversely, if we prove independently (without Theorem 1) that (8) is true, then letting
\( n \to \infty \) in (6), we will get the gamma function expression for Dougall’s \( 2H_2 \)-series displayed in Theorem 1.

**Proposition 3** *(Asymptotic relation for Dougall’s bilateral \( 2H_2 \)-series).*

\[
2H_2\left[\begin{array}{c} a - n, b - n \\ c + n, d + n \end{array}\right] \approx \sqrt{\frac{n\pi}{2}} \quad \text{when} \quad n \to \infty.
\]  

(8)

This can be done by means of the saddle point method (or steepest descent method).

For the sake of brevity, we prove the proposition only for the even natural numbers \( n \). It can be done similarly when \( n \to \infty \) along the odd natural numbers. Splitting the summation into three parts, we can rewrite the bilateral series as

\[
2H_2\left[\begin{array}{c} a - 2n, b - 2n \\ c + 2n, d + 2n \end{array}\right] = \sum_{k=-n}^{n} \left[\begin{array}{c} a - 2n, b - 2n \\ c + 2n, d + 2n \end{array}\right]_k 
\]

(9a)

\[
+ \sum_{k=n+1}^{2n-1} \left\{ \left[\begin{array}{c} a - 2n, b - 2n \\ c + 2n, d + 2n \end{array}\right]_k + \left[\begin{array}{c} a - 2n, b - 2n \\ c + 2n, d + 2n \end{array}\right]_{-k} \right\}
\]

(9b)

\[
+ \sum_{k=2n}^{\infty} \left\{ \left[\begin{array}{c} a - 2n, b - 2n \\ c + 2n, d + 2n \end{array}\right]_k + \left[\begin{array}{c} a - 2n, b - 2n \\ c + 2n, d + 2n \end{array}\right]_{-k} \right\}. 
\]

(9c)

For \( n \to \infty \), we are going to show that the truncated sum in (9a) tends to \( \sqrt{n\pi} \), while two other sums displayed respectively in (9b) and (9c) will be annihilated.

2.1. Denote by \( w_k(n) \) the summand \( w_k(n) = \frac{(a - 2n)_k(b - 2n)_k}{(c + 2n)_k(d + 2n)_k} \). For an indeterminate \( x \) independent of \( m \) and \( n \) with \( 0 \leq m \leq n \), we have two asymptotic relations

\[
\sum_{k=0}^{m-1} \ln\left(1 + \frac{x + k}{2n}\right) \approx 2n \int_{0}^{\frac{m}{2n}} \ln(1 + y) \, dy = 2n \left\{ (1 + y) \ln(1 + y) - y \right\}_0^{\frac{m}{2n}} 
\]

\[
= (m + 2n) \ln \frac{2n + m}{2n} - m
\]

and

\[
\sum_{k=0}^{m-1} \ln\left(1 - \frac{x + k}{2n}\right) \approx 2n \int_{0}^{\frac{m}{2n}} \ln(1 - y) \, dy = -2n \left\{ (1 - y) \ln(1 - y) + y \right\}_0^{\frac{m}{2n}} 
\]

\[
= (m - 2n) \ln \frac{2n - m}{2n} - m.
\]

Then the general hypergeometric term can be estimated as

\[
w_m(n) = \exp\left\{ \ln\left(\frac{(a - 2n)_m(b - 2n)_m}{(c + 2n)_m(d + 2n)_m}\right) \right\} = \exp\left\{ \sum_{k=0}^{m-1} \ln\left(\frac{(1 - \frac{a+k}{2n})(1 - \frac{b+k}{2n})}{(1 + \frac{c+k}{2n})(1 + \frac{d+k}{2n})}\right) \right\} 
\]

\[
\approx \exp\left\{ 2(m - 2n) \ln \frac{2n - m}{2n} - 2(m + 2n) \ln \frac{2n + m}{2n} \right\}
\]
\[
\exp\left\{-4n\left[\left(1 + \frac{m}{2n}\right) \ln\left(1 + \frac{m}{2n}\right) + \left(1 - \frac{m}{2n}\right) \ln\left(1 - \frac{m}{2n}\right)\right]\right\}.
\]

This allows us to estimate further the partial hypergeometric sum displayed in (9a):
\[
\sum_{m=-\infty}^{\infty} w_m(n) \approx \sum_{m=-n}^{n} \exp\left\{-4n\left[\left(1 + \frac{m}{2n}\right) \ln\left(1 + \frac{m}{2n}\right) + \left(1 - \frac{m}{2n}\right) \ln\left(1 - \frac{m}{2n}\right)\right]\right\}
\approx 2n \int_{-1/2}^{1/2} \exp\left\{-4n\left[(1 + y) \ln(1 + y) + (1 - y) \ln(1 - y)\right]\right\} \, dy.
\]

Let \( \phi(y) \) be the function given by
\[
\phi(y) = (1 + y) \ln(1 + y) + (1 - y) \ln(1 - y).
\]

It is trivial to compute its first two derivatives
\[
\phi'(y) = \ln(1 + y) - \ln(1 - y) \quad \text{and} \quad \phi''(y) = \frac{2}{1 - y^2}.
\]

Therefore \( y = 0 \) is the global minimum of \( \phi(y) \) for \( y \in \mathbb{R} \). The saddle point method (also called the steepest descent method, cf. [4, §5.7]) asserts that
\[
\int_{-1/2}^{1/2} \exp\left\{-4n\phi(y)\right\} \, dy \approx \exp\left\{-4n\phi(0)\right\} \sqrt{\frac{\pi}{2n\phi''(0)}} = \sqrt{\frac{\pi}{4n}}
\]
which is equivalent to the following asymptotic relation
\[
\sum_{m=-n}^{n} w_m(n) \approx \sqrt{n\pi} \quad \text{for} \quad n \rightarrow \infty.
\]

2.2. By means of (7), it is not hard to check that
\[
|w_n(n)| = \left|\frac{(a - 2n)n(b - 2n)n}{(c + 2n)n(d + 2n)n}\right| = \left|\frac{(1 - a + n)n(1 - b + n)n}{(c + 2n)n(d + 2n)n}\right|
\]
\[
= \left|\frac{[1 - a, 1 - b, c, d]_{2n}}{[1 - a, 1 - b]_n[c, d]_{3n}}\right| = \mathcal{O}\left(\frac{2^{8n}}{3^{6n}}\right) = \mathcal{O}\left(\frac{16}{27}\right)^{2n}.
\]

Then for \( n < k < 2n \), we can estimate the term ratio
\[
\left|\frac{w_{k+1}(2n)}{w_k(2n)}\right| = \left|\frac{(a - 2n + k)(b - 2n + k)}{(c + 2n + k)(d + 2n + k)}\right| < \left|\frac{(a - n)(b - n)}{(c + 3n)(d + 3n)}\right| \approx \frac{1}{9}.
\]

Therefore we have the following asymptotic relation
\[
\left|\sum_{k=n+1}^{2n-1} w_k(n)\right| \leq |w_n(n)| \sum_{k=n+1}^{\infty} \frac{1}{9^{k-n}} < \frac{|w_n(n)|}{8} = \mathcal{O}\left\{\left(\frac{16}{27}\right)^{2n}\right\} = o(1).
\]

In view of
\[
w_{-k}(n) = \left[\begin{array}{c}
 a - 2n, b - 2n \\
 c + 2n, d + 2n
\end{array}\right]_{-k} = \left[\begin{array}{c}
 1 - c - 2n, 1 - d - 2n \\
 1 - a + 2n, 1 - b + 2n
\end{array}\right]_k
\]
we can analogously show that
\[
\left| \sum_{k=n+1}^{2n-1} w_{-k}(n) \right| = \mathcal{O}\left\{ \left( \frac{16}{27} \right)^{2n} \right\} = o(1).
\]
Consequently the partial sum displayed in (9b) tends to zero for \( n \to \infty \).

2.3. Again by means of (7), it is not difficult to check that
\[
\left| w_{2n}(n) \right| = \frac{(a - 2n)(b - 2n)(c + 2n)(d + 2n)}{(c + 2n)(d + 2n)(d + 2n)} = \mathcal{O}\left\{ \left( \frac{16}{27} \right)^{2n} \right\} = o(1).
\]
Replacing \( k \) by \( k + 2n \), we can also write
\[
\left| \sum_{k=2n}^{\infty} w_k(n) \right| = \left| w_{2n}(n) \right| \times \left| \sum_{k=0}^{\infty} \left[ \begin{array}{c} a, b \\ c + 4n, d + 4n \end{array} \right] \right| = \mathcal{O}\left\{ w_{2n}(n) \right\} = o(1)
\]
where the infinite series with respect to \( k \) in the middle is bounded, whose convergence has been justified by the criterion on generalized hypergeometric series (cf. Bailey [2, §2.1]).

In the similar manner, we can also verify
\[
\left| \sum_{k=2n}^{\infty} w_{-k}(n) g \right| = \mathcal{O}\left\{ w_{-2n}(n) \right\} = \mathcal{O}\left\{ \frac{n^{\Re(1-a-b)}}{28n} \right\} = o(1).
\]
Therefore we have shown the partial sum displayed in (9c) tends to zero for \( n \to \infty \).

This completes the proof of Proposition 3 and so Theorem 1.

3. The very well-poised bilateral \( sH_5 \)-series identity

In the same paper, Dougall (1907, [5]) discovered through contour integration method another fundamental result.

**Theorem 4.** (Douglas, 1907, [5]: see Slater [7, §6.1] also) For five complex numbers \( a, b, c, d, e \) subject to \( \Re(1 + 2a - b - c - d - e) > 0 \) and none of them being integer, there holds the summation formula:

\[
\begin{align*}
5H_5 & \left[ \begin{array}{c} 1 + \frac{a}{2}, b, c, d, e \\ 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e \end{array} \right] \\
= & \sum_{k=-\infty}^{+\infty} \frac{a + 2k}{a} \left[ \begin{array}{c} b, c, d, e \\ 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e \end{array} \right] \\
= & \Gamma\left[ \begin{array}{c} 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 - b, 1 - c, 1 - d, 1 - e, 1 + 2a - b - c - d - e \\ 1 + a, 1 - a, 1 + a - b - c, 1 + a - b - d, 1 + a - b - e, 1 + a - c - d, 1 + a - c - e, 1 + a - d - e \end{array} \right].
\end{align*}
\]

This theorem can be confirmed similarly by means of the modified Abel lemma on summation by parts and asymptotic analysis method.
For two sequences $A_k$ and $B_k$ defined by
\[
A_k = \begin{bmatrix} b, & d \\ 1 + a - b, & 1 + a - d \end{bmatrix}_k \quad \text{and} \quad B_k = \begin{bmatrix} 1 + c, & 1 + e \\ 1 + a - c, & 1 + a - e \end{bmatrix}_k
\]
it is not hard to compute the differences
\[
\triangle A_k = \begin{bmatrix} b, & d \\ 2 + a - b, & 2 + a - d \end{bmatrix}_k \frac{(1 + a + 2k)(1 + a - b - d)}{(1 + a - b)(1 + a - d)},
\]
\[
\nabla B_k = \begin{bmatrix} c, & e \\ 1 + a - c, & 1 + a - e \end{bmatrix}_k \frac{(a + 2k)(c + e - a)}{ce}.
\]

Let $\Omega(a; b, c, d, e)$ stand for the 5$H_5$-series displayed in (10a). By means of the modified Abel lemma on summation by parts, we can manipulate the bilateral Dougall sum as follows:
\[
\Omega(a; b, c, d, e) = \sum_{k=-\infty}^{+\infty} \frac{a + 2k}{a} \begin{bmatrix} b, & c, & d, & e \\ 1 + a - b, & 1 + a - c, & 1 + a - d, & 1 + a - e \end{bmatrix}_k
\]
\[
= \frac{ce}{a(c + e - a)} \sum_{k=-\infty}^{+\infty} A_k \nabla B_k = \frac{ce}{a(c + e - a)} \sum_{k=-\infty}^{+\infty} B_k \triangle A_k
\]
\[
= \frac{(1 + a)ce(1 + a - b - d)}{a(c + e - a)(1 + a - b)(1 + a - d)}
\]
\[
\times 5H_5 \begin{bmatrix} \frac{3 + a}{2}, & b, & 1 + c, & 1 + e \\ \frac{1 + a}{2}, & 2 + a - b, & 1 + a - c, & 2 + a - d, & 1 + a - e \end{bmatrix}.
\]

Therefore we have derived the following recurrence relation:
\[
\Omega(a; b, c, d, e) = \Omega(1 + a; b, 1 + c, d, 1 + e) \frac{(1 + a)ce(1 + a - b - d)}{a(c + e - a)(1 + a - b)(1 + a - d)}. \tag{11}
\]

Iterating $m$-times the last equation, we get further the following recursion formula:
\[
\Omega(a; b, c, d, e) = \Omega(m + a; b, m + c, d, m + e) \begin{bmatrix} 1 + a, & c, & e, & 1 + a - b - d \\ a, & c + e - a, & 1 + a - b, & 1 + a - d \end{bmatrix}_m. \tag{12}
\]

Then the following limiting relation is crucial for proving Theorem 4.

**Proposition 5** (Limiting relation for Dougall’s bilateral 5$H_5$-series).
\[
\lim_{m \to \infty} \Omega(a + m; b, c + m, d, e + m) = 2H_2 \begin{bmatrix} b, & d \\ 1 + a - c, & 1 + a - e \end{bmatrix}
\]
\[
- \Gamma \begin{bmatrix} c, & e, & 1 - b, & 1 - d \\ 1 + a - b, & 1 + a - d, & c - a, & e - a \end{bmatrix}_m 2H_2 \begin{bmatrix} c, & e \\ 1 + a - b, & 1 + a - d \end{bmatrix}.
\]

Denote by $W_k(m)$ the summand
\[
W_k(m) = \frac{a + m + 2k}{a + m} \begin{bmatrix} b, & d, & c + m, & e + m \\ 1 + a - b + m, & 1 + a - d + m, & 1 + a - c, & 1 + a - e \end{bmatrix}_k.
\]

For an arbitrarily small irrational number $\varepsilon > 0$ with $\varepsilon = o(1)$, we divide the summation of $\Omega(a + m; b, c + m, d, e + m)$ into five parts and then evaluate each part separately.
\[ \Omega(a + m; b, c + m, d, e + m) \]
\[ = \sum_{-m \epsilon < k < m \epsilon} W_k(m) + \sum_{-m \epsilon - m < k < -m \epsilon - m} W_k(m) \]
\[ + \sum_{-\infty < k < -m \epsilon - m} W_k(m) + \sum_{m \epsilon - m < k < -m \epsilon} W_k(m) + \sum_{m \epsilon < k < \infty} W_k(m). \]

(13a)

For convenience, we use \( \lambda = \Re(2 + 2a - b - c - d - e) > 1 \) in the following estimations.

3.1. For \( -m \epsilon < k < m \epsilon \), we have from asymptotic relation (7) on \( \Gamma \)-function:

\[ W_k(m) = \frac{a + m + 2k}{a + m} \left[ \begin{array}{cc} b, & d \\ 1 + a - c, & 1 + a - e \end{array} \right] \left[ \begin{array}{cc} c + m, & e + m \\ 1 + a - b + m, & 1 + a - d + m \end{array} \right] \]
\[ = \frac{a + m + 2k}{a + m} \left[ \begin{array}{cc} b, & d \\ 1 + a - c, & 1 + a - e \end{array} \right] \left[ \begin{array}{cc} c, & e \\ 1 + a - b, & 1 + a - d \end{array} \right] \]
\[ \times \left[ \begin{array}{cc} c, & e \\ 1 + a - b, & 1 + a - d \end{array} \right]_{m+k} \]
\[ \approx \frac{a + m + 2k}{a + m} \left[ \begin{array}{cc} b, & d \\ 1 + a - c, & 1 + a - e \end{array} \right] \left( \frac{m}{m+k} \right)^\lambda \]
\[ = \left[ \begin{array}{cc} b, & d \\ 1 + a - c, & 1 + a - e \end{array} \right]_k \{ 1 + \mathcal{O}(\epsilon^\lambda) \}. \]

Therefore we have the following asymptotic relation

\[ \sum_{-m \epsilon < k < m \epsilon} W_k(m) = \sum_{-m \epsilon < k < m \epsilon} \left[ \begin{array}{cc} b, & d \\ 1 + a - c, & 1 + a - e \end{array} \right]_k \{ 1 + \mathcal{O}(\epsilon^\lambda) \} \]
\[ = \mathcal{O}(\epsilon^\lambda) + 2H_2 \left[ \begin{array}{cc} b, & d \\ 1 + a - c, & 1 + a - e \end{array} \right] \]

where the last line has been justified by the convergence of the \( _2H_2 \)-series.

3.2. For \( -m \epsilon < k < m \epsilon \), we have again from asymptotic relation (7) on \( \Gamma \)-function:

\[ W_{k-m}(m) = \frac{a - m + 2k}{a + m} \left[ \begin{array}{cc} b, & d, & c + m, & e + m \\ 1 + a - b + m, & 1 + a - d + m, & 1 + a - c, & 1 + a - e \end{array} \right]_{k-m} \]
\[ = \frac{a - m + 2k}{a + m} \left[ \begin{array}{cc} b, & d \\ 1 + a - b, & 1 + a - d \end{array} \right] \left[ \begin{array}{cc} c - a, & e - a \\ 1 - b, & 1 - d \end{array} \right] \]
\[ \times \left[ \begin{array}{cc} c, & e \\ 1 + a - b, & 1 + a - d \end{array} \right] \]
\[ \approx \frac{a - m + 2k}{a + m} \left( \frac{m}{m-k} \right)^\lambda \left[ \begin{array}{cc} c, & e, & 1 - b, & 1 - d \\ 1 + a - b, & 1 + a - d, & c - a, & e - a \end{array} \right] \]
\[ \times \left[ \begin{array}{cc} c, & e \\ 1 + a - b, & 1 + a - d \end{array} \right]_k \]
\[ = \left\{ \mathcal{O}(\epsilon^\lambda) - \Gamma \left[ \begin{array}{cc} c, & e, & 1 - b, & 1 - d \\ 1 + a - b, & 1 + a - d, & c - a, & e - a \end{array} \right] \right\} \]
Replacing $k$ by $k - m$, we can estimate the following finite sum

$$\sum_{m \leq k < m + m} W_k(m) = \sum_{m \leq k < m + m} W_{k-m}(m)$$

$$= \sum_{m \leq k \leq m} \left[ \begin{array}{c}
1 + a - b, & 1 + a - d, & c, & e
\end{array} \right] k

\times \{ O(\varepsilon^k) - \Gamma \left[ \begin{array}{c}
1 + a - b, & 1 + a - d, & c, & e
\end{array} \right] \right\}

= O(\varepsilon^k) - \Gamma \left[ \begin{array}{c}
1 + a - b, & 1 + a - d, & c, & e
\end{array} \right] 2H_2 \left[ \begin{array}{c}
1 + a - b, & 1 + a - d
\end{array} \right]$

where the last two lines have been justified by the convergence of the $2H_2$-series.

3.3. For $m \epsilon < k < \infty$, we have from (7) that

$$W_{-k-m}(m)$$

$$= \frac{a - m - 2k}{a + m} \left[ \begin{array}{c}
b, & d, & c + m, & e + m
\end{array} \right] k_{-m}$$

$$= \frac{a - m - 2k}{a + m} \left[ \begin{array}{c}
b - a - b + m, & 1 + a - d + m, & c - a, & e - a
\end{array} \right] m_k$$

Replacing $k$ by $-k - m$, we can estimate the following infinite sum

$$\left| \sum_{-\infty < k < -m - \epsilon} W_k(m) \right| \leq \sum_{m \epsilon < k < \infty} \left| W_{-k-m}(m) \right| < \sum_{m \epsilon < k < \infty} O \left( \frac{1}{k^\lambda} \right) = o(1)$$

for the last series results in the tail of a convergent series.

3.4. For $m \epsilon < k < m - m \epsilon$, we have from (7) that

$$W_{-k}(m) = \frac{a + m - 2k}{a + m} \left[ \begin{array}{c}
b, & d, & c + m, & e + m
\end{array} \right] -k$$

$$= \frac{a + m - 2k}{a + m} \left[ \begin{array}{c}
b - a - b + m, & 1 + a - d + m, & c - a, & e - a
\end{array} \right] m_k$$

$$\times \left[ \begin{array}{c}
c, & e
\end{array} \right] m_{-k}$$

$$= \frac{a + m - 2k}{a + m} \left( \frac{m}{k(m-k)} \right)^\lambda \leq O \left( \frac{m^\lambda}{k^\lambda(m-k)^\lambda} \right).$$

Replacing $k$ by $-k$, we can estimate the following finite sum

$$\left| \sum_{m \epsilon - m < k < -m \epsilon} W_k(m) \right| \leq \sum_{m \epsilon < k < m - m \epsilon} \left| W_{-k}(m) \right| < \sum_{m \epsilon < k < m - m \epsilon} O \left( \frac{m^\lambda}{k^\lambda(m-k)^\lambda} \right) = o(1).$$
3.5. For \( m \varepsilon < k < \infty \), we have from (7) that

\[
W_k(m) = \frac{a + m + 2k}{a + m} \left[ \begin{array}{cc} b, & d \\ 1 + a - c, & 1 + a - e \end{array} \right]_k \left[ \begin{array}{cc} c + m, & e + m \\ 1 + a - b + m, & 1 + a - d + m \end{array} \right]_k
\]

\[
= \frac{a + m + 2k}{a + m} \left[ \begin{array}{cc} b, & d \\ 1 + a - c, & 1 + a - e \end{array} \right]_k \left[ \begin{array}{cc} c, & e \\ 1 + a - b, & 1 + a - d \end{array} \right]_m
\]

\[
\times \left[ \begin{array}{cc} c, & e \\ 1 + a - b, & 1 + a - d \end{array} \right]_{m+k}
\]

\[
= \frac{a + m + 2k}{a + m} \mathcal{O} \left( \frac{m}{k(m + k)} \right) = \mathcal{O} \left( \frac{m^{\lambda - 1}}{k^\lambda (m + k)^{\lambda - 1}} \right) < \mathcal{O} \left( \frac{1}{k^\lambda} \right).
\]

Then we can estimate the following infinite sum

\[
\left| \sum_{m \varepsilon < k < \infty} W_k(m) \right| \leq \sum_{m \varepsilon < k < \infty} \left| W_k(m) \right| < \sum_{m \varepsilon < k < \infty} \mathcal{O} \left( \frac{1}{k^\lambda} \right) = o(1)
\]

for the last series results in the tail of a convergent series.

Summing up, we have established the limiting relation displayed in Proposition 5:

\[
H := \lim_{m \to \infty} \Omega(a + m; b, c + m, d, e + m) = 2H_2 \left[ \begin{array}{cc} b, & d \\ 1 + a - c, & 1 + a - e \end{array} \right]
\]

\[
- \Gamma \left[ \begin{array}{cc} c, & e \\ 1 + a - b, & 1 + a - d \end{array} \right] \gamma \left[ \begin{array}{cc} c, & e \\ 1 + a - b, & 1 + a - d \end{array} \right]_2 H_2 \left[ \begin{array}{cc} c, & e \\ 1 + a - b, & 1 + a - d \end{array} \right].
\]

In view of Theorem 1, the last expression may be reduced by replacing both \( 2H_2 \)-series by fractions of \( \Gamma \)-function:

\[
H = \Gamma \left[ \begin{array}{cc} 1 + a - c, & 1 + a - e, 1 - b, 1 - d, 1 + 2a - b - c - d - e \\ 1 + a - b - c, 1 + a - c - d, 1 + a - b - e, 1 + a - d - e \end{array} \right]
\]

\[
- \Gamma \left[ \begin{array}{cc} c, e, 1 - c, 1 - e, 1 - b, 1 - d, 1 + 2a - b - c - d - e \\ c - a, e - a, 1 + a - b - c, 1 + a - c - d, 1 + a - b - e, 1 + a - d - e \end{array} \right]
\]

\[
= \Gamma \left[ \begin{array}{cc} c, e, 1 - c, 1 - e, 1 + a - c, 1 + a - e, 1 - b, 1 - d, 1 + 2a - b - c - d - e \\ 1 + a - b - c, 1 + a - c - d, 1 + a - b - e, 1 + a - d - e \end{array} \right]
\]

\[
\times \left\{ \Gamma^{-1} [c, e, 1 - c, 1 - e] - \Gamma^{-1} [c - a, e - a, 1 + a - c, 1 + a - e] \right\}.
\]

Simplifying the difference displayed in the braces by the reciprocity law

\[
\Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin \pi x}
\]

we find that

\[
\Gamma^{-1} [c, e, 1 - c, 1 - e] - \Gamma^{-1} [c - a, e - a, 1 + a - c, 1 + a - e]
\]

\[
= \frac{\sin \pi c \sin \pi e - \sin \pi (c - a) \sin \pi (e - a)}{\pi^2}
\]

\[
= \frac{\sin \pi a \sin \pi (c + e - a)}{\pi^2} = \Gamma^{-1} [a, 1 - a, c + e - a, 1 + a - c - e]
\]

which leads us to the following expression

\[
H = \Gamma \left[ \begin{array}{cc} c, e, 1 + a - c, 1 + a - e, 1 - b, 1 - c, 1 - d, 1 - e, 1 + 2a - b - c - d - e \\ a, 1 - a, c + e - a, 1 + a - b - c, 1 + a - b - e, 1 + a - c - d, 1 + a - c - e, 1 + a - d - e \end{array} \right].
\]
By invoking (7), we also have

\[
\lim_{m \to \infty} \left[ \frac{c, e, 1 + a, 1 + a - b - d}{a, c + e - a, 1 + a - b, 1 + a - d} \right] = \Gamma \left[ \frac{a, c + e - a, 1 + a - b, 1 + a - d}{c, e, 1 + a, 1 + a - b - d} \right].
\]

Letting \( m \to \infty \) in (12) and canceling the common factors, we finally arrive at

\[
\Omega(a; b, c, d, e) = \Gamma \left[ \frac{1+a-b, 1+a-c, 1+a-d, 1+a-e, 1-b, 1-c, 1-d, 1-e, 1+2a-b-c-d-e}{1+a, 1-a, 1+a-b-c, 1+a-b-d, 1+a-b-e, 1+a-c-d, 1+a-c-e, 1+a-d-e} \right]
\]

which confirms the identity stated in Theorem 4.

References