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On the computational complexity of upper total domination

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Abstract

Let $G=(V,E)$ be an undirected graph. Upper total domination number $\Gamma_t(G)$ is the maximum cardinality over all minimal total dominating sets of G, and upper fractional total domination number $\widetilde{\Gamma}_t(G)$ is the maximum weight over all minimal total dominating functions of G. In this paper we show that: (1) $\widetilde{\Gamma}_t(G)$ is an optimal value of some linear programming and is always a rational number; (2) when G is a tree, $\Gamma_t(G)=\widetilde{\Gamma}_t(G)$; (3) the recognition problems corresponding to the problems of computing $\Gamma_t(G)$ and $\widetilde{\Gamma}_t(G)$ are both NP-complete.

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1. Introduction

Total dominating sets and total dominating functions were first introduced and studied in [4,5]. Let $G=(V,E)$ be an undirected graph with vertex set V and edge set E. The *open neighborhood* of vertex $v \in V$ is given by $N(v)=\{u \in V : uv \in E\}$.

A vertex subset $T \subseteq V$ is a *total dominating set* (TDS) of G, if

$$\forall v \in V: \ N(v) \cap T \neq \emptyset.$$

That is, every vertex in V is adjacent to at least one member of T. A TDS T is minimal if no proper subset of T is also TDS of G. A function $f: V \to [0,1]$ is a *total dominating function* (TDF) of G, if

$$\forall v \in V: \sum_{u \in N(v)} f(u) \geqslant 1.$$

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Given functions $f,g : V \to [0,1]$, say $g \leqslant f$ if $g(v) \leqslant f(v)$ for all $v \in V$. We say $g < f$ if $g \leqslant f$ and $g \neq f$. A TDF f is minimal if no function $g < f$ is also a TDF of G. Integer valued (minimal) TDFs are characteristic functions of (minimal) TDSs, and (minimal) TDFs are the fractional generalizations of (minimal) TDSs.

The *upper total domination number* $\Gamma_{\mathrm{t}}(G)$ is defined by

$$\Gamma_{\mathrm{t}}(G) = \max\{|T| : T \text{ is a minimal TDS of } G\}.$$

For TDFs of G, we similarly define the *upper fractional total domination number* $\widetilde{\Gamma}_{\mathrm{t}}(G)$ as follows:

$$\widetilde{\Gamma}_{\mathrm{t}}(G) = \max\{f(V) : f \text{ is a minimal TDF of } G\}.$$

Here, to simplify notation, we denote $f(U) = \sum_{v \in U} f(v)$, and define the weight of f to mean $\sum_{v \in V} f(v) = f(V)$.

The two parameters $\Gamma_{\mathrm{t}}(G)$ and $\widetilde{\Gamma}_{\mathrm{t}}(G)$ are analogous to the upper domination number UD and UFD which have been studied in [3], and our work is in much spirits as that of Cheston's work. In this paper, we first show that $\widetilde{\Gamma}_{\mathrm{t}}(G)$ is an optimal value of some linear programming and is always a rational number. Next, we prove that $\Gamma_{\mathrm{t}}(G)=\widetilde{\Gamma}_{\mathrm{t}}(G)$ when G is a tree, and give an example of a graph G with $\Gamma_{\mathrm{t}}(G) < \widetilde{\Gamma}_{\mathrm{t}}(G)$. Finally, we claim that the recognition problems corresponding to computing $\Gamma_{\mathrm{t}}(G)$ and $\widetilde{\Gamma}_{\mathrm{t}}(G)$ are both NP-complete.

We remark another related parameter—total domination number $\gamma_{\mathrm{t}}(G)$, the cardinality of a smallest total dominating set of G. Its computational complexity has been much studied in [1,2,7,8]. As to its fractional generalization, let

$$\widetilde{\gamma}_{\mathrm{t}}(G) = \min\{f(V) : f \text{ is a TDF of } G\}.$$

It is easy to show that it can be computed in polynomial time. In fact, $\widetilde{\gamma}_{\mathrm{t}}(G)$ is exactly the optimal value of the following linear programming:

$$\begin{array}{ll} \min & \displaystyle\sum_{i \in V} x_i \\ \text{s.t.} & \begin{cases} Ax \geqslant 1, \\ x \geqslant 0, \end{cases} \end{array}$$

where A is the adjacent matrix of G.

2. Computing $\Gamma_{\mathrm{t}}(G)$ and $\widetilde{\Gamma}_{\mathrm{t}}(G)$

To ensure the existance of TDSs and TDFs, we assume that $G = (V,E)$ is a graph with no isolated vertices. In the rest of the paper, $\sum_{u \in N(v)} f(u)$ will be abbreviated to $f[v]$. For the TDF f, the boundary of f is defined by $B_f = \{v \in V : f[v] = 1\}$. For any subset $U \subseteq V$, denote $N(U) = \bigcup_{v \in U} N(v)$. It was proved in [5] that

A TDF f is minimal if and only if for each $v \notin N(B_f)$, $f(v) = 0$.

Using this characterization, we can reformulate the set of minimal TDFs as the solutions of the following nonlinear inequalities:

$$\begin{cases} 0 \leqslant f(v) \leqslant 1 \\ f[v] \geqslant 1 \\ f(v) \displaystyle\prod_{u \in N(v)} (1 - f[u]) = 0. \end{cases} \quad \text{for all } v \in V. \tag{1}$$

Let $\mathscr{F}$ be the set of all minimal TDFs of G, by (1) we know that $\mathscr{F}$ is closed and bounded in $R^{|V|}$. Hence, there must exist a minimal TDF f of maximum weight $f(V) = \widetilde{\Gamma}_t(G)$. If substituting the constraint $0 \leqslant f(v) \leqslant 1$ by $f(v) = 0, 1$, the solution set of (1) is exactly the set of all minimal TDSs, and the corresponding maximum weight is $\Gamma_t(G)$. Clearly, $\Gamma_t(G) \leqslant \widetilde{\Gamma}_t(G)$.

However, the nonlinear optimization problem described above gives little insight on how to compute $\Gamma_t(G)$ and $\widetilde{\Gamma}_t(G)$. We now look at the problem in a different way. Let f be a minimal TDF and B_f be the boundary of f. For every $v \in V$, since f is a TDF, there exists a vertex $u \in N(v)$ such that $f(u) > 0$; also since f is minimal, $u \in N(B_f)$. It implies that $N(N(B_f)) = V$. We are therefore motivated to define

$$\Theta = \{S : S \subseteq V \text{ and } N(N(S)) = V\}.$$

For each member $S \in \Theta$, we consider the subproblem: finding a minimal dominating function f of maximum weight with the additional constraint that B_f contains S. That is, each vertex $v \in S$ has the property $f[v] = 1$. Thus the subproblem can be solved using linear programming (2):

$$\begin{aligned} \max \quad & \sum_{v \in V} f(v) \\ \text{s.t.} \quad & \begin{cases} 0 \leqslant f(v) \leqslant 1 & v \in N(S), \\ f(v) = 0 & v \in V \setminus N(S), \\ f[v] \geqslant 1 & v \in V \setminus S, \\ f[v] = 1 & v \in S. \end{cases} \end{aligned} \tag{2}$$

Note that the conditions of (2) guarantee that a feasible solution of (2) is a minimal TDF of G. Conversely, if f is a minimal TDF such that $f(V) = \widetilde{\Gamma}_t(G)$, then f must be the optimal solution of (2) for some $S = B_f \in \Theta$. Therefore, we have

Theorem 1. *For any graph G, (1) $\widetilde{\Gamma}_t(G)$ is a computable function and is always rational; and (2) there exists a minimal TDF f of weight $f(V) = \widetilde{\Gamma}_t(G)$ having rational values, and such that the length of the representation of f is bounded by some polynomial of $|G|$.*

We note that $\widetilde{\Gamma}_t(G)$ is the largest solution obtained among those subproblems. This number must be rational since each subproblem involves only rational numbers. The detailed proof of this theorem is omitted, it is similar to that of Theorem 3.1 in [3]. However, the above algorithm for computing $\widetilde{\Gamma}_t(G)$ may not be efficient. For a complete bipartite graph $K_{m,n}$ with bipartition (X, Y), since every subset S of vertices with $S \cap X \neq \emptyset$ and $S \cap Y \neq \emptyset$ is a member of Θ, the algorithm makes $O(2^{m+n})$ "calls" to perform linear programming.

Now we apply the results on unimodular and totally unimodular matrices to prove that $\Gamma_t(T) = \widetilde{\Gamma}_t(T)$, when T is a tree. A matrix A is *totally unimodular* if each subdeterminant of A is 0, +1 or −1 (in particular, each entry of A is 0, +1 or −1). A matrix A of full row rank is *unimodular* if A is integral and each basis of A has determinant +1 or −1. Schrijver [9] has shown the relations between modularity and integral polyhedron. We give the related result in Lemma 1.

Lemma 1 (Schrijver [9]). *Let A be an integral matrix of full row rank. Then the polyhedron $\{x : Ax = b,\ x \geqslant 0\}$ is integral for each integral vector b, if and only if A is unimodular.*

Lemma 2. *The adjacent matrix of a tree $T = (V, E)$ is totally unimodular.*

Proof. Let $A(T)$ be the adjacent matrix of T. We show each subdeterminant of $A(T)$ is 0 , +1 or −1 by induction on the order k of subdeterminant. This is certainly true when $k = 1$ since $A(T)$ is a {0,1}-matrix. Suppose it is true for any subdeterminant of $A(T)$ with order k, $1 \leqslant k < |V|$.

Let A' be any square submatrix of $A(T)$ with order $k + 1$, and let V_1, V_2 denote the subset of V corresponding to the rows and columns of A', respectively . Then there exists a row or column which contains at most one nonzero element. Otherwise, since V_1, V_2 are finite set, there exists a circuit in the subgraph $G[V_1 \cup V_2]$, a contradiction. By the properties of determinant and the induction hypothesis, we have $\det A'$ is 0, +1 or −1. □

Theorem 2. *Let T be a tree, then $\Gamma_t(T) = \widetilde{\Gamma}_t(T)$.*

Proof. Assume that $\widetilde{\Gamma}_t(T)$ is the optimal value of linear programming (2) corresponding to set $S_0 \in \Theta$. When $v \in V \setminus N(S_0)$, $f(v) = 0$. So the constraints of (2) can be simplified to be $A^* f = 1$, $f \geqslant 0$. Here A^* is a matrix of full row rank with following form:

$$A^* = \begin{pmatrix} I & I & 0 \\ A_1 & 0 & -I \\ A_2 & 0 & 0 \end{pmatrix},$$

where $(A_1^t A_2^t)^t$ is a submatrix of $A(T)$. By Lemma 2, $(A_1^t A_2^t)^t$ is totally unimodular, hence A^* is unimodular. Also by Lemma 1, linear programming (2) must have integral optimal solutions, it implies that $\Gamma_t(T) = \widetilde{\Gamma}_t(T)$. □

We end this section by giving an example of a graph G with $\Gamma_t(G) < \widetilde{\Gamma}_t(G)$. Consider the graph in Fig. 1, it consists of four parts, H, H_1, H_2 and H_3. H is a 6-circuit, with vertex set $\{v_1, v_2, v_3, t_1, t_2, t_3\}$; H_i $(i = 1, 2, 3)$ is composed by two complete graphs

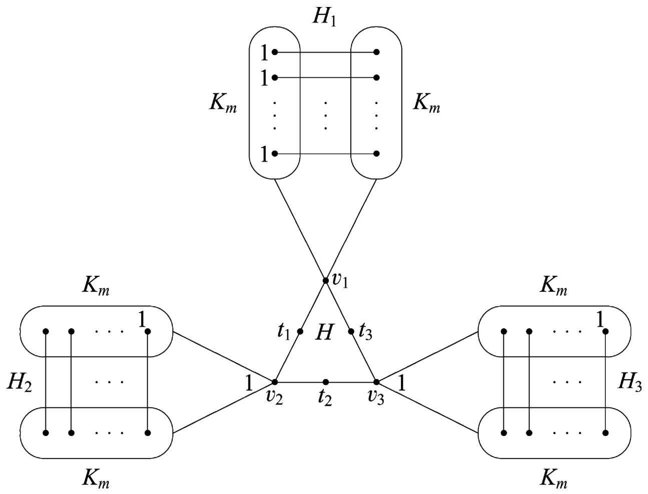

Fig. 1. Graph $G = (V, E)$.

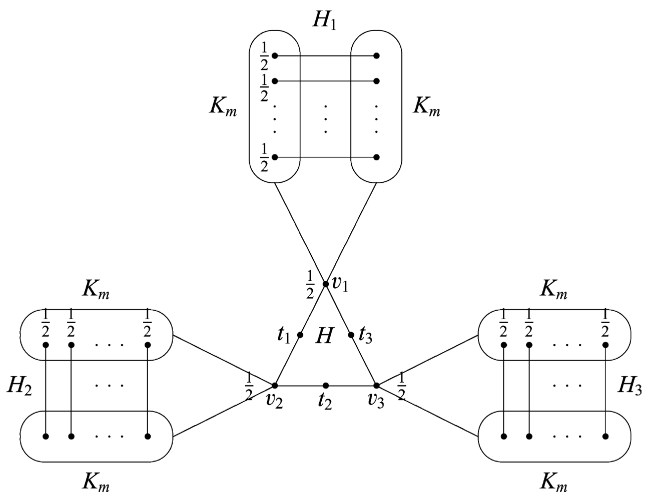

Fig. 2. Graph $G = (V, E)$.

K_m, and m edges between corresponding vertices of these two K_m, $m \geqslant 6$; v_i is adjacent to all $2m$ vertices in H_i, $i = 1, 2, 3$.

First we investigate the minimal TDS T of G with maximum cardinality. Since every t_i $(i = 1, 2, 3)$ is only adjacent to two vertices in $\{v_1, v_2, v_3\}$, so there are at least two vertices in $\{v_1, v_2, v_3\}$ belonging to T. Besides, if the vertices in T is required to be as many as possible, there must be a vertex in $\{v_1, v_2, v_3\}$ not belonging to T. Thus T is constructed in Fig. 1: the vertices labeled with 1 belong to T, the others belong to $V - T$. It is easy to see that T is a minimal TDS of G, and $\Gamma_t(G) = |T| = m + 4$. For the fractional case, a minimal TDF f is given in Fig. 2: the label beside each vertex is the corresponding value of f, and $f(v)=0$ for vertices with no labels. Obviously, $\widetilde{\Gamma}_{\rm t}(G) \geqslant f(V) = \frac{3}{2}m + \frac{3}{2}$. When $m \geqslant 6$, it must be the case $\Gamma_{\rm t}(G) < \widetilde{\Gamma}_{\rm t}(G)$.

3. Complexity

Let us consider the following two recognition problems:

Upper total domination (UTD)
Instance: A graph G and an integer k.
Question: Is $\Gamma_{\rm t}(G) \geqslant k$?
Upper fractional total domination (UFTD)
Instance: A graph G and a rational number q.
Question: Is $\widetilde{\Gamma}_{\rm t}(G) \geqslant q$?

In this section, we will prove that both UTD and UFTD are NP-complete by establishing a polynomial time transformation from the well-known NP-complete problem 3-SAT [6]. 3-SATISFIABILITY (3-SAT) is given as follows:

Instance: Collection $C=\{C_1, C_2, \ldots, C_m\}$ of clauses on a finite set $U=\{x_1, x_2, \ldots, x_n\}$ of variables such that $|C_i| = 3$, $i = 1, 2, \ldots, m$.

Question: Is there a truth assignment for U that satisfies all the clauses in C?

First, for an arbitrary instance of 3-SAT, we construct a graph $G=(V,E)$:

- Corresponding to each variable x_i is a subgraph H_i of G, which is shown in Fig. 3. Vertices v_i and $\bar{v}_i$ represent literals x_i and $\bar{x}_i$, respectively, $i = 1, 2, \ldots, n$. Let $X = \{v_i, \bar{v}_i : i = 1, 2, \ldots, n\}$, $A = \{a_i, \bar{a}_i : i = 1, 2, \ldots, n\}$.
- Corresponding to each clause $C_j = (\tilde{x}_{j1}, \tilde{x}_{j2}, \tilde{x}_{j3})$ is a subgraph F_j of G, which is shown in Fig. 4. Vertex w_j represents clause C_j, $j = 1, 2, \ldots, m$. Let $W = \{w_j : j = 1, 2, \ldots, m\}$.

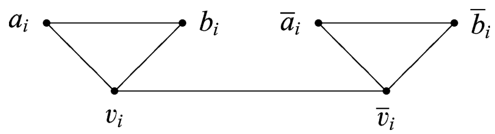

Fig. 3. Subgraph H_i.

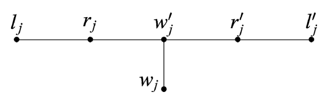

Fig. 4. Subgraph F_j.

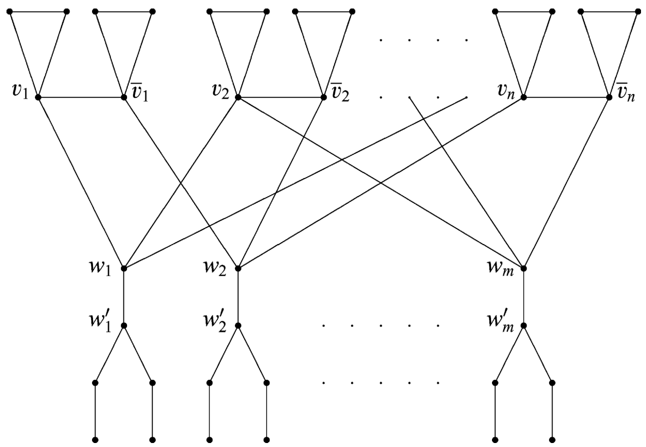

Fig. 5. Graph $G = (V, E)$.

- The edges between subgraphs H_i and F_j are defined as follows:

Vertex w_j is connected to all vertices corresponding to literals in clause C_j. For example, $C_1 = (x_1, \bar{x}_2, x_3)$, then in graph G vertex w_1 is adjacent to vertices v_1, $\bar{v}_2$, v_3. The structure of graph G is depicted in Fig. 5.

In the following, we will discuss some properties of minimal TDFs of graph G. Let f be a minimal TDF of G, subgraphs H_i, F_j and their vertices set are defined as above, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$.

Lemma 3. *In subgraph F_j, $f(r_j) = f(r_j') = 1$ and $f[r_j] = f[r_j'] = 1$.*

Lemma 4. *For some vertex w_j, if $f(w_j) > 0$, then there exists another minimal TDF g of G, such that $g(w_j) = 0$ and $f(V) \leqslant g(V)$.*

Proof. Assume that $X_j = N(w_j) \cap X = \{\tilde{v}_{j1}, \tilde{v}_{j2}, \ldots, \tilde{v}_{jk}\}$, and the vertex in A connected to $\tilde{v}_{jl}$ is $\tilde{a}_{jl}$, $l = 1, 2, \ldots, k$. Except the vertices in X_j, w_j is only connected to w_j', so by Lemma 3, $w_j' \notin B_f$. Since f is minimal, there is a vertex in $X_j \cap B_f$. Without loss of generality, let this vertex to be v_{j1}. Define

$$\theta(\tilde{v}_{jl}) = \max\{0, 1 - (f[\tilde{v}_{jl}] - f(w_j))\}, \quad l = 1, 2, \ldots, k.$$

Obviously, $\theta(\tilde{v}_{j1}) = f(w_j)$. Construct another TDF $g : V \to [0, 1]$,

$$g(v) = \begin{cases} 0, & v = w_j, \\ f(\tilde{a}_{jl}) + \theta(\tilde{v}_{jl}), & v = \tilde{a}_{jl}, \ l = 1, 2, \ldots, k, \\ f(v), & \text{otherwise.} \end{cases}$$

It can be verified that g is a minimal TDF of G with weight $g(V) \geqslant f(V)$. $\square$

According to Lemma 4, we can only investigate the minimal TDFs with zero values on subset W.

Lemma 5. *In subgraph H_i, if $v_i \notin B_f$ and $f(a_i), f(b_i)$ are both positive, then $f(a_i)+f(b_i) = 1 - 2f(v_i)$; for the 3-circuit containing $\bar{v}_i$, the conclusion also holds.*

Proof. By the minimality of f, if $f(a_i), f(b_i)$ are positive and $v_i \notin B_f$, then a_i and b_i belong to B_f. That is, $f(b_i)+f(v_i)=1$, $f(a_i)+f(v_i)=1$, and $f(a_i)+f(b_i)=1-2f(v_i)$. For the 3-circuit containing $\bar{v}_i$, the proof is similar. □

Lemma 6. *In subgraph H_i, the following conclusions hold*:

(1) $f(H_i) \leqslant 4$;
(2) *If $f(H_i) = 4$, then at most one member of $f(v_i)$ and $f(\bar{v}_i)$ is not zero.*

Proof. (1) If $f(H_i) > 4$, without loss of generality, let $f(v_i) + f(a_i) + f(b_i) > 2$. So we have

(i) $f(v_i), f(a_i), f(b_i)$ are all positive;
(ii) the sum of any two members in $\{f(v_i), f(a_i), f(b_i)\}$ is greater than 1.

So v_i, a_i, b_i do not belong to B_f, which is contrary to the minimality of f.

(2) If $f(H_i) = 4$, according to the analysis of (1),

$$f(a_i) + f(b_i) + f(v_i) = f(\bar{a}_i) + f(\bar{b}_i) + f(\bar{v}_i) = 2. \tag{3}$$

Suppose that $f(v_i)$ and $f(\bar{v}_i)$ are both not equal to zero, then v_i and $\bar{v}_i$ are not in B_f.

For the 3-circuit containing v_i, if $f(a_i) \geqslant f(b_i)$, then by (3) we have $f(a_i) > 0$ and $f(a_i) + f(v_i) > 1$, it implies that $b_i \notin B_f$, $a_i \notin N(B_f)$. It is a contradiction with minimality of f. Similar discussion can be given for the 3-circuit containing $\bar{v}_i$. □

Lemma 7. *In subgraph F_j, let $F'_j = F_j \setminus \{w_j\}$, the following conclusions hold*:

(1) $f(F'_j) \leqslant 4$;
(2) *If $f(F'_j) = 4$, then $f(w'_j) = 0$.*

Proof. (1) If $f(F'_j) > 4$, then there is at least one among $f[r_j]$ and $f[r'_j]$ greater than 1, which is contrary to Lemma 3.

(2) Now we assume $f(F'_j) = 4$. If $f(w'_j) > 0$, then by $f[r_j] = f[r'_j] = 1$, $f(F'_j) = 2 + f[r_j] + f[r'_j] - f(w'_j) < 4$, which is a contradiction. □

Since the characteristic functions of minimal TDSs are integer valued minimal TDFs, the conclusions in Lemma 3–Lemma 7 can also be applied to minimal TDSs.

Theorem 3. *UTD is* NP-*complete.*

Proof. It is obvious that UTD is a member of NP since we can, in polynomial time, guess at a subset of vertices, verify that its cardinality is at least k, and then verity that it is a minimal total dominating set.

For an arbitrary instance of 3-SAT: n variables $x_1, x_2, \ldots, x_n$ and a Boolean formula $F = C_1C_2 \cdots C_m$ with three literals per clause, we construct an instance of UTD: graph $G = (V, E)$ depicted in Fig. 5, and $k = 4(n+m)$. We now claim that $\Gamma_t(G) \geqslant 4(n+m)$ if and only if Boolean formula F is satisfiable.

Sufficiency: Assume that there is a truth assignment t satisfying F; then based on t, we obtain a vertex subset T as follows:

1. For any $i \in \{1, 2, \ldots, n\}$,
   If $t(x_i)$="true", then choose $v_i, a_i \in T$, $\bar{a}_i, \bar{b}_i \in T$.
   If $t(x_i)$="false", then choose $a_i, b_i \in T$, $\bar{v}_i, \bar{a}_i \in T$.
2. For any $j \in \{1, 2, \ldots, m\}$,
   Choose $r_j, r'_j, l_j, l'_j \in T$.
3. The other vertices of G are not in T.
   Obviously, T is a minimal TDS of G, and $|T| = 4(n+m)$. That is, $\Gamma_t(G) \geqslant 4(n+m)$.

Necessity. Suppose F is not satisfiable. Let T_0 be any minimal TDS of G and there is no vertices of subset W in T_0. Then T_0 must be one of the two cases:

Case 1: There is a subgraph H_i $(i = 1, 2, \ldots, n)$, such that v_i and $\bar{v}_i$ belong to T_0 at the same time. Thus by the minimality of T_0, the other vertices of H_i are not in T_0. Consider the characteristic function of T_0, by Lemma 6 we have $|T_0| \leqslant 4m + 4n - 2 < k - 1$.

Case 2: There is a subgraph F_j $(j = 1, 2, \ldots, m)$, such that $w'_j \in T_0$. Also by the minimality of T_0, the vertices l_j and l'_j are both not in T_0. Consider the characteristic function of T_0, by Lemma 7 we have $|T_0| \leqslant 4n + 4m - 1 \leqslant k - 1$.

Therefore, by the conclusion of Lemma 4, the cardinality of any minimal TDS of G can not be greater than $k - 1$. That is, $\Gamma_t(G) \leqslant k - 1 < k$. □

Theorem 4. *UFTD is* NP-*complete*.

Proof. Follow from the conclusion (2) of Theorem 1, for any *yes* instance of UFTD, there exists a minimal TDF of weight $\widetilde{\Gamma}_t(G)$ such that the length of the representation of this function is bounded by a polynomial in $|G|$; and also we can demonstrate this fact in polynomial time. So UFTD is a member of NP. Similar to the proof of Theorem 3, for any instance of 3-SAT: n variables $x_1, x_2, \ldots, x_n$ and a Boolean formula $F = C_1C_2 \cdots C_m$ with three literals per clause, we construct a same instance of UFTD as that of UTD: graph $G = (V, E)$ depicted in Fig. 5, and $q = 4(n+m)$. We can also show that for this graph G, $\Gamma_t(G) \geqslant 4(n+m)$ if and only if Boolean formula F is satisfiable. □

In this paper, we do not give further discussion on upper total dominations for trees. We guess for trees, there exist polynomial time algorithms for computing upper total dominations, that is, UTD and UFTD are both members of P.

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