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Stability of some set-valued functional equations

Choonkil Park [a], Donal O'Regan [b], Reza Saadati [c,*]

[a] Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, South Korea
[b] Department of Mathematics, National University of Ireland, Galway, Ireland
[c] Department of Mathematics, Science and Research Branch, Islamic Azad University, Post Code 14778, Ashrafi Esfahani Ave, Tehran, Iran

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ABSTRACT

In this paper, we prove the Hyers–Ulam stability of some set-valued functional equations.

1. Introduction and preliminaries

Set-valued functions in Banach spaces have received a lot of attention in the literature (see [1–3]). The pioneering papers by Aumann [1] and Debreu [2] were inspired by problems arising in Control Theory and Mathematical Economics. We also refer the reader to the papers by Arrow and Debreu [3], McKenzie [4] and the monographs by Hindenbrand [5], Aubin and Frankowska [6], Castaing and Valadier [7], Klein and Thompson [8] and the survey by Hess [9].

The paper of Rassias [10] has motivated the development of what we call *Hyers–Ulam stability* or the *Hyers–Ulam–Rassias stability* of functional equations (also see [11,12]). A generalization of the Rassias theorem was obtained by Găvruta [13] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers–Ulam stability problem for the quadratic functional equation was discussed by Skof [14] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [15] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [16] discussed the generalized Hyers–Ulam stability of the quadratic functional equation.

In [17], Jun and Kim considered the cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1) on $\mathbb{R}$, which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

* Corresponding author.
E-mail addresses: baak@hanyang.ac.kr (C. Park), donal.oregan@nuigalway.ie (D. O'Regan), rsaadati@eml.cc, rezas720@yahoo.com (R. Saadati).

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In [18], Lee et al. considered the quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2) on $\mathbb{R}$, which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Let Y be a real normed space. The family of all closed subsets, containing 0, of Y will be denoted by $cz(Y)$.

Let A, B be nonempty subsets of a real vector space X and λ a real number. We define

$$A+B=\{x\in X : x=a+b, \quad a\in A, b\in B\},$$
$$\lambda A=\{x\in X : x=\lambda a, \quad a\in A\}.$$

Lemma 1.1 (*[19]*). *Let λ and μ be real numbers. If A and B are nonempty subset of a real vector space X, then*

$$\lambda(A+B)=\lambda A+\lambda B,$$
$$(\lambda+\mu)A\subseteq\lambda A+\mu A.$$

Moreover, if A is a convex set and $\lambda\mu\geq 0$, then we have

$$(\lambda+\mu)A=\lambda A+\mu A.$$

A subset $A\subseteq X$ is said to be a *cone* if $A+A\subseteq A$ and $\lambda A\subseteq A$ for all $\lambda>0$. If the zero vector in X belongs to A, then we say that A is a *cone with zero*.

Set-valued functional equations have been investigated by a number of authors and there are many interesting results concerning this problem (see [20–25]).

In this paper, we define the Jensen additive set-valued functional equation, the quadratic set-valued functional equation, the cubic set-valued functional equation and the quartic set-valued functional equation, and prove the Hyers–Ulam stability of some set-valued functional equations.

Throughout this paper, let X be a real vector space, $A\subseteq X$ a cone with zero and Y a Banach space.

2. Stability of the Jensen additive set-valued functional equation

In this section, we prove the Hyers–Ulam stability of the Jensen additive set-valued functional equation.

Theorem 2.1. *If $F : A\to cz(Y)$ is a set-valued map satisfying $F(0)=\{0\}$,*

$$F(x)+F(y)\subseteq 2F\left(\frac{x+y}{2}\right) \tag{2.1}$$

and

$$\sup\{diam(F(x)) : x\in A\}<+\infty$$

for all $x, y\in A$, then there exists a unique additive map $g : A\to Y$ (which we call the Jensen additive map) such that $g(x)\in F(x)$ for all $x\in A$.

Proof. For $x\in A$, letting $y=0$ in (2.1), we get

$$F(x)+F(0)=F(x)\subseteq 2F\left(\frac{x}{2}\right) \tag{2.2}$$

and if we replace x by $2^{n+1}x$, $n\in\mathbb{N}$, in (2.2), then we obtain

$$F(2^{n+1}x)\subseteq 2F(2^n x)$$

and

$$\frac{F(2^{n+1}x)}{2^{n+1}}\subseteq\frac{F(2^n x)}{2^n}.$$

Let $F_n(x)=\frac{F(2^n x)}{2^n}$, $x\in A$, $n\in\mathbb{N}$ and we obtain that $(F_n(x))_{n\geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. We have also

$$diam(F_n(x))=\frac{1}{2^n}diam(F(2^n x)).$$

Now since $\sup\{diam(F(x)) : x\in A\}<+\infty$, we get that $\lim_{n\to+\infty} diam(F_n(x))=0$ for all $x\in A$.

Using the Cantor theorem for the sequence $(F_n(x))_{n\geq 0}$, we obtain that the intersection $\bigcap_{n\geq 0} F_n(x)$ is a singleton set and we denote this intersection by $g(x)$ for all $x \in A$. Thus we obtain a map $g : A \to Y$. Then $g(x) \in F_0(x) = F(x)$ for all $x \in A$.

Now we show that g is additive. We have (note Lemma 1.1)

$$F_n(x) + F_n(y) = \frac{F(2^n x)}{2^n} + \frac{F(2^n y)}{2^n} \subseteq \frac{1}{2^n} \cdot 2F\left(\frac{2^n x + 2^n y}{2}\right) = 2F_n\left(\frac{x+y}{2}\right).$$

By the definition of g, we get for all $x, y \in A$,

$$g(x) + g(y) = \bigcap_{n=0}^{\infty} F_n(x) + \bigcap_{n=0}^{\infty} F_n(y) \subseteq \bigcap_{n=0}^{\infty} \left(2F_n\left(\frac{x+y}{2}\right)\right).$$

Thus

$$g(x) + g(y) = 2g\left(\frac{x+y}{2}\right)$$

for all $x, y \in A$ and so g is additive.

Therefore, we conclude that there exists an additive map $g : A \to Y$ such that $g(x) \in F(x)$ for all $x \in A$.

Next, let us prove the uniqueness of g.

Suppose that F have two additive selections $g_1, g_2 : A \to Y$. We have

$$ng_i(x) = g_i(nx) \in F(nx)$$

for all $n \in \mathbb{N}$, $x \in A$, $i \in \{1, 2\}$. Then we get

$$n\|g_1(x) - g_2(x)\| = \|ng_1(x) - ng_2(x)\| = \|g_1(nx) - g_2(nx)\| \leq 2 \cdot diam(F(nx))$$

for all $x \in A$, $n \in \mathbb{N}$. It follows from $\sup\{diam(F(x)) : x \in A\} < +\infty$ that $g_1(x) = g_2(x)$ for all $x \in A$, as desired. □

3. Stability of the quadratic set-valued functional equation

In this section, we prove the Hyers–Ulam stability of the quadratic set-valued functional equation.

Theorem 3.1. *If $F : A + (-1)A \to cz(Y)$ is a set-valued map satisfying $F(0) = \{0\}$,*

$$F(x+y) + F(x-y) \subseteq 2F(x) + 2F(y) \tag{3.1}$$

and

$$\sup\{diam(F(x)) : x \in A\} < +\infty$$

for all $x, y \in A$, then there exists a unique quadratic map $g : A + (-1)A \to Y$ such that $g(x) \in F(x)$ for all $x \in A$.

Proof. Letting $x = y$ in (3.1), we get

$$F(2x) + F(0) = F(2x) \subseteq 4F(x). \tag{3.2}$$

Replacing x by $2^n x$, $n \in \mathbb{N}$, in (3.2), we obtain

$$F(2 \cdot 2^n x) \subseteq 4F(2^n x)$$

and

$$\frac{F(2^{n+1}x)}{4^{n+1}} \subseteq \frac{F(2^n x)}{4^n}.$$

Let $F_n(x) = \frac{F(2^n x)}{4^n}$, $x \in A$, $n \in \mathbb{N}$, we obtain that $(F_n(x))_{n\geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. We have also

$$diam(F_n(x)) = \frac{1}{4^n} diam(F(2^n x)).$$

Taking into account that $\sup\{diam(F(x)) : x \in A\} < +\infty$, we get

$$\lim_{n\to\infty} diam(F_n(x)) = 0.$$

Using the Cantor theorem for the sequence $(F_n(x))_{n\geq 0}$, we obtain that the intersection $\cap_{n\geq 0} F_n(x)$ is a singleton set and we denote this intersection by $g(x)$ for all $x \in A$. Thus we get a map $g : A + (-1)A \to Y$ and $g(x) \in F_0(x) = F(x)$ for all $x \in A$.

We now show that g is quadratic. For all $x, y \in A$ and $n \in \mathbb{N}$,

$$\begin{aligned} F_n(x+y) + F_n(x-y) &= \frac{F(2^n(x+y))}{4^n} + \frac{F(2^n(x-y))}{4^n} \subseteq \frac{2F\,(2^n x)}{4^n} + \frac{2F\,(2^n y)}{4^n} \\ &= 2F_n\,(x) + 2F_n(y). \end{aligned}$$

By the definition of g, we obtain

$$g(x+y) + g(x-y) = \bigcap_{n=0}^{\infty} F_n(x+y) + \bigcap_{n=0}^{\infty} F_n(x-y) \subseteq \bigcap_{n=0}^{\infty} (2F_n\,(x) + 2F_n(y)),$$

$2g\,(x) \in 2F_n\,(x)$ and $2g(y) \in 2F_n(y)$. Thus we get

$$\|g(x+y) + g(x-y) - 2g\,(x) - 2g(y)\| \le 2 \cdot diam(F_n\,(x)) + 2 \cdot diam(F_n(y)),$$

which tends to zero as n tends to ∞. Thus

$$g(x+y) + g(x-y) = 2g(x) + 2g(y)$$

for all $x, y \in A$.

Next, let us prove the uniqueness of g.

Suppose that F have two quadratic selections $g_1, g_2 : A + (-1)A \to Y$. We have

$$(2n)^2 g_i(x) = g_i(2nx) \in F(2nx)$$

for all $n \in \mathbb{N}, x \in A, i \in \{1, 2\}$. Then we get

$$(2n)^2 \|g_1(x) - g_2(x)\| = \|(2n)^2 g_1(x) - (2n)^2 g_2(x)\| = \|g_1(2nx) - g_2(2nx)\| \le 2 \cdot diam(F(2nx))$$

for all $x \in A$, $n \in \mathbb{N}$. It follows from $\sup\{diam(F(x)) : x \in A\} < +\infty$ that $g_1(x) = g_2(x)$ for all $x \in A$, as desired. □

4. Stability of the cubic set-valued functional equation

In this section, we prove the Hyers–Ulam stability of the cubic set-valued functional equation.

Theorem 4.1. *If $F : A + (-1)A \to cz(Y)$ is a set-valued map satisfying,*

$$F(2x+y) + F(2x-y) \subseteq 2F\,(x+y) + 2F(x-y) + 12F(x) \tag{4.1}$$

and

$$\sup\{diam(F(x)) : x \in A\} < +\infty$$

for all $x, y \in A$, then there exists a unique cubic map $g : A + (-1)A \to Y$ such that $g(x) \in F(x)$ for all $x \in A$.

Proof. Letting $y = 0$ in (4.1), we get

$$2F(2x) \subseteq 16F(x). \tag{4.2}$$

Replacing x by $2^n x$, $n \in \mathbb{N}$, in (4.2), we obtain

$$F(2 \cdot 2^n x) \subseteq 8F(2^n x)$$

and

$$\frac{F(2^{n+1}x)}{8^{n+1}} \subseteq \frac{F(2^n x)}{8^n}.$$

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

5. Stability of the quartic set-valued functional equation

In this section, we prove the Hyers–Ulam stability of the quartic set-valued functional equation.

Theorem 5.1. *If $F : A + (-1)A \to cz(Y)$ is a set-valued map satisfying $F(0) = \{0\}$,*

$$F(2x+y) + F(2x-y) + 6F(y) \subseteq 4F\,(x+y) + 4F(x-y) + 24F(x) \tag{5.1}$$

and

$$\sup\{diam(F(x)) : x \in A\} < +\infty$$

for all $x, y \in A$, then there exists a unique quartic map $g : A + (-1)A \to Y$ such that $g(x) \in F(x)$ for all $x \in A$.

Proof. Letting $y = 0$ in (5.1), we get

$$2F(2x) \subseteq 32F(x). \quad (5.2)$$

Replacing x by $2^n x$, $n \in \mathbb{N}$, in (5.2), we obtain

$$F(2 \cdot 2^n x) \subseteq 16F(2^n x)$$

and

$$\frac{F(2^{n+1}x)}{16^{n+1}} \subseteq \frac{F(2^n x)}{16^n}.$$

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

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