Stability of some set-valued functional equations

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\textbf{A B S T R A C T}

In this paper, we prove the Hyers–Ulam stability of some set-valued functional equations.

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\textbf{1. Introduction and preliminaries}

Set-valued functions in Banach spaces have received a lot of attention in the literature (see [1–3]). The pioneering papers by Aumann [1] and Debreu [2] were inspired by problems arising in Control Theory and Mathematical Economics. We also refer the reader to the papers by Arrow and Debreu [3], McKenzie [4] and the monographs by Hindenbrand [5], Aubin and Frankowska [6], Castaing and Valadier [7], Klein and Thompson [8] and the survey by Hess [9].

The paper of Rassias [10] has motivated the development of what we call Hyers–Ulam stability or the Hyers–Ulam–Rassias stability of functional equations (also see [11,12]). A generalization of the Rassias theorem was obtained by Găvruta [13] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.

The functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \]
is called a \textit{quadratic functional equation}. In particular, every solution of the quadratic functional equation is said to be a \textit{quadratic mapping}. A generalized Hyers–Ulam stability problem for the quadratic functional equation was discussed by Skof [14] for mappings \( f : X \to Y \), where \( X \) is a normed space and \( Y \) is a Banach space. Cholewa [15] noticed that the theorem of Skof is still true if the relevant domain \( X \) is replaced by an Abelian group. Czerwik [16] discussed the generalized Hyers–Ulam stability of the quadratic functional equation.

In [17], Jun and Kim considered the cubic functional equation
\[ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \] (1.1)

It is easy to show that the function \( f(x) = x^3 \) satisfies the functional equation (1.1) on \( \mathbb{R} \), which is called a \textit{cubic functional equation} and every solution of the cubic functional equation is said to be a \textit{cubic mapping}.

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In [18], Lee et al. considered the quartic functional equation
\begin{equation}
    f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y).
\end{equation}

It is easy to show that the function \( f(x) = x^4 \) satisfies the functional equation (1.2) on \( \mathbb{R} \), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping.

Let \( Y \) be a real normed space. The family of all closed subsets, containing \( 0 \), of \( Y \) will be denoted by \( cz(Y) \).

Let \( A, B \) be nonempty subsets of a real vector space \( X \) and \( \lambda \) a real number. We define
\begin{align*}
    A + B &= \{ x \in X : x = a + b, \ a \in A, \ b \in B \}, \\
    \lambda A &= \{ x \in X : x = \lambda a, \ a \in A \}.
\end{align*}

\textbf{Lemma 1.1} ([19]). Let \( \lambda \) and \( \mu \) be real numbers. If \( A \) and \( B \) are nonempty subset of a real vector space \( X \), then
\begin{align*}
    \lambda (A + B) &= \lambda A + \lambda B, \\
    (\lambda + \mu)A &\subseteq \lambda A + \mu A.
\end{align*}
Moreover, if \( A \) is a convex set and \( \lambda \mu \geq 0 \), then we have
\begin{equation}
    (\lambda + \mu)A = \lambda A + \mu A.
\end{equation}

A subset \( A \subseteq X \) is said to be a cone if \( A + A \subseteq A \) and \( \lambda A \subseteq A \) for all \( \lambda > 0 \). If the zero vector in \( X \) belongs to \( A \), then we say that \( A \) is a cone with zero.

Set-valued functional equations have been investigated by a number of authors and there are many interesting results concerning this problem (see [20–25]).

In this paper, we define the Jensen additive set-valued functional equation, the quadratic set-valued functional equation, the cubic set-valued functional equation and the quartic set-valued functional equation, and prove the Hyers–Ulam stability of some set-valued functional equations.

Throughout this paper, let \( X \) be a real vector space, \( A \subseteq X \) a cone with zero and \( Y \) a Banach space.

\section{Stability of the Jensen additive set-valued functional equation}

In this section, we prove the Hyers–Ulam stability of the Jensen additive set-valued functional equation.

\textbf{Theorem 2.1.} If \( F : A \to cz(Y) \) is a set-valued map satisfying \( F(0) = \{ 0 \} \),
\begin{equation}
    F(x) + F(y) \subseteq 2F \left( \frac{x + y}{2} \right)
\end{equation}
and
\[ \sup \{ \text{diam}(F(x)) : x \in A \} < +\infty \]
for all \( x, y \in A \), then there exists a unique additive map \( g : A \to Y \) (which we call the Jensen additive map) such that \( g(x) \in F(x) \) for all \( x \in A \).

\textbf{Proof.} For \( x \in A \), letting \( y = 0 \) in (2.1), we get
\begin{equation}
    F(x) + F(0) = F(x) \subseteq 2F \left( \frac{x}{2} \right)
\end{equation}
and if we replace \( x \) by \( 2^n x \), \( n \in \mathbb{N} \), in (2.2), then we obtain
\begin{equation}
    F(2^{n+1}x) \subseteq 2F(2^n x)
\end{equation}
and
\[ \frac{F(2^{n+1}x)}{2^{n+1}} \subseteq \frac{F(2^n x)}{2^n} \] .

Let \( F_n(x) = \frac{F(2^n x)}{2^n} , x \in A, n \in \mathbb{N} \) and we obtain that \( (F_n(x))_{n \geq 0} \) is a decreasing sequence of closed subsets of the Banach space \( Y \). We have also
\[ \text{diam}(F_n(x)) = \frac{1}{2^n} \text{diam}(F(2^n x)) \] .
Now since \( \sup \{ \text{diam}(F(x)) : x \in A \} < +\infty \), we get that \( \lim_{n \to +\infty} \text{diam}(F_n(x)) = 0 \) for all \( x \in A \).
Using the Cantor theorem for the sequence \((F_n(x))_{n \geq 0}\), we obtain that the intersection \(\bigcap_{n \geq 0} F_n(x)\) is a singleton set and we denote this intersection by \(g(x)\) for all \(x \in A\). Thus we obtain a map \(g : A \rightarrow Y\). Then \(g(x) \in F_0(x) = F(x)\) for all \(x \in A\).

Now we show that \(g\) is additive. We have (note Lemma 1.1)

\[
F_n(x) + F_n(y) = \frac{F(2^n x)}{2^n} + \frac{F(2^n y)}{2^n} \leq \frac{1}{2^n} 2F\left(\frac{2^n x + 2^n y}{2}\right) = 2F\left(\frac{x+y}{2}\right).
\]

By the definition of \(g\), we get for all \(x, y \in A\),

\[
g(x) + g(y) = \bigcap_{n=0}^{\infty} F_n(x) + \bigcap_{n=0}^{\infty} F_n(y) \subseteq \bigcap_{n=0}^{\infty} \left(2F\left(\frac{x+y}{2}\right)\right).
\]

Thus

\[
g(x) + g(y) = 2g\left(\frac{x+y}{2}\right)
\]

for all \(x, y \in A\) and so \(g\) is additive.

Therefore, we conclude that there exists an additive map \(g : A \rightarrow Y\) such that \(g(x) \in F(x)\) for all \(x \in A\).

Next, let us prove the uniqueness of \(g\).

Suppose that \(F\) have two additive selections \(g_1, g_2 : A \rightarrow Y\). We have

\[
ng_i(x) = g_i(nx) \in F(nx)
\]

for all \(n \in \mathbb{N}, x \in A, i \in \{1, 2\}\). Then we get

\[
n\|g_1(x) - g_2(x)\| = \|ng_1(x) - ng_2(x)\| = \|g_1(nx) - g_2(nx)\| \leq 2 \cdot \text{diam}(F(nx))
\]

for all \(x \in A, n \in \mathbb{N}\). It follows from \(\sup\{\text{diam}(F(x)) : x \in A\} < +\infty\) that \(g_1(x) = g_2(x)\) for all \(x \in A\), as desired. \(\square\)

3. Stability of the quadratic set-valued functional equation

In this section, we prove the Hyers–Ulam stability of the quadratic set-valued functional equation.

**Theorem 3.1.** If \(F : A + (-1)A \rightarrow cz(Y)\) is a set-valued map satisfying \(F(0) = \{0\}\),

\[
F(x + y) + F(x - y) \subseteq 2F(x) + 2F(y)
\]

and

\[
\sup\{\text{diam}(F(x)) : x \in A\} < +\infty
\]

for all \(x, y \in A\), then there exists a unique quadratic map \(g : A + (-1)A \rightarrow Y\) such that \(g(x) \in F(x)\) for all \(x \in A\).

**Proof.** Letting \(x = y\) in (3.1), we get

\[
F(2x) + F(0) = F(2x) \subseteq 4F(x).
\]

Replacing \(x\) by \(2^n x, n \in \mathbb{N}\), in (3.2), we obtain

\[
F(2 \cdot 2^n x) \subseteq 4F(2^n x)
\]

and

\[
F(2^n + 1 x) \subseteq \frac{F(2^n x)}{4^n}.
\]

Let \(F_n(x) = \frac{F(2^n x)}{4^n}, x \in A, n \in \mathbb{N}\), we obtain that \((F_n(x))_{n \geq 0}\) is a decreasing sequence of closed subsets of the Banach space \(Y\). We have also

\[
\text{diam}(F_n(x)) = \frac{1}{4^n} \text{diam}(F(2^n x)).
\]

Taking into account that \(\sup\{\text{diam}(F(x)) : x \in A\} < +\infty\), we get

\[
\lim_{n \rightarrow \infty} \text{diam}(F_n(x)) = 0.
\]

Using the Cantor theorem for the sequence \((F_n(x))_{n \geq 0}\), we obtain that the intersection \(\bigcap_{n \geq 0} F_n(x)\) is a singleton set and we denote this intersection by \(g(x)\) for all \(x \in A\). Thus we get a map \(g : A + (-1)A \rightarrow Y, g(x) \in F_0(x) = F(x)\) for all \(x \in A\).
We now show that $g$ is quadratic. For all $x, y \in A$ and $n \in \mathbb{N}$,
\[
F_n(x + y) + F_n(x - y) = \frac{F(2^n(x + y))}{4^n} + \frac{F(2^n(x - y))}{4^n} \leq \frac{2F(2^n x)}{4^n} + \frac{2F(2^n y)}{4^n} = 2F_n(x) + 2F_n(y).
\]
By the definition of $g$, we obtain
\[
g(x + y) + g(x - y) = \lim_{n \to \infty} F_n(x + y) + \lim_{n \to \infty} F_n(x - y) \leq \lim_{n \to \infty} (2F_n(x) + 2F_n(y)),
\]
which tends to zero as $n$ tends to $\infty$. Thus
\[
g(x + y) + g(x - y) = 2g(x) + 2g(y)
\]
for all $x, y \in A$.

Next, let us prove the uniqueness of $g$.

Suppose that $F$ have two quadratic selections $g_1, g_2 : A \to Y$. We have
\[
(2n)^2 g_i(x) = g_i(2nx) \in F(2nx)
\]
for all $n \in \mathbb{N}, x \in A, i \in \{1, 2\}$. Then we get
\[
(2n)^2 \| g_1(x) - g_2(x) \| = \| (2n)^2 g_1(x) - (2n)^2 g_2(x) \| = \| g_1(2nx) - g_2(2nx) \| \leq 2 \cdot diam(F(2nx))
\]
for all $x \in A, n \in \mathbb{N}$. It follows from $\sup \{diam(F(x)) : x \in A \} < +\infty$ that $g_1(x) = g_2(x)$ for all $x \in A$, as desired. $\square$

4. Stability of the cubic set-valued functional equation

In this section, we prove the Hyers–Ulam stability of the cubic set-valued functional equation.

**Theorem 4.1.** If $F : A \to \mathbb{C}(Y)$ is a set-valued map satisfying,
\[
F(2x + y) + F(2x - y) \leq 2F(x + y) + 2F(x - y) + 12F(x)
\]
and
\[
\sup \{diam(F(x)) : x \in A \} < +\infty
\]
for all $x, y \in A$, then there exists a unique cubic map $g : A \to Y$ such that $g(x) \in F(x)$ for all $x \in A$.

**Proof.** Letting $y = 0$ in (4.1), we get
\[
2F(2x) \leq 16F(x).
\]
Replacing $x$ by $2^n x, n \in \mathbb{N}$, in (4.2), we obtain
\[
F(2 \cdot 2^n x) \leq 8F(2^n x)
\]
and
\[
\frac{F(2^{n+1} x)}{8^{n+1}} \leq \frac{F(2^n x)}{8^n}.
\]
The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. $\square$

5. Stability of the quartic set-valued functional equation

In this section, we prove the Hyers–Ulam stability of the quartic set-valued functional equation.

**Theorem 5.1.** If $F : A \to \mathbb{C}(Y)$ is a set-valued map satisfying $F(0) = \{0\}$,
\[
F(2x + y) + F(2x - y) + 6F(y) \leq 4F(x + y) + 4F(x - y) + 24F(x)
\]
and
\[
\sup \{diam(F(x)) : x \in A \} < +\infty
\]
for all $x, y \in A$, then there exists a unique quartic map $g : A \to Y$ such that $g(x) \in F(x)$ for all $x \in A$. 
Proof. Letting $y = 0$ in (5.1), we get
\[ 2F(2x) \subseteq 32F(x). \]  
(5.2)
Replacing $x$ by $2^n x$, $n \in \mathbb{N}$, in (5.2), we obtain
\[ F(2 \cdot 2^n x) \subseteq 16F(2^n x) \]
and
\[ \frac{F(2^{n+1} x)}{16^{n+1}} \subseteq \frac{F(2^n x)}{16^n}. \]

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

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