Note

On the triangle vertex Folkman numbers

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Abstract

For a graph $G$ the symbol $G \rightarrow (3, \ldots, 3)$ means that in every $r$-colouring of the vertices of $G$
there exists a monochromatic triangle. The triangle vertex Folkman numbers $F_r(3) = \min \{|V(G)| : G \rightarrow (3, \ldots, 3) \text{ and } \cl(G) < 2r\}$ are considered. We prove that $F_r(3) = 2r + 7$, $r \geq 3$.

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1. Notation

We consider only finite, non-oriented graphs, without loops and multiple edges. We
call a $p$-clique of the graph $G$ a set of $p$ vertices, each two of which are adjacent.
The largest positive integer $p$ such that the graph $G$ contains a $p$-clique is denoted by
$
\cl(G).
$
In this paper we shall use also the following notation:

$V(G)$—vertex set of the graph $G$;
$E(G)$—edge set of the graph $G$;
$G - v, v \in V(G)$—subgraph of $G$ obtained from $G$ by the removal of $v$ and all edges
adjacent to $v$;
$G - e, e \in E(G)$—subgraph of $G$ such that $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$;
$G + e, e \in E(G)$—supergraph of $G$ such that $V(G + e) = V(G)$ and $E(G + e) = E(G) \cup \{e\}$;
$C_n$—simple cycle on $n$ vertices;

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Let $G_1$ and $G_2$ be two graphs without common vertices. We denote by $G_1 + G_2$ the graph $G$ for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y]: x \in V(G_1), y \in V(G_2)\}$.

2. Vertex Folkman numbers

**Definition.** Let $G$ be a graph and $a_1, \ldots, a_r$ be positive integers. The symbol $G \to (a_1, \ldots, a_r)$ means that for every $r$-colouring of the vertices of

$$V(G) = V_1 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j,$$

there exists $i \in \{1, 2, \ldots, r\}$ such that the graph $G$ contains a monochromatic $a_i$-clique $K$ of colour $i$, i.e. $V(K) \subseteq V_i$.

Define

$$H(a_1, \ldots, a_r; q) = \{G : G \to (a_1, \ldots, a_r) \text{ and } \text{cl}(G) < q\};$$

$$F(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \in H(a_1, \ldots, a_r; q)\}.$$  

It is clear that $G \to (a_1, \ldots, a_r)$ implies $\text{cl}(G) \geq \max\{a_1, \ldots, a_r\}$. Folkman [2], proves that there exists a graph $G$ such that $G \to (a_1, \ldots, a_r)$ and $\text{cl}(G) = \max\{a_1, \ldots, a_r\}$. Therefore the numbers $F(a_1, \ldots, a_r; q)$ exist only if $q > \max\{a_1, \ldots, a_r\}$. These numbers are called vertex Folkman numbers. The numbers $F(3, \ldots, 3; q)$ are called triangle vertex Folkman numbers.

Let $a_1, \ldots, a_r$ be positive integers. We define

$$m = \sum_{i=1}^{r} (a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, \ldots, a_r\}.$$  

Obviously, $K_m \to (a_1, \ldots, a_r)$ and $K_{m-1} \to (a_1, \ldots, a_r)$. Hence, if $q \geq m + 1$, then $F(a_1, \ldots, a_r; q) = m$.

The following propositions hold:

**Proposition 1** ([13]). If $G \to (a_1, \ldots, a_r)$, then $\chi(G) \geq m$.

The numbers $F(a_1, \ldots, a_r; m)$ exist only if $m \geq p + 1$. Łuczak and Urbański proved that $F(a_1, \ldots, a_r; m) = m + p$ [5]. The numbers $F(a_1, \ldots, a_r; m-1)$ exist only if $m \geq p+2$. The exact values of very few of the numbers $F(a_1, \ldots, a_r; m - 1)$ are known. All of these known values are given in [14]. We need only the following two numbers.

**Proposition 2** ([14]). $F(2, 3, 3; 5) = 12$.

**Proposition 3.** $F(3, 3; 4) = 14$. 

$K_n$—complete graph on $n$ vertices;  
$\chi(G)$—chromatic number of the graph $G$. 

Inequality $F(3, 3; 4) \leq 14$ is proved in [9] and the opposite inequality $F(3, 3; 4) \geq 14$ is verified by means of computer in [15].

3. Main result

Consider the vertex Folkman numbers $F(a_1, \ldots, a_r; m-1)$ in the case $a_1 = \cdots = a_r = p$. Define

$$H_r(p) = H(p, \ldots, p; (p - 1)r) \quad \text{and} \quad F_r(p) = F(p, \ldots, p; (p - 1)r).$$

For the numbers $F_r(2)$ it is known that:

**Proposition 4.**

$$F_r(2) = \begin{cases} 11, & r = 3 \text{ or } r = 4, \\ r + 5, & r \geq 5. \end{cases}$$

In [7] Mycielski presents an 11-vertex graph $G$ such that $G \rightarrow (2, 2, 2)$ and $\text{cl}(G) = 2$, proving that $F_3(2) \leq 11$. Chvátal [1], proved that the Mycielski graph is the smallest such graph, and hence $F_3(2) = 11$. The inequality $F_4(2) \geq 11$ was proved in [11], and the inequality $F_4(2) \leq 11$ in [8,10] (see also [12]). Since $G \rightarrow (2, \ldots, 2) \iff \chi(G) \geq r + 1$, the equality $F_r(2) = r + 5$ for $r \geq 5$ follows from the next theorem.

**Theorem A ([8,10]).** Let $G$ be a graph such that $\text{cl}(G) < r$ and $\chi(G) \geq r + 1$, $r \geq 5$. Then $|V(G)| \geq r + 6$ or $G = K_{r-5} + C_5 + C_5$.

The equality $G = K_{r-5} + C_5 + C_5$ means that $G$ is isomorphic to $K_{r-5} + C_5 + C_5$. The equality $F_r(2) = r + 5$, $r \geq 5$, is proved also in [6]. In this paper, we shall give a new proof of Theorem A in Section 7.

Our main result is the following:

**Theorem.** $F_r(3) = 2r + 7$, $r \geq 3$.

Łuczak et al. [6] proved that $2r + 5 \leq F_r(3) \leq 2r + 10$ for $r \geq 4$, and $11 \leq F_3(3) \leq 20$. The bound $F_r(3) \geq 2r + 7$ for $r \geq 4$ is also announced in [6]. However, a proof of this inequality published by these authors is unknown to us. In [13] we prove that $2r + 6 \leq F_r(3) \leq 2r + 8$, $r \geq 3$.

4. Some facts about critical chromatic graphs

A graph $G$ is called edge-critical $k$-chromatic if $\chi(G) = k$ and $\chi(G') < k$ for each proper subgraph $G'$ of $G$. It is clear that $G$ is an edge-critical $k$-chromatic graph if and
only if \( G \) is connected, \( \chi(G) = k \) and \( \chi(G - e) < k \), \( \forall e \in E(G) \). A graph \( G \) is defined to be vertex-critical \( k \)-chromatic if \( \chi(G) = k \) and \( \chi(G - v) < k \), \( \forall v \in V(G) \).

We shall use the following two theorems in the proof of the main result:

**Theorem B** ([3], see also [4]). Let \( G \) be a vertex-critical \( k \)-chromatic graph, \( k \geq 2 \). If \( |V(G)| < 2k - 1 \), then \( G = G_1 + G_2 \), where \( V(G_i) \neq \emptyset \), \( i = 1, 2 \).

**Theorem C** ([3], see also [4]). Let \( G \) be a vertex-critical \( k \)-chromatic graph, \( |V(G)| = n \) and \( k \geq 3 \). Then there exist \( \geq \lceil \frac{3}{2}(\frac{2}{3}k - n) \rceil \) vertices with the property that each of them is adjacent to all the other \( n - 1 \) vertices.

**Remark 1.** In the original statement of Theorems B and C, the graph \( G \) is edge-critical \( k \)-chromatic (and not vertex-critical \( k \)-chromatic). Since each vertex-critical \( k \)-chromatic graph \( G \) contains an edge-critical \( k \)-chromatic subgraph \( H \) such that \( V(G) = V(H) \), the above statements of these theorems are equivalent to the original ones. They are also more convenient for the proof of the main result.

**Remark 2.** The original statement of Theorem C includes also the condition \( n \geq \frac{2}{3}k \). This condition is redundant, since if \( n \geq \frac{2}{3}k \) then the claim of Theorem C is trivial.

5. Lemmas

**Lemma 1.** Let \( G \) be a graph such that
\[
G \rightarrow (\underbrace{3, \ldots, 3}_r), \quad r \geq 2.
\]
Then \( K_1 + G \rightarrow (\underbrace{3, \ldots, 3}_r) \).

**Proof.** Let \( V_1 \cup \cdots \cup V_{r+1} \) be an \( (r + 1) \)-colouring of \( V(K_1 + G) \), and let \( V(K_1) = \{ a \} \). Suppose that \( V_1 \) is an independent set. Then \( V_i = \{ a \} \) or \( a \notin V_i \). If \( V_1 = \{ a \} \), then \( V_2 \cup \cdots \cup V_{r+1} \) is an \( r \)-colouring of \( V(G) \). By (1), for some \( i \geq 2 \), \( V_i \) contains a 3-clique. If \( a \notin V_1 \), then we may assume that \( a \in V_2 \). Let \( V'_2 = V_2 \setminus \{ a \} \). Suppose that each \( V_i \), \( i \geq 3 \), does not contain a 3-clique. Consider the \( r \)-colouring \( (V_1 \cup V'_2) \cup V_3 \cup \cdots \cup V_{r+1} \) of \( V(G) \). By (1), \( V_1 \cup V'_2 \) contains a 3-clique. Since \( V_1 \) is an independent set, \( V'_2 \) is not an independent set. Thus, \( V_2 \) contains a 3-clique. \( \square \)

**Lemma 2.** Let \( G \) be a graph such that
\[
G \rightarrow (\underbrace{2, 3, \ldots, 3}_{r-1}), \quad r \geq 2.
\]
Then \( K_1 + G \rightarrow (\underbrace{3, \ldots, 3}_r) \).
Proof. Let \( V(K_1) = \{ a \} \). Assume the contrary and let \( V_1 \cup \cdots \cup V_r \) be an \( r \)-colouring of \( V(K_1 + G) \) without monochromatic 3-cliques. Let \( a \in V_i \). Then \( V'_i = V_i \setminus \{ a \} \) is an independent set. From (2) it follows that

\[
G \to (3, \ldots, 2, \ldots, 3). \tag{3}
\]

Consider the \( r \)-colouring \( V_1 \cup \cdots \cup V'_1 \cup \cdots \cup V_r \) of \( V(G) \). Since \( V'_i \) is an independent set, (3) implies that for some \( j \neq i \) the set \( V_j \) contains a 3-clique, which is a contradiction.

Lemma 3. Let \( G \) be a graph such that \( G \to (3, \ldots, 3) \). Then \( K_{2r} + G \to (3, \ldots, 3) \).

Proof. We prove this lemma by induction on \( r \). The base \( r = 0 \) is clear. Assume that \( r \geq 1 \). Applying the inductive hypothesis for \( K_{2r-2} + G \), we conclude that \( K_{2r-2} + G \to (3, \ldots, 3) \). By Lemma 1, \( K_{2r-1} + G \to (2, 3, \ldots, 3) \). From Lemma 2 it follows that

\[
K_{2r} + G \to (3, \ldots, 3). \tag{r+1}
\]

Lemma 4. Let \( G \) be a graph and \( G \in H_r(3) \), \( r \geq 3 \). If for some \( v \in V(G) \), \( \chi(G - v) \geq 2r + 1 \), then \( |V(G)| \geq 2r + 7 \).

Proof. Suppose the contrary, i.e. \( |V(G)| \leq 2r + 6 \). Then \( |V(G - v)| \leq 2r + 5 \). Since \( \text{cl}(G - v) < 2r \) and \( \chi(G - v) \geq 2r + 1 \), from Theorem A it follows that \( G - v = K_{2r-4} + C_5 + C_5 \). Thus, \( G \) is a subgraph of \( K_{2r-4} + C_5 + C_5 \). From the obvious equation

\[
K_{2r-4} + C_5 + C_5 = K_2 + \cdots + K_2 + C_5 + C_5
\]

it becomes clear that \( K_{2r-4} + C_5 + C_5 \to (3, \ldots, 3) \). Hence, \( G \to (3, \ldots, 3) \). This contradicts \( G \in H_r(3) \).

6. Proof of the Theorem

6.1. Proof of the inequality \( F_r(3) \geq 2r + 7, r \geq 3 \)

We prove this inequality by induction on \( r \). The base of the induction is \( r = 3 \). We need to prove that \( F_3(3) \geq 13 \). Assume the opposite. Let \( G \in H_3(3) \) and \( |V(G)| \leq 12 \). By Proposition 1 we have \( \chi(G) \geq 7 \). From Lemma 4 it follows that \( G \) is a vertex-critical 7-chromatic graph. By Theorem B, we have \( G = G_1 + G_2 \). Obviously, it is enough to consider only the situation when \( \text{cl}(G_1) \leq \text{cl}(G_2) \). Since \( \text{cl}(G) < 6 \), we must have \( \text{cl}(G_1) = 1 \) or \( \text{cl}(G_1) = 2 \). If\( \text{cl}(G_1) = 1 \), then from \( G \in H_3(3) \) we deduce that \( G_2 \in H(2, 3, 3; 5) \). This contradicts Proposition 2. If \( \text{cl}(G_1) = 2 \), then from \( G \in H_3(3) \) it follows that \( G_2 \in H_2(3) \), which contradicts Proposition 3.
Let $r \geq 4$ and $G \in H_r(3)$. We need to prove that $|V(G)| \geq 2r + 7$. Assume that \( \frac{1}{2}(\frac{5}{3}(2r+1) - n) \leq 1 \), where $n = |V(G)|$. Then $n \geq \left\lceil \frac{(10r+3)}{3} \right\rceil$. Since $r \geq 4$, we have $\left\lceil \frac{(10r+3)}{3} \right\rceil \geq 2r + 7$. Thus, $n = |V(G)| \geq 2r + 7$. Hence we can assume that

\[
\left\lceil \frac{3}{2} \left( \frac{5}{3}(2r+1) - n \right) \right\rceil \geq 2.
\] (4)

By Proposition 1, $\chi(G) \geq 2r + 1$. If for some $v \in V(G)$, $\chi(G - v) \geq 2r + 1$, then by Lemma 4, $|V(G)| \geq 2r + 7$. Hence, we need to consider only the case when $G$ is a vertex-critical $(2r + 1)$-chromatic graph. This, together with (4) and Theorem C, implies that $G = K_2 + G_1$. From $G \in H_r(3)$ it follows that $G_1 \in H_{r-1}(3)$. By the inductive hypothesis, $|V(G_1)| \geq 2r + 5$. Therefore, $|V(G)| \geq 2r + 7$.

6.2. Proof of the inequality $F_r(3) \leq 2r + 7$

Consider the graph $P$, whose complementary graph $\overline{P}$ is given in Fig. 1. In [14] it is proved that

\[
P \to (2, 3, 3).
\] (5)

By (5) and Lemma 2, $K_1 + P \to (3, 3, 3)$. Lemma 3 gives

\[
K_{2r-6} + (K_1 + P) = K_{2r-5} + P \to 3, \ldots, 3.
\]

Since $\text{cl}(P) = 4$, $\text{cl}(K_{2r-5} + P) = 2r - 1$. Hence $K_{2r-5} + P \in H_r(3)$. From $|V(K_{2r-5} + P)| = 2r + 7$ it follows that $F_r(3) \leq 2r + 7$. This completes the proof of the theorem. \( \square \)
7. Proof of Theorem A

Define $G_r = K_r - 5 + C_4 + C_5$, $r \geq 5$. Let $G$ be a graph such that $\text{cl}(G) < r$, $\chi(G) \geq r+1$ and $|V(G)| \leq r + 5$, $r \geq 5$. We need to prove that $G = G_r$. The proof starts by observing that

If $G_r$ is a subgraph of $G$ then $G = G_r$. \hspace{1cm} (6)

Indeed, the relations $|V(G)| \leq r + 5$ and $|V(G_r)| = r + 5$ imply $V(G_r) = V(G)$. Since $\text{cl}(G) < r$ and $\text{cl}(G_r + e) = r$, $\forall e \in E(G_r)$, we have $G = G_r$.

It follows from $\chi(G) \geq r + 1$ that $G$ contains a vertex-critical $(r + 1)$-chromatic subgraph. This fact and (6) imply that it suffices to prove the following statement:

If $G$ is a vertex-critical $(r + 1)$-chromatic graph such that $\text{cl}(G) < r$ and $|V(G)| \leq r + 5$ then $G = G_r$.

The proof is by induction on $r$, with induction base $r = 5$. Let $G$ be a vertex-critical 6-chromatic graph satisfying $\text{cl}(G) < 5$ and $|V(G)| \leq 10$. We claim that $G = G_5$. By Theorem B, we have $G = G_1 + G_2$. We will prove that

$$\text{cl}(G_1) = \text{cl}(G_2) = 2$$ \hspace{1cm} (7)

and

$$\chi(G_i) \geq 3, \quad i = 1, 2.$$ \hspace{1cm} (8)

Since $\text{cl}(G) = \text{cl}(G_1) + \text{cl}(G_2) \leq 4$, (7) will follow from the inequalities $\text{cl}(G_i) \geq 2$, $i = 1, 2$, which we are about to establish. Assume on the contrary that, for instance, $\text{cl}(G_1) = 1$. Clearly, $\chi(G_1) = 1$. We infer from $\text{cl}(G_1) = \chi(G_1) = 1$ that $\text{cl}(G_2) < 4$ and $\chi(G_2) = 5$. Since $|V(G_2)| \leq 9$, this is impossible by $F_4(2) = 11$ [11]. The contradiction proves (7). Assume that (8) is false. Let, for instance, $\chi(G_1) \leq 2$. Then $\chi(G_2) \geq 4$. Since $\text{cl}(G_2) = 2$, Chvátal’s result in [1] implies that $|V(G_2)| \geq 11$, contradicting $|V(G_2)| \leq 9$.

Now (7) and (8) yield $|V(G_i)| \geq 5$, $i = 1, 2$. Because $|V(G)| \leq 10$, we obtain

$$|V(G_i)| = 5, \quad i = 1, 2.$$ \hspace{1cm} (9)

It follows from (7) to (9) that $G_1 = G_2 = C_5$, i.e. $G = C_5 + C_5 = G_5$. We are done with the base case $r = 5$.

Let $r \geq 6$. Then

$$\frac{5}{3}(r + 1) - |V(G)| \geq \frac{5}{3}(r + 1) - (r + 5) > 0.$$

By Theorem C, we have $G = K_1 + G'$. It is clear that $G'$ is a vertex-critical $r'$-chromatic graph satisfying $\text{cl}(G') < r - 1$ and $|V(G')| \leq r + 4$. We obtain $G' = G_{r - 1}$ by the inductive hypothesis. Hence $G = G_r$, and Theorem A follows.

References