J. Math. Anal. Appl. 380 (2011) 865–881

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces

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ARTICLE INFO

Article history:
Received 16 September 2010
Available online 2 March 2011
Submitted by Steven G. Krantz

Keywords:
Gromov hyperbolicity
Gromov hyperbolic graph
Hyperbolic geometry
Riemannian surface
Pinched negative curvature

ABSTRACT

In this paper we obtain the equivalence of the Gromov hyperbolicity between an extensive class of complete Riemannian surfaces with pinched negative curvature and certain kind of simple graphs, whose edges have length 1, constructed following an easy triangular design of geodesics in the surface.

Contents

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[1] Supported in part by three grants from Ministerio de Ciencia e Innovación (MTM 2009-07800, MTM 2009-12740-C03-01 and MTM 2008-02829-E).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.02.067

1. Introduction

What is a Gromov hyperbolic space? A geodesic metric space is called *hyperbolic in the Gromov sense* if there exists an upper bound of the distance from every point in a side of any geodesic triangle to the union of the two other sides (see Definition 2.2). This condition is known as *Rips condition*. The underlying idea with regard to triangles is that in a Gromov hyperbolic space the geodesic triangles are thin, i.e., the Rips condition is another way to understand the negative curvature that the one traditionally formulated as the sum of the internal angles of any geodesic triangles is less than π.

The theory of Gromov hyperbolic spaces was introduced by Mikhail Gromov in the 1980s, cf. [14,15], and from then it has thereafter been studied and developed by many authors, e.g. [7,13,27,36]. It is specially remarkable the fact that this "new" theory grasps the connections between graphs and Potential Theory on Riemannian manifolds (see e.g. [4,18,21]).

The study of Gromov hyperbolicity of a Riemann surface with its Poincaré metric is non-trivial. An obvious reason is that homological "obstacles" may be surrounded by geodesic triangles which are not thin, as in the case of the two-dimensional jungle gym (a $\mathbb{Z}^2$-covering of a torus with genus two). An even stronger reason is the result, proved in [35], that the usual classes O_G, O_{HP}, O_{HB}, O_{HD}, and surfaces with linear isoperimetric inequality, are independent of the Gromov hyperbolic class. More precisely, in each of these classes, as well as in its complement, some surfaces are Gromov hyperbolic and some are not (even in the case of plane domains). This has stimulated a good number of works on the subject, e.g. [17,28,29,31,35] for negative constant curvature and [30,32] with negative variable curvature.

We are interested in studying conditions which determine when a given complete Riemannian surface S is Gromov hyperbolic. In order to do it, the main goal of this work is to get graph-structures $\mathcal{G}$, which are good models for surfaces and, in this way, moving the study of Gromov hyperbolicity from the surface to its associated graph, whose structure is very much simpler and, therefore, to study Rips condition shall be easier. Gromov hyperbolicity is of quite interest in metric graphs theory since it is closely related to concepts arising in the study of trees: in fact, we can consider hyperbolic graphs as a generalization of metric trees.

To replace surfaces, manifolds or even metric spaces by graphs (ε-nets) in order to study Gromov hyperbolicity, and other properties, has been a fruitful idea with many different applications (see [1,16,21]). In recent years, numerous techniques have been developed for the polygonization of surfaces, usually in triangles and quadrilaterals, like the triangulation for the protein design. The advantage of our results is that we use very simple graphs instead of ε-nets.

There are many applications which rely on the concept of Gromov hyperbolic graphs, for instance, measurements on the Internet indicate that it is negatively curved in the sense of Gromov (see [5]), the celebrated growth/preferential attachment process as a mean to construct a scale-free graph leads to a (scaled) Gromov negatively curved graph (see [19,20]), the greedy geographical routing is based on embedding the network graph in the Gromov hyperbolic Poincaré disk, such an embedding is accomplished with minimal distortion if the graph is Gromov hyperbolic (see [22]).

In Section 4, it will be presented a very simple technique for construction of appropriate grids in an extensive class of Riemannian surfaces. The idea is to get a suitable "discretization" of S, selecting particular points in it and connecting them by geodesics, obtaining a polygonization of the surface into hexagons, quadrilaterals and triangles. The important objects in this polygonization are the geodesic triangles, which will grasp all the necessary information about S from the point of view of the Gromov hyperbolicity. From this "triangulation" in the surface it is possible to get a graph $\mathcal{G}$, called *skeleton* (see Definition 4.14), and to obtain the equivalence of the hyperbolicity between both metric spaces.

The main result in this paper is Theorem 4.22, which can be stated in an informal way as follows:

An appropriate complete Riemannian surface, with curvature K satisfying $-k_2^2 \leqslant K \leqslant -k_1^2 < 0$, is hyperbolic if and only if its 1-skeleton, a graph whose edges have length exactly 1, is hyperbolic.

In [35] we obtained a result in the same line for negative constant curvature, $K = -1$. In that work we constructed graphs using strongly the constant curvature. Hence, in this current paper, it has been necessary to prove alternative new results that are valid for negative variable curvature. These new arguments have let us improve the previous paper, also in the following sense: we replace the graphs in [35] by simpler graphs whose edges have length exactly 1.

In order to prove our main theorem, we need technical results which are interesting by themselves. So Theorems 4.3 and 4.7 give some metric inequalities for Y-pieces with variable negative curvature.

The value of Theorem 4.22 is strengthened for the increasing interest of the study of Gromov hyperbolic graphs (see e.g. [6,23,25,26,34]).

Notation and terminology. We denote by X or X_n geodesic metric spaces. By d_X, L_X and diam_X we shall denote, respectively, the distance, the length and the diameter with the metric of X. We denote by S or S_n complete Riemannian surfaces, and by A_K the area in a simply connected Riemannian surface with curvature K.

As usual, we denote by x_+ the positive part of x: $x_+ := x$ if $x \geqslant 0$ and $x_+ := 0$ if $x < 0$.

For brevity we use the following notation: we write $A \lesssim B$, where A, B depend on some parameters, if there exists a constant c such that $A \leqslant cB$ for every value of the parameters. We write $A \approx B$ if $A \lesssim B \lesssim A$.

We say that a claim holds *quantitatively*, if it holds for parameters depending only on the constants in the assumptions. For instance, the first part of Theorem 2.13 says that if Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which just depends on δ, a and b.

2. Background on Gromov hyperbolic spaces

In general, Gromov hyperbolicity can be defined in non-geodesic spaces, but the definition which we use in this paper (which involves thin triangles definition) is valid only in geodesic spaces. Furthermore, it has the virtue of being intuitively simple. We refer to [15] for more background and further results about Gromov hyperbolic spaces.

2.1. The notion of Gromov hyperbolicity

Definition 2.1. A *geodesic* γ on a metric space X is an isometry between an interval $I \subset \mathbb{R}$ and X, i.e., $length(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t-s|$ for every $s, t \in I$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[x, y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). A *geodesic ray* in a space X is an isometric image of the half-line $[0, \infty)$.

Definition 2.2. If X is a geodesic metric space and J is a polygon whose sides are $J_1, J_2, \dots, J_n$, we say that J is *δ-thin* if for every $x \in J_i$ we have that $d(x, \bigcup_{j\neq i} J_j) \leqslant \delta$. We say that a polygon is *geodesic* if all of its sides are geodesics.

A geodesic metric space (X, d) is said to be *Gromov δ-hyperbolic*, if every geodesic triangle in X is δ-thin. We say that X is *hyperbolic* (in the Gromov sense) if it is δ-hyperbolic for some $\delta \geqslant 0$.

Definition 2.3. We define the *Gromov boundary of X*, ∂X, as the set of all geodesic rays emanating from some fixed point $w \in X$, modulo the equivalence relation given by taking any two rays which lie within a bounded Hausdorff distance of each other as equivalent; this definition is independent of w.

Remark 2.4. If X is δ-hyperbolic, it is easy to check that every geodesic polygon with n sides is $(n-2)\delta$-thin. We also have that every "ideal" geodesic polygon with n_1 sides in X and n_2 vertices in the Gromov boundary ∂X is $(n_1+n_2-2)\delta$-thin, i.e., a vertex in ∂X plays a similar role to an additional side.

Examples.

(1) Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic (see e.g. [15, p. 29]).
(2) Every complete simply connected Riemannian manifold with sectional curvature $K \leqslant -k^2$ is δ_1-hyperbolic, with $\delta_1 := \frac{1}{k}\log(1+\sqrt{2})$ (see e.g. [3, p. 130] and [15, p. 52]).
(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [15, p. 29]).

Definition 2.5. A *tripod*, $T := (V, E)$, is a tree (a graph with no cycles) with vertices $V := \{v, v_1, v_2, v_3\}$ and edges $E := \bigcup_{i=1}^{3}[v, v_i]$; hence the vertex v has degree 3 (i.e., $\deg(v) = 3$) and the vertices v_1, v_2, v_3 have degree 1 (i.e., $\deg(v_i) = 1$ for every $i = 1, 2, 3$).

Definition 2.6. Given a geodesic triangle $T = \{x_1, x_2, x_3\}$ in a geodesic metric space X, let T_E be a Euclidean triangle whose sides have the same lengths as those of T. Since there is no possible confusion, we will use the same notation for the corresponding points in T and T_E. The maximum inscribed circle in T_E meets the side $[x_j, x_k]$ in a point y_i, for every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, such that $d_X(x_i, y_j) = d_X(x_i, y_k)$ for every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$. We call the points y_1, y_2, y_3 the *internal points* of T. There is a unique local isometry f of T onto a tripod T_0, with z the vertex of degree 3, and z_1, z_2, z_3 the vertices of degree 1, such that $d_{T_0}(z, z_i) = d_X(y_j, x_i) = d_X(y_k, x_i)$ for every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.

The triangle T is *δ-fine* if $f(p) = f(q)$ implies that $d_X(p, q) \leqslant \delta$. The space X is *δ-fine* if every geodesic triangle in X is δ-fine.

A basic result is that hyperbolicity is equivalent to the property of being fine:

Theorem 2.7. *(See [15, p. 41].) Let us consider a geodesic metric space X.*

(1) *If X is δ-hyperbolic, then it is 4δ-fine.*
(2) *If X is δ-fine, then it is δ-hyperbolic.*

2.2. Auxiliary results on metric spaces

For Cartan–Hadamard manifolds, it is possible to generalize the concept of "fine" to triangles with vertices in the boundary, i.e., to ideal geodesic triangles. Recall that a Cartan–Hadamard manifold is a complete, connected and simply connected Riemannian n-manifold, $n \geqslant 2$, of non-positive sectional curvature. We shall need the following two lemmas.

Lemma 2.8. *Let M be a Cartan–Hadamard manifold with sectional curvatures $K \leqslant -k^2$, and T be a right-angled geodesic triangle in M with sides A, B, C, of respective lengths a, b, c, and opposite angles θ_1, θ_2, $\pi/2$. Then $a + b - \frac{2}{k}\log 2 \leqslant c \leqslant a + b$.*

Proof. It is easily seen by the triangle inequality that $c \leqslant a + b$. Now, by Aleksandrov's Comparison Theorem, it is known that $\cosh kc \geqslant \cosh ka \cosh kb$; hence, $e^{kc} \geqslant \frac{1}{4}e^{ka}e^{kb}$, and we have the required inequality. □

Lemma 2.9. *Let M be a Cartan–Hadamard manifold with sectional curvatures $K \leqslant -k^2$, and Q be a geodesic quadrilateral in M with consecutive sides A, B, C and D, of respective lengths a, b, c and a. Let us assume also that A and C meet orthogonally the sides B and D, respectively. Then we have that $|b - c| \leqslant \frac{2}{k}\log 2$.*

Proof. Notice that $L_M(A) = L_M(D) = a$ by hypothesis. We can split the quadrilateral into two right-angled geodesic triangles with a common side, the hypotenuse (with length r), whose sides have lengths a, b, r and a, c, r respectively. By Lemma 2.8 it holds $c \geqslant r - a$ and $b \leqslant r - a + \frac{2}{k}\log 2$; hence, $|b - c| \leqslant \frac{2}{k}\log 2$. □

We can give now a definition of fine ideal triangles.

Definition 2.10. Given a Cartan–Hadamard surface M, let us consider an ideal geodesic triangle $T = \{x_1, x_2, x_3\}$ in $M \cup \partial M$ with some $x_i \in \partial M$, and an inscribed circle C contained in T, which is tangent to the side $[x_j, x_k]$ at some point y_i', for every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$. We call the *internal points* of T (with respect to C) to those points $y_i \in [x_j, x_k]$ satisfying:

(1) If $x_i \in \partial M$ for every $i = 1, 2, 3$, then $y_i = y_i'$ for every $i = 1, 2, 3$.
(2) If $x_1, x_2 \in \partial M$ and $x_3 \in M$, then $y_1 = y_1'$, $y_3 = y_3'$ and $d_M(x_3, y_2) = d_M(x_3, y_1)$.
(3) If $x_1 \in \partial M$ and $x_2, x_3 \in M$, then $y_1 = y_1'$, $d_M(x_2, y_1) = d_M(x_2, y_3)$ and $d_M(x_3, y_1) = d_M(x_3, y_2)$.

There is a unique local isometry f of T onto a tripod T_0, with z the vertex of degree 3, and z_1, z_2, z_3 the vertices of degree 1, such that $d_{T_0}(z, z_i) = d_M(y_j, x_i) = d_M(y_k, x_i)$ for every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$. Note hat $L_{T_0}([z, z_i]) \in (0, \infty]$.

The triangle T is *δ-fine* if $f(p) = f(q)$ implies that $d_X(p, q) \leqslant \delta$ for some choice of circle C.

Theorem 2.11. *Any geodesic triangle (ideal or not) in a Cartan–Hadamard surface M with curvature $K \leqslant -k^2$ is $4\delta_1$-fine, with $\delta_1 := \frac{1}{k}\log(1 + \sqrt{2})$.*

Proof. Let us consider a geodesic triangle $T = \{x_1, x_2, x_3\}$ in $M \cup \partial M$. If T is not ideal, i.e., $T \subset M$, since M is δ_1-hyperbolic (see [3, p. 130] and [15, p. 52]), by Theorem 2.7 we can conclude T is $4\delta_1$-fine.

If T is an ideal geodesic triangle, let us consider the set E bounded by T. Next, we shall draw an inscribed ball $B(z, r)$ contained in E, which is tangent to the side $[x_j, x_k]$ in the points y_i', for every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$. Since $K \leqslant -k^2$, we have that

$$\frac{4\pi}{k^2}\sinh^2(kr/2) \leqslant A_K\big(B(z, r)\big) \leqslant A_K(E).$$

Denoting by θ_1, θ_2, θ_3 the internal angles of T, and according to Gauss–Bonnet formula, it holds

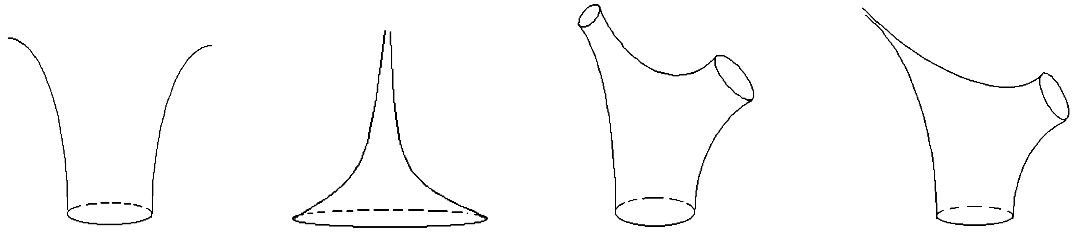
$$k^2 A_K(E) \leqslant -\iint_E K\,dA = \pi - \sum_{i=1}^{3}\theta_i \leqslant \pi.$$

Therefore, $4\sinh^2(kr/2) \leqslant 1$ and $r \leqslant \frac{2}{k}\operatorname{ar\,sinh}(1/2)$.

Let us consider the ideal geodesic quadrilateral $[x_3, y_1'] \cup [y_1', z] \cup [z, y_2'] \cup [y_2', x_3]$, where the geodesics $[y_1', z]$ and $[z, y_2']$ meet orthogonally the geodesics $[x_3, y_1']$ and $[y_2', x_3]$, respectively. Notice first that $L_M([y_1', z]) = L_M([z, y_2']) = r$ and, since the curvature can be non-constant, it is possible to have $L_M([x_3, y_1']) \neq L_M([y_2', x_3])$. Denoting by $t := L_M([x_3, y_1'])$ and $s := L_M([y_2', x_3])$, we can apply Lemma 2.9 and conclude that $|s - t| \leqslant \frac{2}{k}\log 2$. We shall repeat the argument with the other quadrilaterals and, if we denote by y_1, y_2, y_3 the internal points of T, it holds that $d_M(y_i, y_i') \leqslant \frac{2}{k}\log 2$ for $i = 1, 2, 3$.

According to Definition 2.10, it is easily seen that T is $\frac{2}{k}(\log 2 + 2\operatorname{ar\,sinh}(1/2))$-fine. Therefore, and taking into account that $\frac{2}{k}(\log 2 + 2\operatorname{ar\,sinh}(1/2)) \leqslant 4\delta_1$, we can assert that every ideal geodesic triangle in M is $4\delta_1$-fine. □

We present now the class of maps which play the main role in the theory.



(a) Funnel (b) Puncture (c) Y-piece (d) Generalized Y-piece

Fig. 1.

Definition 2.12. A function between two metric spaces $f : X \to Y$ is a *quasi-isometry* if there are constants $a \geqslant 1$, $b \geqslant 0$ with

$$\frac{1}{a} d_X(x_1, x_2) - b \leqslant d_Y\big(f(x_1), f(x_2)\big) \leqslant a d_X(x_1, x_2) + b, \quad \text{for every } x_1, x_2 \in X;$$

such a function is called an (a, b)-*quasi-isometry*.

An (a, b)-*quasigeodesic* in X is an (a, b)-quasi-isometry between an interval of $\mathbb{R}$ and X.

Notice that quasi-isometries are a very flexible kind of maps (they can be even discontinuous); however they are an important tool in Gromov theory, since they preserve the hyperbolicity:

Theorem 2.13. *(See [15, p. 88].) Let us consider an (a, b)-quasi-isometry between two geodesic metric spaces $f : X \to Y$. If Y is hyperbolic, then X is hyperbolic, quantitatively. Furthermore, if f is onto, then X is hyperbolic if and only if Y is hyperbolic, quantitatively.*

The following result will be useful in the next sections (see Theorems 3.14 and 4.22) in order to determine whether a given Riemannian surface is not hyperbolic.

Theorem 2.14. *(See [28, Theorem 2.1].) Let us consider a geodesic metric space X, and $X_1, X_2 \subset X$ two geodesic metric spaces such that $X_1 \cap X_2 = \eta_1 \cup \eta_2$, with η_i compact sets, $d_{X_2}(\eta_1, \eta_2) \geqslant c_2$ and $\operatorname{diam}_{X_i}(\eta_j) \leqslant c_1$ for $i, j = 1, 2$. If X is δ-hyperbolic, then $\delta \geqslant c_2/2 - c_1$.*

3. Background and previous results on Riemannian surfaces

3.1. Background

From now on, we will work just with orientable Riemannian surfaces and we always assume that the Riemannian metric is C^∞ unless perhaps in some simple closed geodesics, each of them bounding a funnel (see Definition 3.2), where we allow the metric to be C^1 and piecewise C^∞, with the "singularities" along these geodesics. Then the curvature is a (possibly discontinuous) function along these geodesics. There is a natural way to define a distance in subsets of Riemannian surfaces.

Definition 3.1. If S_0 is a path-connected subset of a Riemannian surface S, we can consider the *inner distance* as follows:

$$d_{S_0}(x, y) := d_S|_{S_0}(x, y) := \inf\big\{L_{S_0}(\gamma)\colon\ \gamma \subset S_0 \text{ is a continuous curve joining } x \text{ and } y\big\} \geqslant d_S(x, y).$$

Definition 3.2. A bordered or nonbordered surface is *doubly connected* if its fundamental group is isomorphic to $\mathbb{Z}$.

A *funnel* is a doubly connected bordered Riemannian surface whose border is a simple closed geodesic γ. If the curvature verifies $K \leqslant -k^2 < 0$ then there is no other simple closed geodesic freely homotopic to the border of the funnel, and γ minimizes the length in its free homotopy class. (See Fig. 1(a).)

A *puncture* is a doubly connected bordered Riemannian surface whose fundamental group is generated by a simple closed curve σ and there is no closed geodesic $\gamma \in [\sigma]$. If the curvature verifies $K \leqslant -k^2 < 0$ then $L([\sigma]) = \inf_{\gamma \in [\sigma]} L(\gamma) = 0$. (See Fig. 1(b).)

A *Y-piece* is a bordered Riemannian surface which is homeomorphic to a sphere minus three open disks and whose boundary curves are simple closed geodesics. They are a standard tool for constructing Riemannian surfaces with negative curvature. A clear description of these Y-pieces and their use are given in [8, Chapter 1] and [9, Chapter X.3]. (See Fig. 1(c).)

A *generalized Y-piece* is a Riemannian surface (with or without boundary) which is homeomorphic to a sphere without n open disks and m points, with integers $n, m \geqslant 0$ such that $n + m = 3$, the n boundary curves are simple closed geodesics and the m deleted points are punctures. Notice that a generalized Y-piece is topologically the union of a Y-piece and m cylinders. (See Fig. 1(d).)

Definition 3.3. Given a Riemannian surface S, a geodesic γ in S, and a continuous unit vector field ξ along γ, orthogonal to γ, we define the *Fermi coordinates* based on γ as the map $Y(\theta, r) := \exp_{\gamma(\theta)} r\xi(\theta)$.

It is well known that the Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dr^2 + G(\theta, r)^2\, d\theta^2$, where $G(\theta, r)$ is the solution of the scalar equation

$$\frac{\partial^2 G}{\partial r^2}(\theta, r) + K(\theta, r)G(\theta, r) = 0, \qquad G(\theta, 0) = 1, \qquad \frac{\partial G}{\partial r}(\theta, 0) = 0 \tag{3.4}$$

(see e.g. [9, p. 247]).

Lemma 3.5. *(See [32, Lemma 3.1].) Let us consider the positive function $G(\theta, r)$ which is the solution of Eq.* (3.4)*. The following inequalities hold*:

(1) *If $K(\theta, r) \leqslant -k^2 < 0$, then $G(\theta, r) \geqslant \cosh kr$ for every $\theta, r \in \mathbb{R}$.*
(2) *If $K(\theta, r) \geqslant -k^2$, then $G(\theta, r) \leqslant \cosh kr$ for every $\theta, r \in \mathbb{R}$.*

As a consequence of this previous lemma we obtain the following results.

Lemma 3.6. *(See [32, Lemma 3.2].) Let us consider $\mathbb{R}^2 = \{(\theta, r):\ \theta, r \in \mathbb{R}\}$ with two different metrics given in Fermi coordinates as $ds_1^2 = dr^2 + G_1(\theta, r)^2\, d\theta^2$ and $ds_2^2 = dr^2 + G_2(\theta, r)^2\, d\theta^2$, such that their respective curvatures, K_1 and K_2, satisfy $K_1(\theta, r) \leqslant K_2(\theta, r) = -k^2 < 0$. Let us consider two curves σ_1 and σ_2 in $\mathbb{R}^2$ with the same endpoints, such that σ_i is a geodesic for ds_i $(i = 1, 2)$. Then, $L_{ds_1}(\sigma_1) \geqslant L_{ds_2}(\sigma_2)$.*

Lemma 3.7. *Let us consider $\mathbb{R}^2 = \{(\theta, r):\ \theta, r \in \mathbb{R}\}$ with two different metrics given in Fermi coordinates as $ds_1^2 = dr^2 + G_1(\theta, r)^2\, d\theta^2$ and $ds_2^2 = dr^2 + G_2(\theta, r)^2\, d\theta^2$, such that their respective curvatures, K_1 and K_2, verify $K_1(\theta, r) \leqslant K_2(\theta, r) = -k^2 < 0$. Let us consider the simply connected right-angled quadrilateral $Q := \{(\theta, r):\ 0 \leqslant \theta \leqslant c,\ 0 \leqslant r \leqslant a\}$ in $\mathbb{R}^2$, then*

$$A_{K_1}(Q) \geqslant A_{K_2}(Q) = \frac{c}{k} \sinh ka.$$

Remark 3.8. Note that Q is not a geodesic quadrilateral, although three of its sides are geodesics.

Proof. Notice that $G_2(\theta, r) = \cosh kr$; by Lemma 3.5, we have that $G_1(\theta, r) \geqslant G_2(\theta, r)$ for every $(\theta, r) \in \mathbb{R}^2$. Since $dA_{K_i} = G_i(\theta, r)\, dr\, d\theta$ for $i = 1, 2$, we deduce

$$A_{K_1}(Q) = \int_0^c \int_0^a G_1(\theta, r)\, dr\, d\theta \geqslant \int_0^c \int_0^a G_2(\theta, r)\, dr\, d\theta = \int_0^c \int_0^a \cosh kr\, dr\, d\theta = \frac{c}{k} \sinh ka = A_{K_2}(Q). \qquad \square$$

In [10] Chavel and Feldman have proved the following theorem, which generalizes to negative variable curvature the Randol's Collar Lemma about the existence of collars centered on simple closed geodesics with constant curvature $K = -1$ (see [33]).

Definition 3.9. A *collar* in a Riemannian surface S about a simple closed geodesic γ is a doubly connected domain in S bounded by two Jordan curves (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from γ; such a collar is equal to $\{p \in S:\ d_S(p, \gamma) < d\}$, for some positive constant d. The constant d is called the *width* of the collar.

Theorem 3.10. *(See [10, p. 446].) Let S be a Riemannian surface with curvature satisfying $-k^2 \leqslant K \leqslant 0$, and γ a simple closed geodesic on S of length L_γ. Then there exists a collar whose width d satisfies*

$$d \geqslant \frac{1}{k} \operatorname{ar\,cosh}\left(\coth\left(\frac{kL_\gamma}{2}\right)\right). \tag{3.11}$$

Let η_1, η_2 be the boundary curves of such cylinder centered on γ and of width $d = \frac{1}{k} \operatorname{ar\,cosh}(\coth(\frac{kL_\gamma}{2}))$; Lemma 3.5 and $-k^2 \leqslant K$ allow to deduce

$$L_S(\eta_i) \leqslant L_\gamma \coth\left(\frac{kL_\gamma}{2}\right) \quad \text{for } i = 1, 2. \tag{3.12}$$

3.2. Criteria to deduce the non-hyperbolicity of surfaces

Definition 3.13. If c is a positive constant, we say that a complete Riemannian surface S has *c-wide genus* if every simple closed geodesic $\gamma \subset S$ such that $S \setminus \gamma$ is connected, verifies $L_S(\gamma) \geqslant c$. We say that S has *narrow genus* if there is not $c > 0$ such that S has c-wide genus.

Notice that any planar domain has c-wide genus for every c, and that any Riemannian surface with finite genus has c-wide genus for some c.

We will need the following general criteria which guarantees that many surfaces are not hyperbolic.

Theorem 3.14. *Let us consider any complete Riemannian surface S (with or without boundary) with pinched curvature $-k_2^2 \leqslant K \leqslant -k_1^2 < 0$; if S has boundary, we also require that ∂S is the union of simple closed geodesics. If S has narrow genus then it is not hyperbolic.*

Proof. We perform the proof in three steps. We first show that there is always a complete Riemannian surface, R, without boundary and with pinched curvature, containing S. Then, in order to prove the theorem, it suffices to consider only complete Riemannian surfaces without boundary and with pinched curvature. Finally, the result follows then by Theorem 2.14.

Step 1. Let us assume that S has boundary, the hypothesis implies that ∂S is the union of pairwise disjoint simple closed geodesics. In this case, we can construct a complete Riemannian surface R without boundary and with $-k_2^2 \leqslant K \leqslant -k_1^2 < 0$ by pasting to S a cylinder along each simple closed geodesic as follows: If $\gamma_0 \subseteq \partial S$ is a closed geodesic with length l, we can consider the Fermi coordinates based on γ_0. The Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dr^2 + G(\theta, r)^2\, d\theta^2$, with $G(\theta, r)$ satisfying (3.4), with $G(\theta, r)$ an l-periodic function in θ defined in $\mathbb{R} \times [-r_0, 0]$, for some $r_0 > 0$. We have $G(\theta, 0) = 1$ and $\partial G/\partial r(\theta, 0) = 0$ for every $\theta \in \mathbb{R}$. If we define $G(\theta, r) := \cosh k_1 r$ in $\mathbb{R} \times (0, \infty)$, then it is C^1 (and even piecewise C^∞) in $\mathbb{R} \times [-r_0, \infty)$, and l-periodic in θ; furthermore, we have that $K(\theta, r) = -k_1^2$ in $\mathbb{R} \times (0, \infty)$. These coordinates $(\theta, r) \in \mathbb{R} \times [-r_0, \infty)$, with the Riemannian metric $ds^2 = dr^2 + G(\theta, r)^2\, d\theta^2$, attach a funnel to γ_0.

This allows to attach a funnel to S along each simple closed geodesic $\gamma_0 \subseteq \partial S$ and to get a complete Riemannian surface R containing S and with pinched curvature.

Step 2. Since S is geodesically convex in R (every geodesic connecting two points of S is contained in S), then $d_R(z, w) = d_S(z, w)$ for every $z, w \in S$, and any simple closed geodesic in R is contained in S. Therefore, it is sufficient to prove the theorem for surfaces without boundary.

Step 3. Let S be a complete Riemannian surface without boundary, with $-k_2^2 \leqslant K \leqslant -k_1^2 < 0$ and narrow genus. Hence, there exists a sequence of simple closed geodesics $\{\gamma_n\}_n$ in S with $S \setminus \gamma_n$ connected such that $\lim_{n\to\infty} L_S(\gamma_n) = 0$.

The point is to apply Theorem 2.14. By Theorem 3.10, since $-k_2^2 \leqslant K \leqslant -k_1^2 < 0$, it is known that there exists a collar centered on γ_n of width

$$d_n = \frac{1}{k_2} \operatorname{ar}\cosh\left(\coth\left(\frac{k_2 L_S(\gamma_n)}{2}\right)\right).$$

We will divide the surface S into bordered surfaces in the following way. Let us define the bordered Riemannian surfaces S_1^n as the cylinder centered on γ_n of width d_n, and $S_2^n := \overline{S \setminus S_1^n}$, which is connected since $S \setminus \gamma_n$ is connected. We have that $\partial S_1^n = \partial S_2^n = S_1^n \cap S_2^n = \eta_1^n \cup \eta_2^n$ and $d_S(\eta_1^n, \eta_2^n) = d_{S_1^n}(\eta_1^n, \eta_2^n) = 2d_n$. Since $\operatorname{diam}_S(\eta_i^n) \leqslant L_S(\eta_i^n)$, by Theorem 2.14, (3.11) and (3.12), if S is δ-hyperbolic, then

$$\delta \geqslant d_n - L_S(\eta_i^n) \geqslant \frac{1}{k_2} \operatorname{ar}\cosh\left(\coth\left(\frac{k_2 L_S(\gamma_n)}{2}\right)\right) - L_S(\gamma_n) \coth\left(\frac{k_2 L_S(\gamma_n)}{2}\right).$$

Since $\lim_{n\to\infty} L_S(\gamma_n) = 0$, we deduce that $\delta = \infty$ and, therefore, S is not hyperbolic. □

3.3. Technical results

In the next section, hexagons will be quite useful; therefore it is crucial the following result of P. Buser, which shows that hexagons satisfy hyperbolic trigonometric inequalities on surfaces of variable negative curvature.

Theorem 3.15. *(See [8, Theorem 2.5.11].) Let S be a complete simply connected Riemannian surface of negative curvature $-k_2^2 \leqslant K \leqslant -k_1^2$. For any right-angled convex geodesic hexagon in S with consecutive sides of lengths a, γ, b, α, c, β, the following inequalities hold*

$$\sinh k_1 a \sinh k_1 b \cosh k_1 \gamma \leqslant \cosh k_1 c + \cosh k_1 a \cosh k_1 b,$$

$$\sinh k_2 a \sinh k_2 b \cosh k_2 \gamma \geqslant \cosh k_2 c + \cosh k_2 a \cosh k_2 b.$$

These inequalities also hold if some sides of the hexagon have length zero, i.e., if we have an ideal right-angled hexagon.

A closely related result concerning to the estimation of the lengths of the sides in a right-angled hexagon has been obtained recently in [29].

Proposition 3.16. *(See [29, Proposition 4.8].)*

(1) *For every $x, y, t \geqslant 0$ it holds*

$$f(x,y,t) := \operatorname{ar\,cosh} \frac{\cosh t + \cosh x \cosh y}{\sinh x \sinh y} \gtrsim e^{-x} + e^{-y} + e^{-\frac{1}{2}(x+y-t)_+} + (t-x-y)_+.$$

(2) *Given $l_0 > 0$, for every $x, y \geqslant l_0$ and $t \geqslant 0$ it holds*

$$\operatorname{ar\,cosh} \frac{\cosh t + \cosh x \cosh y}{\sinh x \sinh y} \lesssim e^{-x} + e^{-y} + e^{-\frac{1}{2}(x+y-t)_+} + (t-x-y)_+.$$

Remark 3.17. Note that if H is a right-angled hexagon in the unit disk for which three pairwise non-adjacent sides X, Y, T are given (with respective lengths x, y, t), then the opposite side of T in H has length $f(x,y,t)$ (see e.g. [11, p. 86]), and if in addition $x, y \geqslant l_0$, then

$$f(x,y,t) \approx e^{-x} + e^{-y} + e^{-\frac{1}{2}(x+y-t)_+} + (t-x-y)_+.$$

Furthermore, the two following technical lemmas show that, under a few metric restrictions, every point in either a geodesic hexagon or a geodesic quadrilateral is near a side.

Lemma 3.18. *Let S be a complete Riemannian surface with $K \leqslant -k^2 < 0$, and let H be a simply connected right-angled geodesic hexagon in S with three alternate sides, A_i, such that $L_S(A_i) \leqslant L$ for $i = 1, 2$ and $L_S(A_3) > 8\delta_1$. Denoting by η the side which joins A_1, A_2, then $d_S(z,\eta) \leqslant 12\delta_1 + L$ for every $z \in \partial H$, with $\delta_1 := \frac{1}{k}\log(1+\sqrt{2})$.*

Proof. Let us denote by η_1, η_2 the geodesic sides joining the geodesic A_3 and the geodesics A_1, A_2, respectively. Without loss of generality we can assume that S is simply connected, since otherwise we can lift H to the universal covering of S (recall that H is simply connected and that the distances in the universal cover are greater than in the surface). Since S is a simply connected complete Riemannian surface with $K \leqslant -k^2 < 0$, it is δ_1-hyperbolic (see [3, p. 130] and [15, p. 52]) and hence H is $4\delta_1$-thin.

We can consider A_3 as an oriented curve from η_1 to η_2; since $L_S(A_3) > 8\delta_1$, we can assert that there exist two points α and β in the oriented geodesic A_3 defined as $\alpha := \max\{z \in A_3\colon\ d_S(z,\eta_1) \leqslant 4\delta_1\}$ and $\beta := \min\{z \in A_3\colon\ d_S(z,\eta_2) \leqslant 4\delta_1\}$. If $z \in (\alpha,\beta)$ it holds $d_S(z, A_1 \cup \eta \cup A_2) \leqslant 4\delta_1$, since H is $4\delta_1$-thin. If $z \notin (\alpha,\beta)$ then $d_S(z,[\alpha,\beta]) \leqslant 4\delta_1$ and $d_S(z, A_1 \cup \eta \cup A_2) \leqslant 8\delta_1$. Taking into account that $L_S(A_i) \leqslant L$ for $i = 1, 2$, it holds $d_S(z,\eta) \leqslant 8\delta_1 + L$ for every $z \in \bigcup_{i=1}^{3} A_i$. Since $d_S(\eta_1,\eta_2) = L_S(A_3) > 8\delta_1$, if $z \in \eta_1$, then $d_S(z, \partial H \setminus \eta_1) = d_S(z, \partial H \setminus \{\eta_1,\eta_2\}) \leqslant 4\delta_1$ and $d_S(z,\eta) \leqslant 12\delta_1 + L$. Similarly, if $z \in \eta_2$, then $d_S(z,\eta) \leqslant 12\delta_1 + L$. □

Lemma 3.19. *Let S be a complete Riemannian surface with $K \leqslant -k^2 < 0$, and let Q be a simply connected geodesic quadrilateral in S with consecutive sides, A, B, C and η such that $L_S(A) \leqslant L$ and B hits orthogonally the sides A and C. Then $d_S(z,\eta) \leqslant 4\delta_1 + L$ for every $z \in \partial Q$, with $\delta_1 := \frac{1}{k}\log(1+\sqrt{2})$.*

Proof. Without loss of generality we can assume that S is simply connected, since otherwise we can lift Q to the universal covering of S (recall that Q is simply connected and that the distances in the universal cover are greater than in the surface). Since S is a simply connected complete Riemannian surface with $K \leqslant -k^2 < 0$, it is δ_1-hyperbolic (see [3, p. 130] and [15, p. 52]) and hence Q is $2\delta_1$-thin.

If $z \in A$, then $d_S(z,\eta) \leqslant L$. If $z \in C$ there are two possibilities. If $L_S(C) \leqslant 2\delta_1$ then $d_S(z,\eta) \leqslant 2\delta_1$. If $L_S(C) > 2\delta_1$, let us consider the geodesic C as an oriented curve from B to η; therefore, we can assert that there exists a point α in the oriented geodesic C defined as $\alpha := \max\{z \in C\colon\ d_S(z,B) \leqslant 2\delta_1\}$. If $z > \alpha$ it holds $d_S(z,B) > 2\delta_1$ and, since Q is $2\delta_1$-thin, therefore $d_S(z, A \cup \eta \cup B) = d_S(z, A \cup \eta) \leqslant 2\delta_1$; consequently, for every $z \geqslant \alpha$ we have that $d_S(z,\eta) \leqslant 2\delta_1 + L$. If $z < \alpha$, then $d_S(z,\alpha) \leqslant 2\delta_1$ and, therefore, $d_S(z,\eta) \leqslant 4\delta_1 + L$.

Finally, if $z \in B$ we repeat the previous argument; if $L_S(B) \leqslant 2\delta_1$, then $d_S(z,\eta) \leqslant d_S(z,A) + L \leqslant 2\delta_1 + L$. If $L_S(B) > 2\delta_1$, let us consider the geodesic B as an oriented curve from A to C; therefore, we can assert that there exists a point β in the oriented geodesic B defined as $\beta := \min\{z \in B\colon\ d_S(z,C) \leqslant 2\delta_1\}$. If $z < \beta$, then $d_S(z,C) > 2\delta_1$ and, since Q is $2\delta_1$-thin, therefore $d_S(z, A \cup \eta \cup C) = d_S(z, A \cup \eta) \leqslant 2\delta_1$; consequently, for every $z \leqslant \beta$ we have that $d_S(z,\eta) \leqslant 2\delta_1 + L$. If $z > \beta$, then $d(z,\beta) \leqslant 2\delta_1$ and, therefore, $d_S(z,\eta) \leqslant 4\delta_1 + L$. □

4. Generating graphs in Riemannian surfaces: skeletons

In this section we obtain the equivalence of the hyperbolicity of an extensive class of Riemannian surfaces and some simple graphs (see Theorems 4.17 and 4.22). The kind of surfaces which we are going to study is the set of complete Riemannian surfaces S (with or without boundary), which can be decomposed in a union of funnels and generalized Y-pieces; this is a large class of surfaces (see [2], [12, Theorem 4.1] and [30]).

Since the proofs of Theorems 4.17 and 4.22 are long, we have decided to split them into several steps (the results appearing in the two following subsections are, in fact, tools for the proof of these theorems; nevertheless, some of these results, as Theorems 4.3 and 4.7, have their own interest).

We start by constructing a particular kind of trees, $\mathcal{T} := (\mathcal{V}, \mathcal{E})$, associated to a generalized Y-piece, with $\mathcal{V} := \mathcal{V}(\mathcal{T})$ the set of vertices and $\mathcal{E} := \mathcal{E}(\mathcal{T})$ the set of edges.

4.1. Definitions and metric estimations in Y-pieces

In order to construct these trees associated to the generalized Y-pieces let us introduce some definitions.

Definition 4.1. Let us consider any generalized Y-piece, Y, with simple closed geodesics (or punctures) $\gamma_1, \gamma_2, \gamma_3 \in \partial Y$; if $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$, we shall call *seams* of Y to the geodesics Γ_i in Y joining γ_j and γ_k (and orthogonal to both of them) such that $L_Y(\Gamma_i) = d_Y(\gamma_j, \gamma_k)$.

We shall call *related hexagons to* Y to the two ideal right-angled geodesic hexagons, H and H', obtained by splitting Y along its seams, such that $Y = H \cup H'$ and $\partial H \cap \partial H' = \bigcup_{i=1}^{3} \Gamma_i$.

The geodesics $\eta_i := \gamma_i \cap H$ and $\eta_i' := \gamma_i \cap H'$ will be called *geodesics related to the geodesic* γ_i in H and H', respectively; therefore, $\gamma_i = \eta_i \cup \eta_i'$ for $i = 1, 2, 3$.

We shall call *related triangles to* Y to the two ideal geodesic triangles, T and T', contained in the related hexagons, H and H' respectively, constructed in the following way. If we denote by w_i the middle point in the related geodesic η_i and by w_i' the middle point in the related geodesic η_i', for $i = 1, 2, 3$; we define $T = \{w_1, w_2, w_3\} \subset H$ and $T' = \{w_1', w_2', w_3'\} \subset H'$ (see Fig. 2).

If $L_Y(\gamma_i) = 0$ (i.e., if γ_i is a puncture) for some i, both related triangles have, at least, two sides with infinity length.

Remark 4.2. The generalized Y-piece, Y, has been splitting into the union of two right-angled geodesic hexagons H and H', which are, as well, the union of four simply connected sets bounded by three geodesics quadrilaterals and its related triangles, respectively.

Notice that, due to the variable curvature, H and H' can be non-isometric; therefore, in general, it is possible to have $L_H(\eta_i) \neq L_{H'}(\eta_i')$ for every $i = 1, 2, 3$. This fact makes more complicated our work.

The following theorem shows that given any generalized Y-piece with pinched negative curvature, the lengths of its seams are bounded in terms of the bounds of the lengths of its boundary geodesics; furthermore, we have got explicit expressions for these bounds. Next Theorem 4.3 is interesting by itself, since it generalizes the result proved in [35] with constant negative curvature to pinched negative curvature, and we will use it in the following results of this section.

Theorem 4.3. *Let us consider a generalized Y-piece, Y, with $-k_2^2 \leqslant K \leqslant -k_1^2 < 0$ and $l \leqslant L_Y(\gamma_i) \leqslant L$ for every closed geodesic $\gamma_i \subseteq \partial Y$, for $i = 1, 2, 3$. There exist constants c_0, m and M, which just depend on l, L, k_1 and k_2, with the following properties.*

(1) *The seams verify*

$$m \leqslant L_Y(\Gamma_i) \leqslant M \quad \textit{for every } i = 1, 2, 3.$$

(2) *There exists a hexagon related to Y, H, whose sides are the seams Γ_i and the related geodesics η_i for every $i = 1, 2, 3$, such that*

$$c_0 \leqslant L_Y(\eta_i) \leqslant L \quad \textit{for every } i = 1, 2, 3.$$

Furthermore, we have explicit formulas for the constants:

$$m := \frac{1}{k_2} \operatorname{ar}\cosh\bigl(\coth(Lk_2/2)\bigr), \qquad C := \frac{1}{k_1} \operatorname{ar}\cosh \frac{\cosh Lk_1 (1 + \cosh Lk_1)}{\sinh^2(lk_1/2)},$$
$$c_0 := \min\left\{ \frac{l}{2}, \frac{1}{k_2} \operatorname{ar}\coth\left(\cosh\left(\frac{k_2}{k_1} \max\left\{ \operatorname{ar}\cosh\left(\frac{\sinh(Ck_1/2)}{\sinh(lk_1/8)} \right), \operatorname{ar}\sinh\left(\frac{4\pi}{lk_1} \right) \right\} \right) \right) \right\},$$
$$M := \frac{1}{k_1} \operatorname{ar}\cosh \frac{\cosh Lk_1 (1 + \cosh Lk_1)}{\sinh^2(c_0 k_1)}.$$

Proof. We perform the proof in three steps. First, we prove that there exists a positive constant m such that $m \leqslant L_Y(\Gamma_i)$ for every $i = 1, 2, 3$. Then, we prove that there exist two positive constants C and c_0 such that C is an upper bound for the length of at least one of the seams, and $c_0 \leqslant L_Y(\eta_i)$ for every $i = 1, 2, 3$. This second step implies that there exists a positive constant M such that $L_Y(\Gamma_i) \leqslant M$ for every $i = 1, 2, 3$.

Step 1. The lower bound for the length of the seams is easily obtained from Theorem 3.10: since $-k_2^2 \leqslant K \leqslant 0$, for any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$ there exist two collars, each of them centered on γ_j and γ_k, of widths d_j, d_k respectively, verifying (3.11). Therefore,

$$L_Y(\Gamma_i) \geqslant d_j + d_k \geqslant \max\{d_j, d_k\} \geqslant \frac{1}{k_2}\left(\operatorname{ar}\cosh\left(\coth\left(\frac{k_2 \min\{L_S(\gamma_j), L_S(\gamma_k)\}}{2}\right)\right)\right) \geqslant m.$$

Step 2. It is easily seen that at least two related geodesics in either H or H' have lengths greater or equal than $l/2$; without loss of generality we can assume that this happens in H (if this happens in H' we shall rename it as H). Hence, we have $L_Y(\eta_1), L_Y(\eta_2) \geqslant l/2$ and, applying Theorem 3.15, we deduce the inequality $L_Y(\Gamma_3) \leqslant C$. If $L_Y(\eta_3) \geqslant l/2$, then we obtain the same inequality for $L_Y(\Gamma_1)$ and $L_Y(\Gamma_2)$, and we finish the proof with $M := C$ and $c_0 := l/2$.

Therefore, let us deal with the case $L_Y(\eta_3) < l/2$. We shall prove that $L_Y(\eta_1), L_Y(\eta_2) \geqslant l/2$ implies $L_Y(\eta_3) \geqslant c_0$. Let us consider a hexagon H^* isometric to H; pasting the seams of H and H^* we obtain a new Y-piece, Y^*. Since $-k_2^2 \leqslant K < 0$, by Theorem 3.10, there exists a collar in Y^*, centered on η_3 (and its symmetric geodesic in H^*), with width d satisfying

$$d \geqslant \frac{1}{k_2} \operatorname{ar}\cosh\bigl(\coth L_Y(\eta_3) k_2\bigr). \tag{4.4}$$

Then we deduce $L_Y(\Gamma_1), L_Y(\Gamma_2) \geqslant d$.

If we prove that d has an upper bound, then we will have a lower bound for $L_Y(\eta_3)$. In order to find an upper bound for d we shall construct quadrilaterals completely contained in H' in the following way. Let us define the set Q as the intersection of H' with the neighborhood of γ_3 of radius d. Let us consider the following geodesics contained in ∂Q: η_3', with length $L_Y(\eta_3') \geqslant l - L_Y(\eta_3) > l/2$, and the oriented geodesics $\alpha \subset \Gamma_1$ and $\beta \subset \Gamma_2$, emanating from η_3' and with length exactly d. We denote by σ the set of points in ∂Q at distance d from $\eta_3' \subset \gamma_3$.

We are going to estimate the area of Q. Notice that d is the width for the simple closed geodesic $\eta_3 \cup \eta_3^* \subset \partial Y^*$, not for $\gamma_3 = \eta_3 \cup \eta_3' \subset \partial Y$; therefore it is possible that Γ_3 enters in the neighborhood of η_3' of radius d; in this case there are points in the geodesic side Γ_3 at distance less than d from η_3' (otherwise the argument is simpler); note that in this case σ has two connected components. Hence, Γ_3 hits σ in two points, x and y; we shall call x' and y' to their respective projections into η_3'.

Let $\tilde{Q} \subset Q$ be the geodesic quadrilateral with sides $[x, x'], [y, y'], [x', y'] \subseteq \eta_3'$ and $[x, y] \subset \Gamma_3$. Notice that $L_Y([x, x']) = L_Y([y, y']) = d$ and we shall write $s := L_Y([x', y'])$.

Therefore $Q \setminus \tilde{Q} \subset H'$ contains two disjoint quadrilaterals Q_1, Q_2, both of them with height d and the sum of the lengths of their bases is $L_Y(\eta_3') - L_Y([x', y']) > l/2 - s$.

Since $K \leqslant -k_1^2$, by Lemma 3.7, we have

$$\frac{l/2 - s}{k_1} \sinh dk_1 \leqslant A_K(Q_1) + A_K(Q_2) \leqslant A_K\bigl(H'\bigr).$$

By Gauss–Bonnet formula, $-\iint_{H'} K\, dA = \pi$, and taking into account that $K \leqslant -k_1^2$, it holds

$$k_1^2 A_K\bigl(H'\bigr) \leqslant -\iint\limits_{H'} K\, dA = \pi.$$

Joining both inequalities, we get

$$(l/2 - s) \sinh dk_1 \leqslant \frac{\pi}{k_1}. \tag{4.5}$$

Next, let us estimate the length s using hyperbolic trigonometry. In order to do it, we shall consider the geodesic quadrilateral $\tilde{Q}$. The geodesic $[x, y]$ is contained in Γ_3 and, using $L_Y(\Gamma_3) \leqslant C$, we deduce $t := L_Y([x, y]) \leqslant L_Y(\Gamma_3) \leqslant C$. Notice that standard hyperbolic trigonometry and Lemma 3.6 give $\sinh(tk_1/2) \geqslant \sinh(sk_1/2) \cosh dk_1$. Hence, $\sinh(Ck_1/2) \geqslant \sinh(tk_1/2) \geqslant \sinh(sk_1/2) \cosh dk_1$. Therefore,

$$s \leqslant \frac{2}{k_1} \operatorname{ar}\sinh\left(\frac{\sinh(Ck_1/2)}{\cosh dk_1}\right).$$

Using this above inequality in (4.5) we get

$$\left[\frac{l}{2} - \frac{2}{k_1} \operatorname{ar}\sinh\left(\frac{\sinh(Ck_1/2)}{\cosh dk_1}\right)\right] \sinh dk_1 \leqslant \frac{\pi}{k_1}. \tag{4.6}$$

If

$$\frac{2}{k_1}\operatorname{ar\,sinh}\left(\frac{\sinh(Ck_1/2)}{\cosh dk_1}\right) \geqslant \frac{l}{4} \quad \Rightarrow \quad d \leqslant \frac{1}{k_1}\operatorname{ar\,cosh}\left(\frac{\sinh(Ck_1/2)}{\sinh(lk_1/8)}\right).$$

From this last inequality and (4.4), we deduce

$$L_Y(\eta_3) \geqslant \varepsilon_0 := \frac{1}{k_2}\operatorname{ar\,cosh}\left(\cosh\left(\frac{k_2}{k_1}\operatorname{ar\,cosh}\left(\frac{\sinh(Ck_1/2)}{\sinh(lk_1/8)}\right)\right)\right).$$

In other case, if

$$\frac{2}{k_1}\operatorname{ar\,sinh}\left(\frac{\sinh(Ck_1/2)}{\cosh dk_1}\right) < \frac{l}{4},$$

from (4.6) we get

$$\frac{l}{4}\sinh dk_1 \leqslant \frac{\pi}{k_1} \quad \Rightarrow \quad d \leqslant \frac{1}{k_1}\operatorname{ar\,sinh}\left(\frac{4\pi}{lk_1}\right).$$

From this last inequality and (4.4) we deduce

$$L_Y(\eta_3) \geqslant \varepsilon_1 := \frac{1}{k_2}\operatorname{ar\,cosh}\left(\cosh\left(\frac{k_2}{k_1}\operatorname{ar\,sinh}\left(\frac{4\pi}{lk_1}\right)\right)\right).$$

Therefore, $L_Y(\eta_3) \geqslant \min\{\varepsilon_0, \varepsilon_1\}$ and we can conclude that $L_Y(\eta_i) \geqslant c_0$ for $i = 1, 2, 3$, since $c_0 = \min\{l/2, \varepsilon_0, \varepsilon_1\}$.

Step 3. In order to obtain the upper bound of the length of the other two seams, Γ_1, Γ_2, we shall repeat the previous argument in the first part of the second step for Γ_3, since $L_Y(\eta_i) \geqslant c_0$ for $i = 1, 2, 3$, we have $L_Y(\Gamma_i) \leqslant M$ for $i = 1, 2$. Notice that this inequality also holds for $i = 3$, since $c_0 \leqslant l/2$. □

Next, let us give an alternative proof of Theorem 4.3 without explicit constants. However, taking into account the important role of the Y-pieces in the study of surfaces (see e.g. [2,30]), we consider useful to provide of explicit constants appearing in Theorem 4.3.

In the proof of Theorem 4.3, in order to get upper bounds for the lengths of the seams, we have used strongly that the lengths of the three simple closed geodesics in the boundary have a positive lower bound. In fact, the length of a given seam which joins two simple closed geodesics in the boundary, whose lengths do have a lower bound, has an upper bound, with no restriction about the lower bound of the length of the third simple closed geodesic.

Theorem 4.7. *Let us consider a generalized Y-piece, Y, with $-k_2^2 \leqslant K \leqslant -k_1^2 < 0$, and the simple closed geodesics $\gamma_i \subseteq \partial Y$ for $i = 1, 2, 3$, such that $L_Y(\gamma_i) \leqslant L$ for every $i = 1, 2, 3$ and $L_Y(\gamma_i) \geqslant l$ for $i = 1, 2$. There exist constants m and M, which just depend on l, L, k_1 and k_2, such that the seam Γ_3, joining γ_1 and γ_2, verifies*

$$m \leqslant L_Y(\Gamma_3) \leqslant M.$$

In fact, m is the constant in Theorem 4.3.

Proof. The lower bound for the length of the seam is easily obtained from Theorem 3.10, since $-k_2^2 \leqslant K \leqslant 0$ and $L_Y(\gamma_i) \leqslant L$ for $i = 1, 2$, it is known that there exist two collars, each of them centered on γ_1 and γ_2, of width d_1, d_2 respectively, verifying (3.11). Therefore

$$L_Y(\Gamma_3) \geqslant d_1 + d_2 \geqslant \max\{d_1, d_2\} \geqslant \frac{1}{k_2}\left(\operatorname{ar\,cosh}\left(\coth\left(\frac{k_2 \min\{L_S(\gamma_1), L_S(\gamma_2)\}}{2}\right)\right)\right) \geqslant m.$$

Next, let us get an upper bound for the length of the seam. Let us assume that $0 < L_Y(\gamma_3) \leqslant L$, since if $L_Y(\gamma_3) = 0$, i.e., if γ_3 is a puncture, we obtain the same result by a limit process (see [8, Chapter 4.4]). Taking into account that $l \leqslant L_Y(\gamma_i)$ for $i = 1, 2$, then either $l/2 \leqslant L_Y(\eta_i)$ or $l/2 \leqslant L_Y(\eta_i')$ for some $i = 1, 2$.

Without loss of generality, we can assume that $l/2 \leqslant L_Y(\eta_1)$ in the hexagon H. If it holds $l/2 \leqslant L_Y(\eta_2)$ as well, applying Theorem 3.15 in H we have

$$L_Y(\Gamma_3) \leqslant C := \frac{1}{k_1}\operatorname{ar\,cosh}\frac{\cosh Lk_1(1 + \cosh Lk_1)}{\sinh^2(lk_1/2)},$$

and we have finished the proof with $M := C$.

If it holds $L_Y(\eta_2) < l/2$, then $l/2 \leqslant L_Y(\eta_2')$ and we shall apply Theorem 3.15 in H and H' in the following way:

$$l/2 \leqslant L_Y(\eta_1) \leqslant \frac{1}{k_1} \operatorname{ar}\cosh \frac{\cosh(k_1 L_Y(\Gamma_1)) + \cosh(k_1 L_Y(\Gamma_3)) \cosh(k_1 L_Y(\Gamma_2))}{\sinh(k_1 L_Y(\Gamma_3)) \sinh(k_1 L_Y(\Gamma_2))},$$
$$l/2 \leqslant L_Y(\eta_2') \leqslant \frac{1}{k_1} \operatorname{ar}\cosh \frac{\cosh(k_1 L_Y(\Gamma_2)) + \cosh(k_1 L_Y(\Gamma_3)) \cosh(k_1 L_Y(\Gamma_1))}{\sinh(k_1 L_Y(\Gamma_3)) \sinh(k_1 L_Y(\Gamma_1))}.$$

Since $L_Y(\Gamma_i) \geqslant \frac{1}{k_2} \operatorname{ar}\cosh(\coth(Lk_2/2))$ for every $i = 1, 2, 3$, by Proposition 3.16 it holds

$$l/2 \lesssim e^{-k_1 L_Y(\Gamma_3)} + e^{-k_1 L_Y(\Gamma_2)} + e^{-\frac{k_1}{2}(L_Y(\Gamma_3)+L_Y(\Gamma_2)-L_Y(\Gamma_1))_+} + k_1\big(L_Y(\Gamma_1) - L_Y(\Gamma_2) - L_Y(\Gamma_3)\big)_+,$$
$$l/2 \lesssim e^{-k_1 L_Y(\Gamma_3)} + e^{-k_1 L_Y(\Gamma_1)} + e^{-\frac{k_1}{2}(L_Y(\Gamma_3)+L_Y(\Gamma_1)-L_Y(\Gamma_2))_+} + k_1\big(L_Y(\Gamma_2) - L_Y(\Gamma_1) - L_Y(\Gamma_3)\big)_+.$$

Combining all cases, just one of the following possibilities holds:

(1) $L_Y(\Gamma_1) - L_Y(\Gamma_2) - L_Y(\Gamma_3), L_Y(\Gamma_2) - L_Y(\Gamma_1) - L_Y(\Gamma_3) \geqslant 0$; then $L_Y(\Gamma_3) \leqslant 0$, which is a contradiction.
(2) $L_Y(\Gamma_1) - L_Y(\Gamma_2) - L_Y(\Gamma_3) \geqslant 0$ and $L_Y(\Gamma_2) - L_Y(\Gamma_1) - L_Y(\Gamma_3) < 0$; then $L_Y(\Gamma_2) + L_Y(\Gamma_3) \leqslant L_Y(\Gamma_1)$.
(3) $L_Y(\Gamma_1) - L_Y(\Gamma_2) - L_Y(\Gamma_3) < 0$ and $L_Y(\Gamma_2) - L_Y(\Gamma_1) - L_Y(\Gamma_3) \geqslant 0$; then $L_Y(\Gamma_1) + L_Y(\Gamma_3) \leqslant L_Y(\Gamma_2)$.
(4) $L_Y(\Gamma_1) - L_Y(\Gamma_2) - L_Y(\Gamma_3), L_Y(\Gamma_2) - L_Y(\Gamma_1) - L_Y(\Gamma_3) < 0$.

From (2) we obtain that

$$l/2 \lesssim e^{-k_1 L_Y(\Gamma_3)} + e^{-k_1 L_Y(\Gamma_1)} + e^{-\frac{k_1}{2}(L_Y(\Gamma_3)+L_Y(\Gamma_1)-L_Y(\Gamma_2))},$$

which implies that $L_Y(\Gamma_3)$, $L_Y(\Gamma_1)$ and $L_Y(\Gamma_3) + L_Y(\Gamma_1) - L_Y(\Gamma_2)$ cannot be large simultaneously; hence, there exists a constant M_1, which just depends on l, k_1 and k_2, such that $L_Y(\Gamma_3) \leqslant M_1$ or $L_Y(\Gamma_1) \leqslant M_1$ or $L_Y(\Gamma_3) + L_Y(\Gamma_1) - L_Y(\Gamma_2) \leqslant 2M_1$. Consequently, if $L_Y(\Gamma_3) \leqslant M_1$ we have finished. If $L_Y(\Gamma_1) \leqslant M_1$, then $L_Y(\Gamma_3) \leqslant L_Y(\Gamma_2) + L_Y(\Gamma_3) \leqslant L_Y(\Gamma_1) \leqslant M_1$. If $L_Y(\Gamma_3) + L_Y(\Gamma_1) - L_Y(\Gamma_2) \leqslant 2M_1$ we can use that $L_Y(\Gamma_1) - L_Y(\Gamma_2) - L_Y(\Gamma_3) \geqslant 0$ and then $L_Y(\Gamma_3) \leqslant M_1$. Therefore, in case (2) we always have $L_Y(\Gamma_3) \leqslant M_1$.

By symmetry, from (3) we obtain that $L_Y(\Gamma_3) \leqslant M_1$ for the previous constant M_1.

Finally, from (4) we obtain that

$$l/2 \lesssim e^{-k_1 L_Y(\Gamma_3)} + e^{-k_1 L_Y(\Gamma_2)} + e^{-\frac{k_1}{2}(L_Y(\Gamma_3)+L_Y(\Gamma_2)-L_Y(\Gamma_1))},$$
$$l/2 \lesssim e^{-k_1 L_Y(\Gamma_3)} + e^{-k_1 L_Y(\Gamma_1)} + e^{-\frac{k_1}{2}(L_Y(\Gamma_3)+L_Y(\Gamma_1)-L_Y(\Gamma_2))}.$$

Therefore, following a similar argument, we deduce, from the first inequality, that it holds $L_Y(\Gamma_3) \leqslant M_1$ or $L_Y(\Gamma_2) \leqslant M_1$ or $L_Y(\Gamma_3) + L_Y(\Gamma_2) - L_Y(\Gamma_1) \leqslant 2M_1$ and, from the second, that it holds $L_Y(\Gamma_3) \leqslant M_1$ or $L_Y(\Gamma_1) \leqslant M_1$ or $L_Y(\Gamma_3) + L_Y(\Gamma_1) - L_Y(\Gamma_2) \leqslant 2M_1$. There are the following possibilities.

(4.1) If $L_Y(\Gamma_3) \leqslant M_1$, then we have finished.
(4.2) If $L_Y(\Gamma_1), L_Y(\Gamma_2) \leqslant M_1$, then triangle inequality in H implies $L_Y(\Gamma_3) \leqslant \sum_{i=1}^3 L_Y(\eta_i) + \sum_{i=1}^2 L_Y(\Gamma_i) \leqslant 3L + 2M_1$.
(4.3) If $L_Y(\Gamma_i) \leqslant M_1$ and $L_Y(\Gamma_3) + L_Y(\Gamma_j) - L_Y(\Gamma_i) \leqslant 2M_1$ for some permutation $\{i, j\}$ of $\{1, 2\}$, then $L_Y(\Gamma_3) \leqslant L_Y(\Gamma_3) + L_Y(\Gamma_j) \leqslant 3M_1$.
(4.4) Finally, if $L_Y(\Gamma_3) + L_Y(\Gamma_1) - L_Y(\Gamma_2) \leqslant 2M_1$ and $L_Y(\Gamma_3) + L_Y(\Gamma_2) - L_Y(\Gamma_1) \leqslant 2M_1$, then we can deduce $L_Y(\Gamma_3) \leqslant 2M_1$.

Writing $M := \max\{C, 3L + 2M_1, 3M_1\}$ we have the result. $\square$

4.2. Skeletons of Y-pieces

We can already construct the trees associated to those generalized Y-pieces which appear into the decomposition of the surface (see Definition 4.12).

Definition 4.8. Given a positive constant L, let us consider:

- a generalized Y-piece, Y, with $L_Y(\gamma_i) \leqslant L$ for at least two simple closed geodesics $\gamma_i \subseteq \partial Y$,
- its related triangles, $T = \{w_1, w_2, w_3\}$ and $T' = \{w_1', w_2', w_3'\}$,
- their respective internal points $u_i \in [w_j, w_k]$ and $u_i' \in [w_j', w_k']$ (see Definitions 2.6 and 2.10) for every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.

We shall say that a tree $\mathcal{T} := (\mathcal{V}, \mathcal{E})$ is a *skeleton* of Y (see Fig. 2) if it has tripod structure (see Definition 2.5) with vertices $\mathcal{V} = \{v, v_1, v_2, v_3\}$ and edges $\mathcal{E} := \bigcup_{i=1}^3 [v, v_i]$ satisfying one of the following properties:

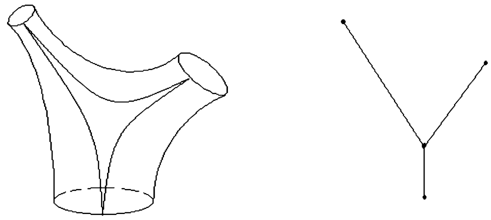


Fig. 2.

(1) $L_{\mathcal{T}}[v, v_i] = d_H(u_j, w_i) = d_H(u_k, w_i)$ for any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.
(2) $L_{\mathcal{T}}[v, v_i] = d_{H'}(u'_j, w'_i) = d_{H'}(u'_k, w'_i)$ for any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.

Remark 4.9. There exist two skeletons of Y, associated to both related triangles. Furthermore, one of the following properties holds:

(1) $L_{\mathcal{T}}[v_i, v_j] = d_H(w_i, w_j)$ for every $i, j \in \{1, 2, 3\}$.
(2) $L_{\mathcal{T}}[v_i, v_j] = d_{H'}(w'_i, w'_j)$ for every $i, j \in \{1, 2, 3\}$.

Lemma 4.10. *Given any generalized Y-piece, Y, with $K \leqslant -k^2 < 0$, $L_Y(\gamma) \leqslant L$ for at least two simple closed geodesics $\gamma \subseteq \partial Y$, and T a related triangle to Y, then $d_Y(z, T) \leqslant 2(9\delta_1 + L)$ for every $z \in Y$, with $\delta_1 := \frac{1}{k}\log(1 + \sqrt{2})$.*

Proof. Let us denote by γ_1, γ_2, γ_3, the simple closed geodesics in ∂Y. Without loss of generality we can assume that $L_Y(\gamma_1), L_Y(\gamma_2) \leqslant L$, and there is no restriction about the upper bound for $L_Y(\gamma_3)$.

We denote by H the related hexagon to Y with $T \subset H$; notice that the related triangles splits their related hexagons associated into four simply connected ideal geodesic polygons: the related triangle and three geodesic quadrilaterals. All of them are isometrics to subsets in the universal covering of S, which is δ_1-hyperbolic since $K \leqslant -k^2$ (see [3, p. 130] and [15, p. 52]).

Now, we shall show that every point in H is near to the related triangle T. The triangle T is the boundary of a simply connected set, E; therefore, by Theorem 2.11, it is $4\delta_1$-fine and for every $z \in E$ it holds $d_Y(z, T) \leqslant 4\delta_1$.

In order to obtain a bound for z in the quadrilaterals in H, since these polygons are simply connected, it will be sufficient to check that every point $z \in \partial H$ is near to T. Since these three quadrilaterals satisfy the conditions in Lemma 3.19, we have $d_H(z, T) \leqslant 4\delta_1 + L$ for every $z \in \partial H$. Consequently, $d_Y(z, T) \leqslant 4\delta_1 + L$ for every $z \in H$.

Next, let us prove that $d_Y(z, T) \leqslant 2(9\delta_1 + L)$ for every $z \in H'$. In order to do it, let us distinguish two cases.

If $L_Y(\eta'_3) > \max\{L, 8\delta_1\} \geqslant 8\delta_1$, since $L_Y(\eta'_i) \leqslant L_Y(\gamma_i) \leqslant L$ for $i = 1, 2$, the hexagon H' satisfies conditions in Lemma 3.18; then $d_{H'}(z, \Gamma_3) \leqslant 12\delta_1 + L$ holds for every $z \in \partial H'$. By the previous argument, we have that $d_Y(z, T) \leqslant d_{H'}(z, \Gamma_3) + 4\delta_1 + L \leqslant 16\delta_1 + 2L$ for every $z \in H'$.

If $L_Y(\eta'_3) \leqslant \max\{L, 8\delta_1\}$, then we shall consider the related triangle $T' \subset H'$, which is the boundary of a simply connected set E'. By Theorem 2.11, T' is $4\delta_1$-fine; therefore, for every $z \in E'$ it holds $d_{H'}(z, T') \leqslant 4\delta_1$. As geodesic quadrilaterals are $2\delta_1$-thin, for every $z \in T'$ it holds $d_{H'}(z, \partial H') \leqslant 2\delta_1$. Then $d_{H'}(z, \partial H') \leqslant 6\delta_1$ for every $z \in H'$. If $d_{H'}(z, \partial H') = d_{H'}(z, \bigcup_{i=1}^3 \Gamma_i)$, since $\Gamma_i \subset \partial H$ for every $i = 1, 2, 3$, by the previous argument, then $d_Y(z, T) \leqslant d_{H'}(z, \bigcup_{i=1}^3 \Gamma_i) + 4\delta_1 + L \leqslant 10\delta_1 + L$ for every $z \in H'$. If $d_{H'}(z, \partial H') = d_{H'}(z, \bigcup_{i=1}^3 \eta'_i)$, since $L_Y(\eta'_i) \leqslant L_Y(\gamma_i) \leqslant \max\{L, 8\delta_1\}$ for $i = 1, 2, 3$, then $d_Y(z, T) \leqslant d_Y(z, \bigcup_{i=1}^3 \eta_i) + 4\delta_1 + L + \max\{L, 8\delta_1\} \leqslant 10\delta_1 + L + \max\{L, 8\delta_1\}$ for every $z \in H'$.

Consequently, $d_Y(z, T) \leqslant \max\{4\delta_1 + L, 16\delta_1 + 2L, 10\delta_1 + L + \max\{L, 8\delta_1\}\} \leqslant 2(9\delta_1 + L)$ for every $z \in Y$. □

Proposition 4.11. *Given any generalized Y-piece, Y, with $K \leqslant -k^2 < 0$ and $L_Y(\gamma) \leqslant L$ for at least two simple closed geodesics $\gamma \subseteq \partial Y$, there exists a $(1, 4(11\delta_1 + L))$-quasi-isometry of Y onto its skeleton $\mathcal{T}$, with $\delta_1 := \frac{1}{k}\log(1 + \sqrt{2})$.*

Proof. Let us denote by γ_1, γ_2, γ_3, the simple closed geodesics in ∂Y. Consider a related triangle $T = \{w_1, w_2, w_3\} \subset H$, its internal points u_1, u_2, u_3, and the skeleton of Y corresponding to T, $\mathcal{T} := (\mathcal{V}, \mathcal{E})$ given by Definition 4.8, with vertices $\mathcal{V} := \{v, v_1, v_2, v_3\}$ and edges $\mathcal{E} := \bigcup_{i=1}^3 [v, v_i]$.

We shall construct now the required quasi-isometry of Y onto its skeleton $\mathcal{T}$. Let $f : Y \to \mathcal{T}$ be a map verifying $f(u_i) = v$ and $f(w_i) = v_i$ for $i = 1, 2, 3$, f is an isometry from the geodesics $[u_i, w_j]$, $[u_k, w_j]$ in H onto the edge $[v, v_j]$ of $\mathcal{T}$ for every permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, $f(\gamma_i) = v_i$ for every $\gamma_i \subseteq \partial Y$ such that $L_Y(\gamma_i) \leqslant L$ and, for every point $x \in Y$ which is not in T or in those geodesics with $L_Y(\gamma_i) \leqslant L$, then $f(x)$ is the image by f of the nearest point to x in the triangle T.

The related triangle T is the boundary of a simply connected set; therefore, according to Theorem 2.11, it is $4\delta_1$-fine (see Definitions 2.6 and 2.10). Then for every $x, y \in T$ it holds $d_{\mathcal{T}}(f(x), f(y)) \leqslant d_Y(x, y) \leqslant d_{\mathcal{T}}(f(x), f(y)) + 8\delta_1$.

According to Lemma 4.10, it holds $d_Y(z, T) \leqslant 2(9\delta_1 + L)$ for every $z \in Y$; consequently, given any two points $x, y \in Y$ and their respective projections, x', y', into T, it holds

$$d_Y(x, y) \leqslant d_Y\big(x', y'\big) + 4(9\delta_1 + L) \leqslant d_{\mathcal{T}}\big(f\big(x'\big), f\big(y'\big)\big) + 4(9\delta_1 + L) + 8\delta_1 = d_{\mathcal{T}}\big(f(x), f(y)\big) + 4(11\delta_1 + L).$$

The other inequality follows in a similar way. □

4.3. Skeletons of surfaces

Many complete Riemannian surfaces can be decomposed in a union of funnels and generalized Y-pieces (see [2], [12, Theorem 4.1] and [30]). This is the kind of surfaces which we are going to study. The following results use this decomposition in order to obtain the associated skeletons $\mathcal{G}$.

Definition 4.12. Let us consider a positive constant L and a complete Riemannian surface S (with or without boundary) with $-k_2^2 \leqslant K \leqslant -k_1^2 < 0$; if S has boundary, we also require that ∂S is the union of simple closed geodesics. We say that S is *L-decomposable* if there exists a decomposition of S as a union of funnels $\{F_m\}_{m\in M}$ and generalized Y-pieces $\{Y_n\}_{n\in N}$, such that $L_S(\gamma) \leqslant L$ for at least two simple closed geodesics $\gamma \subset \partial Y_n$ for each n and, if $L_S(\gamma) > L$ for some simple closed geodesic $\gamma \subset \bigcup \partial Y_n$, then γ is in the boundary of just one generalized Y-piece, i.e., $\gamma \subset \bigcup_m \partial F_m \cup \partial S$.

Remark 4.13. Notice that if S is L-decomposable, then its Y-pieces Y_n are connected each other through simple closed geodesics $\gamma \subset \partial Y_n$ verifying $L_S(\gamma) \leqslant L$.

Definition 4.14. Let us consider an L-decomposable complete Riemannian surface S (with or without boundary). We say that a graph $\mathcal{G}$ is a *skeleton* of S if it is the union of $\{\mathcal{T}_n\}_{n\in N}$ with the following properties:

(1) $\mathcal{T}_n$ is a skeleton of Y_n for each $n \in N$.
(2) If $Y_n \cap Y_m = \bigcup_{i\in I_{nm}} \gamma_{nm}^i$ (with $\gamma_{nm}^i = \gamma_{mn}^i$), then $\mathcal{T}_n \cap \mathcal{T}_m = \bigcup_{i\in I_{nm}} v_{nm}^i$, where v_{nm}^i is the vertex associated to γ_{nm}^i, and we identify v_{nm}^i with v_{mn}^i in order to obtain $\mathcal{G}$.

A 1-*skeleton* $\mathcal{G}$ of S is a graph isomorphic to a skeleton of the surface, such that every edge has length 1.

Remark 4.15. Notice that $\operatorname{card} I_{nm} \leqslant 3$, and $\mathcal{T}_n \cap \mathcal{T}_m = \emptyset$ if and only if $Y_n \cap Y_m = \emptyset$.

As the following result shows, in order to study Gromov hyperbolicity of a Riemannian surface with variable negative curvature, one can "forget" the funnels, i.e., funnels do not influence Gromov hyperbolicity of Riemannian surfaces with $K \leqslant -k^2 < 0$.

Theorem 4.16. *(See [32, Theorem 5.5].) Let us consider a complete Riemannian surface S (with or without boundary) with $K \leqslant -k^2 < 0$; if S has boundary, we also require that ∂S is the union of simple closed geodesics. Let us denote by F the union of the funnels of S. If S_0 is the bordered complete Riemannian surface obtained by deleting from S the interior of F, then S is hyperbolic if and only if S_0 is hyperbolic, quantitatively.*

Theorem 4.17 below lets us move the study of the hyperbolicity of a complete Riemannian surface S to its skeleton $\mathcal{G}$, with much simpler structure.

Theorem 4.17. *Let us consider an L-decomposable complete Riemannian surface S (with or without boundary) with $-k_2^2 \leqslant K \leqslant -k_1^2 < 0$, and let $\mathcal{G}$ be a skeleton of S. Then S is hyperbolic if and only if $\mathcal{G}$ is hyperbolic, quantitatively.*

Proof. Firstly, in order to study the hyperbolicity of S, by Theorem 4.16, we can assume that S does not have funnels. Therefore S is the union of generalized Y-pieces $\{Y_n\}$, such that if $L_S(\gamma) > L$ for some simple closed geodesic $\gamma \subset \bigcup \partial Y_n$, then $\gamma \subset \partial S$. Furthermore, according to Definition 4.14, removing funnel does not impact the skeleton.

By Theorem 2.13, it suffices to show that there exists an (a, b)-quasi-isometry of S onto a skeleton $\mathcal{G}$ of the surface, with a, b constants depending just on k_1, k_2 and L.

Proposition 4.11 gives that, for each $n \in N$, there exists a surjective $(1, 4(11\delta_1 + L))$-quasi-isometry $f_n : Y_n \to \mathcal{T}_n$ with $\delta_1 := \frac{1}{k_1} \log(1 + \sqrt{2}\,)$.

Let us define $f : S \to \mathcal{G}$ such that $f|_{Y_n} := f_n$; we will show now that this map f is a surjective $(1 + \kappa, 8(11\delta_1 + L))$-quasi-isometry, with $\kappa := 4k_2(11\delta_1 + L)/\operatorname{ar}\cosh(\coth(Lk_2/2))$.

First, let us consider two points $x, y \in S$ which are not in the same Y_n, and an oriented geodesic σ from x to y in S. Notice that σ meets at most a finite number of Y_n's, since it is a compact curve crossing simple closed geodesics $\gamma \subset \bigcup \partial Y_n$ with $L_S(\gamma) \leqslant L$ and, by (3.11), these geodesics are far away each other, hence there cannot be an infinitely many Y-pieces involved. In order to simplify the notation, we shall denote them by $Y_1, Y_2, \dots, Y_r$, where $x \in Y_1$, $y \in Y_r$ and the geodesic σ meets Y_{k+1} after Y_k. We shall denote by $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_r$ the skeletons associated to the these Y-pieces; therefore, $f(x) \in \mathcal{T}_1$ and $f(y) \in \mathcal{T}_r$.

For every generalized Y-piece Y_k let us define the points $\sigma_k := \sigma \cap \partial Y_k \cap \partial Y_{k+1}$ for every $k = 1, \dots, r-1$; notice that the vertices $v_k = f(\sigma_k)$ belong to the skeletons associated to each Y_k. Since $\sigma := [x, \sigma_1] \cup [\sigma_1, \sigma_2] \cup \dots \cup [\sigma_{r-1}, y]$, it holds

$$d_S(x, y) = L_S(\sigma) = d_{Y_1}(x, \sigma_1) + \sum_{k=1}^{r-2} d_{Y_k}(\sigma_k, \sigma_{k+1}) + d_{Y_r}(\sigma_{r-1}, y).$$

According to (3.11), it holds $d_{Y_k}(\sigma_k, \sigma_{k+1}) \geqslant \frac{1}{k_2} \operatorname{ar}\cosh(\coth(Lk_2/2))$. Now, by Proposition 4.11, for every $k = 1, \dots, r-1$ we have

$$\begin{aligned} d_{\mathcal{T}_k}(v_k, v_{k+1}) &\leqslant d_{Y_k}(\sigma_k, \sigma_{k+1}) + 4(11\delta_1 + L) \\ &= d_{Y_k}(\sigma_k, \sigma_{k+1}) + \kappa \frac{1}{k_2} \operatorname{ar}\cosh\bigl(\coth(Lk_2/2)\bigr) \leqslant (1+\kappa) d_{Y_k}(\sigma_k, \sigma_{k+1}). \end{aligned}$$

Taking into account the above inequalities, it holds

$$\begin{aligned} d_{\mathcal{G}}\bigl(f(x), f(y)\bigr) &\leqslant d_{\mathcal{T}_1}\bigl(f(x), v_1\bigr) + \sum_{k=1}^{r-2} d_{\mathcal{T}_k}(v_k, v_{k+1}) + d_{\mathcal{T}_r}\bigl(v_{r-1}, f(y)\bigr) \\ &\leqslant d_{Y_1}(x, \sigma_1) + 4(11\delta_1 + L) + (1+\kappa) \sum_{k=1}^{r-2} d_{Y_k}(\sigma_k, \sigma_{k+1}) + d_{Y_k}(\sigma_{r-1}, y) + 4(11\delta_1 + L) \\ &\leqslant (1+\kappa)\left(d_{Y_1}(x, \sigma_1) + \sum_{k=1}^{r-2} d_{Y_k}(\sigma_k, \sigma_{k+1}) + d_{Y_k}(\sigma_{r-1}, y) \right) + 8(11\delta_1 + L) \\ &= (1+\kappa)\, d_S(x, y) + 8(11\delta_1 + L). \end{aligned}$$

If x and y are in the same Y_n, we have two cases. If the geodesic which joins x and y in S is contained in Y_n, then $d_S(x, y) = d_{Y_n}(x, y)$ and we can apply Proposition 4.11. In other case, we can apply the previous argument.

In order to get the other inequality let us follow a similar argument. Let g be the oriented geodesic in $\mathcal{T}$ from $f(x) \in \mathcal{T}_1$ to $f(y) \in \mathcal{T}_s$; g meets $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s$, and meets $\mathcal{T}_{k+1}$ after $\mathcal{T}_k$. Let us denote by v_k the vertex of g in $\mathcal{T}_k \cap \mathcal{T}_{k+1}$. Let $Y_1, Y_2, \dots, Y_s$ be the Y-pieces associated to these skeletons (therefore $x \in Y_1$ and $y \in Y_s$) and $T_1, T_2, \dots, T_s$ be the related triangles to them. Let us denote by w_k^i for $i = 1, 2$ the vertices belonging to the related triangle T_k which satisfy $w_k^1 \in \partial Y_{k-1} \cap \partial Y_k$, $v_{k-1} = f(w_k^1)$ for every $k = 2, \dots, s$, $w_k^2 \in \partial Y_k \cap \partial Y_{k+1}$ and $v_k = f(w_k^2)$ for every $k = 1, \dots, s-1$.

Next, according to Definition 4.8, $d_{\mathcal{T}_k}(v_{k-1}, v_k) = d_{Y_k}(w_k^1, w_k^2)$ for $k = 2, \dots, s-1$, and applying Proposition 4.11 in Y_1 and Y_s, it holds

$$d_S(x, y) \leqslant d_{Y_1}\bigl(x, w_2^1\bigr) + \sum_{k=2}^{s-1} d_{Y_k}\bigl(w_k^1, w_k^2\bigr) + \sum_{k=2}^{s-2} d_{Y_k}\bigl(w_k^2, w_{k+1}^1\bigr) + d_{Y_s}\bigl(w_{s-1}^2, y\bigr).$$

Since $d_{Y_k}(w_k^2, w_{k+1}^1) \leqslant L \leqslant \frac{Lk_2}{2 \operatorname{ar}\cosh(\coth(Lk_2/2))} d_{Y_k}(w_k^1, w_k^2) \leqslant \kappa d_{Y_k}(w_k^1, w_k^2) = \kappa d_{\mathcal{T}_k}(v_{k-1}, v_k)$, we obtain

$$\begin{aligned} d_S(x, y) &\leqslant (1+\kappa)\left(d_{\mathcal{T}_1}\bigl(f(x), v_1\bigr) + \sum_{k=2}^{s-1} d_{\mathcal{T}_k}(v_{k-1}, v_k) + d_{\mathcal{T}_s}\bigl(v_{s-1}, f(y)\bigr) \right) + 8(11\delta_1 + L) \\ &= (1+\kappa)\, d_{\mathcal{G}}\bigl(f(x), f(y)\bigr) + 8(11\delta_1 + L). \end{aligned}$$

Therefore, f is a $(1+\kappa, 8(11\delta_1 + L))$-quasi-isometry of S onto $\mathcal{G}$, and Theorem 2.13 finishes the proof. □

4.4. The main result

In order to state the main result in this paper, Theorem 4.22, we need previously a definition and two lemmas.

Definition 4.18. An edge e in a graph $\mathcal{G}$ is a *tree-edge* if one of the following properties holds:

(1) The graph obtained from $\mathcal{G}$ by removing e is not connected.
(2) The edge e is isometric to the half-line $[0,\infty)$.

Remark 4.19. Notice that (1) covers the case of edges with a vertex of degree 1, since a vertex with degree 1 will have degree 0 after the edge is removed and so be disconnected from the rest.

It is easy to check that the following lemmas hold.

Lemma 4.20. *Let us consider two graphs $\mathcal{G}_1$, $\mathcal{G}_2$, and a graph isomorphism $f:\mathcal{G}_1\to\mathcal{G}_2$, such that $L_{\mathcal{G}_1}(e)=L_{\mathcal{G}_2}(f(e))$ for every non-tree-edge $e\in\mathcal{G}_1$. Then $\mathcal{G}_1$ is δ-hyperbolic if and only if $\mathcal{G}_2$ is δ-hyperbolic.*

Lemma 4.21. *Let $\{T_n\}$ be a family of tripods and G any graph obtained by pasting the tripods by identifying pairwise disjoint couples of vertices of $\{v\in V(T_n)\colon \deg(v)=1\}$. Let G_0 be the graph with every edge of length 1 which is isomorphic to G. If every T_n has edges with lengths x_n, y_n, z_n verifying $c_1\leqslant x_n+y_n, x_n+z_n, y_n+z_n\leqslant c_2$ for every n, then there exists an (a,b)-quasi-isometry of G onto G_0, where a, b depend just on c_1 and c_2.*

We can already state the main result in this paper.

Theorem 4.22. *Let us consider an L-decomposable complete Riemannian surface S with $-k_2^2\leqslant K\leqslant -k_1^2<0$, and let $\mathcal{G}$ be its 1-skeleton. If we define*

$$\alpha:=\inf\left\{L_S(\gamma)\colon\ \gamma\subseteq\left(\bigcup_n\partial Y_n\right)\setminus\left(\bigcup_m\partial F_m\cup\partial S\right)\text{ and } S\setminus\gamma \text{ is connected}\right\},$$
$$\beta:=\sup\left\{L_S(\gamma)\colon\ \gamma\subseteq\bigcup_m\partial F_m\cup\partial S,\ \gamma\subseteq\partial Y_n\text{ for some } n,\text{ and } S\setminus\left(\bigcup_m\partial F_m\cup Y_n\right)\text{ is connected}\right\},$$

the following hold:

(1) *If $\alpha=0$ or $\beta=\infty$, then S is not hyperbolic.*
(2) *If $\alpha>0$ and $\beta<\infty$, then S is hyperbolic if and only if $\mathcal{G}$ is hyperbolic, quantitatively.*

Proof. Firstly, by Theorem 4.16, in order to study the hyperbolicity of S, we can assume that S does not have funnels.

If $\alpha=0$, then S has narrow genus and Theorem 3.14 gives that S is not hyperbolic. If $\beta=\infty$, then there exist generalized Y-pieces Y_n (which do not disconnect S; recall that S does not have funnels), with $\gamma_n^1,\gamma_n^2\subset\partial Y_n$, $L_S(\gamma_n^1), L_S(\gamma_n^2)\leqslant L$ and $d_{Y_n}(\gamma_n^1,\gamma_n^2)\to\infty$; then Theorem 2.14 gives that S is not hyperbolic.

We deal now with the case $\alpha>0$ and $\beta<\infty$. Let $\mathcal{G}_1$ be a skeleton of S; by Theorem 4.17, S is hyperbolic if and only if $\mathcal{G}_1$ is hyperbolic, quantitatively. Let us write $\mathcal{G}_1=\bigcup_n\mathcal{T}_n^1$. Note that if there exists a simple closed geodesic $\gamma\subset\partial Y_n$ with $L_S(\gamma)>\max\{L,\beta\}$, then $\mathcal{T}_n^1$ has three tree-edges. Let us define

$$\mathcal{Y}:=\left\{Y_n\colon\ \mathcal{T}_n^1\text{ has some non-tree-edge}\right\},$$
$$\mathcal{Y}_1:=\left\{Y_n\in\mathcal{Y}\colon\ \mathcal{T}_n^1\text{ has three non-tree-edges}\right\},$$
$$\mathcal{Y}_2:=\left\{Y_n\in\mathcal{Y}\colon\ \mathcal{T}_n^1\text{ has two non-tree-edge}\right\}.$$

Note that $\mathcal{Y}=\mathcal{Y}_1\cup\mathcal{Y}_2$. Then, if $Y_n\in\mathcal{Y}$, then $L_S(\gamma)\leqslant\max\{L,\beta\}$ for every simple closed geodesic $\gamma\subset\partial Y_n$. If $Y_n\in\mathcal{Y}_1$, then $L_S(\gamma)\geqslant\alpha$ for every simple closed geodesic $\gamma\subset\partial Y_n$. If $Y_n\in\mathcal{Y}_2$, then $L_S(\gamma)\geqslant\alpha$ for two simple closed geodesics $\gamma\subset\partial Y_n$. Denote by x_n^1, y_n^1, z_n^1 the lengths of the edges of $\mathcal{T}_n^1$. Notice that, since $\alpha>0$ and $\beta<\infty$, if $\mathcal{G}_1=\bigcup_n\mathcal{T}_n^1$, according to Theorems 4.3 and 4.7 there exist constants m and M which only depend on k_1, k_2, L, α and β, verifying $m\leqslant x_n^1+y_n^1, x_n^1+z_n^1, y_n^1+z_n^1\leqslant M+2L$ for every $Y_n\in\mathcal{Y}_1$, and $m\leqslant x_n^1+y_n^1\leqslant M+2L$ and the edge with length z_n^1 is a tree-edge for every $Y_n\in\mathcal{Y}_2$.

Next, let us consider the graph $\mathcal{G}_2$ obtained from $\mathcal{G}_1$ by replacing the tree-edges in $\mathcal{G}_1$ by tree-edges with length exactly 1. Hence, there exists a graph isomorphism $f_1:\mathcal{G}_1\to\mathcal{G}_2$ such that for every non-tree-edges $e\in\mathcal{G}_1$ it holds $L_{\mathcal{G}_1}(e)=L_{\mathcal{G}_2}(f_1(e))$. By Lemma 4.20, we have that $\mathcal{G}_1$ is δ-hyperbolic if and only if $\mathcal{G}_2$ is δ-hyperbolic. Notice now that, if $\mathcal{G}_2=\bigcup_n\mathcal{T}_n^2$ and every $\mathcal{T}_n^2$ has edges with lengths x_n^2, y_n^2, z_n^2, then $\min\{m,1\}\leqslant x_n^2+y_n^2, x_n^2+z_n^2, y_n^2+z_n^2\leqslant M+2L+2$ for every n.

Finally, the 1-skeleton $\mathcal{G}$ is obtained from $\mathcal{G}_2$ by replacing the non-tree-edges in $\mathcal{G}_2$ by non-tree-edges with length exactly 1; hence, by Lemma 4.21, there exists a surjective (a,b)-quasi-isometry $f_2:\mathcal{G}_2\to\mathcal{G}$, where a, b depend just on k_1, k_2, L, α and β; then Theorem 2.13 finishes the proof. $\square$

As a consequence of Theorem 4.22, we obtain that hyperbolicity is a property stable under significant metric changes (even with non-quasi-isometric deformations), as long as the topology is preserved, for Riemannian surfaces with skeletons. The result is not true without this hypothesis (even with curvature $K=-1$) as Matsuzaki and Rodríguez have proved in [24].

Next we prove that the hyperbolicity is stable under twist.

Theorem 4.23. *Let S be any L-decomposable complete Riemannian surface with $-k_2^2 \leqslant K \leqslant -k_1^2 < 0$ and let S' be any surface obtained from S with any amount of twist around the geodesics in $\bigcup_n \partial Y_n$. Then S is hyperbolic if and only if S' is hyperbolic, quantitatively.*

Proof. Note that S and S' have isomorphic 1-skeletons, $\alpha(S)=\alpha(S')$ and $\beta(S)=\beta(S')$. Then, Theorem 4.22 gives the result. □

Acknowledgments

I would like to thank Professor José Manuel Rodríguez for sharing many ideas with me and for some useful discussions. Also, I would like to thank the referee for the careful reading of the manuscript and for all helpful suggestions.

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