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Topological Morita Equivalences Induced by Ideals Generated by Dense Idempotents*

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Given two complete right linearly topologized rings (R, ρ) and (S, σ), and a bimodule ${}_R B_S$ endowed with a complete topology β, in such a way that $(B_S, \beta) \in$ **CLT**-(S, σ) and there be a continuous ring homomorphism $(R, \rho) \to \mathrm{CEnd}^u_S(B, \beta)$, we define a functor $- \hat{\otimes}_R B$: **CLT**-$(R, \rho) \to$ **CLT**-(S, σ) which is left adjoint to the functor $\mathrm{CHom}^u_S((B, \beta), -)$: **CLT**-$(S, \sigma) \to$ **CLT**-(R, ρ). Then we consider the particular case in which $(S, \sigma) = eRe$ with its induced topology, where e is a dense idempotent of R (that is, ReR is dense in (R, ρ)). Under these hypotheses we show that the pair of functors $- \hat{\otimes}_R Re$: **CLT**-$(R, \rho) \to$ **CLT**-(S, σ) and $- \hat{\otimes}_S eR$: **CLT**-$(S, \sigma) \to$ **CLT**-(R, ρ) is an equivalence of categories. As an application of this result, we re-obtain a theorem of Xu, Shum, and Turner-Smith on similarities between infinite matrix subrings. © 1998 Academic Press

INTRODUCTION

Presently, Morita equivalence has become a basic topic in abstract algebra: sufficiently thorough accounts of it can be found also in fairly basic textbooks such as [6]. The main aim of Morita's theory is to characterize the equivalences between two categories of modules **Mod**-R and **Mod**-S. The prototype of such a situation is the case in which S is any ring (with unit) and R is the ring of $n \times n$ matrices with coefficients in S: in this case it is possible to find bimodules ${}_R P_S$ and ${}_S Q_R$ of the form $P = Re$ and $Q = eR$, where e is an idempotent matrix having a 1 in a single entry and 0's elsewhere, such that the pair of functors $- \otimes_R P$:

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$\mathbf{Mod}\text{-}R \to \mathbf{Mod}\text{-}S$ and $-\otimes_S Q$: $\mathbf{Mod}\text{-}S \to \mathbf{Mod}\text{-}R$ is an equivalence of categories. Two rings R and S such that $\mathbf{Mod}\text{-}R$ and $\mathbf{Mod}\text{-}S$ are equivalent are called *similar*.

Morita's original paper [7] appeared in 1958. Generalizations and extensions of it have been proposed and developed by many authors since the 1970s: besides the already classical work by Fuller [8], it is particularly important for us to cite E. Gregorio's paper [9], in which Morita's theory is generalized to the context of linearly topologized rings and torsion modules over them, a context that incorporates Fuller's one as a special case. In these approaches, great importance is given to categorical ideas, and matrices are no longer mentioned. But more recently some authors tried to recover a more elementary point of view, and reintroduced matrices proving results similar to those of the classical Morita theory, but extended to rings of matrices indexed by infinite sets. In this line of development, a very general result, that is formally analogous to the one which holds in the finite case, but removes *any* restriction on the indexing set, was achieved by Yonghua Xu, Kar-Ping Shum, and R. F. Turner-Smith in [1].

The motivation for the present work initially arose from this last paper [1]. Our first idea was to try to generalize the result found there to the topological rings of matrices with summable rows described in [2]. While trying to do this, we realized that the hypothesis of dealing with rings of matrices played no essential role, and that the theory we were working on could be developed in a much more general context. In fact, our proofs, originally thought of for a topological matrix ring, worked more or less unchanged if such a ring was substituted with *any* topological ring (R, ρ) which were linearly topologized, Hausdorff, and complete! We then entered upon a deeper study of the subject, reorganizing it thoroughly: this work is an account of this reorganization.

This paper is structured as follows (for terminology and notation see *Conventions and Notation*). In Section 1, we consider two right linearly topologized rings (R, ρ) and (S, σ), a linearly topologized right (R, ρ)-module (A, α), and a linearly topologized right (S, σ)-module (B, β) which is also a left topological (R, ρ)-module, in such a way that the canonical ring homomorphism ω: $(R, \rho) \to \mathrm{CEnd}_S(B, \beta)$ be continuous when $\mathrm{CEnd}_S(B, \beta)$ is endowed with the topology of uniform convergence. We then define a topology on $A \otimes_R B$, denoted by $\tau(\alpha, \beta)$, and we consider the Hausdorff space $(A \tilde{\otimes}_R B, \tilde{\tau}(\alpha, \beta))$ canonically associated to $(A \otimes_R B, \tau(\alpha, \beta))$; in this way we obtain a functor $-\tilde{\otimes}_R B$: $\mathbf{LT}\text{-}(R, \rho) \to \mathbf{LT}\text{-}(S, \sigma)$ that we prove to be left adjoint to the functor $\mathrm{CHom}_S^u((B, \beta), -)$: $\mathbf{LT}\text{-}(S, \sigma) \to \mathbf{LT}\text{-}(R, \rho)$.

In Section 2 we specialize to the case in which (R, ρ) is still an arbitrary right linearly topologized ring, but (S, σ) has the form $S = eRe$, with the induced topology, where e is an idempotent of R, and the bimodule (B, β)

is an ideal of the form Re or eR, again with its induced topology. This section is the core of the paper, and contains a set of technical propositions that, in the subsequent section, lead almost directly to the proof of the main theorem. In order to obtain the most significant of these results, we need to assume that e is a dense idempotent of (R, ρ), that is, that ReR is dense in (R, ρ).

All the rings and modules considered in Section 3 are complete. In this section we introduce the completed tensor products $- \hat{\otimes}_R Re$: **CLT**-$(R, \rho) \to$ **CLT**-(S, σ) and $- \hat{\otimes}_S eR$: **CLT**-$(S, \sigma) \to$ **CLT**-(R, ρ), and then we prove the following

MAIN THEOREM. *Let (R, ρ) be a right linearly topologized, Hausdorff, and complete ring; let $e \in R$ be a dense idempotent of (R, ρ); put $(S, \sigma) = eRe$ with the induced topology; then the pair of functors*

$$- \hat{\otimes}_R Re\colon \mathbf{CLT}\text{-}(R, \rho) \to \mathbf{CLT}\text{-}(S, \sigma) \qquad \text{and}$$

$$- \hat{\otimes}_S eR\colon \mathbf{CLT}\text{-}(S, \sigma) \to \mathbf{CLT}\text{-}(R, \rho)$$

is an equivalence of categories.

The section continues studying some properties of this equivalence; in particular, it is shown that the equivalence of the Main Theorem induces an equivalence between **Mod**-(R, ρ) and **Mod**-(S, σ). Finally, another version of the Main Theorem is given, which uses CHom functors instead of tensor products.

To conclude the paper, we want of course to show how the result on infinite matrix subrings that originated all this can be deduced from the theory we have developed. Before attending to this task, we recall in Section 4 those elements of the theory on topological rings of infinite matrices (due to H. Leptin, see [2]) which are strictly necessary in the subsequent section; no proofs are given. Finally, in Section 5 we give, along with another minor application, the following reformulation of the main result of [1]:

COROLLARY (cf. [1, Theorem 3.2]). *Let R be a ring, Λ a non-empty set, R' the ring of row-finite Λ-indexed matrices with coefficients in R, $R^{(0)}$ the subring of finite-rank matrices in R', $l \in R'$ and idempotent such that $R^{(0)} l R^{(0)} = R^{(0)}$, $S = lR^{(0)}l$, $\overline{S} = lR'l$, and $\mathscr{C}_{R^{(0)}}$ (resp. $\mathscr{C}_S$) the full subcategory of* **Mod**-$R^{(0)}$ *(resp.* **Mod**-S*) consisting of those modules M such that $MR^{(0)} = M$ (resp. $MS = M$). Then*:

(1) *$\mathscr{C}_{R^{(0)}}$ and $\mathscr{C}_S$ are hereditary pretorsion classes*;

(2) *the pair of functors*

$$- \hat{\otimes}_{R'} R'l \qquad \text{and} \qquad - \hat{\otimes}_{\overline{S}} lR'$$

is an equivalence of categories between $\mathscr{C}_{R^{(0)}}$ and $\mathscr{C}_S$.

Clearly, this work owes very much to Gregorio's paper [9] (and also to Bourbaki, of course!). However, our treatment is different from Gregorio's in that we use a different topology on the tensor product, which enables us to represent directly the equivalence between the categories of the complete modules using the completed tensor products.

Conventions and Notation. All linearly topologized modules and rings will be supposed to be Hausdorff. When we need to speak of a module which is endowed with a linear topology but is not Hausdorff, we shall say that its topology is linear, but we shall not call it "linearly topologized." We shall often abbreviate "linearly topologized" as l.t. With the exception of Section 5—where, anyway, we shall give explicit advice—all rings have unit $1 \neq 0$; if $R, S, \dots$ is the ring, its unit is usually denoted by $1_R, 1_S, \dots,$ and its zero by $0_R, 0_S, \dots$. Given a ring R, a right R-module M, and an element $x \in M$ we denote by $\mathrm{Ann}_R(x)$ the *right* annihilator of x in R. If (R, ρ) is a right l.t. (hence Hausdorff) ring, we denote by **LT**-(R, ρ) the category of right (R, ρ)-topological l.t. (Hausdorff) modules and continuous R-linear applications, and by **Mod**-(R, ρ) the full subcategory of **LT**-(R, ρ) consisting of the discrete modules; if, moreover, (R, ρ) is complete, we denote by **CLT**-(R, ρ) the full subcategory of **LT**-(R, ρ) which has as objects the complete modules. Note that the objects of **Mod**-(R, ρ) are precisely those (abstract) right R-modules M such that, for all $x \in M$, $\mathrm{Ann}_R(x)$ is open in (R, ρ). Finally, for $(A, \alpha), (B, \beta) \in$ **LT**-(R, ρ), $\mathrm{CHom}_R((A, \alpha), (B, \beta))$ is the set of continuous R-linear applications $f: (A, \alpha) \to (B, \beta)$, and $\mathrm{CHom}^u_R((A, \alpha), (B, \beta))$ denotes this same set endowed with the topology of uniform convergence; also, $\mathrm{CEnd}_R(A, \alpha) = \mathrm{CHom}_R((A, \alpha), (A, \alpha))$ and $\mathrm{CEnd}^u_R(A, \alpha) = \mathrm{CHom}^u_R((A, \alpha), (A, \alpha))$.

1. A TOPOLOGY ON THE TENSOR PRODUCT

In this section we put a topology on the tensor product $A \otimes_R B$ in such a way that the functor $- \otimes_R B$ be a left adjoint of the functor $\mathrm{CHom}^u_S(B, -)$. To be more precise, we need to introduce some notation.

1.1. DEFINITION. Let (R, ρ) and (S, σ) be two right l.t. rings; we denote by (R, ρ)-**UT-LT**-(S, σ) the following category. The objects of (R, ρ)-**UT-LT**-(S, σ) are the l.t. (Hausdorff) abelian groups (A, α) such that:

(1) A has an (R, S)-bimodule structure ${}_R A_S$;

(2) $(A_S, \alpha) \in$ **LT**-(S, σ);

(3) R acts on (A, α) by continuous S-endomorphisms, that is, there exists a ring homomorphism $\omega: R \to \mathrm{CEnd}_S(A, \alpha)$;

(4) $\omega : (R, \rho) \to \mathrm{CEnd}_S^u(A, \alpha)$ is continuous (for the displayed topologies).

A morphism f in (R, ρ)-**UT**-**LT**-(S, σ) from (A, α) to (B, β) is a continuous homomorphism of abelian groups $f : (A, \alpha) \to (B, \beta)$ which is simultaneously R-linear to the left and S-linear to the right.

Throughout this section we shall put ourselves in the following setting: (R, ρ) and (S, σ) are two right l.t. rings, $(A, \alpha) \in$ **LT**-(R, ρ), and $(B, \beta) \in (R, \rho)$-**UT**-**LT**-(S, σ). It should be noted, however, that the definitions of Subsection 1.2, as well as some of the propositions below, make sense, or hold, under more relaxed hypotheses; but in this paper we are not interested in pursuing our treatment up the fullest possible generality.

1.2. Let $\tau_1(\alpha)$ be the topology on $A \otimes_R B$ having as a basis of neighbourhoods of zero the family of submodules

$$\{\mathrm{Im}(A' \otimes_R B) : A' \text{ is an open submodule of } (A, \alpha)\},$$

where $\mathrm{Im}(A' \otimes_R B)$ denotes the image in $A \otimes_R B$ of the obvious morphism $A' \otimes_R B \to A \otimes_R B$. Then, let $\tau_2(\beta)$ be the inductive topology of the family $\{``a \otimes -" : a \in A\}$, where $``a \otimes -" : B \to A \otimes_R B$ is the function sending b to $a \otimes b$. Finally, let $\tau(\alpha, \beta) = \tau_1(\alpha) \wedge \tau_2(\beta)$ be the topology having as a basis of neighbourhoods of zero the family of submodules of $A \otimes_R B$ which are open both in $\tau_1(\alpha)$ and in $\tau_2(\beta)$, and endow $A \otimes_R B$ with the topology $\tau(\alpha, \beta)$.

1.3. PROPOSITION. *$(A \otimes_R, B, \tau(\alpha, \beta))$ is a right topological (S, σ)-module, and its topology is S-linear.*

Proof. It is clear that $\tau(\alpha, \beta)$ is S-linear; hence to show that $(A \otimes_R B, \tau(\alpha, \beta))$ is topological over (S, σ) it suffices to show that for all $a_0 \in A$, for all $b_0 \in B$, and for all open submodules W of $(A \otimes_R B, \tau(\alpha, \beta))$ there exists an open right ideal U of (S, σ) such that $(a_0 \otimes b_0) \cdot U \subseteq W$. In fact, since by hypothesis $``a_0 \otimes -"^{\leftarrow}(W)$ is open in (B, β), which in turn is topological over (S, σ), there exists an open right ideal U of (S, σ) such that

$$s \in U \Rightarrow b_0 s \in ``a_0 \otimes -"^{\leftarrow}(W) \Rightarrow a_0 \otimes (b_0 s) = (a_0 \otimes b_0)s \in W,$$

as required. ∎

1.4. PROPOSITION. *Let $f : (A, \alpha) \to (C, \gamma)$ and $g : (B, \beta) \to (D, \delta)$ be two morphisms in* **LT**-(R, ρ) *and* (R, ρ)-**UT**-**LT**-(S, σ), *respectively; then $f \otimes_R g : (A \otimes_R B, \tau(\alpha, \beta)) \to (C \otimes_R D, \tau(\gamma, \delta))$ is continuous.*

Proof. Let V be a neighbourhood of zero in $(C \otimes_R D, \tau(\gamma, \delta))$; we want to show that $(f \otimes_R g)^{\leftarrow}(V)$ is open in $\tau(\alpha, \beta)$, that is, that it is open both in $\tau_1(\alpha)$ and $\tau_2(\beta)$ (see Subsection 1.2). Since V is open in $\tau_1(\gamma)$, there exists an open submodule C' of (C, γ) such that $\mathrm{Im}(C' \otimes_R D) \subseteq V$. Let $A' = f^{\leftarrow}(C')$, so that A' is an open submodule of (A, α); it is clear that $(f \otimes_R g)(\mathrm{Im}(A' \otimes_R B)) \subseteq V$, so $(f \otimes_R g)^{\leftarrow}(V)$ is open in $\tau_1(\alpha)$. Next we show that for all $a \in A$ "$a \otimes -$"$^{\leftarrow}((f \otimes_R g)^{\leftarrow}(V))$ is open in (B, β), so that $(f \otimes_R g)^{\leftarrow}(V)$ will be open in $\tau_2(\beta)$: in fact,

$$\begin{aligned} \text{“}a \otimes -\text{”}^{\leftarrow}\big((f \otimes_R g)^{\leftarrow}(V)\big) &= \big\{b \in B : a \otimes b \in (f \otimes_R g)^{\leftarrow}(V)\big\} \\ &= \{b \in B : f(a) \otimes g(b) \in V\} \\ &= \{b \in B : g(b) \in \text{“}f(a) \otimes -\text{”}^{\leftarrow}(V)\} \\ &= g^{\leftarrow}(\text{“}f(a) \otimes -\text{”}^{\leftarrow}(V)), \end{aligned}$$

and since V is open in $\tau_2(\delta)$ and g is continuous, the last set is open in (B,β). ∎

1.5. We denote by $(A \tilde{\otimes}_R B, \tilde{\tau}(\alpha, \beta))$ the Hausdorff space canonically associated with $(A \otimes_R B, \tau(\alpha, \beta))$; from Proposition 1.3 it follows that $(A \tilde{\otimes}_R B, \tilde{\tau}(\alpha, \beta)) \in$ **LT**-(S, σ). If, moreover, $f : (A, \alpha) \to (C, \gamma)$ and $g : (B, \beta) \to (D, \delta)$ are two morphisms in **LT**-(R, ρ) and (R, ρ)-**UT-LT**-(S, σ), respectively, we denote by

$$f \tilde{\otimes}_R g : \big(A \tilde{\otimes}_R B, \tilde{\tau}(\alpha, \beta)\big) \to \big(C \tilde{\otimes}_R D, \tilde{\tau}(\gamma, \delta)\big)$$

the continuous morphism canonically associated with $f \otimes_R g$. We have thus defined a functor

$$- \tilde{\otimes}_R - : \mathbf{LT}\text{-}(R, \rho) \times (R, \rho)\text{-}\mathbf{UT}\text{-}\mathbf{LT}\text{-}(S, \sigma) \to \mathbf{LT}\text{-}(S, \sigma).$$

In particular, for a fixed $(B,\beta) \in (R, \rho)$-**UT-LT**-(S, σ) we obtain a functor

$$- \tilde{\otimes}_R B : \mathbf{LT}\text{-}(R, \rho) \to \mathbf{LT}\text{-}(S, \sigma). \tag{1.6}$$

A fixed $(B, \beta) \in (R, \rho)$-**UT-LT**-(S, σ) also yields a functor

$$\mathrm{CHom}_S^u((B, \beta), -) : \mathbf{LT}\text{-}(S, \sigma) \to \mathbf{LT}\text{-}(R, \rho). \tag{1.7}$$

We claim that (1.6) is left adjoint to (1.7); the following proposition actually tells more than this.

1.8. PROPOSITION. *Let* $(B, \beta) \in (R, \rho)$-**UT-LT**-(S, σ); *for all* $(A, \alpha) \in$ **LT**-(R, ρ), $(C, \gamma) \in$ **LT**-(S, σ) *there are natural and topological isomor-*

phisms (*of topological abelian groups*)

$$\Phi_C^A : \mathrm{CHom}_S^u\big((A \otimes_R B, \tau(\alpha, \beta)), (C, \gamma)\big)$$
$$\to \mathrm{Chom}_R^u\big((A, \alpha), \mathrm{CHom}_S^u((B, \beta), (C, \gamma))\big)$$
$$f \mapsto [a \mapsto [b \mapsto f(a \otimes b)]]$$

and

$$\Psi_C^A : \mathrm{CHom}_R^u\big((A, \alpha), \mathrm{CHom}_S^u((B, \beta), (C, \gamma))\big)$$
$$\to \mathrm{CHom}_S^u\big((A \otimes_R B, \tau(\alpha, \beta)), (C, \gamma)\big)$$
$$g \mapsto \left[\sum_i a_i \otimes b_i \mapsto \sum_i g(a_i)(b_i)\right]$$

which are inverse one of each other.

Proof. The matter is proving that Φ_C^A and Ψ_C^A are well-defined, because then it will be obvious that they are homomorphisms of abelian groups, that they are inverse one of each other, that they are both continuous, and that they are both natural.

Let us show that Φ_C^A is well-defined. For each $a \in A$ the morphism $b \mapsto f(a \otimes b)$ is indeed continuous, because it is the composition of "$a \otimes -$" followed by f. Moreover, if C' is an open submodule of (C, γ), and if A' is an open submodule of (A, α) such that $\mathrm{Im}(A' \otimes_R B) \subseteq f^{\leftarrow}(C')$, then for all $a' \in A'$ and all $b \in B$ one has $f(a' \otimes b) \in C'$, and this shows that the map $[a \mapsto [b \mapsto f(a \otimes b)]]$ is continuous for the topology of uniform convergence on $\mathrm{CHom}_S((B, \beta), (C, \gamma))$. Thus Φ_C^A is well-defined.

Let us prove that Ψ_C^A is well-defined. Given an open submodule C' of (C, γ), we put $V = (\Psi_C^A(g))^{\leftarrow}(C')$ and show that V is open both in $\tau_1(\alpha)$ and $\tau_2(\beta)$ (see Subsection 1.2). If A' is an open submodule of (A, α) such that $\forall a' \in A'$, $\forall b \in B$ $g(a')(b) \in C'$, then $\mathrm{Im}(A' \otimes_R B) \subseteq V$ and V is open in $\tau_1(\alpha)$. Next, let $a \in A$ be arbitrary but fixed, and let $B'(a)$ be an open submodule of (B, β) such that $b' \in B'(a) \Rightarrow g(a)(b') \in C'$; if $b' \in B'(a)$, then $\Psi_C^A(g)(a \otimes b') = g(a)(b') \in C'$, that is, $b' \in$ "$a \otimes -$"$^{\leftarrow}(V)$, so that $B'(a) \subseteq$ "$a \otimes -$"$^{\leftarrow}(V)$ and V is open in $\tau_2(\beta)$ as well. Thus V is open in $\tau(\alpha, \beta)$ and Ψ_C^A is well-defined. ∎

In the previous proposition (C, γ) is Hausdorff, so we have a natural topological isomorphism

$$\mathrm{CHom}_S^u\big((A \otimes_R B, \tau(\alpha, \beta)), (C, \gamma)\big)$$
$$\cong \mathrm{CHom}_S^u\Big(\big(A \tilde{\otimes}_R B, \tilde{\tau}(\alpha, \beta)\big), (C, \gamma)\Big);$$

therefore:

1.9. COROLLARY. *Let* $(B, \beta) \in (R, \rho)$-**UT-LT**-(S, σ); *for all* $(A, \alpha) \in$ **LT**-(R, ρ), $(C, \gamma) \in$ **LT**-(S, σ)

$$\mathrm{CHom}_S^u\Big(\big(A \tilde{\otimes}_R B, \tilde{\tau}(\alpha, \beta)\big), (C, \gamma)\Big) \cong \mathrm{CHom}_R^u\big((A, \alpha), \mathrm{CHom}_S^u((B, \beta), (C, \gamma))\big),$$

the isomorphism being natural and topological.

2. IDEMPOTENT GENERATED IDEALS

We now specialize our study to tensor products of a particular form. Throughout this section we use the following notation: (R, ρ) is a right (Hausdorff) l.t. right, e is an idempotent of R, and $s = eRe$; note that S is a subring of R, but its unit is $1_S = e \neq 1_R$ (in general); we shall endow S with its induced topology, which we denote by σ. Moreover, if we have a module $(A, \alpha) \in$ **LT**-(R, ρ), we shall denote by αe the topology induced by α on the S-submodule Ae; this notation is motivated by the fact that a typical neighbourhood of zero in the topology αe has the form $A'e$, where A' is a submodule of A open in the topology α. We shall extend this notation to analogous situations without further notice.

Let $(A, \alpha) \in$ **LT**-(R, ρ), and consider the homomorphism (of right S-modules)

$$\mu : A \otimes_R Re \to Ae \subseteq A$$

$$a \otimes x \mapsto ax;$$

it is clear that indeed $\mathrm{Im}\,\mu = Ae$, and that μ is injective ($\sum_i a_i x_i = 0 \Rightarrow \sum_i a_i \otimes x_i = \sum_i a_i x_i \otimes e = 0$), so that μ is an isomorphism between $A \otimes_R Re$ and Ae. Next, since $Re \in (R, \rho)$-**UL-LT**-(S, σ) we can endow $A \otimes_R Re$ with the topology $\tau(\alpha, \rho e)$ defined in Subsection 1.2. It is readily proved that then μ is a topological isomorphism:

2.1. PROPOSITION. *For all open submodules A' of (A, α) and for all $a \in A$, "$a \otimes -$"$^{\leftarrow}(A' \otimes_R Re)$ is open in $(Re, \rho e)$.*

Remark. Obviously in this case $\mathrm{Im}(A' \otimes_R Re) = A' \otimes_R Re$, because Re is a direct summand of R.

Proof. Let U be an open right ideal of (R, ρ) such that $aU \subseteq A'$. If $x \in U \cap Re$ one has

$$\mu(a \otimes x) = ax \in A' \cap Ae = A'e;$$

say that $\mu(a \otimes x) = a'e = \mu(a' \otimes e)$ for a suitable $a' \in A'$; then $a \otimes x = a' \otimes e \in A' \otimes_R Re$. ∎

2.2. COROLLARY. *$\mu : (A \otimes_R Re, \tau(\alpha, \rho e)) \to (Ae, \alpha e)$ is a topological isomorphism.*

Proof. This is trivial, because $\mu(A' \otimes_R Re) = A'e$ for every submodule A' of A. ∎

2.3. COROLLARY. *The canonical isomorphism*

$$(A \otimes_R R, \tau(\alpha, \rho)) \to (A, \alpha)$$
$$a \otimes r \mapsto ar$$

is a topological isomorphism as well.

Proof. Put $e = 1_R$ in Corollary 2.2 ∎

If $(A, \alpha) \in$ **LT**-(R, ρ), then of course $(A, \alpha) \in$ **LT**-(S, σ) too, and since $R \in (S, \sigma)$-**UT**-**LT**-(R, ρ) we can form the tensor product $A \otimes_S R$ and endow it with the topology $\tau(\alpha, \rho)$. The following lemma is needed in the proof of Proposition 2.10 below.

2.4. LEMMA. *The canonical map*

$$\phi : (A \otimes_S R, \tau(\alpha, \rho)) \to (A, \alpha)$$
$$a \otimes r \mapsto ar$$

is continuous.

Proof. Let A' be an open submodule of (A, α); since, obviously, $\phi(A' \otimes_S R) = A'$, $\phi^{\leftarrow}(A')$ is open in $\tau_1(\alpha)$. To show that $\phi^{\leftarrow}(A')$ is open in $\tau_2(\rho)$, fix an arbitrary element $a \in A$, and pick an open right ideal V of (R, ρ) such that $aV \subseteq A'$; if $v \in V$, we have $\phi(a \otimes v) = av \in A'$, that is, "$a \otimes -$"$^{\leftarrow}(\phi^{\leftarrow}(A')) \supseteq V$. ∎

We now study a partially symmetric situation: given $(A, \alpha) \in$ **LT**-(R, ρ), we consider the homomorphism (of right R-modules)

$$\nu : Ae \otimes_S eR \to AeR \subseteq A \qquad (2.5)$$
$$a \otimes x \mapsto ax;$$

it is clear that indeed $\operatorname{Im} \nu = AeR$. This situation is only "partially symmetric" because, although Re is replaced by eR and the tensor product is over S rather than over R, we still have to start with an R-module A. Nevertheless, this is not really restrictive, since any $(B, \beta) \in$ **LT**-(S, σ) can be viewed as $(B, \beta) = (Ae, \alpha e)$ for a suitable $(A, \alpha) \in$ **LT**-(R, ρ), as the following well-known remark points out.

2.6. PROPOSITION. *Given* $(B, \beta) \in \mathbf{LT}\text{-}(S, \sigma)$, *put* $(A, \alpha) = \mathrm{CHom}^u_S((Re, \rho e), (B, \beta))$; *then the two maps*

$$\begin{array}{ccc} (B, \beta) \to (Ae, \alpha e) & & (Ae, \alpha e) \to (B, \beta) \\ b \mapsto [x \mapsto bex] & \text{and} & f \mapsto f(e) \end{array}$$

are topological S-linear isomorphisms, inverse one of each other.

Proof. This is straightforward (cf. [5, Exercise 4.9] for a similar statement). ∎

In order to obtain results similar to those of the first part of this section, we need to introduce a hypothesis on e:

2.7. DEFINITION. $e \in R$ will be called a *dense idempotent of* (R, ρ) if it is an idempotent and ReR, the two-sided ideal of R generated by e, is dense in (R, ρ).

Notice. From now on, we shall assume that e be a dense idempotent of (R, ρ).

The density hypothesis on ReR implies a similar statement for all $(A, \alpha) \in \mathbf{LT}\text{-}(R, \rho)$:

2.8. PROPOSITION. *If* $(A, \alpha) \in \mathbf{LT}\text{-}(R, \rho)$, *then* AeR *is dense in* (A, α).

Proof. Let $a \in A$, and let A' be an open submodule of (A, α). Choose an open right ideal V of (R, ρ) such that $aV \subseteq A'$. Since ReR is dense in (R, ρ), we can write $1_R = \sum_i r_i e s_i + v$ for suitable $r_i, s_i \in R$ and $v \in V$; then

$$a - \sum_i a r_i e s_i = a \cdot 1_R - a \cdot \sum_i r_i e s_i$$
$$= a \cdot \sum_i r_i e s_i + av - a \cdot \sum_i r_i e s_i = av \in A',$$

and $\sum_i a r_i e s_i \in AeR$. ∎

For the rest of this section it is understood that $(A, \alpha) \in \mathbf{LT}\text{-}(R, \rho)$ and that ν is the map of (2.5); we also call α' the topology induced by α on AeR. Moreover, since $eR \in (S, \sigma)\text{-}\mathbf{UT}\text{-}\mathbf{LT}\text{-}(R, \rho)$ we can endow $Ae \otimes_S eR$ with the topology $\tau(\alpha e, e\rho)$. The following lemma is crucial.

2.9. LEMMA. *If* W *is an open submodule of* $(Ae \otimes_S eR, \tau(\alpha e, e\rho))$, *then* $\nu(W)$ *is open in* (AeR, α').

Proof. Let A' be an open submodule of (A, α) such that $A'e \otimes_S eR \subseteq W$; we claim that $\nu(W) \supseteq A' \cap AeR$. In fact, let $a' \in A' \cap AeR$; then

$a' = \sum_i a_i e r_i$ for suitable elements $a_i \in A$ and $r_i \in R$. Moreover, since W is open in $\tau_2(e\rho)$, we can find an open right ideal V of R such that $x \in eV \Rightarrow \forall i\ a_i e \otimes x \in W$; then, we can pick an open right ideal U of (R, ρ) such that $u \in U \Rightarrow \forall_i\ r_i u \in V$; finally, since ReR is dense in (R, ρ), we can write $1_R = \sum_j s_j e t_j + u$ with $s_j, t_j \in R$ and $u \in U$. Now

$$a' = a' \cdot 1_R = a'\Big(\sum_j s_j e t_j\Big) + a'u = \sum_j a' s_j e t_j + \Big(\sum_i a_i e r_i\Big)u,$$

and clearly

$$\sum_j a' s_j e t_j = \nu\Big(\sum_j a' s_j e \otimes e t_j\Big) \in \nu(A'e \otimes_S eR),$$

$$\Big(\sum_i a_i e r_i\Big)u = \nu\Big(\sum_i a_i e \otimes e r_i u\Big).$$

Since $u \in U$, $er_i u \in eV$ for all i; thus $a_i e \otimes e r_i u \in W$ for all i. On the other hand, $\nu(A'e \otimes_S er) \subseteq \nu(W)$, and therefore $a' \in \nu(W)$, as had to be proved. ∎

In the following proposition, Z is the closure of zero in $(Ae \otimes_S eR, \tau(\alpha e, e\rho))$:

$$Z = \bigcap \big\{W : W \text{ is an open submodule of } \big(Ae \otimes_S eR, \tau(\alpha e, e\rho)\big)\big\}.$$

2.10. PROPOSITION. *$\nu : (Ae \otimes_S eR, \tau(\alpha e, e\rho)) \to (AeR, \alpha')$ is continuous and open, and* Ker $\nu = Z$.

Proof. ν is continuous by Lemma 2.4 and Proposition 1.4, and is open by Lemma 2.9. Since (A, α) is Hausdorff, it follows that $Z \subseteq \operatorname{Ker} \nu$; let us prove the reverse inclusion. We suppose that $\sum_i x_i y_i = 0$ for $x_i \in Ae$, $y_i \in eR$, and show that $\sum_i x_i \otimes y_i \in W$ for each open submodule W of $(Ae \otimes_S eR, \tau(\alpha e, e\rho))$. So, let W be arbitrary but fixed, and write $x_i = a_i e$, $y_i = e r_i$ with $a_i \in A, r_i \in R$. Let V be an open right ideal of (R, ρ) such that $z \in eV \Rightarrow \forall i\ x_i \otimes z \in W$; let U be an open right ideal of (R, ρ) such that $\forall i\ r_i U \subseteq V$; let $u \in U$, $s_j, t_j \in R$ such that $1_R = \sum_j s_j e t_j + u$; then

$$\begin{aligned}\sum_i x_i \otimes y_i &= \sum_i a_i e \otimes (e r_i \cdot 1_R) = \sum_i a_i e \otimes \Big(\sum_j e r_i s_j e t_j\Big) + \sum_i a_i e \otimes e r_i u \\ &= \sum_{ij} a_i e r_i s_j e \otimes e t_j + \sum_i a_i e \otimes e r_i u \\ &= \sum_j \Big(\sum_i a_i e r_i\Big) s_j e \otimes e t_j + \sum_i a_i e \otimes e r_i u = \sum_i a_i e \otimes e r_i u.\end{aligned}$$

Now

$$u \in U \Rightarrow \forall i\, r_i u \in V \Rightarrow \forall i\, er_i u \in eV \Rightarrow \forall i\, a_i e \otimes er_i u \in W$$

and the claim follows. ∎

We call $\tilde{\nu}$: $(Ae \tilde{\otimes}_S eR, \tilde{\tau}(\alpha e, e\rho)) \to (A, \alpha)$ the continuous morphism canonically associated with ν. By Corollary 2.2, $(A \tilde{\otimes}_R Re, \tilde{\tau}(\alpha, \rho e)) = (A \otimes_R Re, \tau(\alpha, \rho e))$, so, for elegance, we put $\tilde{\mu} = \mu$ as well. The results of this section can be summarized as follows:

2.11. THEOREM. *Let* (R, ρ), e, *and* (S, σ) *be as above, and let* $(A, \alpha) \in$ **LT**-(R, ρ); *then*

$$\tilde{\mu}: \left(A \tilde{\otimes}_R Re, \tilde{\tau}(\alpha, \rho e)\right) \to (Ae, \alpha e)$$

is a topological isomorphism, and

$$\tilde{\nu}: \left(Ae \tilde{\otimes}_S eR, \tilde{\tau}(\alpha e, e\rho)\right) \to (A, \alpha)$$

is a dense topological embedding.

3. EQUIVALENCES

We now turn our attention to the case of complete rings and modules. For the beginning of this section, until a different advice is given, (R, ρ) and (S, σ) will be two complete right l.t. rings.

3.1. Let $(A, \alpha) \in$ **LT**-(R, ρ), $(B, \beta) \in (R, \rho)$-**UT-LT**-(S, σ). We denote by $(A \hat{\otimes}_R B, \hat{\tau}(\alpha, \beta))$ the Hausdorff completion of $(A \otimes_R B, \tau(\alpha, \beta))$: in this way we obtain a functor

$$- \hat{\otimes}_R - : \mathbf{LT}\text{-}(R, \rho) \times (R, \rho)\text{-}\mathbf{UT}\text{-}\mathbf{LT}\text{-}(S, \sigma) \to \mathbf{CLT}\text{-}(S, \sigma)$$

which acts on morphisms in the obvious way. In particular, fixing a $(B, \beta) \in (R, \rho)$-**UT-LT**-(S, σ) and restricting the first variable to **CLT**-(R, ρ) we obtain a functor

$$- \hat{\otimes}_R B : \mathbf{CLT}\text{-}(R, \rho) \to \mathbf{CLT}\text{-}(S, \sigma). \tag{3.2}$$

It is well known that $\mathrm{CHom}_S^u((B, \beta), (C, \gamma))$ is complete as soon as (C, γ) is complete; therefore a fixed $(B, \beta) \in (R, \rho)$-**UT-LT**-(S, σ) also yields a functor

$$\mathrm{CHom}_S^u((B, \beta), -) : \mathbf{CLT}\text{-}(S, \sigma) \to \mathbf{CLT}\text{-}(R, \rho), \tag{3.3}$$

which is right adjoint to the functor (3.2) (the following proposition actually tells more than this):

3.4. PROPOSITION. *Let* $(B, \beta) \in (R, \rho)$-**UT-LT**-(S, σ); *for all* $(A, \alpha) \in$ **CLT**-$(R, \rho), (C, \gamma) \in$ **CLT**-(S, σ)

$$\mathrm{CHom}_S^u\Big(\big(A \mathbin{\hat{\otimes}_R} B, \hat{\tau}(\alpha, \beta)\big), (C, \gamma)\Big) \cong \mathrm{CHom}_R^u\big((A, \alpha), \mathrm{CHom}_S^u((B, \beta), (C, \gamma))\big),$$

the isomorphism being natural and topological.

Proof. Since (C, γ) is complete,

$$\mathrm{CHom}_S^u\Big(\big(A \mathbin{\hat{\otimes}_R} B, \hat{\tau}(\alpha, \beta)\big), (C, \gamma)\Big) \cong \mathrm{CHom}_S^u\big((A \otimes_R B, \tau(\alpha, \beta)), (C, \gamma)\big)$$

topologically; the result then follows immediately from Proposition 1.8. ∎

For the rest of this section, (R, ρ) is a complete right l.t. ring, e is a dense idempotent of (R, ρ), and (S, σ) is the ring $S = eRe$ endowed with the relative topology of ρ: $\sigma = e\rho e$. Observe that if $(A, \alpha) \in$ **LT**-(R, ρ), then Ae is closed in (A, α), being the subset of A on which the identity function and the right multiplication by e coincide; so, if (A, α) is complete, $(Ae, \alpha e)$ is complete as well. The same is true of Re, eR, and $S = eRe$.

Our equivalence theorem rests entirely on the following obvious corollary of Theorem 2.11:

3.5. LEMMA. *For all* $(A, \alpha) \in$ **CLT**-(R, ρ) *there are natural and topological isomorphisms*

$$\hat{\mu}: \big(A \mathbin{\hat{\otimes}_R} Re, \hat{\tau}(\alpha, \rho e)\big) \to (Ae, \alpha e) \qquad \textit{and}$$

$$\hat{\nu}: \big(Ae \mathbin{\hat{\otimes}_S} eR, \hat{\tau}(\alpha e, e\rho)\big) \to (A, \alpha).$$

It is now almost immediate to prove our main theorem.

3.6. THEOREM. *Let (R, ρ) be a complete right l.t. ring, e a dense idempotent of (R, ρ) and (S, σ) the ring $S = eRe$ endowed with the relative topology of ρ. The pair of functors*

$$- \mathbin{\hat{\otimes}_R} Re: \textbf{CLT-}(R, \rho) \to \textbf{CLT-}(S, \sigma) \qquad \textit{and}$$

$$- \mathbin{\hat{\otimes}_S} eR: \textbf{CLT-}(S, \sigma) \to \textbf{CLT-}(R, \rho)$$

is an equivalence of categories.

Proof. Let $(A, \alpha) \in \mathbf{CLT}\text{-}(R, \rho)$; by Lemma 3.5 there are natural and topological isomorphisms

$$\Big(\big(A \mathbin{\hat{\otimes}_R} Re\big) \mathbin{\hat{\otimes}_S} eR, \hat{\tau}(\hat{\tau}(\alpha, \rho e), e\rho)\Big) \cong \Big(Ae \mathbin{\hat{\otimes}_S} eR, \hat{\tau}(\alpha e, e\rho)\Big) \cong (A, \alpha).$$

Now let $(B, \beta) \in \mathbf{CLT}\text{-}(S, \sigma)$. By Proposition 2.6, $(B, \beta) = (Ae, \alpha e)$ for a suitable $(A, \alpha) \in \mathbf{CLT}\text{-}(R, \rho)$; we can therefore apply again Lemma 3.5 to obtain natural and topological isomorphisms

$$\begin{aligned}\Big(\big(B \mathbin{\hat{\otimes}_S} eR\big) \mathbin{\hat{\otimes}_R} Re, \hat{\tau}(\hat{\tau}(\beta, e\rho), \rho e)\Big) &\cong \Big(\big(Ae \mathbin{\hat{\otimes}_S} eR\big) \mathbin{\hat{\otimes}_R} Re, \hat{\tau}(\hat{\tau}(\alpha e, e\rho), \rho e)\Big)\\ &\cong \Big(A \mathbin{\hat{\otimes}_R} Re, \hat{\tau}(\alpha, \rho e)\Big) \cong (Ae, \alpha e) \cong (B, \beta).\end{aligned}$$

The claim follows by these two calculations. ∎

We want now to show that the pair of functors $- \mathbin{\hat{\otimes}_R} Re$ and $- \mathbin{\hat{\otimes}_S} eR$ induces an equivalence between $\mathbf{Mod}\text{-}(R, \rho)$ and $\mathbf{Mod}\text{-}(S, \sigma)$.

3.7. LEMMA. *Let (R, ρ) be a right l.t. ring, e a dense idempotent of (R, ρ), and $(A, \alpha) \in \mathbf{LT}\text{-}(R, \rho)$; then (A, α) is discrete if (and only if) $(Ae, \alpha e)$ is discrete.*

Proof. Let A' be an open submodule of (A, α) such that $A'e = \{0_A\}$; since e is dense, A' is the closure in (A, α) of $A'eR = \{0_A\}$, and since (A, α) is Hausdorff this implies that $A' = \{0_A\}$. ∎

3.8. COROLLARY. *For every $(A, \alpha) \in \mathbf{CLT}\text{-}(R, \rho)$, $(A \mathbin{\hat{\otimes}_R} Re, \hat{\tau}(\alpha, \rho e))$ is discrete if and only if (A, α) is discrete.*

Proof. This is immediate from Corollary 2.2 and the preceding lemma. ∎

3.9. PROPOSITION. *The following facts hold*:

(1) $A \in \mathbf{Mod}\text{-}(R, \rho) \Rightarrow A \mathbin{\hat{\otimes}_R} Re \in \mathbf{Mod}\text{-}(S, \sigma)$;

(2) $B \in \mathbf{Mod}\text{-}(S, \sigma) \Rightarrow B \mathbin{\hat{\otimes}_S} eR \in \mathbf{Mod}\text{-}(R, \rho)$.

Proof. If M is an (abstract) abelian group, we denote by δ_M the discrete toplogy on M. By Proposition 1.3, it suffices to prove that $(A \mathbin{\hat{\otimes}_R} Re, \hat{\tau}(\delta_A, \rho e))$ and $(B \mathbin{\hat{\otimes}_S} eR, \hat{\tau}(\delta_B, e\rho))$ are discrete. The first one is discrete by Corollary 3.8; for the second one, we observe that (B, δ_B) is topologically isomorphic to $((B \mathbin{\hat{\otimes}_S} eR) \mathbin{\hat{\otimes}_R} Re, \hat{\tau}(\hat{\tau}(\delta_B, e\rho), \rho e))$, and again Corollary 3.8 implies that $(B \mathbin{\hat{\otimes}_S} eR, \hat{\tau}(\delta_B, e\rho))$ is discrete. ∎

3.10. The equivalence between **CLT**-(R, ρ) (resp. **Mod**-(R, ρ)) and **CLT**-(S, σ) (resp. **Mod**-(S, σ)) can also be described by means of CHom functors: for all $(A, \alpha) \in$ **CLT**-(R, ρ) and all $(B, \beta) \in$ **CLT**-(S, σ) there are in fact natural and topological isomorphisms

$$\begin{aligned}
&\mathrm{CHom}^u_S((Re, \rho e), \mathrm{CHom}^u_R((eR, e\rho), (A, \alpha)))\\
&\quad\cong \mathrm{CHom}^u_R\Big(\Big(Re \mathbin{\hat{\otimes}_S} eR, \hat{\tau}(\rho e, e\rho)\Big), (A, \alpha)\Big)\\
&\quad\cong \mathrm{CHom}^u_R((R, \rho), (A, \alpha))\\
&\quad\cong (A, \alpha),\\
&\mathrm{CHom}^u_R((eR, e\rho), \mathrm{CHom}^u_S((Re, \rho e), (B, \beta)))\\
&\quad\cong \mathrm{CHom}^u_S\Big(\Big(eR \mathbin{\hat{\otimes}_R} Re, \hat{\tau}(e\rho, \rho e)\Big), (B, \beta)\Big)\\
&\quad\cong \mathrm{CHom}^u_S((S, \sigma), (B, \beta))\\
&\quad\cong (B, \beta).
\end{aligned}$$

Of course, this implies that the pair of functors $\mathrm{CHom}^u_R((eR, e\rho), -)$: **CLT**-$(R, \rho) \to$ **CLT**-(S, σ) and $\mathrm{CHom}^u_S((Re, \rho e), -)$: **CLT**-$(S, \sigma) \to$ **CLT**-(R, ρ) is an equivalence of categories. Moreover, it is obvious that these two functors send discrete modules into discrete modules.

In particular the functor $\mathrm{CHom}^u_R((eR, e\rho), -)$ (resp. $\mathrm{CHom}^u_S((Re, \rho e), -)$) is left adjoint to the functor $\mathrm{CHom}^u_S((Re, \rho e), -)$ (resp. $\mathrm{CHom}^u_R((eR, e\rho), -)$); by Proposition 3.4 then we have:

3.11. PROPOSITION. *For all* $(A, \alpha) \in$ **CLT**-(R, ρ) *and all* $(B, \beta) \in$ **CLT**-(S, σ) *we have natural and topological isomorphisms*

(1) $(A \mathbin{\hat{\otimes}_R} Re, \hat{\tau}(\alpha, \rho e)) \cong \mathrm{CHom}^u_R((eR, e\rho), (A, \alpha))$ *and*

(2) $(B \mathbin{\hat{\otimes}_S} eR, \hat{\tau}(\beta, e\rho)) \cong \mathrm{CHom}^u_S((Re, \rho e), (B, \beta))$.

Since they are an equivalence of categories, the functors $- \mathbin{\hat{\otimes}_R} Re$ and $- \mathbin{\hat{\otimes}_S} eR$ are fully faithful; to conclude this section, we like to point out that this fact can be strengthened in a "topological" sense.

3.12. Let (A, α), $(C, \gamma) \in$ **CLT**-(R, ρ). We denote by $\omega_A : A \to (A \mathbin{\hat{\otimes}_R} Re) \mathbin{\hat{\otimes}_S} eR$ (to ease the notation we omit the indication of the topologies) the canonical topological isomorphism mentioned in the proof of Theorem 3.6. The functors $- \mathbin{\hat{\otimes}_R} Re$ and $- \mathbin{\hat{\otimes}_S} eR$ yield homomorphisms of abelian groups

$$T_1 : \mathrm{CHom}^u_R(A, C) \to \mathrm{CHom}^u_S\Big(A \mathbin{\hat{\otimes}_R} Re, C \mathbin{\hat{\otimes}_R} Re\Big)$$

$$f \mapsto f \mathbin{\hat{\otimes}_R} Re$$

and

$$T_2 : \mathrm{CHom}_S^u\left(A \hat{\otimes}_R Re, C \hat{\otimes}_R Re\right) \to \mathrm{CHom}_R^u(A, C)$$
$$g \mapsto \omega_C^{-1} \circ \left(g \hat{\otimes}_S eR\right) \circ \omega_A,$$

which we know to be inverse one of each other. But more is true: they are *topological* isomorphisms as well. Let us prove this.

T_1 is continuous. In fact, a typical neighbourhood of zero in $\mathrm{CHom}_S^u(A \hat{\otimes}_R Re, C \hat{\otimes}_R Re)$ is the set of all those functions g such that $\mathrm{Im}\, g \subseteq C' \otimes_R Re = C' \hat{\otimes}_R Re$, with C' a suitable open submodule of C; if $f : A \to C$ satisfies $\mathrm{Im}\, f \subseteq C'$, then $\mathrm{Im}(f \hat{\otimes}_R Re) \subseteq C' \otimes_R Re$.

T_2 is continuous. In fact, let C' be an open submodule of C; then, by Proposition 2.1, $C' \hat{\otimes}_R Re = C' \otimes_R Re$ is open in $C \hat{\otimes}_R Re$. For each $g : A \hat{\otimes}_R Re \to C \hat{\otimes}_R Re$ such that $\mathrm{Im}\, g \subseteq C' \otimes_R Re$ we have $\mathrm{Im}(g \otimes_S eR) \subseteq (C' \otimes_R Re) \otimes_S eR$. We now remember that for $c \in C$, $x \in Re$ and $y \in eR$ it results $\omega_C^{-1}((c \otimes x) \otimes y) = cxy$, and therefore $\omega_C^{-1}((C' \otimes_R Re) \otimes_S eR) \subseteq C'$. Since C' is open, hence closed, hence complete, it follows that $\mathrm{Im}(T_2(g)) \subseteq C'$, whence the claim.

In almost the same way one shows that, for $(B, \beta), (D, \delta) \in$ **CLT-**(S, σ), the isomorphisms

$$T_3 : \mathrm{CHom}_S^u(B, D) \to \mathrm{CHom}_R^u\left(B \hat{\otimes}_S eR, D \hat{\otimes}_S eR\right)$$
$$g \mapsto g \hat{\otimes}_S eR$$

and

$$T_4 : \mathrm{CHom}_R^u\left(B \hat{\otimes}_S eR, D \hat{\otimes}_S eR\right) \to \mathrm{CHom}_S^u(B, D)$$
$$f \mapsto \omega_D^{-1} \circ \left(f \hat{\otimes}_R Re\right) \circ \omega_B$$

(where $\omega_B : B \to (B \hat{\otimes}_S eR) \hat{\otimes}_R Re$ is the canonical topological isomorphism) are topological too.

4. TOPOLOGICAL RINGS OF MATRICES INDEXED BY AN ARBITRARY SET

To conclude this paper, we want to show how the results of [1] can be deduced from our Theorem 3.6. Before doing so, we need to recall the theory of topological rings of matrices indexed by an arbitrary set, that was developed by H. Leptin in 1957 (see [2]). Leptin's work [2] is the classical reference on the subject; in his master thesis [3], the author restated

Leptin's results in modern fashion, sometimes with new proofs; [4] is a revision of [3], with some proofs left to the reader. Since we are not going to give any proof of the results of this section, the interested reader is referred to one of the abovementioned works.

Throughout this section, (R, ρ) is a complete right l.t. ring, and $(A, \alpha) \in$ **CLT**-(R, ρ); moreover, $\Lambda \neq \emptyset$ is a non-empty set, and we denote by $\mathscr{F}$ the filter of the open neighbourhoods of zero in (A, α).

4.1. DEFINITION. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a Λ-indexed family of elements of A; we shall say that the family $(x_\lambda)_{\lambda \in \Lambda}$ is *summable in* (A, α) if and only if for every $A' \in \mathscr{F}$ there is a finite subset $F \subseteq \Lambda$ such that $\lambda \in \Lambda \setminus F \Rightarrow x_\lambda \in A'$.

Since (A, α) is Hausdorff and complete, each summable family $(x_\lambda)_{\lambda \in \Lambda}$ of elements of A has a unique *sum* in A, which will be denoted by $\sum_{\lambda \in \Lambda} x_\lambda$.

Write $\mathrm{Summ}_\Lambda(A, \alpha)$ for the set of all summable Λ-indexed families of elements of A; obviously,

$$A^{(\Lambda)} \subseteq \mathrm{Summ}_\Lambda(A, \alpha) \subseteq A^\Lambda,$$

and if (A, α) is discrete then $A^{(\Lambda)} = \mathrm{Summ}_\Lambda(A, \alpha)$.

The following proposition is obvious.

4.2. PROPOSITION. *Let* $\bar{x} = (x_\lambda)_{\lambda \in \Lambda}$, $\bar{y} = (y_\lambda)_{\lambda \in \Lambda} \in \mathrm{Summ}_\Lambda(A, \alpha)$, $r \in R$, $\bar{r} = (r_\lambda)_{\lambda \in \Lambda} \in R^\Lambda$; *then*:

(1) $\bar{x} + \bar{y} = (x_\lambda + y_\lambda)_{\lambda \in \Lambda} \in \mathrm{Summ}_\Lambda(A, \alpha)$ *and*

$$\sum_{\lambda \in \Lambda} x_\lambda + y_\lambda = \left(\sum_{\lambda \in \Lambda} x_\lambda \right) + \left(\sum_{\lambda \in \Lambda} y_\lambda \right);$$

(2) $\bar{x}r = (x_\lambda r)_{\lambda \in \Lambda} \in \mathrm{Summ}_\Lambda(A, \alpha)$ *and*

$$\sum_{\lambda \in \Lambda} x_\lambda r = \left(\sum_{\lambda \in \Lambda} x_\lambda \right) \cdot r;$$

(3) $\bar{x}\bar{r} = (x_\lambda r_\lambda)_{\lambda \in \Lambda} \in \mathrm{Summ}_\Lambda(A, \alpha)$;

(4) *if* (B, β) *is a second module in* **CLT**-(R, ρ) *and* $f : (A, \alpha) \to (B, \beta)$ *is linear and continuous, then* $f(\bar{x}) = (f(x_\lambda))_{\lambda \in \Lambda} \in \mathrm{Summ}_\Lambda(B, \beta)$ *and*

$$\sum_{\lambda \in \Lambda} f(x_\lambda) = f\left(\sum_{\lambda \in \Lambda} x_\lambda \right).$$

Remark. $\mathrm{Summ}_\Lambda(A, \alpha)$ is hence a right R-module.

If the previous proposition, (2) is actually a particular case of (4); in the same way, if $(A, \alpha) = (R, \rho)$, and if $\bar{r} = (r_\lambda)_{\lambda \in \Lambda} \in \mathrm{Summ}_\Lambda(R, \rho)$, $r \in R$, then $r\bar{r} = (rr_\lambda)_{\lambda \in \Lambda} \in \mathrm{Summ}_\Lambda(R, \rho)$ and

$$\sum_{\lambda \in \Lambda} rr_\lambda = r \cdot \left(\sum_{\lambda \in \Lambda} r_\lambda \right).$$

4.3. We endow A^Λ with the topology $\beta_0(\alpha)$ of uniform convergence over Λ; in other words, a base of neighbourhoods of zero for $\beta_0(\alpha)$ is the family of submodules

$$A'^\Lambda = \left\{ (x_\lambda)_{\lambda \in \Lambda} \in A^\Lambda : \forall \lambda \in \Lambda \ x_\lambda \in A' \right\}$$

indexed by $A' \in \mathscr{F}$. We also endow $\mathrm{Summ}_\Lambda(A, \alpha)$ with the topology induced by this one: it will again be denoted by $\beta_0(\alpha)$.

It can then be proved:

4.4. Proposition. $(\mathrm{Summ}_\Lambda(A, \alpha), \beta_0(\alpha)) \in$ **CLT**-(R, ρ).

4.5. Definition. A *square Λ-matrix with coefficients in* (R, ρ) is a family $\bar{a} = (a_{\lambda\mu})_{(\lambda, \mu) \in \Lambda \times \Lambda}$ of elements of R such that for every $\lambda \in \Lambda$ the family $(a_{\lambda\mu})_{\mu \in \Lambda}$, called the *$\lambda$th row of $\bar{a}$*, is summable in (R, ρ). Λ is called the *index set of $\bar{a}$*, and the family $(a_{\lambda\mu})_{\lambda \in \Lambda}$ is called the *μth column of $\bar{a}$*.

The set of square Λ-matrices with coefficients in (R, ρ) will be denoted by $\mathrm{SM}_\Lambda(R, \rho)$.

4.6. Let $\bar{a} = (a_{\kappa\lambda})_{(\kappa, \lambda) \in \Lambda \times \Lambda}$, $\bar{b} = (b_{\mu\nu})_{(\mu, \nu) \in \Lambda \times \Lambda} \in \mathrm{SM}_\Lambda(R, \rho)$. We define the usual "rows by columns" product $\bar{a}\bar{b}$: for each $(\lambda, \nu) \in \Lambda \times \Lambda$ put

$$c_{\lambda\nu} = \sum_{\mu \in \Lambda} a_{\lambda\mu} b_{\mu\nu},$$

and define in this way the family $\bar{c} = (c_{\lambda\nu})_{(\lambda, \nu) \in \Lambda \times \Lambda}$; it can be shown that the rows of this family are summable, hence $\bar{c} = \mathrm{SM}_\Lambda(R, \rho)$; therefore we can define $\bar{a}\bar{b} = \bar{c}$. This product turns out to be associative and bilinear, so that $R' = \mathrm{SM}_\Lambda(R, \rho)$ is a ring: its unit $1_{R'}$ is defined as

$$1_{R'} = (u_{\lambda\mu})_{(\lambda, \mu) \in \Lambda \times \Lambda} \qquad \text{such that } u_{\lambda\mu} = \begin{cases} 1_R & \text{iff } \lambda = \mu, \\ 0_R & \text{iff } \lambda \neq \mu. \end{cases}$$

Observe that when ρ is the discrete topology, $\mathrm{SM}_\Lambda(R, \rho)$ is precisely the ring of row-finite matrices with coefficients in R, which is the subject of study in [1].

4.7. If F is a finite subset of Λ and V is an open right ideal of (R, ρ), denote by $\mathscr{W}(F; V)$ the following R-submodule of $\mathrm{SM}_\Lambda(R, \rho)$:

$$\mathscr{W}(F; V) = \left\{(a_{\lambda\mu})_{(\lambda,\mu)\in\Lambda\times\Lambda} \in \mathrm{SM}_\Lambda(R, \rho) : \forall\lambda \in F, \forall\mu \in \Lambda\ a_{\lambda\mu} \in V\right\}.$$

Such submodules form a basis of neighbourhoods of zero in $\mathrm{SM}_\Lambda(R, \rho)$ for a topology which we denote by $\beta(\rho)$. Observe that $(\mathrm{SM}_\Lambda(R, \rho), \beta(\rho))$, as a topological (R, ρ)-module, is topologically isomorphic to the topological product of Λ copies of $(\mathrm{Summ}_\Lambda(R, \rho), \beta_0)$; hence it is Hausdorff and complete.

4.8. PROPOSITION. *If* $\mathrm{SM}_\Lambda(R, \rho)$ *is endowed with the topology* $\beta(\rho)$, *then the matrix product*

$$\mathrm{SM}_\Lambda(R, \rho) \times \mathrm{SM}_\Lambda(R, \rho) \to \mathrm{SM}_\Lambda(R, \rho), \qquad (\bar{a}, \bar{b}) \mapsto \bar{a}\bar{b}$$

is continuous.

Remark. $(\mathrm{SM}_\Lambda(R, \rho), \beta(\rho))$ is thus a topological ring (which is l.t. and complete).

Set $(R', \rho') = (\mathrm{SM}_\Lambda(R, \rho), \beta(\rho))$; it can be shown that Summ_Λ is a functor from **CLT**-(R, ρ) to **CLT**-(R', ρ'). In [4] it is proved that Summ_Λ : **CLT**-$(R, \rho) \to$ **CLT**-(R', ρ') is an equivalence of categories; also this result will now follow as a corollary from Theorem 3.6.

4.9. The ring R' contains the following *matrix units* (for $\lambda, \mu \in \Lambda$);

$$\epsilon_{\lambda\mu} = (a_{\kappa\nu})_{(\kappa,\nu)\in\Lambda\times\Lambda} \qquad \text{such that } a_{\kappa\nu} = \begin{cases} 1_R & \text{iff } \kappa = \lambda, \nu = \mu, \\ 0_R & \text{otherwise.} \end{cases}$$

We abbreviate $\epsilon_{\lambda\lambda}$ in ϵ_λ.

4.10. PROPOSITION. *The matrix units* $\epsilon_{\lambda\mu}$ *satisfy the following properties*

(1) $(\epsilon_\lambda)_{\lambda\in\Lambda} \in \mathrm{Summ}_\Lambda(R', \rho')$ *and* $\sum_{\lambda\in\Lambda}\epsilon_\lambda = 1_R$;

(2) *it results*

$$\epsilon_{\kappa\lambda}\epsilon_{\mu\nu} = \begin{cases} \epsilon_{\kappa\nu} & \text{iff } \lambda = \mu, \\ 0_{R'} & \text{iff } \lambda \neq \mu. \end{cases}$$

5. APPLICATIONS TO INFINITE MATRIX RINGS

We finally return to our original subject of study, that is, infinite matrices. We now show how the theory that we have developed can be applied to re-obtain the results of [1, 4].

Application I. *Matrices of Finite Rank*

5.1. Let R be a ring; endow R with the discrete topology δ_R. Let $\Lambda \neq \emptyset$ be a non-empty set, and put $(R', \rho') = \mathrm{SM}_\Lambda(R) = \mathrm{SM}_\Lambda(R, \delta_R)$, where $\rho' = \beta(\delta_R)$, the topology described in Subsection 4.7. Finally, let $R^{(0)}$ be the (*non-unitary*) subring of R' consisting of the matrices of finite rank:

$$R^{(0)} = \Big\{\bar{a} = (a_{\lambda\mu})_{(\lambda, \mu) \in \Lambda \times \Lambda} \in R' : \exists F \subseteq \Lambda,\ F \text{ finite},$$
$$\text{such that } \forall\lambda \in \Lambda, \forall\mu \in \Lambda \setminus F\ a_{\lambda\mu} = 0_R\Big\}.$$

We want to stress that in general $R^{(0)}$ is a ring *without unit*. By Proposition 4.10, $R^{(0)}$ is dense in (R', ρ').

5.2. LEMMA. *If* $\bar{a} \in R^{(0)}$, *then* $\mathrm{Ann}_{R'}(\bar{a})$ *is open in* (R', ρ').

Proof. Write $\bar{a} = (a_{\lambda\mu})_{(\lambda, \mu) \in \Lambda \times \Lambda}$, and let $F \subseteq \Lambda$, F finite, such that $\mu \in \Lambda \setminus F \Rightarrow \forall\lambda \in \Lambda\ a_{\lambda\mu} = 0_R$; put

$$V = \Big\{\bar{b} = (b_{\lambda\mu})_{(\lambda, \mu) \in \Lambda \times \Lambda} \in R' : \forall\lambda \in F, \forall\mu \in \Lambda\ b_{\lambda\mu} = 0_R\Big\};$$

V is an open right ideal of (R', ρ'), and $\mathrm{Ann}_{R'}(\bar{a}) \supseteq V$. ∎

5.3. Let l be an idempotent of R' such that $R^{(0)} l R^{(0)} = R^{(0)}$, and put $S = l R^{(0)} l$; S is a (*non-unitary*) subring of R' (cf. [1]). Consider now $\bar{S}$, the closure of S in (R', ρ'), and compare it with $lR'l$: obviously $S \subseteq lR'l$, and since $lR'l$ is closed in (R', ρ'), $\bar{S} \subseteq lR'l$; on the other hand, if $a \in lR'l$ there exists a net $(a_\gamma)_{\gamma \in \Gamma}$ taking values in $R^{(0)}$ and converging to a (because $R^{(0)}$ is dense in (R', ρ')), and then $(la_\gamma l)_{\gamma \in \Gamma}$ is a net with values in S and converging to $lal = a$, so that $lR'l \subseteq \bar{S}$; hence $\bar{S} = lR'l$. Note that this implies that $\bar{S}$ has l as its unit element.

We endow S and $\bar{S}$ with the topology induced by ρ', and we denote it by σ and $\bar{\sigma}$, respectively.

5.4. PROPOSITION. *Let* (S, σ) *and* $(\bar{S}, \bar{\sigma})$ *be as above, and* $B \in$ **Mod**-S. *The following are equivalent*:

(1) $BS = B$;

(2) $B \in$ **Mod**-(S, σ) *and* $BS = B$;

(3) $B \in$ **Mod**-$(\bar{S}, \bar{\sigma})$, *understanding that B is unitary as an $\bar{S}$-module*;

(4) *for all $x \in B$ there exists $s \in S$ such that $xs = x$.*

Proof. (1) ⇒ (2). Let us show that $\forall x \in B\ \mathrm{Ann}_S(x)$ is open in (S, σ). By (1), $x = \sum_{i=1}^n y_i l a_i l$, for some $y_i \in B$, $a_i \in R^{(0)}$. By Lemma 5.2 $\mathrm{Ann}_{R'}(a_i)$

is open in (R', ρ'), so we can find an open right ideal U_i of (R', ρ') such that $lU_i \subseteq \mathrm{Ann}_{R'}(a_i)$; we put $U = \bigcap_{i=1}^n U_i, V = U \cap S$: V is hence an open right ideal of (S, σ). Moreover,

$$\begin{aligned} b \in V \Rightarrow lb &\in \mathrm{Ann}_{R'}(a_i) && \text{for each } i = 1, \dots, n \\ &\Rightarrow y_i la_i lb = 0_B && \text{for each } i = 1, \dots, n \\ &\Rightarrow xb = 0_B, \end{aligned}$$

so $\mathrm{Ann}_S(x) \supseteq V$.

(2) ⇒ (3). Since (R', ρ') is complete, $(\overline{S}, \overline{\sigma})$ is the (Hausdorff) completion of (S, σ); being discrete, B is therefore a right topological $(\overline{S}, \overline{\sigma})$-module, and we want to prove that it is unitary. If $x \in B$, write $x = \sum_{i=1}^n y_i la_i l$ as above; then

$$xl = \left(\sum_{i=1}^n y_i la_i l \right) \cdot l = \sum_{i=1}^n y_i la_i l = x.$$

(3) ⇒ (4). Let $(l_\gamma)_{\gamma \in \Gamma}$ be a net taking values in S and converging to l; since B is topological over $(\overline{S}, \overline{\sigma})$, if $x \in B$ then $(xl_\gamma)_{\gamma \in \Gamma}$ is a net in B converging to $xl = x$; since B is discrete there exists $\overline{\gamma} \in \Gamma$ such that $xl_{\overline{\gamma}} = x$.

(4) ⇒ (1). This is obvious. ∎

5.5. Let T be a ring, possibly without unit; following [1], we denote by $\mathscr{C}_T$ the full subcategory of **Mod**-T consisting of those right T-modules M such that $MT = M$. Of course, if T has a unit then $\mathscr{C}_T =$ **Mod**-T. If $T = S$, the matrix subring described in Subsection 5.3, then Proposition 5.4 shows that $\mathscr{C}_S$ is closed under arbitrary direct sums, quotients, and submodules, i.e., that $\mathscr{C}_S$ is a hereditary pretorsion class.

The following is a reformulation of Theorem 3.2 of [1].

5.6. THEOREM. *Let R be a ring (with unit), $\Lambda \neq \emptyset$ a non-empty set, $R' = \mathrm{SM}_\Lambda(R)$ the ring of row-finite Λ-indexed matrices with coefficients in R, $R^{(0)}$ the subring of finite-rank matrices in R', $l \in R'$ an idempotent such that $R^{(0)} l R^{(0)} = R^{(0)}$, $S = lR^{(0)}l$, $\overline{S} = lR'l$. Then:*

(1) *$\mathscr{C}_{R^{(0)}}$ and $\mathscr{C}_S$ are hereditary pretorsion classes;*

(2) *the pair of functors*

$$- \hat{\otimes}_{R'} R'l \qquad \text{and} \qquad - \hat{\otimes}_{\overline{S}} lR'$$

is an equivalence of categories between $\mathscr{C}_{R^{(0)}}$ and $\mathscr{C}_S$.

Proof. By Proposition 5.4, $\mathscr{C}_{R^{(0)}} = \mathbf{Mod}\text{-}(R',\rho')$ and $\mathscr{C}_S = \mathbf{Mod}\text{-}(\overline{S}, \overline{\sigma})$; this immediately yields (1). Moreover, since $R^{(0)} l R^{(0)} = R^{(0)}$ is dense in (R',ρ'), by Theorem 3.6 the pair $- \hat{\otimes}_{R'} R'l$ and $- \hat{\otimes}_{\overline{S}} lR'$ is an equivalence between $\mathbf{CLT}\text{-}(R',\rho')$ and $\mathbf{CLT}\text{-}(\overline{S}, \overline{\sigma})$; but Proposition 3.9 tells that this equivalence induces an equivalence between $\mathbf{Mod}\text{-}(R', \rho')$ and $\mathbf{Mod}\text{-}(\overline{S}, \overline{\sigma})$: this proves (2). ∎

Application II. *Generalized Similarity between a Matrix Ring and Its Ring of Coefficients*

As in Section 4, let (R, ρ) be a complete right l.t. ring, $\Lambda \neq \emptyset$ a set, $(R', \rho') = (\mathrm{SM}_\Lambda(R, \rho), \beta(\rho))$. Consider, for a fixed $\lambda \in \Lambda$, the matrix unit ϵ_λ defined in Subsection 4.9; by Proposition 4.10, ϵ_λ is a dense idempotent of (R', ρ'). On the other hand, it is easy to verify (cf. [2, 3]) that $\epsilon_\lambda R' \epsilon_\lambda$, with the topology induced by ρ', is topologically isomorphic, as a ring, to (R, ρ). Then:

5.7. THEOREM. (cf. [4, Sect. 4]). *Let (R, ρ) be a complete right l.t. ring, $\Lambda \neq \emptyset$ a non-empty set, $(R', \rho') = (\mathrm{SM}_\Lambda(R, \rho), \beta(\rho))$; the categories* **CLT**-(R, ρ) *and* **CLT**-(R', ρ') *are equivalent.*

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