Regularity and morphic property of rings

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\textbf{ABSTRACT}

This is a continuation of recent work on the morphic property of rings. The main objective of this article is to study the relationships between regular rings and quasi-morphic rings, between unit-regular rings and morphic rings, and between strongly regular rings and centrally morphic rings.

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\textit{1. Introduction}

Rings $R$ are associative with unity. For an element $a$ in a ring $R$, the left (resp., right) annihilator of $a$ in $R$ is denoted $l_R(a)$ or $I(a)$ (resp., $r_R(a)$ or $r(a)$). The element $a \in R$ is called \textit{left quasi-morphic} if there exist $b, c \in R$ such that $Ra = l(b)$ and $l(a) = Rc$, and $a$ is called \textit{left morphic} if in addition $b$ and $c$ can be chosen to be the same element. The ring $R$ is called left quasi-morphic (resp., left morphic) if each of its elements is left quasi-morphic (resp., left morphic) in $R$. Right quasi-morphic (resp., morphic) rings are defined analogously. A left and right quasi-morphic (resp., morphic) ring is called a quasi-morphic (resp., morphic) ring. In 1976, Ehrlich [5] observed that a ring $R$ is a unit-regular ring (i.e., for each $a \in R$, $a = aua$ for some unit $u$ of $R$) iff it is left morphic and (von Neumann) regular. But it is only in 2004 when Nicholson and Sánchez Campos [10] started a systematic study of morphic rings. The concept of a left quasi-morphic ring was introduced by Camillo and Nicholson [11] in 2007 as a unifying generalization of left morphic rings and regular rings. It turns out that left...
quasi-morphic rings share a number of important properties with regular rings. For instance, among other interesting results, it is shown in [1] and [2] that finite intersections and finite sums of principal left ideals are again principal in a left quasi-morphic ring. These results largely attract our interest in quasi-morphic rings. Observe from [1] and [2] that all the known examples of quasi-morphic rings so far are either morphic rings, or regular rings, or the direct products of these rings. Our starting point is the indispensable question whether there exist quasi-morphic rings that do not belong to any of the three types. In Section 2, after constructing a special type of generators for principal ideals of the power series ring over a regular ring using the technique developed by Herbera [7], we prove that, for any regular ring \( R \), \( R[x]/(x^{n+1}) \) is quasi-morphic for each \( n \geq 0 \). This result is further used in Section 3 to construct a family of semiprimitive quasi-morphic rings. These results give the first known examples of quasi-morphic rings that are neither regular rings, nor morphic rings, nor the direct products of regular rings and morphic rings. It was proved in [8] and [9] that, for an integer \( n \geq 1 \), a ring \( R \) is unit-regular iff \( R[x]/(x^{n+1}) \) is morphic and that, for an endomorphism \( \sigma \) of a unit-regular ring \( R \) with \( \sigma(e) = e \) for all \( e^2 = e \in R \), \( R[x]/(x^{n+1}) \) is left morphic for each \( n \geq 0 \). In Section 4, we are motivated to consider similar questions for quasi-morphic rings: for an integer \( n \geq 1 \), is it true that a ring \( R \) is regular iff \( R[x]/(x^{n+1}) \) is quasi-morphic? For an endomorphism \( \sigma \) of a regular ring \( R \) with \( \sigma(e) = e \) for all \( e^2 = e \in R \), is \( R[x]/(x^{n+1}) \) left quasi-morphic for each \( n \geq 0 \)? Partial answers to these questions are obtained and some more examples of quasi-morphic rings are presented. In Section 5, we define a ring \( R \) to be left centrally morphic if for any \( a \in R \) there exists a central element \( b \) of \( R \) such that \( Ra = I(b) \) and \( I(a) = Rb \). This definition is motivated by the fact that, for an integer \( n \geq 1 \), a ring \( R \) is strongly regular iff \( R[x]/(x^{n+1}) \) is left centrally morphic. Several properties of these rings are proved, including a structure theorem of left (or right) perfect, left centrally morphic rings.

We write \( C(R) \), \( J(R) \) and \( U(R) \) for the center, the Jacobson radical and the group of units of \( R \), respectively. The ring of integers modulo \( n \) is denoted by \( \mathbb{Z}_n \). We write \( M_n(R) \) for the ring of all \( n \times n \) matrices over \( R \). The ring of polynomials in indeterminate \( x \) over a ring \( R \) is denoted by \( R[x] \). For an endomorphism \( \sigma \) of a ring \( R \), \( R[x; \sigma] \) denotes the (left) skew polynomial ring, in which the multiplication is subject to the condition that \( xr = \sigma(r)x \) for all \( r \in R \). For \( r, s \in R \), we say that \( r \) is equivalent to \( s \) if there exist \( u, v \in U(R) \) such that \( s = urv \).

2. The ring \( R[x]/(x^{n+1}) \)

We prove that, for a regular ring \( R \), \( R[x]/(x^{n+1}) \) is quasi-morphic for each \( n \geq 0 \). This gives a family of quasi-morphic rings that are neither regular rings, nor morphic rings (if \( R \) is not unit-regular), nor the direct products of regular rings and morphic rings. Our tool is the technique developed by Herbera [7] in constructing a special type of generators for principal ideals of the power series ring \( R[x] \) over a regular ring \( R \).

First we fix some notation. Following Herbera [7], let

\[
E = \left\{ e(x) \in R[x]: e(x) = e + \sum_{k=1}^{\infty} (1-e)a_k x^k, \text{ where } e^2 = e, \ a_k \in R, \ k = 1, \ldots \right\}.
\]

Fix an integer \( n \geq 0 \) and let \( S = R[x]/(x^{n+1}) \equiv R[x]/(x^{n+1}) \). For any \( \alpha = \sum_{k=0}^{n} a_k x^k \in S \), let \( \overline{\alpha} = \sum_{k=0}^{n} a_k x^k \in S \) be the image of \( \alpha \). We let

\[
E = \left\{ \overline{e(x)}: e(x) \in E \right\}.
\]

If \( R \) is a regular ring, the principal one-sided ideals of \( R[x]/(x^{n+1}) \) are completely described by the next result.

**Proposition 1.** Let \( R \) be a regular ring and let \( S = R[x]/(x^{n+1}) \) where \( n \geq 0 \). For \( \alpha \in S \) the following statements hold:
Lemma 3. Let \( R \) be a ring and let \( \alpha = \sum_{i=0}^{n} a_i x^i \in S := R[x]/(x^{n+1}) \) where \( \{e_i\}_{i=0}^{n} \) is a sequence of orthogonal idempotents of \( R \). Then

\[
S\alpha = I(\alpha^o) \quad \text{and} \quad S\alpha^o = I(\alpha).
\]

Proof. An easy calculation shows that

\[
S\alpha = R e_0 + R(e_0 + e_1)x + \cdots + R(e_0 + \cdots + e_n)x^n = I(\alpha^o).
\]

Since \( 1 - e_0 - \cdots - e_n, e_n, \ldots, e_1 \) are also orthogonal idempotents of \( R \), the second equality follows. \( \square \)
**Theorem 4.** Let $R$ be a regular ring and let $n \geq 0$. Then $R[x]/(x^{n+1})$ is a quasi-morphic ring.

**Proof.** By symmetry, it is enough to show that $S := R[x]/(x^{n+1})$ is left quasi-morphic. Let $\alpha \in S$. By Proposition 1,

\[ S\alpha = S(e_0 + e_1x + \cdots + e_n x^n)u \quad \text{and} \quad \alpha S = v(f_0 + f_1x + \cdots + f_n x^n)S, \]

where $u, v$ are units of $S$ and $\{e_i\}_{i=0}^n, \{f_i\}_{i=0}^n$ are sequences of orthogonal idempotents of $R$. Let $\beta = \sum_{i=0}^n e_i x^i$ and $\gamma = \sum_{i=0}^n f_i x^i$. Then, by Lemma 3,

\[ S\alpha = (S\beta)u = I(\beta^o)u = I(u^{-1}\beta^o), \]
\[ I(\alpha) = I(\nu \gamma) = I(\gamma)\nu^{-1} = (S\gamma^o)v^{-1} = S(\gamma^o v^{-1}). \]

So $\alpha$ is left quasi-morphic in $S$. \qed

**Corollary 5.** If $R$ is regular and $n \geq 0$, then the matrix rings over $R[x]/(x^{n+1})$ are all quasi-morphic.

**Proof.** If $R$ is regular then $\mathbb{M}_k(R)$ is regular for each $k \geq 1$. So $\mathbb{M}_k(R[x]/(x^{n+1})) \cong \mathbb{M}_k(R)[x]/(x^{n+1})$ is quasi-morphic by Theorem 4. \qed

3. The ring $\mathcal{R}[D, C]$

For any regular ring $R$ that is not unit regular and for any $n \geq 1$, $S := R[x]/(x^{n+1})$ is a quasi-morphic ring (by Theorem 4) that is not regular, and it is not morphic (see Theorem 11). Moreover, it can be easily seen that $S$ is not the direct product of morphic rings and regular rings. However, $S$ is not semiprimitive. A natural question is whether there exist semiprimitive quasi-morphic rings that are neither regular, nor morphic, nor the direct product of regular rings and morphic rings. The answer to this question is “Yes”. To explain this, we consider the following “tail ring”.

For a subring $C$ of a ring $D$, the set

\[ \mathcal{R}[D, C] := \{(d_1, \ldots, d_n, c, c, \ldots) : d_i \in D, c \in C, n \geq 1\}, \]

with addition and multiplication defined componentwise, is a ring. A necessary and sufficient condition for $\mathcal{R}[D, C]$ to be left morphic is obtained in [3]. Here we present a necessary and sufficient condition for $\mathcal{R}[D, C]$ to be left quasi-morphic.

**Proposition 6.** $\mathcal{R}[D, C]$ is a left quasi-morphic ring if and only if the following hold:

1. $D$ is left quasi-morphic.
2. For any $r \in C$ there exist $s, t \in C$ such that $Cr = I_C(t)$, $I_C(r) = Cs$, $Dr = I_D(t)$ and $I_D(r) = Ds$. 

**Proof.** It is similar to the proof of [3, Theorem 1]. \qed

**Corollary 7.** $\mathcal{R}[D, D]$ is a left quasi-morphic ring if and only if $D$ is a left quasi-morphic ring.
We regard $R[x]/(x^{n+1})$ as a subring of $M_{n+1}(R)$ by identifying the element $a_0 + a_1 x + \cdots + a_n x^n$ in $R[x]/(x^{n+1})$ with the matrix

$$
\begin{pmatrix}
    a_0 & a_1 & \cdots & a_{n-1} & a_n \\
    a_0 & a_1 & \cdots & a_{n-1} & \cdot \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_0 & a_1 & \cdots & a_{n-1} & a_0 \\
\end{pmatrix}
$$

in $M_{n+1}(R)$.

We will denote by $\varphi : R[x]/(x^{n+1}) \to M_{n+1}(R)$ such ring inclusion.

**Lemma 8.** Let $R$ be a ring and let $(e_i)_i$ be a sequence of orthogonal idempotents of $R$. Set $C = R[x]/(x^{n+1})$ and $D = M_{n+1}(R)$. If $\alpha = \sum_{i=0}^n e_i x^i \in C$, then:

1. $D\varphi(\alpha) = D \begin{pmatrix}
    e_0 & \cdots & 0 \\
    e_0 + e_1 & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & e_0 + e_1 + \cdots + e_n \\
\end{pmatrix}$.
2. $\varphi(\alpha)D = \begin{pmatrix}
    e_0 + e_1 + \cdots + e_n & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & e_0 \\
\end{pmatrix} D$.
3. $D\varphi(\alpha) = I_D(\varphi(\alpha^\circ))$ and $I_D(\varphi(\alpha)) = D\varphi(\alpha^\circ)$.

**Proof.** (1) By row operations, the matrix $e_i \varphi(e_0 + e_1 x + \cdots + e_n x^n)$ can be transformed into the diagonal matrix $A_i$ whose $(j, j)$-entry is $e_i$ provided $i < j \leq n + 1$ and zero otherwise. That is, there exists an invertible matrix $U_i$ such that $U_i e_i \varphi(e_0 + e_1 x + \cdots + e_n x^n) = A_i = e_i A_i$. Hence

$$
D\varphi(e_0 + e_1 x + \cdots + e_n x^n) = \sum_{i=0}^n D e_i \varphi(e_0 + e_1 x + \cdots + e_n x^n) = \sum_{i=0}^n D e_i A_i = D \sum_{i=0}^n A_i.
$$

(2) By a similar proof of (1).

(3) By (2),

$$
I_D(\varphi(\alpha)) = D \begin{pmatrix}
    1 - e_0 - e_1 - \cdots - e_n & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & 1 - e_0 \\
\end{pmatrix}.
$$

The latter is just $D(\varphi(\alpha^\circ))$ by (1), because $1 - e_0 - \cdots - e_n, e_n, \ldots, e_1$ are also orthogonal idempotents of $R$. So $I_D(\varphi(\alpha)) = D(\varphi(\alpha^\circ))$, and the other equality follows by interchanging $\alpha$ with $\alpha^\circ$. \[Q.E.D.\]

**Theorem 9.** Let $S = R[M_{n+1}(R), R[x]/(x^{n+1})]$ where $R$ is a regular ring and $n \geq 0$. Then the following hold:

1. $S$ is a semiprimitive ring that is not regular.
2. The matrix rings over $S$ are all quasi-morphic.
3. If in addition $R$ is not unit-regular, then $S$ is not morphic.
Proposition 1, there exist two sequences \( \{e_i\}_{i=0}^{n} \) into \( D := \mathbb{M}_{n+1}(R) \) as above, and let \( S = R[D, C] \). We first prove that \( S \) is left quasi-morphic. Since \( R \) is regular, \( D \) is regular, so it is quasi-morphic. In view of Proposition 6, it is enough to show that every element \( \alpha \in C \) satisfies Proposition 6(2). By Proposition 1, there exist two sequences \( \{e_i\}_{i=0}^{n}, \{f_i\}_{i=0}^{n} \) of orthogonal idempotents of \( R \) such that

\[
C\alpha = C \left( \sum_{i=0}^{n} e_i x^i \right) u \quad \text{and} \quad \alpha C = v \left( \sum_{i=0}^{n} f_i x^i \right) C,
\]

where \( u, v \) are units of \( C \). Let \( \beta = \sum_{i=0}^{n} e_i x^i \) and \( \gamma = \sum_{i=0}^{n} f_i x^i \). Then \( C\alpha = I_C(u^{-1} \beta) \) and \( I_C(\alpha) = C(y^\circ v^{-1}) \) by the proof of Theorem 4. So by Proposition 6(2) it suffices to show that \( D\phi(\alpha) = D\phi(\beta)\phi(u) = D\phi(\alpha) = D\phi(\gamma)\phi(v) = D\phi(\gamma^\circ v^{-1}) \). By (3.1) and Lemma 8,

\[
D\phi(\alpha) = D\phi(\beta u) = D\phi(\beta)\phi(u) = I_D(\phi(\gamma)\phi(v)^{-1}) = D\phi(\gamma^\circ v^{-1}).
\]

Thus we have proved that \( S \) is a left quasi-morphic ring. By symmetry, \( S \) is right quasi-morphic. For any \( k \geq 1 \),

\[
\mathbb{M}_k(S) = \mathbb{M}_k \left( R \left[ \mathbb{M}_{n+1}(R), \frac{R[x]}{(x^n+1)} \right] \right)
\]

\[
\cong \mathbb{R} \left[ \mathbb{M}_k(\mathbb{M}_{n+1}(R)), \frac{R[x]}{(x^n+1)} \right]
\]

\[
\cong \mathbb{R} \left[ \mathbb{M}_{n+1}(\mathbb{M}_k(R)), \frac{R[k]}{(x^n+1)} \right]
\]

is quasi-morphic as above because \( \mathbb{M}_k(R) \) is regular.

Since \( \mathbb{M}_{n+1}(R) \) is semiprimitive, so is \( S \). Since \( C \) is an image of \( S \), \( S \) cannot be regular. If \( S \) is morphic, then \( \mathbb{M}_{n+1}(R) \) is morphic (being a direct summand of \( S \)), and then \( \mathbb{M}_{n+1}(R) \) is unit-regular by a result of Ehrlich [5] (as \( \mathbb{M}_{n+1}(R) \) is regular already). This clearly shows that \( R \) is unit-regular by [6, Corollary 4.7]. \( \Box \)

4. Two questions

It was proved in [8, Theorem 9] that a ring \( R \) is unit-regular iff \( R[x]/(x^2) \) is morphic. First, we point out that this result can be stated in a more desirable form as Theorem 11 below. We begin with a lemma.

Lemma 10. Let \( n \geq 0 \) be an integer. If \( R[x]/(x^{n+1}) \) is left quasi-morphic (resp., left morphic), then so is \( R \).

Proof. Let \( a \in R \) and let \( \alpha = a \in S := R[x]/(x^{n+1}) \). Since \( \alpha \) is left quasi-morphic in \( S \), \( S\alpha = I(\beta) \) and \( I(\alpha) = S\gamma \), where \( \beta = \sum_{i=0}^{n} b_ix^i \), \( \gamma = \sum_{i=0}^{n} c_ix^i \in S \). But
By computation, one has \( \beta \) is regular in \( R \). Suppose that the implication in one direction is by [8, Corollary 5]. Suppose that \( \gamma \) is finite.

Question 1. Let \( n \geq 1 \) be an integer. Then a ring \( R \) is unit-regular iff \( R[\alpha]/(\alpha^{n+1}) \) is morphic.

Proof. The implication in one direction is by [8, Corollary 5]. Suppose that \( S := R[\alpha]/(\alpha^{n+1}) \) is morphic. Then \( R \) is morphic by Lemma 10. Let \( a \in R \). Then \( Ra = I(b) \) for some \( b \in R \). We next show that \( a \) is regular in \( R \). Thus \( a \) is unit-regular by Ehrlich [5]. Now let \( \alpha = bx^0 \in S \). Since \( S \) is left morphic, there exists \( \beta = \sum_{i=0}^{n} b_{i}x^{i} \in S \) such that \( S \alpha = I(\beta) \) and \( I(\alpha) = S \beta \). Since \( S \) is right morphic, each principal right ideal of \( R \) is a right annihilator by [10, Theorem 24]. Thus, we have \( r(\alpha) = r(\beta) = r(I(\beta)) = \beta S \).

By computation, one has

\[
I(\alpha) = I(b) + Rx + \cdots + Rx^{n},
\]

\[
r(\alpha) = r(b) + Rx + \cdots + Rx^{n},
\]

\[
S \beta = \{ r_{0}b_{0} + (r_{0}b_{1} + r_{1}b_{0})x + \cdots + (r_{0}b_{n} + \cdots + r_{n}b_{0})x^{n} : r_{i} \in R, 0 \leq i \leq n \},
\]

\[
\beta S = \{ b_{0}s_{0} + (b_{0}s_{1} + b_{1}s_{0})x + \cdots + (b_{0}s_{n} + \cdots + b_{n}s_{0})x^{n} : s_{i} \in R, 0 \leq i \leq n \}.
\]

Thus, \( x \in I(\alpha) = S \beta \) and \( x \in r(\alpha) = \beta S \). Hence there exist \( r_{0}, r_{1}, s_{0}, s_{1} \in R \) such that

\[
0 = r_{0}b_{0}, \quad 1 = r_{0}b_{1} + r_{1}b_{0}, \quad 0 = b_{0}s_{0}, \quad 1 = b_{0}s_{1} + b_{1}s_{0}.
\]

So \( r_{0} = r_{0}(b_{0}s_{1} + b_{1}s_{0}) = r_{0}b_{1}s_{0} = (r_{0}b_{1} + r_{1}b_{0})s_{0} = s_{0} \). Thus, \( b_{0} = b_{0}(r_{0}b_{1} + r_{1}b_{0}) = b_{0}r_{0}b_{1} + b_{0}r_{1}b_{0} = b_{0}s_{0}b_{0} + b_{0}r_{1}b_{0} = b_{0}r_{1}b_{0} \). Therefore, \( b_{0} \) is regular in \( R \). But, from \( I(\alpha) = S \beta \) it follows that \( Rb_{0} = I(b) \). Since \( I(b) = Ra \), we have \( Ra = Rb_{0} \); so \( a \) is regular in \( R \). \( \square \)

Because of Theorems 4 and 11, one is motivated to raise the following question.

**Question 1.** Let \( n \geq 1 \) be an integer. Is it true that a ring \( R \) is regular iff \( R[\alpha]/(\alpha^{n+1}) \) is quasi-morphic?

We only have a partial result to this question. A ring \( R \) is called **directly finite** if \( ab = 1 \) in \( R \) implies \( ba = 1 \), and \( R \) is called **reversible** if \( ab = 0 \) in \( R \) implies \( ba = 0 \). Clearly, every reversible ring is directly finite.
Proposition 12. Suppose that $S := R[x]/(x^{n+1})$ is left quasi-morphic where $n \geq 0$. The following hold:

1. $R$ is semiprimitive if in addition $R$ is directly finite.
2. $R$ is regular if in addition $R$ is reversible.

Proof. (1) Let $a \in J(R)$. Since $R$ is left quasi-morphic by Lemma 10, $Ra = I(u)$ for some $u \in R$. Let $\alpha = ux^n \in S$. By hypothesis, there exists $\beta := \sum_{i=0}^{n} b_i x^i \in S$ such that $I(\alpha) = S \beta$. Our computation shows that

$$I(\alpha) = I(u) + Rx + \cdots + Rx^n = Ra + Rx + \cdots + Rx^n \quad \text{and} \quad S \beta = \{r_0 b_0 + (r_0 b_1 + r_1 b_0)x + \cdots + (r_0 b_n + \cdots + r_n b_0)x^n : r_i \in R, \ 0 \leq i \leq n\}.$$ 

It follows from $I(\alpha) = S \beta$ that $Ra = Rb_0$ and $x \in S \beta$. So $b_0 \in J(R)$ and there exist $r_0, r_1 \in R$ such that $r_0 b_0 = 0$ and $r_0 b_1 + r_1 b_0 = 1$. Since $b_0 \in J(R)$, $r_0 b_1 = 1 - r_1 b_0$ is a unit of $R$. Since $R$ is directly finite, it follows that $r_0$ is a unit of $R$. So $0 = r_0^{-1}(r_0 b_0) = b_0$. Hence $a = 0$.

(2) Let $a \in R$. Then $Ra = I(u)$ for some $u \in R$. As done in the proof of (1), there exists $b_0 \in R$ such that $Ra = Rb_0$, $r_0 b_0 = 0$ and $r_0 b_1 + r_1 b_0 = 1$ where $r_0, r_1 \in R$. Since $R$ is reversible, $b_0 r_0 = 0$ and hence $b_0 = b_0(r_0 b_1 + r_1 b_0) = b_0 r_1 b_0$. So $b_0$ is regular and thus $a$ is regular. □

For a morphic ring $R$, $R[x]/(x^2)$ need not be quasi-morphic. In fact, by [4, Corollary 9], $Z_4[x]/(x^2)$ is not morphic, so it is not quasi-morphic (any commutative quasi-morphic ring is morphic by [2, Corollary 7]). However, $Z_4$ is not semiprimitive. Below we give an example of a semiprimitive morphic ring $R$ such that $R[x]/(x^2)$ is not quasi-morphic. The next lemma is used only in Example 14 below.

Lemma 13. For $n \geq 0$, $\mathcal{R}[D(x), C(x)](x) \cong \mathcal{R}[D(x), C(x)](x^{n+1})$.

Proof. It is straightforward to verify that $\theta : [\mathcal{R}[D(x), C(x)](x) \cong \mathcal{R}[D(x), C(x)](x^{n+1})$, given by

$$(a_1^{(0)}, a_2^{(0)}, \ldots) + (a_1^{(1)}, a_2^{(1)}, \ldots) x + \cdots + (a_1^{(n)}, a_2^{(n)}, \ldots) x^n \mapsto (a_1^{(0)} + a_1^{(1)} x + \cdots + a_1^{(n)} x^n, a_2^{(0)} + a_2^{(1)} x + \cdots + a_2^{(n)} x^n)$$

is the required isomorphism. □

Example 14. Let $S = R[x]/(x^2)$ where $R = \mathcal{R}[D, C]$ with $D = M_2(Z_2)$ and $C = Z_2[y]/(y^2)$. Then the following hold:

1. $R$ is a semiprimitive morphic ring that is not regular.
2. $S$ is not a left quasi-morphic ring.

Proof. (1) is by [3, Example 0.1] (or by [8, Theorem 8]).

(2) By Lemma 13, $S = R[x]/(x^2) \cong \mathcal{R}[D(x), C(x)](x^2)$. Since $C = Z_2[y]/(y^2)$ is directly finite but is not semiprimitive, $C[x]/(x^2)$ is not left quasi-morphic by Proposition 12(1). So $S$ is not left quasi-morphic by Proposition 6. □

It was proved in [9] that if $R$ is a unit-regular ring and $\sigma : R \to R$ is an endomorphism with $\sigma(e) = e$ for all $e^2 = e \in R$, then $R[x; \sigma]/(x^{n+1})$ is left morphic for each $n \geq 0$. This is the motivation of the next question.
**Theorem 15.** Suppose that \( I \) is an ideal of a regular ring \( R \) and \( \sigma : R \to R \) is an endomorphism such that \( \sigma(e) = e \) for all \( e^2 = e \in R \). Let

\[
S := \{a + bx : a \in R, b \in I\} \subseteq R[x; \sigma]/(x^2).
\]

Then \( S \) is a left quasi-morphic ring.

**Proof.** Clearly, \( S \) is a subring of \( R[x; \sigma]/(x^2) \). Let \( \alpha = a + bx \in S \) where \( a \in R \) and \( b \in I \). We prove that \( \alpha \) is left quasi-morphic in \( S \). Since \( R \) is regular, write \( a = aa'a \) with \( a' \in R \) and let \( e_0 = aa' \) and \( f_0 = a'a \). Then \( e_0, f_0 \) are idempotents and \( a = e_0a = af_0 \). Because \( 1 - (1 - e_0)b \sigma(a')x - 1 - a'bx \) are both units of \( S \), \( \alpha \) is equivalent to

\[
\left[ 1 - (1 - e_0)b \sigma(a')x \right](a + bx)(1 - a'bx) = \left[ 1 - (1 - e_0)b \sigma(a')x \right][a + (1 - e_0)bx] = a + (1 - e_0)b(1 - f_0)x.
\]

Thus, in view of the notice prior to this theorem, we can assume that \( e_0b = 0 = bf_0 \). Write \( b = bb' \) with \( b' \in R \) and let \( e_1 = bb' \) and \( f_1 = b'b \). Then \( e_1, f_1 \) are idempotents of \( I \) and \( e_0e_1 = 0 = f_1f_0 \). Let

\[
g = e_0 + e_1 - e_1e_0 \quad \text{and} \quad h = f_0 + f_1 - f_0f_1.
\]

Then \( g, h \) are idempotents of \( R \) and the following hold:

\[
e_0, e_1 \in gRg \quad \text{and} \quad f_0, f_1 \in hRh.
\]

It follows that

\[
(1 - g)a = (1 - g)b = a(1 - h) = b(1 - h) = 0.
\]

Now let

\[
\beta = (1 - g) + e_1(1 - e_0)x \in S,
\]

\[
\gamma = (1 - h) + (1 - f_0)f_1x \in S.
\]

By (4.3), one obtains

\[
\beta \alpha = \left[(1 - g) + e_1(1 - e_0)x \right](a + bx) = e_1(1 - e_0)x = e_1 \sigma((1 - e_0)a)x = 0 \quad \text{and} \quad \alpha \gamma = (a + bx)[(1 - h) + (1 - f_0)f_1x] = b \sigma(1 - h)x = b(1 - h)x = 0.
\]
Hence $Sβ \subseteq I(α)$ and $Sα \subseteq I(γ)$. We now verify the following facts:

1. $Sα = Rf_0 + (Rf_1 + If_0)x$.
2. $I(α) = R(1 - g) + I(1 - e_0)x$.
3. $Sβ = R(1 - g) + [Re_1(1 - e_0) + I(1 - g)]x$.

For (1), since $Sα \supseteq (Re_0)α = Ra$, one obtains $Sα = Ra + [Rb + Iσ(a)]x$. But because $Ra = Rf_0$, $Rb = Rf_1$ and $Iσ(a) = If_0$, we see (1) follows.

If $r + sx\in I(α)$, then $ra = 0$ and $rb + σ(a) = 0$; so $0 = [rb + σ(a)]f_0 = sσ(a)f_0 = sσ(αf_0) = sσ(a)$. Hence $rb = 0$ and so $I(α) = \{r + sx\in S: ra = rb = 0, σ(a) = 0\}$. Notice that $ra = rb = 0 \iff re_0 = re_1 = 0 \iff rg = 0$ and $σ(a) = 0 \iff se_0 = 0$, so (2) follows.

For (3), $Sβ = [r(1 - g) + [re_1(1 - e_0) + s(1 - g)]x: r \in R, s \in I]$. Noting that $(lx)β = I(1 - g)x$ and $R(1 - e_1)β = R(1 - g)$ (by (4.2)), one obtains (3).

Thus, to show $I(α) \subseteq Sβ$, it suffices to show that $I(1 - e_0) \subseteq Re_1(1 - e_0) + I(1 - g)$. So, let $c \in I$. Then

$$
c(1 - e_0) = \left[ce_1 + c(1 - e_1)\right](1 - e_0)
= ce_1(1 - e_0) + c(1 - e_1)(1 - e_0)
= ce_1(1 - e_0) + c(1 - g) \in Re_1(1 - e_0) + I(1 - g).
$$

So $I(α) \subseteq Sβ$.

Finally, to see $Sα \supseteq I(γ)$, let $r + sx \in I(γ)$. Then

$$
r(1 - h) + \left[r(1 - f_0)f_1 + s(1 - h)\right]x = 0.
$$

So $r(1 - h) = 0$ and $r(1 - f_0)f_1 + s(1 - h) = 0$. By (4.2), $f_1(1 - h) = 0$, and it follows that $0 = [r(1 - f_0)f_1 + s(1 - h)](1 - h) = s(1 - h)$. So $r(1 - f_0)f_1 = 0$. Thus, $r = rh = r(f_0 + f_1 - f_0f_1) = rf_0 \in Rf_0$ and $s = sh = s(f_0 + f_1 - f_0f_1) \in Rf_1 + If_0$. Hence $r + sx \in Sα$. We have proved that $I(α) = Sβ$ and $Sα \subseteq I(γ)$. So, $α$ is left quasi-morphic in $S$. □

**Corollary 16.** Let $R$ be a regular ring and let $σ : R → R$ be an endomorphism such that $σ(e) = e$ for all $e^2 = e ∈ R$. Then $R[x; σ]/(x^2)$ is a left quasi-morphic ring.

The assumption that $σ(e) = e$ for all $e^2 = e ∈ R$ in Corollary 16 cannot be removed by the next example.

**Example 17.** There exists a Boolean ring $R$ and an automorphism $σ : R → R$ with $σ(1) = 1$, but $R[x; σ]/(x^2)$ is not left quasi-morphic.

**Proof.** Consider the direct product $R = Z_2 × Z_2$ and let $σ : R → R$ be given by $(a_1, a_2) ↦ (a_2, a_1)$. Then $σ$ is an automorphism of $R$ with $σ(1) = 1$. Let $b = (1, 0) ∈ R$ and $S = R[x; σ]/(x^2)$. We next show that there do not exist $c, d ∈ R$ such that $I(bx) = S(c + dx)$. Suppose that $I(bx) = S(c + dx)$ where $c, d ∈ R$. Then

$$
x ∈ I(bx) = S(c + dx)
= \{rc + [rd + sσ(c)]x: r, s ∈ R\}.
$$

So $rc = 0$ and $1 = rd + sσ(c)$ for some $r, s ∈ R$. Thus $c = c(rd + sσ(c)) = sσ(c)$. Since $bx \neq 0$, $c + dx$ cannot be a unit of $S$. So $c \neq 1$. Thus $cc(c) = 0$ and hence $c = 0$. Thus, $rd = 1$, showing that $d = 1$. So $I(bx) = Sx$. But $1 - b ∈ I(bx)$ and $1 - b \notin Sx$. So $I(bx) ≠ Sx$. Hence $S$ is not left quasi-morphic. □
There exists a regular ring $R$ that is not unit-regular and an endomorphism $\sigma \neq 1_R$ such that $\sigma(e) = e$ for all $e^2 = e \in R$. Recall that a ring is called abelian if each of its idempotents is central. An abelian regular ring is called a strongly regular ring.

**Example 18.** Let $R = S \times T$ where $S$ is a strongly regular ring that is not commutative and $T$ is a regular ring that is not unit-regular. Then $R$ is regular, but it is not unit-regular. Take a unit $v$ of $S$ that is not central, and let $u = (v, 1_T)$. Then $u$ is a unit of $R$. Let $\sigma : R \to R$ be the endomorphism given by $\sigma(r) = u^{-1}ru$. Then $\sigma \neq 1_R$, and $\sigma(e) = e$ for all $e^2 = e \in R$.

By [10, Example 8], there exists a regular ring $R$ and an endomorphism $\sigma$ such that $\sigma$ is not onto, but $\sigma(e) = e$ for all $e^2 = e \in R$.

Example 19 below is another corollary of Theorem 15. For an ideal $I$ of a ring $R$, the trivial extension of $R$ by $I$, denoted by $R \rtimes I$, is the abelian group $R \oplus I$ with multiplication defined by $(a, b)(c, d) = (ac, ad + bc)$ for all $a, c \in R$ and $b, d \in I$.

**Example 19.** Let $R$ be a regular ring and let $I$ be an ideal of $R$. Then $R \rtimes I$ is a quasi-morphic ring.

5. Centrally morphic rings

A ring $R$ is called left centrally morphic if, for each $a \in R$, there exists $b \in C(R)$ such that $Ra = I(b)$ and $I(a) = Rb$. Right centrally morphic rings are defined analogously. A left and right centrally morphic ring is called a centrally morphic ring. This notion is motivated by the next fact.

**Theorem 20.** Let $n \geq 1$ be an integer. Then $R$ is strongly regular if and only if $R[x]/(x^{n+1})$ is a left centrally morphic ring.

**Proof.** “⇒”. Let $\alpha \in S := R[x]/(x^{n+1})$. By the proof of [9, Corollary 3], there exist orthogonal idempotents $e_0, \ldots, e_n$ of $R$ such that $\alpha$ is equivalent to $\beta := e_0 + e_1x + \cdots + e_nx^n \in S$. By Lemma 3, $S\beta = I(\beta^o)$ and $S\beta^o = I(\beta)$. Since $R$ is strongly regular, all idempotents of $R$ are central. So $\beta$ and $\beta^o$ are central in $S$. Thus, there exist $u, v \in U(S)$ such that $\alpha = ubv = (uv)\beta$. It follows that $S\alpha = S\beta = I(\beta^o)$ and $I(\alpha) = I(\beta) = S\beta^o$. So $S$ is left centrally morphic.

“⇐”. Let $a \in R$. Since $a$ is left morphic in $R$ by Lemma 10, $Ra = I(b)$ for some $b \in R$. Let $\alpha = bx^n \in S$. Then there exists $\beta = \sum_{i=0}^{n} b_i x^i \in C(S)$ such that $I(\alpha) = S\beta$. Since $\beta \in C(S)$, one has that $b_i \in C(R)$ for $i = 0, \ldots, n$. By computation, one has

$$I(\alpha) = I(b) + Rx + \cdots + Rx^n$$

and

$$S\beta = \{ r_0b_0 + (r_0b_1 + r_1b_0)x + \cdots + (r_0b_n + \cdots + r_nb_0)x^n : r_i \in R, 0 \leq i \leq n \}.$$ 

Thus, $x \in I(\alpha) = S\beta$. Hence there exist $r_0, r_1 \in R$ such that

$$0 = r_0b_0$$

and

$$1 = r_0b_1 + r_1b_0.$$ 

So $b_0 = b_0(r_0b_1 + r_1b_0) = b_0r_0b_1 + b_0r_1b_0 = r_0b_0b_1 + b_0r_1b_0 = b_0r_1b_0$. Therefore, $b_0$ is regular in $R$. But, from $I(\alpha) = S\beta$ it follows that $Rb_0 = I(b)$. Since $I(b) = Ra$, we have $Ra = Rb_0$ is an ideal of $R$. Thus, we have proved that $R$ is regular and every principal left ideal of $R$ is an ideal. Hence $R$ is strongly regular by [6, Theorem 3.2; p. 26].

Thus, centrally morphic rings are a generalization of strongly regular rings. Next, we give some properties of centrally morphic rings.
Lemma 21. Every left centrally morphic ring is abelian.

Proof. Let $R$ be a left centrally morphic ring and let $e^2 = e \in R$. Then there exists $b \in C(R)$ such that $Re = I(b)$ and $R(1-e) = I(e) = Rb$. Thus, since $b \in C(R)$, $Re$ and $R(1-e)$ are ideals of $R$. So, for any $r \in R$, $er \in Re$ and $(1-e)r \in R(1-e)$. Thus, $er = re$ and $(1-e)re = 0$. It follows that $er = re$. □

Examples 22. The following statements hold:

1. Strongly regular rings and commutative morphic rings are all centrally morphic. As Theorem 20 shows, there exists a centrally morphic ring that is neither strongly regular nor commutative.
2. Left centrally morphic rings are left morphic. But the converse does not hold, as any unit-regular ring that is not strongly regular is a morphic ring that is not left centrally morphic by Lemma 21.
3. Let $L = \prod R_i$ be a direct product of rings. Then $L$ is left centrally morphic iff $R_i$ is left centrally morphic for each $i$.
4. Any matrix ring or triangular matrix ring of size greater than 1 is not left centrally morphic.
5. If $R$ is left centrally morphic and $e^2 = e \in R$, then $eRe$ is left centrally morphic.
6. Let $I$ be an ideal of a strongly regular ring $R$. Then $R \cong I$ is a centrally morphic ring.

Proof. (1)–(5) are clear in view of Lemma 21.

(6) Let $\alpha \in S := R \cong I$. As shown in the proof of [8, Theorem 12], there exist orthogonal idempotents $e$ and $f$ in $R$ such that $\alpha$ is equivalent to $\gamma := (e, f) \in S$ and that $S\gamma = I(\beta)$ and $I(\gamma) = S\beta$ where $\beta = (1-e-f, f) \in S$. Since $R$ is strongly regular, idempotents of $R$ are central. So it follows that $\gamma$ and $\beta$ are central in $S$. Hence $S\gamma = S\alpha = I(\beta)$ and $I(\alpha) = I(\gamma) = S\beta$. □

It is still unknown whether a semiprime, left morphic ring is semiprimitive (see [10, Question, p. 402]). But, a semiprimitive morphic ring need not be regular by [8, Theorem 8] (or [3, Example 0.1]). In contrast to these facts, we have the following result.

Proposition 23. A ring $R$ is strongly regular if and only if $R$ is semiprime, left centrally morphic.

Proof. One implication is clear. Suppose that $R$ is a semiprime, left centrally morphic ring. Suppose that $a^2 = 0$ where $a \in R$. Then there exists $b \in C(R)$ such that $Ra = I(b)$. Thus, $Ra$ is an ideal. So $aR \subseteq Ra$ and hence $(aR)^2 \subseteq (Ra)(aR) = 0$. Therefore, $a = 0$ since $R$ is semiprime. We have proved that $R$ is a reduced ring. Thus $R$ is strongly regular by a result of [2, Corollary 8] that a reduced, left quasi-morphic ring is regular. □

Theorem 24. A ring $R$ is semiperfect, left centrally morphic if and only if $R$ is a finite direct product of local, left centrally morphic rings.

Proof. In view of Lemma 21 and Examples 22(3), the claim follows from the well known fact that a ring is semiperfect iff the unity is the sum of orthogonal local idempotents. □

Our concluding result is a structure theorem for left (or right) perfect, left centrally morphic rings, which is proved using several results of Nicholson and Sánchez Campos [10]. In [10], a ring is called left special if it is a local, left morphic ring with nilpotent Jacobson radical. These rings are characterized in [10, Theorem 9], and, in particular, they are precisely the left uniserial rings of finite composition length. The proof of [10, Theorem 9] clearly shows the following result.

Theorem 25. The following are equivalent for a ring $R$:

1. $R$ is left centrally morphic, local and $J(R)$ is nilpotent.
2. $R$ is local and $J(R) = Re$ for some $c \in C(R)$ with $c^n = 0$, $n \geq 1$. 
Thus, $R_i$ is generated by some power of a same central element. The left (resp., right) socle of the ring $R$ is (two-sided) uniserial of finite composition length such that each of its one-sided ideals is generated by some power of a same central element. The left (resp., right) socle of the ring $R$ is denoted by $\text{Soc}_R(R)$ (resp., $\text{Soc}(R_R)$).

**Theorem 26.** The following are equivalent for a ring $R$:

1. $R$ is left perfect, left centrally morphic.
2. $R$ is right perfect, left centrally morphic.
3. $R$ is semiperfect, left centrally morphic in which $J(R)$ is nil and $\text{Soc}(R_R)$ is an essential left ideal.
4. $R$ is semiperfect, left centrally morphic in which $J(R)$ is nil and $\text{Soc}(R_R)$ is an essential left ideal.
5. $R$ is a finite direct product of centrally special rings.
6. $R$ is semiperfect, right centrally morphic in which $J(R)$ is nil and $\text{Soc}(R_R)$ is an essential right ideal.
7. $R$ is semiperfect, right centrally morphic in which $J(R)$ is nil and $\text{Soc}(R_R)$ is an essential left ideal.
8. $R$ is right perfect, right centrally morphic.
9. $R$ is right perfect, right centrally morphic.

**Proof.** Because of the left–right symmetry of Condition (5), it suffices to show the equivalences (1) ⇔ (2) ⇔ (3) ⇔ (4) ⇔ (5). Clearly, (5) ⇒ (1) ⇒ (3) and (5) ⇒ (2) ⇒ (4) hold.

(3) ⇒ (5). Suppose (3) holds. Then $J(R)$ is nilpotent by [10, Lemma 33]. By Theorem 24, $R$ is a finite direct product of local, left centrally morphic rings, where each direct summand has nilpotent Jacobson radical; so it is centrally special by Theorem 25.

(4) ⇒ (5). Suppose (4) holds. By Theorem 24, $R = R_1 \times \cdots \times R_n$ where each $R_i$ is a local, left centrally morphic ring. To show (5), we only need to show that each $R_i$ has a nilpotent Jacobson radical by Theorem 25. Since Condition (4) passes to direct summands, $J(R_i)$ is nil and $\text{Soc}(R_R)$ is an essential left ideal of $R_i$ for each $i$. In particular, $\text{Soc}(R_R) \neq 0$. Let $R_i a \neq 0$ be a minimal left ideal of $R_i$. Since $R_i$ is left centrally morphic, there exists $b \in C(R_i)$ such that $R_i a = R_i(b)$ and $R_i(a) = R_i b$. Thus, $R_i a \cong R_i / R_i(a) = R_i / R_i b$. This shows that $R_i b$ is a maximal left ideal of $R_i$, so $J(R_i) = R_i b$ because $R_i$ is local. Since $R_i$ is nil, $b$ is nil and hence $J(R_i) = R_i b$ is nilpotent.

It is worth noting that, by [8, Example 18], there exists a commutative locally centrally morphic ring $R$ such that $J(R)$ is nil, but not nilpotent. By Proposition 23 and Theorem 26, a semiprime ring or a one-sided perfect ring, being left centrally morphic is the same as being right centrally morphic. But we do not know whether a left centrally morphic ring is always right centrally morphic.

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References