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Procedia Engineering 97 (2014) 1155 – 1166

**Procedia
Engineering**www.elsevier.com/locate/procedia

12th GLOBAL CONGRESS ON MANUFACTURING AND MANAGEMENT, GCMM 2014

A review on approaches for handling Bezier curves in CAD for Manufacturing

Hetal N.Fitter^{a*}, Akash B. Pandey^b, Divyang D. Patel^c, Jitendra M. Mistry^c^a*Mechanical Engineering Department, ITM Universe, Vadodara*^b*Mechanical Engineering Department, MSU of Baroda*^c*Mechanical Engineering Department, SVIT, VASAD*

Abstract

Ever since aesthetics have emerged in modern design, parametric curve like Bezier is widely used in CAD design. Various techniques and methodologies like curve fitting, curve manipulation, blending and merging of curves have been proposed over the years for better handling and enhancing Bezier curve use with every possible application in CAD domain such as image extraction, modelization, profile approximation, fairing and smoothing, etc. These techniques have found its applications in manufacturing as well. A review is done on various approaches for handling Bezier curve in Computer Aided Design for the purpose of manufacturing for example tool path optimization, profile design, reverse engineering, etc.

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Selection and peer-review under responsibility of the Organizing Committee of GCMM 2014

Keywords: Bernstein Basis, Degree of Bezier curve, parametric continuities, curve fitting, curve manipulation, curve blending Parameterization, Blending function

1. Introduction

In the early stages of manufacturing evolvement, products were designed from manufacturer's point of view. The geometric shapes of the products were easily described by elementary or a combination of elementary geometrical entities, for example, a planar patch by its length and width, cylinder by its radius and height, cube by its breadths. But with ever increasing demand for functional, comfort and aesthetics in products, have compelled the designers to deal with complex shapes leading to invention of new techniques to create complicated curves and surfaces. Technological developments in computer hardware and software gradually made it possible to

* Hetal N. Fitter. Tel.: +91-9824006431.

E-mail address: ehenf@yahoo.co.in

automatically create these free form shapes in digital form with the help of mathematical descriptions. Past few decades have witnessed formulations of many novel mathematical representations of free form curves with the need to make them more computers compatible [1-19]. Parameterization of a curve have availed some important advantages of being bounded in parameter range, independent of coordinate system over implicit representation. Furthermore, programming of parametric curves is easier and shorter in length. It was no sooner realised that parametric form of representing curves and surfaces proved to be the most promising shape description methods of all[20].

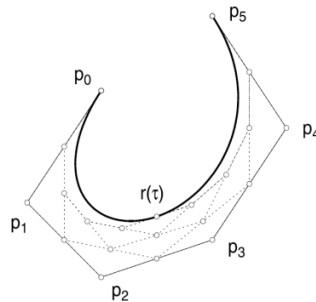


Fig. 1 Subdivision by De Casteljau's algorithm

Bezier curve was named after a French mathematician and engineer, Pierre Bézier, who developed this method of computer drawing using Bernstein basis in the late 1960s while working for the car manufacturer Renault. The curve possesses a characteristic intuitiveness in expressing the desired shape through property of Convexity of control polygon. Interestingly during the same time, de Casteljau too adopted Bernstein Basis for his work which was focused on the property of non-negativity and partition of unity of the basis function associated with the control points. His algorithm was lately acknowledged as de Casteljau's algorithm which evaluates and subdivides the Bezier curve as in fig 1.[21]. Subdivision of the curve is required to break the curve into number of small segments for various applications like curve fitting, segmentation, interpolation, and so. Besides some key Bernstein Basis properties that constraints the behaviour of Bezier curve like symmetry, recursion, non-negativity, unity of partition, unimodality, relation to monomial basis, lower and upper bounds, variation diminishing property, derivatives and integrals there are some algorithms based properties like degree elevation, degree reduction and composition that have consistently been evaluated and analyzed over a period of decades in order to broaden its applications. Based on the applications found, the paper is categorized as follows; curve fitting, curve manipulation and curve blending and merging will be discussed in section 2-4. First an overview of each approach is given in order to understand its applications further.

2. Curve Fitting

2.1. Overview

In the process of design and manufacture of a free form shaped product, some may become difficult to express in the CAD model. Such problems are solved by the process called Reverse Engineering where in digitization of the physical part is done using coordinate measuring machines (CMMs) or various types of non-contact scanning techniques[22]. With the advancement in machine vision techniques, image processing is finding extensive applications in object recognition, facial features extraction, visual inspection of industrial parts, security surveillance, etc[23]. Curve fitting is an essential task in image extraction where the extracted object contours are identified into number of small segments which are further described using lines and curves and the extracted object regions are fitted using surface fitting techniques like triangular patches, least squares, multistage methods, etc. According to Sarbajit Pal[23], polygonal approximation have proved to improve the performance of approximation for smooth curves using circular arcs and higher order curves. Such curves are best approximated by polygons

which can further be optimized to obtain smooth or fair fit. Various algorithms used for fitting have local or global schemes, which follow heuristic, geometric or algebraic considerations.

2.2. Applications

In the early stages of CAD/CAM development, geometric models lacked precision in creating free form shapes due to difficulties in defining geometric expressions when Kawabe S. [24] in 1980 sculpted a solid model and obtained its geometric information through numerically controlled 3 co-ordinate measuring machine. The measured data were processed by fitting Bezier curves and surfaces using Hosaka's expression[24] to generate a geometric model of the surface in the computer and a cutter path for NC machining was generated from it. Approximating the curve to digitized data generally presents some errors. Errors could be because of improper measurement process, geometrical limitations, clay model flaws, human error, instrument error, etc. Geometrical limitations can be specified as approximation tolerance. New approximations to the same data were also tried by slightly modifying the discrete data points but as Mineur Y. [25] observed during modelization of the curve, such empirical manipulations not necessarily give the desired result at once and can become tedious. Assuming a planar Bezier curve with smooth and monotonous curvature variation in mind, fixing degrees to be chosen in the range of 3 to 5, he proposed a curve fitting method with shape characterization of the curve. Shape characterization was achieved by applying geometrical conditions to the Bezier polygon with a constant angle to each vertex and a constant ratio between the lengths of each adjacent edge. By varying the tangent direction through the end points angles modifications to the fit can be made. This typical curve is used in styling CAD system for automotive industry.

The challenges faced in reverse engineering techniques for reconstruction and curve and surface fitting with accuracy and precision can be eliminated by generating tool path directly from the discrete point clouds. In the same light, Yingjie Z. [26] presented a tool path generation algorithm that uses cubic Bezier curves to fit the projected points obtained from initial tool path plan for guide surface. The curve is fitted using piecewise geometrically continuous cubic Bezier curve and the positions on the curve are adjusted to obtain equal arc length spacing for smoother material removal rate and decrease the fluctuation of cutting load. Accuracy is maintained by continuously measuring and evaluating error between the original and approximated data. When error goes beyond predefined value, the piecewise segment is subdivided into two parts and the point with maximum error is made a new breakpoint.

Towards the trend of making the CAD/CAM system more user friendly Loney G. et al [27] developed a software package called computer Interactive Surfaces Pre-APT (CISPA) built upon Computer Aided Sculptured Pre-APT (CASPA) as an enhancement. The system would guide the user through mathematical definitions of curve and free form surface which can be used to generate NC codes. Hochfeld et al [28] discussed the role of Bezier curves and surfaces during the investigations and installations of various computational methods into the Volkswagen car body design. Software development projects were accompanied with the emerging new technologies like large mainframe computers, powerful scanning machines, computer aided drafting and Numerical Control (NC) milling machines.

The initial ideas of curve fitting explored the Bernstein basis polynomial to its best when Rafajlowicz E.[2] studied the application of BB polynomial in mathematical statistics as a nonparametric estimation of the probability density function. He observed that the computational complexity of de Casteljau algorithm depends on the degree of the polynomial and in contrast proposed an algorithm which adapted Horner scheme for generating new probabilistic BB polynomial which was viewed in a different light. The proposed algorithm was based on the law of large numbers with a view of prospective application to generate large degree BB polynomial. The author claimed new potential application of BB polynomial in image processing and curve fitting with this property. Phien H.N. et al, [9] proposed three algorithms in connection to de Casteljau for computing control points on the Bezier curve considered as more efficient. Although the algorithms become more efficient than de Casteljau's only as the degree of the curve increases beyond cubic.

A Voronoi diagram of a polygon has various applications like pocketing tool path, automated mesh generation, computer vision and font generation. Kim D.S. et al [3] describes that bisectors in implicit form like line segment, elliptical (circular) arc, parabolic arc, hyperbolic arc are inconvenient for geometric processing and generating

programming code. An ordinary quadratic rational Bezier curve can easily be converted into elliptical bisector and subdivided but for hyperbolic and parabolic was not an easy fit. Hence, he considered the condition for minimum offset distance at the intersection of the same where when the bisector is subdivided at these points, each bisector can be converted into Bezier segment and each will have a monotonic offset distance. This made the entire Voronoi edge consisting of implicit bisectors to be converted into rational quadratic Bezier curve for unified representation. Author also stated that no extra effort was required to handle special cases for manipulating Voronoi edges.

Imine M. et al [5] developed some explicit expressions to represent an original Bezier curve into a piecewise division of the curve and make curve modelling more precise and localized. The technique was termed Parametric Piecewise Modelling (PPM) where he also presented effects of increasing the number of control points and generalization on PPM, order reduction, error analysis and intermediate curve construction. The motivation behind developing PPM was to manipulate a portion of the curve generated directly from the model and thereby circumvent the need to generate a new model to redefine that particular portion of the curve where intuitive instincts cannot be applied.

On the diverse side, many curve fitting algorithms has been developed for image processing. Ideally it is desirable and most natural to fit a profile contour or cloud points by a single Bezier curve but it is not possible in most of the situations due to some inherent limitations of Bezier curve. The global handling of the curve holds back its application where large curvature changes are required along the curve or number of small curvatures required within small intervals and points where abrupt change in tangent vectors or sudden discontinuity is desired like in sharp corners and turns. Again, defining curvature of the curve for a digital curve is not possible as there is no exact mathematical definition available. Therefore, the curves are more apt to fit by piecewise approximation in which the fitting length is broken into number of small segments and Bezier curve pieces are approximated in each segment connecting end to end. Detection and exact location of break point is crucial to achieve the best fit and many algorithms are essentially dedicated for the selection procedure. These points are identified where tangency and curvature discontinuities are observed and it is based on different criteria fixed by various authors will assist in locating tangent discontinuities and large curvature gaps.

Shao & Zhou's [4] with the purpose of converting bitmapped image into its outlined representation particularly by cubic Bezier curves developed a curve fitting algorithm. First the critical points were identified based on tangential discontinuities termed as corner points and points of big curvature change called as joints points. To locate critical corners chord curve area method was applied at each point on the curve as in fig. 2-a, to compute the curvature values for the corresponding points on the curve. Joint points were identified using area deviation algorithm. Once the exact positions of critical points are achieved fixed chord length was chosen and the Bezier curves segments were fit between the adjacent pair of critical points to fit into the data points. Pal S. et al [23] used chain coding technique to detect break points on the curve. Chain coding is a boundary representation technique that divides the digital curve into sequence of straight line segments of specified length and direction as depicted from fig. 2-b. These segments are assigned integer C_i varying from 0 to 7 according to its direction to the vector. A digital curve can be expressed by a string of codes following the sequence as n chain codes and is denoted as $C_1 C_2$

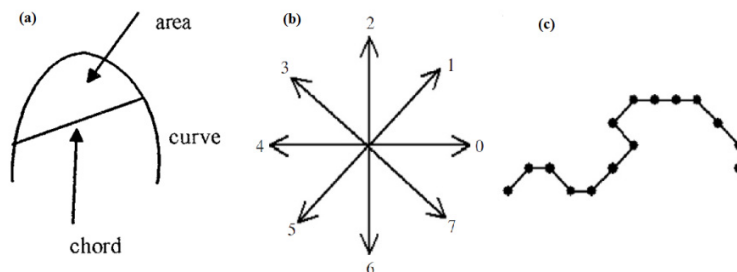


Fig. 2 (a) Chord, area and curve, (b) Chain code direction vector & (c) Curve representing 8 directional chain code 10701131000776

$C_3 \dots C_n$. The break point algorithm which is a rule based algorithm is implemented to obtain local maximum curvature to introduce dominant point. On breakpoint detection, the cubic Bezier curves are approximated in the curve segments. Since two of the four control points of cubic Bezier are anchored at end to end connections on the

segments the fitting of curve can only be done by proper positioning of two intermediate handles. A simple algebraic interpolation technique was used to interpolate the two intermediate control points to minimize the computational expense. To minimize the approximation error of the control points' refinement was done using Two Dimensional Logarithmic Search Algorithm (TDLSA) and Evolutionary Search Algorithm (ESA) schemes.

In machining too there has been an inclination on fit the G01 blocks (linear interpolation) also called continuous short blocks (CSB) into parametric curves before real time interpolation. Parametric form in CAD/CAM and CNC systems does justify its popularity by offering significant benefits like significant reduction in unsmooth motions due to acceleration and deceleration jerks, discontinuities in segmentation and of course the transmission load is reduced. The numerous CSB are unavoidable in the digital CAD model and not desirable for machining time minimization.

Yau and Wang [29] designed a Fast Bezier Interpolator (FBI) with real-time look ahead function with using PC-based control architecture. The need was to develop a real time algorithm which can avail the benefits of smooth interpolation with the parametric form along with the CSB. To maintain the specified accuracy, a CSB criterion is proposed to find the critical corner angle first. This criterion includes two tests, a) the critical corner angle test and b) the bi-chord error test. The angle between two connected NC blocks was defined as the 'corner angle' and as per the criterion, any corner smaller than the critical corner angle will be marked as a 'break point'. Bi-chord error test is used to obtain good accuracy and must satisfy the critical corner test. To restrain the contour error in an algorithm, this test was used to examine whether cubic Bezier curves can be used for curve fitting or not. The FBI interpolates 5 CSBs in fig. 3-a, -b, & -c [29]. However, this algorithm could not guarantee tangential continuity at the joint points.

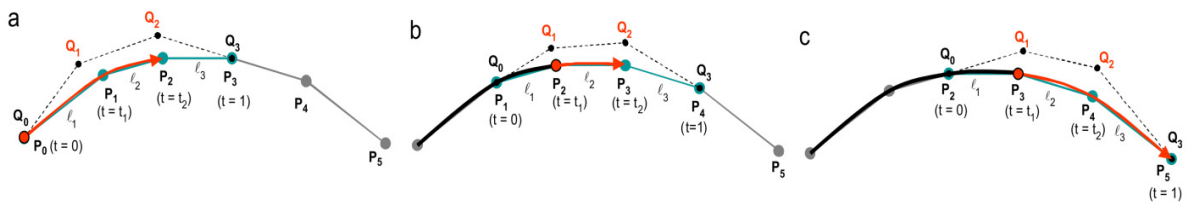


Fig. 3 (a) First and second blocks of CSB, (b) Third block of CSB, (c) Fourth and last block of CSB

Choi Y. -K. et al [30] proposed a tool path generation algorithm using Bezier curves and surfaces. Evaluation of cutter contact point was done by converting the forward step in physical domain into the parametric domain using Taylor's expansion and error compensation technique. The conversion was done for the curve and surface to be machined is in the parametric form. Further, side steps are calculated and algorithm is implemented. Desired surface was generated in CAD software and its coordinates by MATLAB software. This algorithm offered C^1 tangential continuity which means the discontinuities in the tool path still cause fluctuations in feed speed and acceleration. To this, Pateloup et al [31] proposed an algorithm for approximating a series of line segments and circular arcs using B-spline curve (a generalization of Bezier curve) to obtain C^2 continuous curves adapted for high feed rate machining and was achieved. But again, B-spline too does not have any explicit close-form expression and as observed by Q.Z. Bi et al [32] optimizing the evolution of curvature under predefined constraints was not straightforward. Consequently, he opted to use fair transition curves in replacement of linear and circular arcs for tool path smoothing in order to fulfil high speed and high accuracy machining. Advantage of fairing curve is that it has only one curvature extrema. The developed algorithm generates two cubic Bezier transition curves for every segment junction and has been integrated to an open NC system. Cubic Bezier transition curve is a generalization of cubic Bezier with control points B_{10} , B_{11} and B_{12} collinear as in fig.4-a. Explicit analytical expressions have been provided for both the approximation error and maximum curvature. Fig.4-b shows the comparison of linear tool path and smoothed tool path using transition curves. The Bezier transition algorithm generated smoother speed and acceleration curves and reduces machining time.

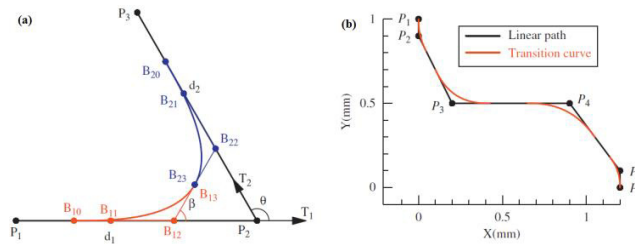


Fig. 4 (a) A pair of cubic Bezier transition curve, (b) G^2 continuous path with transition curve compared with linear path

3. Curve Manipulations

3.1. Overview

Having face challenges in fitting the curve on to the required data in order to achieve the desired shape or fit, researchers' explored manipulation of the parametric curve based on required constraints and shapes. This method will change the mathematical basis of the curve representation by changing the number of control points, knots, adding or removing weights and shape parameters, raising and lowering the order of curve but within the specified constraints such that it does not affect the characteristic properties of the curve. Several techniques and algorithms have been proposed aiming to have better control over the curve and to make them more intuitive and versatile in applications.

3.2. Application

Bezier, B-spline, Nurbs are the control point-based curves whose boundary information like end positions, tangents and higher order derivatives of end points cannot be directly interpolated but instead needs to be approximated for their control polygons. Then there is Hermite curve which can be explicitly represented by its boundary conditions and same for Hermite patches. In the international standard STEP (Standard for The Exchange of Product model data), Nurbs has been accepted as the only form for representation of free form curves and surfaces. Hence efficient geometric data conversion algorithm becomes necessary in order to exchange product data between CAD-systems. Ye X. [33], motivated by the need of conversion between different free form curves and surfaces, presented a method for explicitly expressing Bezier points directly from the boundary information for frequently used Hermite curves, coons-Hermite Cartesian sum patches and Coons-Boolean sum patches.

Parametric quadratic and cubic curves are extensively being used in various applications because of its geometric and numerical properties. These curves are not only specified by the quantities revealed in their mathematical definitions like control points or parametric coefficients but also via different geometric constraints as demonstrated by [3, 7, 8, 11, 17, 34, 35]. Curve manipulation for offsetting, fairing and blending applications has been viewed on a broader horizon in these literatures.

Kim H. O. et al [34] characterized and presented the necessary and sufficient conditions for the curvature of quadratic rational Bezier curve as an application for offset curves to be regular and have same tangent directions with quadratic rational Bezier curve. Approximate rational Bezier curve is of $3n-2$ degree higher than the Bezier curve of degree 'n'. This results in offset approximate by rational Bezier curve of high degree. Ahn Y. J. [12] proposed a method to approximate the offset Bezier curves of high degree using circle approximation method using same degree Bezier curve. In the method, circular arc has been approximated by a plane polynomial curve and bound for exact Hausdorff's distance [12] between two offset curves were derived using Floater's geometric Hermite interpolation method from previous work. Approximating rational functions with polynomial is quite complicated besides dealing with complex form of its derivatives and integrals. For which, Youdu H. et al [16] put forward a method to approximate rational Bezier curve with Bezier curves by constantly elevating the degree of rational Bezier curve and construct Bezier curve using the elevated control points as control points of the Bezier curve.

C-shaped transition curves from two non-enclosing circles Ω_1 and Ω_0 seen in fig. 6, a & b respectively.

[11] On emphasizing fairing curves further, generalized cubic Bezier spiral from cubic Bezier by imposing restrictions on $\theta=0$, $g=h$, $k=6/5 h \cos\phi$ and $0 < \phi < 1/2\pi$ shown in fig. 6-c in order to let the designer specify a point on the circle where transition curve can meet. Such construction worked as a tool to control the curvature and allows G^2 transition (or blending) between curve and circle.

A lot research has been done on Clothoid spirals so far pertaining to its behaviour suitable for fairing. The pro of clothoid is it does not have curvature extrema from aesthetics and CAD point of view, and from design of road and mobile trajectory's view point; it guarantees a constant jerk which minimizes wheel slip which is again a desirable property. But the con side of clothoid is, it is represented by Fresnel Integrals which is a non polynomial and cannot be solved analytically. Montes N. [14] in the search of continuous curve approximation of clothoid with lowest possible degree without the loss of clothoid behavior approximated the Fresnel integral to obtain rational Bezier curves (RBC). The online approach for mobile robot path design guaranteed original clothoidal behaviour when the RBC was used.

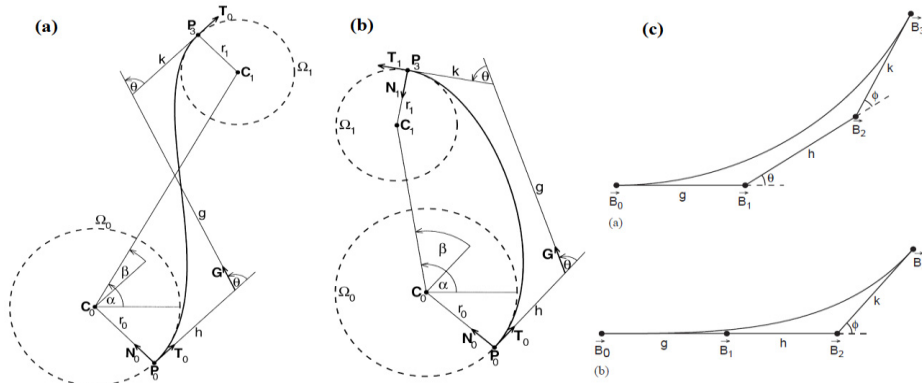


Fig. 6 (a) S-shaped cubic Bezier transition curve, (b) C-shaped cubic Bezier transition curve, (c) a, Cubic Bezier and b, Cubic Bezier spiral

There is a limitation to modifying quadratic and cubic Bezier beyond fairing, blending, representing analytical curve segments in parametric form or satisfying end conditions, etc when applications require large profiles or surfaces. Use of higher order Bezier is suggested as its modification through polygon vertices can be achieved. Also blending end points up to curvature continuity can be achieved and not affecting global property of Bezier significantly. Ntoko N.-M. [1] suggested the use of quintic Bezier curves for such applications with alternative blending functions with shape parameters. This would also circumvent the use of number of curve segments or surface patches.

It is better known that Bezier curves are apt to be approximated by their control polygon and subdivision of the curve makes approximation better. On recursive subdivision, the error bound between length of control polygon and length of chord converges and approximation in terms of arc-length is enhanced. If L_c denotes the chord-length and L_p polygon-length then arc-length of the Bezier curve is in the interval $[L_c, L_p]$ but L Grevsen J. [38] calculated arc-length at the midpoint $[L_c + L_p/2]$ and discovered that this quantity converges much faster under subdivision. He proposed this very good method of approximation through arc-length along with error estimates and tolerance distributions.

All available methods for improved shape approximation like degree elevation, composite Bezier curve or refinement and subdivision ultimately aim at reducing the gap between the curve and its control polygon. But by degree elevation; the number of control point increases, in composite Bezier curve; number of segments increases as the shape gets complex and subdivision increases the number of segments and control points as well. With the same motivation Sohail F. A. et al [15] introduced quasi-Bezier curve as a novel contribution to the Bezier curve theory which would consider local information within the classical Bezier framework, without increasing the number of control points or computational complexity. The gap between the curve and its control polygon is reduced and local

control is attained while retaining the core properties of the classical Bezier curve. Refer fig.7 (a,b).

Han X. –A. [13] achieved similar results through extension of classical Bernstein basis function inheriting most properties of Bezier curve. The basis function was introduced with ‘n’ adjustable shape parameters λ to control the shape of the curve without changing the control polygon as in fig.7 c.

From the view point of Basis function, Bezier curves have also been investigated using trigonometric polynomials. Parametric representation through trigonometric polynomial has been emphasized owing to its benefit in offering local control over the shape through shape parameters and its continuity conditions are being examined consistently under various degrees like Quadratic Trigonometric Bezier curve in [18], Quartic trigonometric Bezier curve in [19], quasi-quintic trigonometric Bezier curves [17]. In [39] Wu X. analyses the effects of shape parameters on the shape feature of the quadratic trigonometric Bezier curve using on theory of envelop and

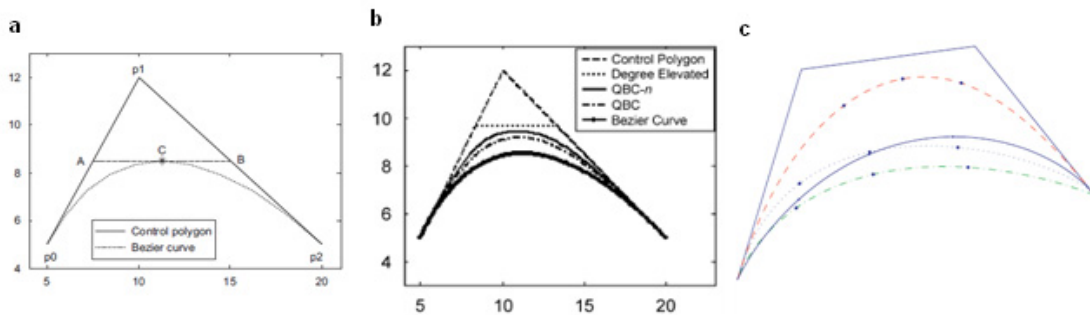


Fig. 7 (a) Quadratic Bezier curve illustrating the gap, (b) QBC-n, QBC and BC comparison; degree elevation, (c) Effect of shape on cubic Quassi Bezier by $\lambda_1, \lambda_2, \lambda_3$

topological mapping.

4. Curve Blending & Merging

4.1. Overview

Designers apply blending in order to avoid sharp edges and vertices to make them visually smooth and aesthetically pleasant. This can be understood as using chamfers and fillets in machining but in CAGD, blends are often used to acquire intricate shapes which otherwise are not possible to obtain by ordinary parametric entities, like a vertex can be substituted with a curve arc. Also it has found application in approximating a given curve. Certain order of continuity needs to be provided at the end points of the blends to ensure the smoothness. First order continuity is necessary for minimum desirable smoothness. Higher the order of continuity more pleasant is the blend. Merging is the term used to join or merge or stitch two adjacent curve segments with either point continuity C^0 or tangential continuity C^1 or curvature continuity C^2 . Although merging in CAGD is also referred to as blending in many cases and there is no proof documented on differentiating them.

4.2. Application

Mesh models used in computer graphics and CAD may sometimes have holes, gaps or vacant areas. This could be because of many reasons like local cloud data loss, some broken part in the model or could be a scanning equipment fault. Such mesh models are completed with hole filling or repair work algorithms. Li Z. et al [40] presented one such algorithm to fill the mesh hole using polynomial blending. The algorithm first searches features from the neighbouring area points to define the feature of the missing part in the hole. Based on the features two blending curves are constructed to complete the missing parts of the hole. These feature curves divides original hole

into small sub holes and Bezier-Lagrange hybrid patch is used to fill each sub hole.

Developable surfaces are more appealing in CAGD because of its ease in product surface development as well as ease in fabrication. Geodesic is referred to as the shortest line or curve segment between two points on a specific surface. It plays an important role in designing developable surface based product like shoemaking. Li C.-Y et al [41] studied Bezier curves as geodesics abutting G^1 connection required in joining two developable surfaces by constraining positions of control points and polynomials associated with each surface.

Integration of CAD & CAM system calls for fast communication modes in data sharing, data conversion, transferring geometric data from one system to another without the loss of accuracy and information in approximate conversion. With a common aim to minimize the loss of information and high degree of accuracy preservation during data transfer, Hui S. M. [10] considered a problem of merging two adjacent Bezier curves into a single Bezier curve. To satisfy more needs of approximate conversion, Cheng M. et al [42] developed a merging technique to merge multiple Bezier curves of different degrees.

5. Conclusion

The curve fitting approaches are found to be used in CNC tool path planning as seen in [24, 26-32, 43]. These adaptations have lead to saving machining time, reducing number of CNC blocks and reducing acceleration and deceleration jerks. Curve manipulation techniques have helped in making Bezier curve more users friendly and adaptive to versatile complex design solutions. Techniques pertaining to quadratic and cubic Bezier curves have been studied more by the researchers for the benefits discussed earlier. Manipulating of the curve has helped generate offsets easy. Also applications of using higher degree curves too are found to satisfy intricate shapes and their offsets which is another use in machining pockets on milling machines. Curve Blending have made it possible to obtain shapes which are difficult to produce directly from a single curve where as curve merging have facilitated in compacting the communication data size during data transfer and data conversion.

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