Numerical investigation on limitation of boussinesq equation for generating focusing waves

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Abstract

In this paper, the limitation of the Boussinesq equation for generating focusing waves is discussed in terms of the vertical asymmetry factor (VAF) and the water depth. The relationship between them is investigated through experiments in numerical wave tank (NWT) based on both the Boussinesq and fully nonlinear models. A fitting curve associating the VAF and the maximum water depth is suggested based on the numerical results. By using the curve, user could determine whether the Boussinesq model is applicable for studying focusing waves if both the parameters are given.

Keywords: Boussinesq equation; Focusing wave; Dispersion; NWT

1. Introduction

The dynamics of shallow water waves play an important role in the history of gravity water wave theories. Big challenges are faced especially when rogue waves occur, which have been overlooked in the past due to their extremely rare in-situ observations. In fact, these hazards happen much more often than expected, and should be paid great attention. A detailed review on rogue wave studies is given in [1]. The investigations on shallow water waves could be traced back to Scott Russell's observation of a solitary wave [2], and Airy's study on long waves [3]. In order to better understand the phenomenon, the Boussinesq equation was derived to study the solitary wave [2]. Not until 1895, systematic study of nonlinear shallow water waves was carried out by Korteweg & de Vries [4], who...
obtained the well-known KdV equation. Due to the fact that the KdV equation only considers waves propagating in one direction, its applications are limited. Therefore, the Boussinesq model becomes a more effective tool to study the wave dynamics numerically. A breakthrough was made by Wei et al. [5] on Boussinesq type equation, who had allowed for the fully nonlinearities in its derivation. Madsen et al [6] made one step further extended the Boussinesq equation to relatively deep water. A detailed review about the KdV and Boussinesq equation is presented in [7].

In order to generate rogue waves, focusing technique based on the linear dispersion is widely used. Sun et al. [8] have successfully generated the focusing waves and studied the dynamics acting on the cylinder by using Boussinesq model. Besides, Fuhrman and Madsen [9] also simulated focusing waves based on their numerical Boussinesq model. Note that both of their studies only involved cases with the cut-off frequency \( \omega_c < 2\omega_0 \) (subscript ‘0’ denotes the peak values). However, the cut-off frequency determines the vertical asymmetry of the focusing wave profile. Here we introduce the vertical asymmetry factor by estimating the ratio of the focusing wave crest height over its total wave height, i.e., \( VAF = \frac{H_{cr}}{H} \). The larger the cut-off frequency is, the larger vertical asymmetry the profile exhibits. Note that if considering strong nonlinear waves, horizontal asymmetry factor should be introduced, which is not the emphasis in this study and will not be discussed. In order to consider highly asymmetric focusing wave, it is necessary to investigate cases with cut-off frequency \( \omega_c \geq 2\omega_0 \). However, one should be aware that the Boussinesq equation has limitation on water depth. The reason is that the linear dispersion relation for high frequency components will not be fulfilled in relatively deep water. Meanwhile, the involvement of the high frequency components depends on the VAF, the more high VAF is, the more high frequency components are involved. Therefore, the success for generating a focusing wave by using the Boussinesq model lies in choosing both proper VAF and the water depth. However, the study about the relationship between VAF and water depth has not been carried out.

To shed light on the question raised above, we will study the relationship between water depth and VAF based on the numerical simulations by validating the results of the Boussinesq model with that of the fully nonlinear model. Supposed that the VAF is known, the water depth will be given based on this study, beyond which the Boussinesq model is not applicable for simulating focusing waves. So that it will provide a guideline for the users to study the focusing waves by using the Boussinesq model.

2. Mathematical formulation

2.1. The fully nonlinear model

The fully nonlinear model, referred as the Spectral Boundary Integral (SBI) method, has been suggested by Fructus et al. [10], and is later improved by Wang & Ma [11]. For completeness, only the formulations are presented here and details of the derivation are omitted. Introducing \( V = \frac{\partial \bar{\phi}}{\partial \eta} \sqrt{1 + |\nabla \eta|^2} \) to the boundary integral leads to

\[
\int \frac{V'}{R} \frac{1}{(1+D^2)^{3/2}} \, dX' + \int \frac{V'}{r} \, dX' = 2\pi \bar{\phi} - \int \bar{\phi}' \cdot \left[ (\eta' - \eta) \bar{V}' \frac{1}{R} \right] \, dX' + \int \bar{\phi}' \frac{R \bar{V}' \eta' - \eta' - \eta - 2h}{r^3} \, dX' \tag{1}
\]

and \( \nabla \) is the horizontal gradient operator, \( \eta \) is the the free surface elevation, \( \bar{\phi} \) denotes the velocity potential at free surface, \( r = \sqrt{R^2 + (\eta' + \eta + 2h)^2} \), \( R = |\mathbf{R}| = |\mathbf{X}' - \mathbf{X}| \), \( D = (\eta' - \eta)/R \). \( h \) is water depth. Based on that, the solution to the vertical velocity is reformulated as \( V = V_1 + V_2 + V_{3,s} + V_{3,b} + V_{4,s} + V_{4,b} \), where \( V_{3,s} \) and \( V_{4,s} \) are the surface integral parts and \( V_{3,b} \) and \( V_{4,b} \) are the bottom integral parts. Fructus et al [10] has rewritten \( V_{4,s} \) and transformed the dominant part into the third order convolutions. Grue [12] made one step further, expanded the kernels of \( V_{3,s} \) and \( V_{4,s} \) and wrote the dominant parts into the convolutions up to the sixth and seventh order respectively. Both the remaining integration parts of \( V_{3,b} \) and \( V_{4,b} \) are neglected.

In the study by Wang & Ma [11], some numerical techniques on improving the computational efficiency of the SBI model have been proposed. First, a new numerical de-singularity technique is introduced. It is found that to reach the same level of accuracy, one could use less points if the new method is applied. Therefore, it costs less CPU
time compared with the original method when considering the same level of accuracy. The other contribution is that a critical gradient has been suggested. When the waves are moderate, the boundary integrals will be evaluated through estimating convolutions up to the seventh order and the rest integration parts are so small that could be neglected. Due to that the most time consuming part is neglected, the scheme is very fast. Otherwise, when the waves are violent, the boundary integral solver recovers to calculating the convolutions up to the third order through estimating convolutions up to the seventh order and the rest integration parts are so small that could be neglected. Due to that the most time consuming part is neglected, the scheme is very fast. Otherwise, when the waves are violent, the boundary integral solver recovers to calculating the convolutions up to the third order through estimating convolutions up to the seventh order and the rest integration parts are so small that could be neglected. Due to that the most time consuming part is neglected, the scheme is very fast. Otherwise, when the waves are violent, the boundary integral solver recovers to calculating the convolutions up to the third order through estimating convolutions up to the seventh order and the rest integration parts are so small that could be neglected. Due to that the most time consuming part is neglected, the scheme is very fast. Otherwise, when the waves are violent, the boundary integral solver recovers to calculating the convolutions up to the third order through estimating convolutions up to the seventh order and the rest integration parts are so small that could be neglected. Due to that the most time consuming part is neglected, the scheme is very fast.

2.2. The Boussinesq model

The Boussinesq model adopted here is developed by Shi et al. [13], in which the estimation to the horizontal velocity is up to the second order, i.e., $\mathbf{u} = \mathbf{u}_a + \mathbf{u}_2(Z)$, where $\mathbf{u}_a$ denotes the velocity at a reference level $Z = Z_a$, $\mathbf{u}_2(Z) = (Z_a - Z)\nabla A + (Z_a^2 - Z^2)\frac{\nabla B}{2}$

and $Z_a = -0.53h$, $A = \nabla \cdot (h \mathbf{u}_a)$, $B = \nabla \cdot \mathbf{u}_a$. Note that a higher order Boussinesq model is suggested by Madsen et al. [6], in which the estimation to the horizontal velocity is up to the fifth order after introducing the Padé approximation. A multi-layer model is also suggested by Lynett & Liu [14], which divides the domain in several layers along the vertical direction. Although such improved models could be applied in relatively deep water, they cost more computational effort. Thus, only the second order version is discussed here considering the efficiency.

3. Numerical simulation

The domain of the NWT covers 32 peak wave lengths ($L_0$) and is resolved into 4096 points in both the numerical models. The water depth is fixed to $h = 1.5m$. The waves are generated in the middle of the domain, and absorbed at both ends. The duration lasts for 35 peak periods ($T_0$). The random wave spectrum adopted here is the Bretschneider-Mitsuyasu (B-M) spectrum, which is based on large number of fully developed wave records in shallow water. Thus, it could well represent the characteristics of shallow water waves. The spectrum is reformulated in terms of the significant wave height $H_s$ and peak frequency $\omega_0$ by Goda [15] and follows as

$$S(\omega) = \frac{1.25H_s^2\omega_0^4}{\pi^2} e^{-1.25(\omega_0/\omega)^4}$$

Based on that, the focusing wave is generated by linear superposition of all the Fourier components, i.e., $\eta(X,T) = \sum_{n=1}^{N} a_n \cos(k_nX - \omega_nT + \theta_n)$, where $a_n$ and $k_n$ are the amplitude and wave number of the wave component to be generated, $\omega_n = \sqrt{gk_n \tanh(k_n h)}$ is the corresponding circular frequency, $\alpha_n = \sqrt{25(\omega_n)\Delta\omega}$, $\Delta \omega = \omega_0/50$, $N = 50\omega_c/\omega_0$, $\theta_n = -k_nX_f + \omega_nT_f$, $X_f = 3L_0$ and $T_f = 2T_0$ are the focusing location and time respectively. We introduce a new form of Ursell number for random waves, i.e., $U_r = H_s L_0^2/h^5$. For a specific dimensionless water depth $k_0h$ and Ursell number $U_r$, one will obtain the corresponding spectrum.

To aid understanding the problem, the free surface time history at focusing location is shown in Fig. 1. It indicates that when $k_0h$ is relatively small, the focusing wave is successfully generated in the Boussinesq model (Fig. 1a); while $k_0h$ becomes larger, the waves are not fully focused in the Boussinesq model (Fig. 1b). It implies that for a specific $VAF$, the validity of the Boussinesq equation depends on the dimensionless water depth $k_0h$, and it is only accurate when $k_0h$ is smaller than a certain value.

Before we carry out further numerical simulations, the equation for estimating the error is introduced by using $Error = |\eta_{SB}(X_0,T_f) - \eta_{SBI}(X_0,T_f)|/|\eta_{SB}(X_0,T_f)|$, where $\eta_{SB}$ and $\eta_{SBI}$ are the free surface time history obtained by using the SBI and Boussinesq model respectively. We have found that only free surface time history with $Error \leq 2\%$ is acceptable. Meanwhile, the $VAF = 0.55\sim0.72$ is chosen in the following study, which corresponds to $\omega_c/\omega_0 = 1.5\sim10$. 

Next, we carry out a series of numerical experiments by attempting on different combinations of the vertical asymmetry factor, dimensionless water depth and Ursell number. The error against $k_\theta h$ with respect to different $VAF$ and $U_r$ is shown in Fig. 2. It should be noted that the errors are in the same level when considering the same $k_\theta h$, despite of the magnitude of $U_r$. For a specific $VAF$, the error is insignificant when $k_\theta h$ is small, i.e., Error $\leq 2\%$; However, the error keeps growing when $k_\theta h$ increases, and becomes larger than 2\% at about the same $k_\theta h$. Now we define the maximum depth in dimensionless form, which is a function of $VAF$ and independent of $U_r$, by using $h_m(VAF) = \max(k_\theta h(VAF, \text{error} < 2\%))$. Although the maximum depth $h_m$ is determined for a given $VAF$ as aforementioned, the value of $VAF$ selected by the user could be different from that we discussed in this study. Therefore, we will fit the data by a curve representing the relationship between $VAF$ and $h_m$ using least square method, which is given by

$$h_m = 0.0758 \times VAF^{-5.0044}$$

and the fitting results are shown in Fig. 3. The maximum error between the numerical results (in squares) and the fitted results (solid line) is only 1.7\%.

4. Discussion

The curve described in Fig. 3 indicates that the area underneath denotes the valid zone, while area above denotes the invalid zone for simulating focusing waves by using the Boussinesq model. Note that this graph is independent of the specific values of $H_s$ and $\omega_0$, the variables appearing in it are dimensionless. While $VAF$ increases, the maximum depth within which the Boussinesq model could be applied becomes smaller. This is because more high frequency components are involved if $VAF$ becomes larger, and the accuracy of the dispersion relation of these components requires relatively smaller water depth.

![Fig. 1. Time history of the free surface elevation ($VAF = 0.69$, $U_r = 0.4$)](image1)

![Fig. 2. Error against $k_\theta h$ with respect to different $VAF$ and $U_r$ (dash line corresponds to Error = 2\%)](image2)
5. Conclusion

The relationship between the maximum depth $h_m$ and the vertical asymmetry factor $VAF$ for simulating focusing waves by using the Boussinesq model is obtained, i.e., $h_m = 0.0758 \times VAF^{-0.0044}$. With the help of this curve, user is able to find the maximum water depth if the $VAF$ is given. So that one can determine whether the Boussinesq model is applicable for studying focusing waves based on the given information. If the experimental water depth is less than $h_m$, the Boussinesq model should be adopted as it costs less computational efforts. Otherwise, the Boussinesq model cannot be applied and fully nonlinear models, such as SBI, should be employed.

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