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Stability of a Class of Nonlinear Difference Equations

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In this paper using KAM theory we investigate the stability nature of the zero equilibrium of the system of two nonlinear difference equations

$$\left.\begin{aligned} x_{n+1} &= a_1x_n + b_1y_n + f(c_1x_n + c_2y_n) \\ y_{n+1} &= a_2x_n + b_2y_n + f(c_1x_n + c_2y_n) \end{aligned}\right\}, \qquad n = 0, 1, \dots,$$

where a_i, b_i, c_i, $i = 1, 2$, are real constants and $f\colon \mathbb{R} \to \mathbb{R}$ is a C^∞ function.

Key Words: nonlinear difference equations; stability; area preserving map; KAM theory.

1. INTRODUCTION

The main goal in this paper is to generalize the results concerning the stability nature of the zero equilibrium of the systems considered in [2]–[4]. So we consider the system of two nonlinear difference equations

$$\left.\begin{aligned} x_{n+1} &= a_1x_n + b_1y_n + f(c_1x_n + c_2y_n) = F_1(x_n, y_n) \\ y_{n+1} &= a_2x_n + b_2y_n + f(c_1x_n + c_2y_n) = F_2(x_n, y_n) \end{aligned}\right\}, \qquad n = 0, 1, \dots, \tag{1}$$

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where a_i, b_i, c_i, $i = 1, 2$, are real constants and $f\colon \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that

$$f \neq 0, \qquad f(0) = 0, \qquad f'(0) = 0. \tag{2}$$

Moreover, the map $F = (F_1, F_2)$ satisfies the following condition:

"F is an area preserving map with an elliptic fixed point at the origin and eigenvalues $\lambda, \bar{\lambda}$ such that $\lambda^k \neq 1, \quad k = 3, 4$." (H)

In the first proposition of this paper we prove that F satisfies the condition (H) if and only if

$$\begin{gathered} a_1 b_2 - a_2 b_1 = 1, \qquad |a_1 + b_2| < 2, \qquad a_1 + b_2 \neq -1, \\ (a_1 - a_2) c_2 = (b_1 - b_2) c_1. \end{gathered} \tag{3}$$

It is known (see [1–6]) that if F satisfies the condition (H), KAM theory can be applied to study the stability of the zero equilibrium of (1). In Proposition 2 we find conditions on a_i, b_i, c_i, $i = 1, 2$, and the function f so that the zero equilibrium of (1) is stable.

2. MAIN RESULTS

We now prove our main results.

In the first proposition we find necessary and sufficient conditions so that the map $F = (F_1, F_2)$ satisfies the condition (H).

PROPOSITION 1. *Consider the map $F = (F_1, F_2)$ where F_1, F_2 are defined in* (1) *where $f\colon \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that* (2) *are satisfied. Then F satisfies the condition* (H) *if and only if* (3) *hold.*

Proof. Suppose that F satisfies the condition (H). Then from (2) it is obvious that the point $(0, 0)$ is a fixed point for F. Let, for any $x, y \in \mathbb{R}$,

$$DF = \begin{pmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} a_1 + c_1 f' & b_1 + c_2 f' \\ a_2 + c_1 f' & b_2 + c_2 f' \end{pmatrix}. \tag{4}$$

Then, using (2), we can see that the characteristic equation of DF with $x = y = 0$ is the following:

$$\lambda^2 - \lambda(a_1 + b_2) + a_1 b_2 - a_2 b_1 = 0. \tag{5}$$

From the condition (H) we have that (5) has roots λ and $\bar{\lambda}$ where $\lambda^k \neq 1$, $k = 3, 4$. Then we have

$$a_1 b_2 - a_2 b_1 = 1, \qquad |a_1 + b_2| < 2, \qquad a_1 + b_2 \neq -1. \tag{6}$$

Moreover, since from the condition (H), F is an area preserving map (see [1]) we have for all $x, y \in \mathbb{R}$

$$\det DF(x, y) = 1. \tag{7}$$

Therefore, relations (4), (6), and (7) imply that

$$f'(u)\big((a_1 - a_2)c_2 - (b_1 - b_2)c_1\big) = 0 \tag{8}$$

for all $u \in \mathbb{R}$. Since (2) holds, it is obvious that there exists a $u \in \mathbb{R}$ such that $f'(u) \neq 0$. Then, from (8),

$$(a_1 - a_2)c_2 = (b_1 - b_2)c_1. \tag{9}$$

Hence from (6) and (9), relations (3) are satisfied.

Suppose now that (3) holds. Then we can easily prove that F satisfies the condition (H). This completes the proof of the proposition. ∎

In the following proposition using KAM theory we study the stability of the zero equilibrium of (1) where (2) and (3) hold.

PROPOSITION 2. *Consider the system of two nonlinear difference equations* (1) *where* $f: \mathbb{R} \to \mathbb{R}$ *is a* C^∞ *function such that* (2) *hold and* a_i, b_i, c_i, $i = 1, 2$, *are real constants such that* (3) *are satisfied. Moreover, suppose that*

$$\begin{aligned} c_2 &\neq 0 \quad \text{if } b_1 \neq b_2, \\ c_1 &\neq 0 \quad \text{if } b_1 = b_2. \end{aligned} \tag{10}$$

Then the zero equilibrium of (1) *is stable if the following conditions are satisfied*:

$$f'''(0) \neq p(f''(0))^2,$$

$$p = \begin{cases} \dfrac{c_2(2c+1)g(a_1 - a_2)}{(2-c)(c+1)(b_1 - b_2)a_2}, & \text{if } b_1 \neq b_2, \\ \dfrac{-c_1(1+2c)}{(2-c)(c+1)}, & \text{if } b_1 = b_2, \end{cases} \tag{11}$$

where $c = a_1 + b_2$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *is a function such that*

$$g(v) = v^2 - cv + 1. \tag{12}$$

Proof. We may proceed to find the Birkhoff Normal Form of (5) by using the transforms of KAM theory described in [2].

First Transformation. First we find the matrix P such that

$$P^{-1}AP = \operatorname{diag}(\lambda, \bar{\lambda}), \qquad A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix},$$

where $\lambda, \bar{\lambda}$ are the eigenvalues of the matrix A which satisfy (12). Since λ satisfies (12), we get

$$P = \begin{pmatrix} \dfrac{a_1 - \bar{\lambda}}{a_2} & \dfrac{a_1 - \lambda}{a_2} \\ 1 & 1 \end{pmatrix}, \qquad P^{-1} = \frac{a_2}{\lambda - \bar{\lambda}} \begin{pmatrix} 1 & \dfrac{\lambda - a_1}{a_2} \\ -1 & \dfrac{a_1 - \bar{\lambda}}{a_2} \end{pmatrix}.$$

Now the change of variables

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = P \begin{bmatrix} u_n \\ v_n \end{bmatrix}$$

transforms (1) into the system

$$\left.\begin{aligned} u_{n+1} &= \lambda u_n + \frac{a_2 - a_1 + \lambda}{\lambda - \bar{\lambda}} f(\mu u_n + \bar{\mu} v_n) \\ v_{n+1} &= \bar{\lambda} v_n + \frac{a_2 - a_1 + \bar{\lambda}}{\bar{\lambda} - \lambda} f(\mu u_n + \bar{\mu} v_n) \end{aligned}\right\}, \qquad n = 0, 1, \ldots, \tag{13}$$

where

$$\mu = \frac{c_1(a_1 - \bar{\lambda})}{a_2} + c_2. \tag{14}$$

Since (2) holds, by writing the Taylor expansion for $f(\mu u_n + \bar{\mu} v_n)$ about $(0, 0)$ and keeping terms up to order 3, from (13) we take

$$\begin{aligned} u_{n+1} &= \lambda u_n + \frac{a_2 - a_1 + \lambda}{\lambda - \bar{\lambda}} \\ &\quad \times \left(\frac{A}{2} (\mu u_n + \bar{\mu} v_n)^2 + \frac{B}{6} (\mu u_n + \bar{\mu} v_n)^3 \right) + O_4, \\ v_{n+1} &= \bar{\lambda} v_n + \frac{a_2 - a_1 + \bar{\lambda}}{\bar{\lambda} - \lambda} \\ &\quad \times \left(\frac{A}{2} (\mu u_n + \bar{\mu} v_n)^2 + \frac{B}{6} (\mu u_n + \bar{\mu} v_n)^3 \right) + O_4, \end{aligned} \tag{15}$$

where $n = 0, 1, \ldots$ and $A = f''(0)$, $B = f'''(0)$.

Second Transformation. In (15) we make the change of variables

$$\begin{aligned} u_n &= \xi_n + \phi_2(\xi_n, \eta_n) + \phi_3(\xi_n, \eta_n), \\ v_n &= \eta_n + \psi_2(\xi_n, \eta_n) + \psi_3(\xi_n, \eta_n), \end{aligned} \tag{16}$$

with

$$\begin{aligned} \phi_k(\xi, \eta) &= \sum_{j=0}^{k} \alpha_{kj} \xi^{k-j} \eta^j, \\ \psi_k(\xi, \eta) &= \sum_{j=0}^{k} \bar{\alpha}_{kj} \xi^j \eta^{k-j}, \end{aligned} \qquad k = 2, 3 \tag{17}$$

to obtain

$$\xi_{n+1} = \lambda \xi_n + \alpha_2 \xi_n^2 \eta_n + O_4, \qquad \eta_{n+1} = \bar{\lambda} \eta_n + \bar{\alpha}_2 \xi_n \eta_n^2 + O_4. \quad (18)$$

In order to calculate the coefficients α_{kj}, α_2 we substitute (16), (17), (18) to (15). Then the resulting equations will show an equality of two power series in ξ_n and η_n whose coefficients can be identified recursively up to order 3. Hence we get

$$\begin{aligned}
\alpha_{20} &= \frac{A(a_2 - a_1 + \lambda)\mu^2}{2(\lambda - \bar{\lambda})(\lambda^2 - \lambda)}, \\
\alpha_{21} &= \frac{A(a_2 - a_1 + \lambda)\mu\bar{\mu}}{(\lambda - \bar{\lambda})(1 - \lambda)}, \\
\alpha_{22} &= \frac{A(a_2 - a_1 + \lambda)\bar{\mu}^2}{2(\lambda - \bar{\lambda})(\bar{\lambda}^2 - \lambda)}, \\
\alpha_{30} &= \frac{(a_2 - a_1 + \lambda)\mu}{(\lambda^3 - \lambda)(\lambda - \bar{\lambda})}\left(A(\alpha_{20}\mu + \bar{\alpha}_{22}\bar{\mu}) + \frac{\mu^2 B}{6}\right), \\
\alpha_{31} &= \text{arbitrary}, \\
\alpha_{32} &= \frac{a_1 - a_2 - \lambda}{(\lambda - \bar{\lambda})^2}\left(A\mu\bar{\mu}(\alpha_{21} + \bar{\alpha}_{20}) + A\bar{\mu}^2\bar{\alpha}_{21}\right. \\
&\qquad \left. + A\mu^2\alpha_{22} + \frac{B\mu\bar{\mu}^2}{2}\right), \\
\alpha_{33} &= \frac{(a_2 - a_1 + \lambda)\bar{\mu}}{(\bar{\lambda}^3 - \lambda)(\lambda - \bar{\lambda})}\left(A(\alpha_{22}\mu + \bar{\alpha}_{20}\bar{\mu}) + \frac{\bar{\mu}^2 B}{6}\right),
\end{aligned} \quad (19)$$

and

$$\begin{aligned}
\alpha_2 = {} & \frac{A\mu\bar{\mu}\alpha_{20}(a_2 - a_1 + \lambda)}{\lambda - \bar{\lambda}} + \frac{A\bar{\mu}\mu\bar{\alpha}_{21}(a_2 - a_1 + \lambda)}{\lambda - \bar{\lambda}} \\
& + \frac{A\mu^2\alpha_{21}(a_2 - a_1 + \lambda)}{\lambda - \bar{\lambda}} + \frac{A\bar{\mu}^2\bar{\alpha}_{22}(a_2 - a_1 + \lambda)}{\lambda - \bar{\lambda}} \\
& + \frac{B\mu^2\bar{\mu}(a_2 - a_1 + \lambda)}{2(\lambda - \bar{\lambda})}.
\end{aligned} \quad (20)$$

We suppose first that $b_1 \neq b_2$. Then from (3), (14), and since $\bar{\lambda}$ satisfies (12), we get

$$\begin{aligned}\mu &= \frac{(a_1 - \bar{\lambda})(a_1 - a_2)c_2}{a_2(b_1 - b_2)} + c_2 \\ &= \frac{c_2\big((a_1 - \bar{\lambda})(a_1 - a_2) + b_2(a_1 - a_2) - 1\big)}{a_2(b_1 - b_2)} \\ &= \frac{c_2\big((a_1 - a_2)(a_1 - \bar{\lambda} + b_2) - 1\big)}{a_2(b_1 - b_2)} \\ &= \frac{c_2((a_1 - a_2)\lambda - 1)}{a_2(b_1 - b_2)}. \end{aligned} \tag{21}$$

Then relations (19), (20), and (21) imply that

$$\begin{aligned}\alpha_2 = {} & \frac{c_2^3|(a_1 - a_2)\lambda - 1|^2}{a_2^3(b_1 - b_2)^3} \\ & \times \left(\frac{c_2 A^2}{a_2(b_1 - b_2)} \left(\frac{(a_2 - a_1 + \lambda)^2((a_1 - a_2)\lambda - 1)^2}{2(\bar{\lambda} - \lambda)^2(\lambda^2 - \lambda)} \right.\right. \\ & \qquad + \frac{|a_2 - a_1 + \lambda|^2|(a_1 - a_2)\lambda - 1|^2}{(\bar{\lambda} - \lambda)^2(\bar{\lambda} - 1)} \\ & \qquad + \frac{(a_2 - a_1 + \lambda)^2((a_1 - a_2)\lambda - 1)^2}{(\bar{\lambda} - \lambda)^2(1 - \lambda)} \\ & \qquad \left. + \frac{|a_2 - a_1 + \lambda|^2|(a_1 - a_2)\lambda - 1|^2}{2(\bar{\lambda} - \lambda)^2(\bar{\lambda} - \lambda^2)} \right) \\ & \left. + \frac{B(a_2 - a_1 + \lambda)((a_1 - a_2)\lambda - 1)}{2(\lambda - \bar{\lambda})} \right). \end{aligned}$$

Then, for $b_1 \neq b_2$ we take

$$\Re(a_2) = \frac{c_2^3(g(a_1 - a_2))^2}{4a_2^3(b_1 - b_2)^3} \left(\frac{c_2 A^2(2c + 1)g(a_1 - a_2)}{(2 - c)(c + 1)a_2(b_1 - b_2)} - B \right). \tag{22}$$

Suppose now that $b_1 = b_2$. Then using (9) and since from (6), $a_1 \neq a_2$, we have $c_2 = 0$. So, from (14), it is obvious that

$$\mu = \frac{c_1\left(a_1 - \bar{\lambda}\right)}{a_2}. \tag{23}$$

Therefore, from (19), (20), and (23), we get

$$\begin{aligned}
\alpha_2 = {} & \frac{c_1^3 |a_1 - \bar{\lambda}|^2}{a_2^3} \\
& \times \left(\frac{c_1 A^2}{a_2} \left(\frac{(a_2 - a_1 + \lambda)^2 \left(a_1 - \bar{\lambda}\right)^2}{2(\bar{\lambda} - \lambda)^2 (\lambda^2 - \lambda)} + \frac{|a_2 - a_1 + \lambda|^2 |a_1 - \bar{\lambda}|^2}{(\bar{\lambda} - \lambda)^2 (\bar{\lambda} - 1)} \right.\right. \\
& \left. + \frac{(a_2 - a_1 + \lambda)^2 \left(a_1 - \bar{\lambda}\right)^2}{(\bar{\lambda} - \lambda)^2 (1 - \lambda)} + \frac{|a_2 - a_1 + \lambda|^2 |a_1 - \bar{\lambda}|^2}{2(\bar{\lambda} - \lambda)^2 (\bar{\lambda} - \lambda^2)} \right) \\
& \left. + \frac{B(a_2 - a_1 + \lambda)\left(a_1 - \bar{\lambda}\right)}{2(\lambda - \bar{\lambda})} \right).
\end{aligned}$$

Then, since $\lambda + \bar{\lambda} = a_1 + b_2$, $a_2 = (a_1 b_2 - 1)/b_2$, we have

$$\Re(\alpha_2) = \frac{c_1^3 b_2^2}{4(1 - a_1 b_2)} \left(\frac{c_1 A^2 (1 + 2c)}{(2 - c)(1 + c)} + B \right). \tag{24}$$

Third Transformation. The change of variables

$$\xi_n = r_n + i s_n, \qquad \eta_n = r_n - i s_n,$$

transforms (18) into the system

$$\begin{aligned}
r_{n+1} &= \Re(\lambda) r_n - \Im(\lambda) s_n + r_n\left(r_n^2 + s_n^2\right)\Re(\alpha_2) \\
&\quad - s_n\left(r_n^2 + s_n^2\right)\Im(\alpha_2), \\
s_{n+1} &= \Im(\lambda) r_n + \Re(\lambda) s_n + s_n\left(r_n^2 + s_n^2\right)\Re(\alpha_2) \\
&\quad + r_n\left(r_n^2 + s_n^2\right)\Im(\alpha_2).
\end{aligned} \tag{25}$$

Let $w = \gamma_0 + \gamma_1(r_n^2 + s_n^2)$, $\gamma_0, \gamma_1 \in \mathbb{R}$. Then

$$\begin{aligned}\cos w &= \cos\gamma_0 \cos\left(\gamma_1\left(r_n^2 + s_n^2\right)\right) - \sin\gamma_0 \sin\left(\gamma_1\left(r_n^2 + s_n^2\right)\right)\\ &= \left(1 - \frac{\gamma_1^2\left(r_n^2 + s_n^2\right)^2}{2!} + \cdots\right)\cos\gamma_0\\ &\quad - \left(\gamma_1\left(r_n^2 + s_n^2\right) - \frac{\gamma_1^3\left(r_n^2 + s_n^2\right)^3}{3!} + \cdots\right)\sin\gamma_0\\ &= \cos\gamma_0 - \gamma_1\left(r_n^2 + s_n^2\right)\sin\gamma_0 + O_4. \end{aligned} \tag{26}$$

Working similarly, we can find

$$\sin w = \sin\gamma_0 + \gamma_1\left(r_n^2 + s_n^2\right)\cos\gamma_0 + O_4. \tag{27}$$

We write now (25) into the Birkhoff Normal Form

$$\begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{pmatrix}\begin{pmatrix} r_n \\ s_n \end{pmatrix} + \begin{pmatrix} O_4 \\ O_4 \end{pmatrix}. \tag{28}$$

Then, from (25), (26), (27), and (28), we take

$$\Re(\lambda) = \cos\gamma_0, \qquad \gamma_1 = -\frac{\Re(\alpha_2)}{\sin\gamma_0}. \tag{29}$$

Hence, since λ is a root of (12), and (22), (24), (29) hold, it follows that

$$\cos\gamma_0 = \frac{c}{2},$$

$$\gamma_1 = \begin{cases} -\dfrac{c_2^3\left(g(a_1 - a_2)\right)^2}{2\sqrt{4 - c^2}\, a_2^3 (b_1 - b_2)^3} \\ \quad\times\left(\dfrac{c_2 A^2 (2c + 1) g(a_1 - a_2)}{(2 - c)(c + 1) a_2 (b_1 - b_2)} - B\right), & \text{if } b_1 \neq b_2, \\ -\dfrac{c_1^3 b_2^2}{2\sqrt{4 - c^2}\,(1 - a_1 b_2)}\left(\dfrac{c_1 A^2 (1 + 2c)}{(2 - c)(1 + c)} + B\right), & \text{if } b_1 = b_2. \end{cases} \tag{30}$$

Since $a_1 - a_2 \in \mathbb{R}$ and $-2 < c < 2$ hold, it follows that $a_1 - a_2$ does not satisfy (12). Hence if (11) is satisfied, from (10) and (30) we get $\gamma_0, \gamma_1 \neq 0$. Therefore, from Theorem 2.13 of [5], the zero equilibrium of (1) is stable. This completes the proof of the proposition. ▮

Remark 1. We see now that our system (1) includes systems considered in [2], [3], and [4] and Proposition 2 holds for all of them.

First consider the system in [3]:

$$\left.\begin{aligned} u_{n+1} &= 2u_n - v_n + \ln\left(\frac{a}{1 + (a-1)e^{u_n}}\right), \quad a > 1 \\ v_{n+1} &= u_n \end{aligned}\right\}, \qquad n = 0, 1, \ldots . \tag{31}$$

Then by adding both sides of (31) and setting $u_n = x_n$, $y_n = u_n + v_n$, $f(x) = \ln(a/[1 + (a-1)e^x]) - [(1-a)/a]x$, we take the following system:

$$\left.\begin{aligned} x_{n+1} &= \frac{2a+1}{a}x_n - y_n + f(x_n) \\ y_{n+1} &= \frac{3a+1}{a}x_n - y_n + f(x_n) \end{aligned}\right\}, \qquad n = 0, 1, \ldots, \tag{32}$$

which is equivalent to (31). It is obvious that (32) is included in (1) where $a_1 = (2a+1)/a$, $a_2 = (3a+1)/a$, $b_1 = b_2 = -1$, $c_1 = 1$, $c_2 = 0$. We can easily prove that (2), (3), and (10) are satisfied. Moreover,

$$\begin{gathered} f''(0) = \frac{1-a}{a^2}, \qquad f'''(0) = \frac{(1-a)(2-a)}{a^3}, \\ p = -\frac{a(3a+2)}{(a-1)(2a+1)}, \end{gathered} \tag{33}$$

where the constant p is defined in Proposition 2. Therefore, from (33) and since $a \neq 0$, relation (11) is satisfied. Hence, from Proposition 2, the zero equilibrium of (31) is stable.

Next consider the system defined in [2]:

$$\left.\begin{aligned} u_{n+1} &= -v_n + \ln\left(e^{u_n} + \frac{\beta}{E}\right) - \ln E \\ v_{n+1} &= u_n \end{aligned}\right\}, \qquad n = 0, 1, \ldots, \tag{34}$$

where $\beta \in (0, \infty)$, $\beta \neq 1$, and E is the positive solution of the equation $E^2 - E - \beta = 0$. Working similarly, we can prove that (34) is equivalent to the system

$$\left.\begin{aligned} x_{n+1} &= \frac{E+1}{E} x_n - y_n + f(x_n) \\ y_{n+1} &= \frac{2E+1}{E} x_n - y_n + f(x_n) \end{aligned}\right\}, \qquad n = 0, 1, \ldots, \tag{35}$$

where $f(x) = \ln(e^x + \beta/E) - \ln E - (1/E)x$. It is obvious that (35) is included in (1) where $a_1 = (E+1)/E$, $a_2 = (2E+1)/E$, $b_1 = b_2 = -1$, $c_1 = 1$, $c_2 = 0$. We can prove that (2), (3), and (10) hold. Furthermore,

$$\begin{gathered} f''(0) = \frac{E-1}{E^2}, \qquad f'''(0) = \frac{(E-1)(E-2)}{E^3}, \\ p = -\frac{E(E+2)}{(2E-1)(E+1)}, \end{gathered} \tag{36}$$

where p is defined in Proposition 2. Then since $\beta \in (0, \infty)$ and $\beta \neq 1$, relations (36) imply that (11) holds and so the zero equilibrium of (34) is stable.

Finally, consider the system defined in [4]:

$$\left.\begin{aligned} x_{n+1} &= x_n - a y_n + a(1 - e^{y_n} + y_n) \\ y_{n+1} &= x_n + (1-a) y_n + a(1 - e^{y_n} + y_n) \end{aligned}\right\}, \qquad n = 0, 1, \ldots, \tag{37}$$

where a is a constant, $1 < a < 4$, $a \neq 2, 3$. It is obvious that (37) is included in (1) where $a_1 = a_2 = 1$, $b_1 = -a$, $b_2 = 1 - a$, $c_1 = 0$, $c_2 = 1$. We can easily prove that (2), (3), and (10) are satisfied. Moreover,

$$f''(0) = -a, \qquad f'''(0) = -a, \qquad p = \frac{5 - 2a}{a(a-3)}, \tag{38}$$

where p is defined in Proposition 2. Then since $a \neq 2$, relations (38) imply that (11) holds and so the zero equilibrium of (37) is stable.

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