Ideals, bifiltered modules and bivariate Hilbert polynomials

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Abstract

Let $R$ be a ring of polynomials in $m+n$ variables over a field $K$ and let $I$ be an ideal in $R$. Furthermore, let $(R_{rs})_{r,s\in\mathbb{Z}}$ be the natural bifiltration of the ring $R$ and let $(M_{rs})_{r,s\in\mathbb{Z}}$ be the corresponding natural bifiltration of the $R$-module $M = R/I$ associated with the given set of generators introduced by Levin. The author shows an algorithm for constructing a characteristic set $G = \{g_1, \ldots, g_s\}$ of $I$ with respect to a special type of reduction introduced by Levin, that allows one to find the Hilbert polynomial in two variables of the bifiltered and bigraded $R$-module $R/I$. This algorithm can be easily extended to the case of bifiltered $R$-submodules of free $R$-modules of finite rank $p$ over $R$.

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1. Introduction

Bifiltered and bigraded rings $R$, bifiltered and bigraded ideals and $R$-modules are useful tools for the study of Segre products of $K$-algebras, tensor products of graded algebras, Rees rings and symmetric algebras associated with homogeneous ideals in graded rings (blow-up algebras), affine and projective elimination theory, Weyl algebras and linear partial differential equations with coefficients in a polynomial ring.

Recently Levin (1999) has studied Hilbert polynomials in two variables of a bifiltered submodule of a free module of finite rank over a bifiltered commutative ring of polynomials with
coefficients in a field $K$, while Robbiano and Valla (1998) have studied Hilbert–Poincaré series in two variables of a bigraded $K$-subalgebra of a bigraded commutative ring of polynomials with coefficients in a field $K$.

Levin has shown that given a bifiltered polynomial ring $R = \mathbb{K}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$, a bifiltered $R$-module $M$ represented as a factor module of a free $R$-module $E$ of finite rank, and a set of generators $\{ f_1, \ldots, f_r \}$ of the relation module $R(M)$ of $M$ it is possible to find another ("characteristic") set of generators $G$ of $R(M)$ such that the Hilbert polynomial in two variables of $M$ is the numerical polynomial in two variables associated with the set of pairs of maximal terms $\{(u_{g1}, v_{g1}), \ldots, (u_{g2}, v_{g2})\}$ with respect to two term orderings, that are respectively degree preserving with respect to the sets of variables $\{X_1, \ldots, X_m\}$ and $\{Y_1, \ldots, Y_n\}$. Definitions and results in Levin (1999) are different from the ones in Robbiano and Valla (1998).

The existence of such a set of generators is proved by using Ritt’s characteristic set theory as in Ritt (1950) and a notion of reduction, which is similar to the one used in Gröbner bases theory as in Buchberger (1976a,b), while the invariants of such a polynomial with respect to the excellent bifiltrations are used for studying some properties of the corresponding linear systems of PDE’s.

Let $\sigma_X$ and $\sigma_Y$ be term orderings on the set of terms in $\mathbb{K}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$, that are respectively degree preserving with respect the sets of variables $\{X_1, \ldots, X_m\}$ and $\{Y_1, \ldots, Y_n\}$, and let $I$ be an ideal with an excellent bifiltration in the bifiltered ring $\mathbb{K}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$. The author shows an algorithm for constructing a characteristic set $G = \{g_1, \ldots, g_k\}$ of $I$, that allows one to find the Hilbert polynomial in two variables of the bifiltered and bigraded $K$-algebra $\mathbb{K}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]/I$.

This algorithm can be easily extended to the case of bifiltered submodules of free modules of finite rank $p$ over bifiltered rings of polynomials.

2. Preliminaries

Let $K$ be a field and let $R = \mathbb{K}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ be the polynomial ring in $m + n$ variables with coefficients in $K$. Let $\mathbb{N}_0 = \{0, 1, \ldots, n, \ldots\}$.

Definition. $T_X = \{X_1^{a_1} \cdots X_m^{a_m} : (a_1, \ldots, a_m) \in \mathbb{N}_0^m\}$ is the set of terms in $R$ in the variables $X_1, \ldots, X_m$.

$T_Y = \{Y_1^{b_1} \cdots Y_n^{b_n} : (b_1, \ldots, b_n) \in \mathbb{N}_0^n\}$ is the set of terms in $R$ in the variables $Y_1, \ldots, Y_n$.

$T_{XY} = \{X_1^{a_1} \cdots X_m^{a_m} Y_1^{b_1} \cdots Y_n^{b_n} : (a_1, \ldots, a_m, b_1, \ldots, b_n) \in \mathbb{N}_0^{m+n}\}$ is the set of terms in $R$ in the variables $X_1, \ldots, X_m, Y_1, \ldots, Y_n$.

$T_{XY}$ is a monoid and it is the product of the monoids $T_X$ and $T_Y$.

If $t \in T_{XY}$, then deg$_X t = \sum_{i=1, \ldots, m} a_i$, deg$_Y t = \sum_{j=1, \ldots, n} b_j$, while deg($t$) = $\sum_{i=1, \ldots, m} a_i + \sum_{j=1, \ldots, n} b_j$.

If $r, s \in \mathbb{Z}$, then $T(r, s) = \{t \in T_{XY} : \text{deg}_X t \leq r \text{ and } \text{deg}_Y t \leq s\}$.

2.1. Bifiltered rings

Here some basic properties of bifiltered rings are introduced.

Definition. Let $R = \mathbb{K}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$. A bisquence $(R_{rs})_{r, s \in \mathbb{Z}}$ of vector $K$-subspaces of $R$ is called a bifiltration of $R$ if:

(i) $R_{rs} = 0$ if either $r$ or $s$ is negative.
In this paper \( R = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n] \) is bifiltered with respect to the natural bifiltration defined by \( R_{rs} = T(r,s) \).

**Definition.** Let \( R = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n] \) and let \( M \) be a \( R \)-module. A bisequence \((M_{rs})_{r,s \in \mathbb{Z}}\) of vector \( K \)-subspaces of \( M \) is called a bifiltration of \( M \) if:

(i) \( M_{rs} = 0 \) if either \( r \leq r_0 \) or \( s \leq s_0 \) (\( r_0, s_0 \in \mathbb{Z} \) are fixed) and \( \cup \{ M_{rs} : r, s \in \mathbb{Z} \} = M \).

(ii) \( M_{rs} \subseteq M_{r+1,s} \) and \( M_{rs} \subseteq M_{r,s+1} \) for all \( r \) and \( s \).

(iii) \( M_{rs} \cap M_{rk} = M_{rh} \) for all \( r, s, h, k \).

A bifiltration is called excellent if:

(i) \( M_{rs} \) is a finitely generated vector \( K \)-space for all \( r \) and \( s \).

(ii) \( R_{rs}M_{hk} \subseteq M_{r+h,s+k} \) for all \( r \geq r^*, s \geq s^* \) and all \( h \) and \( k \) nonnegative.

An ideal \( I \) of \( R \) is bifiltered if it is bifiltered as an \( R \)-module. If \( M \) is a finitely generated \( R \)-module generated by \( \{ f_1, \ldots, f_p \} \), then \( (M_{rs} = \sum_{i=1}^{p} R_{rs}f_i)_{(r,s) \in \mathbb{Z}^2} \) is an excellent bifiltration of \( M \). This bifiltration will be called a natural bifiltration of \( I \) associated with the set of generators \( \{ f_1, \ldots, f_p \} \).

### 2.2. Term orderings

Here some basic properties of bidegree preserving term orderings are introduced.

**Definition.** A term ordering \( \sigma \) on \( T_{XY} \) is a total order such that:

(i) \( 1 <_\sigma t \) for all \( t \in T_{XY} \setminus \{ 1 \} \);

(ii) \( t_1 <_\sigma t_2 \) implies \( t_1t' <_\sigma t_2t' \) for all \( t' \in T_{XY} \).

**Definition (Levin, 1999).** A term ordering \( \sigma \) on the set of terms \( T_{XY} \) is \( X \)-bidegree preserving if \( t_1 <_\sigma t_2 \) when either \( \deg_X t_1 <_\sigma \deg_X t_2 \) or \( \deg_X t_1 = \deg_X t_2 \) and \( \deg_Y t_1 <_\sigma \deg_Y t_2 \).

A term ordering \( \sigma \) on the set of terms \( T_{XY} \) is \( Y \)-bidegree preserving if \( t_1 <_\sigma t_2 \) when either \( \deg_Y t_1 <_\sigma \deg_Y t_2 \) or \( \deg_Y t_1 = \deg_Y t_2 \) and \( \deg_X t_1 <_\sigma \deg_X t_2 \).

**Example 1.** Let \( t_1, t_2 \in T_{XY} \). Let \( t_1 = X_1^{a_1} \cdots X_m^{a_m} Y_1^{b_1} \ldots Y_n^{b_n} \) and let \( t_2 = X_1^{c_1} \cdots X_m^{c_m} Y_1^{d_1} \cdots Y_n^{d_n} \). If \( \sigma \) is defined by \( t_1 <_\sigma t_2 \) if either \( (\sum_{i=1}^{m} a_i, \sum_{j=1}^{n} b_j) <_{lex} (\sum_{i=1}^{m} c_i, \sum_{j=1}^{n} d_j) \) or \( (\sum_{i=1}^{m} a_i, \sum_{j=1}^{n} b_j) = (\sum_{i=1}^{m} c_i, \sum_{j=1}^{n} d_j) \) and \( (a_1, \ldots, a_m, b_1, \ldots, b_n) <_\tau (c_1, \ldots, c_m, d_1, \ldots, d_n) \), where \( \tau \) is a term ordering on \( T_{XY} \), then \( \sigma \) is an \( X \)-bidegree preserving term ordering on \( T_{XY} \). In a similar way if \( \sigma \) is defined by \( t_1 <_\sigma t_2 \) if either \( (\sum_{j=1}^{n} b_j, \sum_{i=1}^{m} a_i) <_{lex} (\sum_{j=1}^{n} d_j, \sum_{i=1}^{m} c_i) \) or \( (\sum_{j=1}^{n} b_j, \sum_{i=1}^{m} a_i) = (\sum_{j=1}^{n} d_j, \sum_{i=1}^{m} c_i) \) and \( (b_1, \ldots, b_n, a_1, \ldots, a_m) <_\tau (d_1, \ldots, d_n, c_1, \ldots, c_m) \), where \( \tau \) is a term ordering on \( T_{XY} \), then \( \sigma \) is a \( Y \)-bidegree preserving term ordering on \( T_{XY} \).

Now let \( \sigma_X \) and \( \sigma_Y \) be respectively an \( X \)-bidegree preserving term ordering and a \( Y \)-bidegree preserving term ordering.

If \( f \in K[X_1, \ldots, X_m, Y_1, \ldots, Y_n] \), then \( f = \sum_{h=1}^{w} a_{ih}t_{ih} = \sum_{h=1}^{w} a_{jh}t_{jh} \) with \( t_{ih} >_{\sigma_X} t_{ih} >_{\sigma_X} \cdots >_{\sigma_X} t_{iw} \) and \( t_{j1} >_{\sigma_Y} t_{j2} >_{\sigma_Y} \cdots >_{\sigma_Y} t_{jw} \) with nonzero \( a_{ih}, a_{jh} \in K \) and \( t_{ih}, t_{jh} \in T_{XY} \) for all \( h \).
Remark. Given \( f \in K[x_1, \ldots, x_m, y_1, \ldots, y_n] \) we have \( u_f = v_f \) if and only if \( f = \sum_{h=1, \ldots, w} a_h t_h \) with \( t_1 >_{\sigma_X} t_h \) and \( t_1 >_{\sigma_Y} t_h \) for all \( h = 2, \ldots, w \) with nonzero \( a_h \in K \) and \( t_h \in T_{XY} \). In particular if \( f \) is homogeneous and \( u_f = v_f \), then \( \deg_X t_i(b) = \deg_X t_i(h+1) \) and \( \deg_Y t_i(h) = \deg_Y t_i(h+1) \) for all \( h \), i.e. \( f \) is bihomogeneous. Conversely there exist bihomogeneous polynomials \( f \) with \( u_f \neq v_f \) as in the following example.

Example 2. Let \( R = K[x_1, x_2, y_1, y_2] \) and let \( t_1, t_2 \in T_{XY} \). Let \( t_1 = x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2} \) and let \( t_2 = x_1^{c_1} x_2^{c_2} y_1^{d_1} y_2^{d_2} \). Let \( \sigma_X \) be defined by \( t_1 <_{\sigma_X} t_2 \) when \((a_1 + b_1 + b_2, a_1, a_2, b_1, b_2) <_{\text{lex}} (c_1 + c_2, d_1 + d_2, c_1, c_2, d_1, d_2)\). Let \( \sigma_Y \) be defined by \( t_1 <_{\sigma_Y} t_2 \) when \((b_1 + b_2, a_1 + a_2, b_1, b_2, a_1, a_2) <_{\text{lex}} (d_1 + d_2, c_1 + c_2, d_1, d_2, c_1, c_2)\). Let \( f = x_1 y_2 + x_2 y_1, u_f = x_1 y_2 \) and \( v_f = x_2 y_1 \).

### 3. L-reduction

Here the definitions and properties following from the notion of reduction given in Levin (1999) are introduced. Such reduction will be called L-reduction.

**Definition (Levin, 1999).** Let \( f, g \in K[x_1, \ldots, x_m, y_1, \ldots, y_n] \) and let \( \sigma_X \) and \( \sigma_Y \) be respectively an X-bidegree preserving and a Y-bidegree preserving term ordering on \( T_{XY} \). \( f \) is L-reduced with respect to \( g \) if \( f \) does not contain any term \( tu_g \) such that \( \deg_Y tv_g \leq \deg_Y v_f \).

A subset \( F \) of \( R = K[x_1, \ldots, x_m, y_1, \ldots, y_n] \) is L-autoreduced if each element of \( F \) is L-reduced with respect to the other ones.

**Example 3.** Let \( R = K[x_1, y_1] \). Let \( \sigma_X \) and \( \sigma_Y \) be as above with \( \tau = \text{lexicographic} \). Let \( f = x_1^2 - y_1 \) and \( g = x_1 - y_1^2, u_f = x_1^2, v_f = y_1, u_g = x_1 \) and \( v_g = y_1^2 \). \( f \) is L-reduced with respect to \( g \), because \( u_f \neq tu_g \) and \( 2 = \deg_Y x_1 v_g = \deg_Y x_1 y_1^2 > 1 = \deg_Y v_f = \deg_Y y_1 \).

In Levin (1999) it is shown that each L-autoreduced subset \( F \) of \( R \) is finite and each \( f \in R \) can be L-reduced in a finite number of steps with respect to an L-autoreduced subset \( F = \{ f_1, \ldots, f_r \} \) of \( R \). Moreover \( f \) L-reduces to a polynomial \( g \) with respect to \( F \) and there exist \( g_1, \ldots, g_r \in R \) such that \( f = \sum_{i=1, \ldots, r} g_i f_i + g \) and \( g \) is L-reduced with respect to \( F \). \( g \) is called a L-normal form of \( f \) with respect to \( F \).

**L-Reduction Algorithm (Levin).**

**Input** \( f \in R \), a positive integer \( r \), \( F = \{ f_1, \ldots, f_r \} \), where \( f_i \neq 0 \) for all \( i = 1, \ldots, r \).

**Output** \( g \in R \) and \( g_1, \ldots, g_r \in R \) such that \( f = \sum_{i=1, \ldots, r} g_i f_i + g \) and \( g \) is L-reduced with respect to \( F \).

**Begin**

\( g_1 := 0, \ldots, g_r := 0, g := f \)

**While** there exists \( i, \ i = 1, \ldots, r \), and a term \( t \), that appears in \( g \) with a nonzero coefficient \( a_i \), such that \( u_{f_i} \) divides \( t \) and \( \deg_Y (\frac{L}{u_{f_i}} v_{f_i}) \leq \deg_Y v_f \) **do**


Let \( I \) be an ideal in \( K[X_1, \ldots, X_m, Y_1, \ldots, Y_n] \) and let \( \sigma_X \) and \( \sigma_Y \) be respectively an \( X \)-bidegree preserving and a \( Y \)-bidegree preserving term ordering on \( T_{XY} \). Let \( G = \{g_1, \ldots, g_s\} \) be a Gröbner basis of \( I \) with respect to \( \sigma \). Then \( G \) is an \( X \)-bidegree preserving and a \( Y \)-bidegree preserving term ordering on \( T_{XY} \).

Let \( \sigma \) be a term ordering on the set of all polynomials, which is a pre-order, and the definition of \textit{ranking} on the set of all L-autoreduced subsets of polynomials. In Levin (1999) it is also given the definition of \textit{ranking} on the set of all polynomials, which is a pre-order, and the definition of \textit{ranking} on the set of all L-autoreduced subsets of polynomials.

**Remark.** The definition of L-reduction given by Levin is weaker than the one used in Gröbner bases theory (Buchberger, 1976a,b).
(i) $G$ is a Gröbner basis of $I$ with respect to $\sigma_X$.

(ii) If $f$ is a nonzero element of $I$ and $f = \sum_{i=1,\ldots,r} a_i t_i g_j(i)$ where $a_i \in K$ with $a_i \neq 0$, $t_i \in T_X$, $j(i) \in \{1,\ldots,s\}$ for all $i$ by the L-reduction algorithm, then $\deg_Y v_f \geq \deg_Y t_i v_{g_j(i)}$ for all $i = 1, \ldots, r$.

(iii) Let $d = \min(\deg_Y f : f \in I, f \neq 0)$. For every $f \in I$ with $\deg_Y f = d$ there exists at least one $g \in G$ such that $u_g$ is a term of $f$ and $\deg_Y v_g = d$.

(iv) Let $g \in G$ with $\deg_Y g = d$. If $F$ is a Gröbner basis of $I$ with respect to $\sigma_Y$, then there exists $f' \in F$ with $\deg_Y f' = d$ such that $g = a f'$ for some nonzero $a \in K$.

**Proof.** (i) By Levin (1999), Theorem 4.4, if $f \in I$, then $f$ is L-reduced with respect to $G$ if and only if $f = 0$. Since $G \subseteq I$, then $M_{\sigma_X}(G) \subseteq M_{\sigma_X}(I)$. Let $f$ be a nonzero element of $I$. Since $f$ L-reduces to zero with respect to $G$, then it reduces to zero with respect to $G$ and with respect to $\sigma_X$. Therefore $M_{\sigma_X}(f) = \deg_X(f) u_f = r t M_{\sigma_X}(g)$ for some $r \in R, t \in T_X$ and $g \in G$. It follows that $M_{\sigma_X}(I) \subseteq M_{\sigma_X}(G)$ and then (i) follows by definition of Gröbner basis.

(ii) Now let $G = \{g_1, \ldots, g_s\}$ with $r_k(g_1) < \cdots < r_k(g_s)$ and let $f$ be a nonzero element of $I$. By the L-reduction algorithm $f = \sum_{i=1,\ldots,r} a_i t_i g_j(i)$ where $a_i \in K$ with $a_i \neq 0$, $t_i \in T_X$, $j(i) \in \{1,\ldots,s\}$ for all $i$ and $t_i u_{g_j(i)} \geq \sigma_X t_{u_{g_j(i)}}$. Furthermore $u_f = t_i u_{g_j(i)}$. If $r = 1$, then $v_f = t_i v_{g_j(i)}$, and (ii) is proved. Now suppose that $r > 1$. Let us suppose that $\deg_Y v_f \geq \deg_Y t_i v_{g_j(i)}$ for all $i = 1, \ldots, r'$ and $\deg_Y v_f = \deg_Y t_i v_{g_j(i)}$ for some $i$, whenever $f = \sum_{i=1,\ldots,r'} a_i t_i g_j(i)$ and $r' \leq r - 1$. Let $f_1 = f - a_i t_i g_j(i)$. $f_1 \in I$ and by the induction hypothesis $\deg_Y v_{f_1} = \deg_Y t_i v_{g_j(i)}$ for some $i = 2, \ldots, r$. Since $f$ is L-reducible with respect to $G$, then $\deg_Y v_f \geq \deg_Y t_i v_{g_j(i)}$. So either $v_f = v_{f_1}$, when $t_i v_{g_j(i)} < \sigma_Y v_f$ or $v_f = t_i v_{g_j(i)}$ or $v_{f_1} = t_i v_{g_j(i)}$, when $\deg_Y v_f = \deg_Y t_i v_{g_j(i)}$ and $v_f < \sigma_Y t_i v_{g_j(i)}$. Now (ii) follows by the induction hypothesis.

(iii) By (ii) if $f$ is a nonzero element of $I$ and $\deg_Y f = d$, then $f$ L-reduces to zero with respect to $G$. So there exists a $g \in G$ such that $t_i u_g$ is a term of $f$ and $d = \deg_Y f = \deg_Y v_f \geq \deg_Y v_g = d + \deg_Y v_g$. So $\deg t = 0$ and $\deg_Y v_g = d$, e.g. $t \in T_X$.

(iv) If $f$ is a Gröbner basis of $I$ with respect to $\sigma_Y$ and $f \in F$ with $\deg_Y f = \deg_Y f' = d$, then by (iii) there exists at least one $g \in G$ with $\deg_Y g = \deg_Y v_g = d$. Since $g \in I$, there exists at least one $f' \in F$ with $\deg_Y f' = \deg_Y v' = d$ such that $v_g = t' v'$ and $t' \in T_X$. If $t c_\sigma(f') g - \deg_Y t c_\sigma(g) f' \neq 0$, then $\deg_Y (t c_\sigma(f') g - \deg_Y t c_\sigma(g) f') < d$. So we have a contradiction by minimality of $d$. If $a = \frac{lc_{\sigma_Y}(g)}{lc_{\sigma_Y}(f')}$, then $g = a t' f'$. By (iii) there exists $g' \in G$ such that $f' \in G$ contains a term $t u_{g'}$ with $t \in T_X$ and $\deg_Y f' = \deg_Y t u_{g'} = \deg_Y t v_{g'} = \deg_Y v_{g'} = d$. If $g' \neq g$, then $g$ is not L-reduced with respect to $g'$ and we have a contradiction by definition of characteristic set. So $g = g'$ and then $t = t' = 1$, i.e. $g = a f'$.

**Remark.** If $F$ is a reduced Gröbner basis of $I$ with respect to $\sigma_X$, then it is an L-autoreduced subset of $I$ by definition of L-reduction and rank($F$) is greater than or equal to rank($G$), where $G$ is a characteristic set of $I$, by (i) of the Theorem 3.1.

4. An algorithm for the characteristic set

In this section we present an algorithm for the characteristic set of an ideal $I$ in $R$ with respect to the L-reduction.

Since Levin’s theory is an extension of Ritt’s theory, one can try procedures analogous to Ritt’s procedure (Ritt, 1950) for partial differential equations by using the Fourier transform. The usual Ritt’s procedures find a characteristic set of a finite set and an extended characteristic
set of an ideal. If only linear polynomials are considered, then such an extended characteristic set is a characteristic set.

Furthermore Ritt’s notions of reduction and characteristic sets in the case of linear partial differential equations with constant coefficients coincide respectively with the notions of reduction in Gröbner basis theory and reduced Gröbner bases as in Carrá Fero (2001) and Kondratieva et al. (1999). So an algorithm similar to Buchberger’s algorithm for Gröbner bases with the new definition of L-reduction can be used in order to find such Levin characteristic sets.

**Definition.** Let \( f, g \in K[X_1, \ldots, X_m, Y_1, \ldots, Y_n] \) and let \( \sigma_X \) and \( \sigma_Y \) be respectively an \( X \)-bidegree preserving and a \( Y \)-bidegree preserving term ordering on \( T_{XY} \). \( S(f, g) = \text{lcm}(lcm(g), \frac{f}{\text{lcm}(g)} f - \text{lcm}(g) \frac{l}{a_j} f - \text{lcm}(g) \frac{l}{a_g} g) \), where \( l = l.c.m.(u_j, u_g) \).

**Buchberger’s Algorithm for L-Reduction.**

**Input** \( F = \{f_1, \ldots, f_r\} \), a basis of the ideal \( I \) in \( R \), an \( X \)-bidegree preserving term ordering \( \sigma_X \) and a \( Y \)-bidegree preserving term ordering \( \sigma_Y \) on \( T_{XY} \).

**Output** \( G = \{g_1, \ldots, g_s\} \), an L-autoreduced basis of \( I \).

\[
\begin{align*}
s & := r; \\
H & := F; \\
P & := \{(i, j) : 1 \leq i < j \leq r\}; \\
\text{while } P \text{ is nonempty do} & \text{ Choose } (i, j) \in P; \\
& \quad f := S(f_i, f_j); \\
& \quad P := P \setminus \{(i, j)\}; \\
& \quad g := \text{L-normal form of } f \text{ with respect to } H; \\
& \quad \text{if } g \neq 0 \text{ then} \\
& \quad \quad P := P \cup \{(i, s + 1) : 1 \leq i \leq s\}; \\
& \quad \quad f_{s+1} := g; \\
& \quad \quad H := H \cup \{g\}; \\
& \quad \quad s := s + 1; \\
& \quad \text{begin } G = \emptyset, E = H; \\
& \quad \text{while } E \neq \emptyset \text{ do} \\
& \quad \quad \text{select } e_0 \text{ from } E; \\
& \quad \quad E := E \setminus e_0; \\
& \quad \quad \text{if } e_0 \text{ is L-reduced with respect to all } e \in E \text{ and} \\
& \quad \quad \quad e_0 \text{ is L-reduced with respect to all } g \in G \text{ then} \\
& \quad \quad \quad G := G \cup e_0 \text{ end}; \\
& \text{end}
\end{align*}
\]

**Theorem 4.1.** Let \( F = \{f_1, \ldots, f_r\} \subset K[X_1, \ldots, X_m, Y_1, \ldots, Y_n] \). Let \( I = (f_1, \ldots, f_r) \) be an ideal in \( K[X_1, \ldots, X_m, Y_1, \ldots, Y_n] \) and let \( \sigma_X \) and \( \sigma_Y \) be respectively an \( X \)-bidegree preserving and a \( Y \)-bidegree preserving term ordering on \( T_{XY} \). Let \( F_1 \) be a Gröbner basis of \( I \) with respect to \( \sigma_X \), given by the usual Buchberger algorithm with input \( F \). Let \( F_2 \) be a Gröbner basis of \( I \) with respect to \( \sigma_Y \), given by the usual Buchberger algorithm with input \( F_1 \). Let \( G \) be the output of the Buchberger algorithm for L-reduction with input \( F_2 \). \( G \) is a characteristic set of \( I \).

**Proof.** \( F \subseteq F_1 \subseteq F_2 \) by definition of the Gröbner basis with respect to a term ordering and \( I = (F) = (F_1) = (F_2) \). Moreover \( F_2 \) is a Gröbner basis of \( I \) with respect to \( \sigma_X \) and it contains every polynomial \( g \) in a characteristic set of \( I \) with \( \deg_Y v_g = d \) up to a nonzero element \( a \in K \).
by (iv) of Theorem 3.1. Let $H$ be as in the Buchberger algorithm for $L$-reduction with input $F := F_2$, $F_2 \subseteq H$ and then $I = (H)$, because each element $h$ of $H$ is in $I = (F_2)$ by its own definition. Let $G$ be the output of the Buchberger algorithm for $L$-reduction with input $F := F_2$. $G \subseteq H$ so $(G) \subseteq I$. On the other hand every $h \in H \setminus G$ is in $(G)$ by definition of $L$-reduction. It follows that $I = (H) \subseteq (G)$, i.e. $I = (G)$. $G$ is $L$-autoreduced by its own definition. In order to prove that $G$ is a characteristic set of $I$ it is sufficient to prove that each nonzero $f \in I$ $L$-reduces to zero with respect to $G$ by Levin (1999), Theorem 4.4. If $g, g' \in G$, then $S(g, g') \in H$ and it $L$-reduces to zero with respect to $H$, by definition of the Buchberger algorithm for $L$-reduction. Since each $h \in H \setminus G$ $L$-reduces to zero with respect to $G$, then $S(g, g')$ $L$-reduces to zero with respect to $G$. By repeating the proof in Becker and Weispfenning (1993), Lemma 5.44 p. 210 and Theorem 5.48 p. 211 ((iii) $\Rightarrow$ (i)), for $L$-reduction each $f \in I$ $L$-reduces to a unique element $f_0 \in I$, because the $L$-reduction is also a reduction with respect $\sigma_X$. Finally we have to show that $f_0 = 0$. Since $I = (G)$, by repeating the proof in Becker and Weispfenning (1993), Theorem 5.35 p. 206 (iv) $\Rightarrow$ (v)) and Lemma 5.26 p. 202, for $L$-reduction $f$ and zero $L$-reduce to the same $f_0 \in I$. But zero $L$-reduces to zero and then $f_0 = 0$, because each $f \in I$ $L$-reduces to a unique element of $I$. Now $G$ is a characteristic set by Levin (1999), Theorem 4.4.

Example 4. Let $F = \{Y_1^2 - 1, X_1 - Y_1\}$ and let $I = (F)$. $\sigma_X$ is a lexicographic term ordering with $Y_1 <_{\sigma_X} X_1$ and $\sigma_Y$ is a lexicographic term ordering with $X_1 <_{\sigma_Y} Y_1$. $F$ is a reduced Gröbner basis of $I$ with respect to $\sigma_X$ and then $F = F_1$. $F_2 = \{Y_1^2 - 1, X_1 - Y_1, X_1^2 - 1\}$ is the Gröbner basis of $I$ with respect to $\sigma_Y$ given by the Buchberger algorithm with input $F_1$. $F_1 \subseteq F_2$. $G = \{Y_1^2 - 1, X_1 - Y_1, X_1 Y_1 - 1, X_1^2 - 1\}$, because $S(X_1 Y_1 - 1, X_1^2 - 1)$ $L$-reduces to $X_1 Y_1 - 1$ with respect to $F_2$.

Remark. Example 4 shows that it is necessary to have a Gröbner basis of $I$ with respect to $\sigma_X$ and to $\sigma_Y$ as input of the Buchberger’s algorithm for $L$-reduction in order to find a characteristic set of $I$. In fact the Buchberger’s algorithm for $L$-reduction with input $F := F = F_1$ has as output $G = F = F_1$, which is not a characteristic set of $I$.

5. Hilbert polynomials in two variables

Here the notion and some properties of the Hilbert polynomial in two variables given by Levin are introduced. Furthermore some examples of such polynomials are shown.

Theorem 5.1 (Levin, 1999). Let $I$ be an ideal in $R = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ and let $\sigma_X$ and $\sigma_Y$ be respectively an $X$-bidgree preserving and a $Y$-biddegree preserving term ordering on $T_{XY}$. Let $G = \{g_1, \ldots, g_k\}$ be a characteristic set of $I$ and let $V = R/I$.

(i) $V$ is a bifiltered $R$-module with the bifiltration $(V_{rs})_{(r,s)\in\mathbb{Z}^2}$, where $V_{rs}$ as a vector $K$-space is generated by $U_{rs}^t = U_{rs}^t \cup U_{rs}^{ts}$, where $U_{rs}^t = \{t \in T : \deg_X t \leq r, \deg_Y t \leq s\}$ and $t$ is not a multiple of any $u_{g_j}$, $j = 1, \ldots, k\} and $U_{rs}^{ts} = \{t \in T : \deg_X t \leq r, \deg_Y t \leq s\}$ and $t = wu_{g_j}$ with $\deg_Y \, wu_{g_j} > s$ for all $w \in T_{XY}$ and $j = 1, \ldots, k\}.

(ii) $(V_{rs})_{(r,s)\in\mathbb{Z}^2}$ is an excellent bifiltration of $V$.

Theorem 5.2 (Levin, 1999). Let $I$ be an ideal in $R = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ and let $\sigma_X$ and $\sigma_Y$ be respectively an $X$-bidgree preserving and a $Y$-biddegree preserving term ordering on $T_{XY}$. Let $G = \{g_1, \ldots, g_k\}$ be a characteristic set of $I$ and let $V = R/I$. Then there exists a numerical polynomial $\omega_I(t_1, t_2)$ in two variables $t_1$ and $t_2$ such that:

(i) $\omega_I(t_1, t_2) = \dim_K(V_{t_1, t_2})$ for all $t_1 \geq t_1^*$ and $t_2 \geq t_2^*$. 


(ii) \( \deg_{\xi}(\omega_I(t_1, t_2)) \leq m \) and \( \deg_{\zeta}(\omega_I(t_1, t_2)) \leq n \), and \( \omega_I(t_1, t_2) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{ij} \binom{t_1+i}{j} \binom{t_2+j}{n-j} \), where \( a_{ij} \in \mathbb{Z} \) for all \( i \) and \( j \).

(iii) Let \( A = \{(i, j) : i = 0, \ldots, m, j = 0, \ldots, n, a_{ij} \neq 0\} \); let \( \mu = (\mu_1, \mu_2) \) and \( v = (v_1, v_2) \) be the maximal elements in \( A \) respectively with respect to the lexicographic and reverse-lexicographic term orderings on \( \mathbb{N}_0^2 \). \( \mu, v, a_{mn}, a_{\mu_1\mu_2}, a_{v_1v_2} \) do not depend on the excellent bifiltration of \( V = R/I \).

The Hilbert polynomial in two variables of \( I \) as in Levin (1999) is the numerical polynomial in two variables \( H_I(t_1, t_2) = \omega_I(t_1, t_2) = \dim_K(V_{t_1t_2}) = \omega_1(t_1, t_2) + \omega_2(t_1, t_2) \), where \( \omega_1(t_1, t_2) = \text{card}(U'_{t_1t_2}) \) and \( \omega_2(t_1, t_2) = \text{card}(U''_{t_1t_2}) \), when \( t_1 \geq t_0 \) \( e \ t_2 \geq t_0 \), while it is defined as \( \omega_1(t_1, t_2) = \text{card}(U'_{t_1t_2}) \) in Caboara et al. (1996) and Robbiano and Valla (1998).

The Hilbert polynomial in two variables can be found by using either algorithms in Kondrateva et al. (1992) and Levin (1999) or algorithms in Caboara et al. (1996).

Example 5 (Levin, 1999). \( I = (X_1 + Y_1^2 + 1) \). \( \omega_1(t_1, t_2) = \binom{t_1+1}{1} \binom{t_2+1}{1} = t_2 + 1 \).
\( \omega_2(t_1, t_2) = \binom{t_1+1}{1} \binom{t_2+1}{1} - \binom{t_1}{1} \binom{t_2}{1} = 2t_1 \).
\( H_I(t_1, t_2) = \omega_1(t_1, t_2) + \omega_2(t_1, t_2) = 2(t_1 + 1) + (t_2 + 1) - 2 \).
\( A = \{(1, 0), (0, 1), (0, 0)\}, \mu = (1, 0), v = (0, 1), a_{11} = 0, a_{10} = 2 \) and \( a_{01} = 1 \).

The next examples show the influence of the maximal degree with respect to \( Y \) of a set of generators of an ideal \( I \) on the Hilbert polynomial of \( I \) in two variables.

Example 6. Let \( X = \{X_1, X_2\} \) and let \( Y = \{Y_1\} \). Let \( \sigma_X \) and \( \sigma_Y \) be as in Example 1 with \( X_1 >_{\sigma_X} X_2 \). Let \( I = (f_1 = X_1^2 - Y_1^2, f_2 = X_2^2 - Y_1^2) \). \( \{f_1, f_2\} \) is L-autoreduced but it is not a characteristic set of \( I \). \( \{f_1 = X_1^2 - X_2Y_1, f_2 = X_2^2 - X_1Y_1 \} \) is L-autoreduced and it is a characteristic set of \( I \).
\( u_{f_1} = X_1^2, u_{f_2} = X_2^2, v_{f_1} = X_1^2 \) and \( v_{f_2} = Y_1^2 \).
\( |U'_{t_1, t_2}| = 4t_2 + 4 \) and \( |U''_{t_1, t_2}| = 2\binom{t_1+2}{2} - 4t_2 - 2 \) when \( t_1 \geq 2 \) and \( t_2 \geq 0 \). So \( \omega_1(t_1, t_2) = 4t_2 + 4 \) and \( \omega_2(t_1, t_2) = 2\binom{t_1+2}{2} - 4t_2 - 2 \).
\( H_I(t_1, t_2) = \omega_1(t_1, t_2) + \omega_2(t_1, t_2) = 2\binom{t_1+2}{2} + 4(t_2 + 1) - 4(t_1 + 1) + 2 \).
\( A = \{(2, 0), (1, 0), (0, 1)\}, \mu = (2, 0), v = (0, 1), a_{21} = 0, a_{20} = 2 \) and \( a_{01} = 4 \).

Example 7. Let \( X = \{X_1, X_2\} \) and let \( Y = \{Y_1\} \). Let \( \sigma_X \) and \( \sigma_Y \) be as in Example 1 with \( X_1 >_{\sigma_X} X_2 \). Let \( I = (f_1 = X_1^2 - X_2Y_1, f_2 = X_3^2 - X_1Y_1) \). \( \{f_1, f_2\} \) is L-autoreduced but it is not a characteristic set of \( I \). \( \{f_1 = X_1^2 - X_2Y_1, f_2 = X_3^2 - X_1Y_1 \} \) is L-autoreduced and it is a characteristic set of \( I \).
\( u_{f_1} = X_1^2, u_{f_2} = X_2^2, u_{f_3} = v_{f_3} = X_1, v_{f_1} = X_2Y_1 \) and \( v_{f_2} = X_1Y_1 \).
\( |U'_{t_1, t_2}| = 4t_2 + 4 \) and \( |U''_{t_1, t_2}| = 2\binom{t_1+2}{2} - 4 \) when \( t_1 \geq 2 \) and \( t_2 \geq 0 \). So \( \omega_1(t_1, t_2) = 4t_2 + 4 \) and \( \omega_2(t_1, t_2) = 2\binom{t_1+2}{2} - 4 \).
\( H_I(t_1, t_2) = \omega_1(t_1, t_2) + \omega_2(t_1, t_2) = 2\binom{t_1+2}{2} + 4(t_2 + 1) - 4(t_1 + 1) + 2 \).
\( A = \{(2, 0), (1, 0), (0, 1)\}, \mu = (2, 0), v = (0, 1), a_{21} = 0, a_{20} = 2 \) and \( a_{01} = 4 \).

Example 8. Let \( X = \{X_1, X_2\} \) and let \( Y = \{Y_1\} \). Let \( \sigma_X \) and \( \sigma_Y \) be as in Example 1 with \( X_1 >_{\sigma_X} X_2 \). Let \( I = (f_1 = X_1^2, f_2 = X_2^2) \). \( u_{f_1} = v_{f_1} = X_1^2 \) and \( u_{f_2} = v_{f_2} = X_2^2 \). \( \{f_1, f_2\} \) is L-autoreduced and it is a characteristic set of \( I \).
\( |U'_{t_1, t_2}| = 4t_2 + 4 \) and \( |U''_{t_1, t_2}| = 0 \) when \( t_1 \geq 0 \). So \( \omega_1(t_1, t_2) = 4t_2 + 4 \) and \( \omega_2(t_1, t_2) = 0 \).
\( H_I(t_1, t_2) = \omega_1(t_1, t_2) + \omega_2(t_1, t_2) = 4(t_2 + 1) \).
\( A = \{(0, 1)\}, \mu = v = (0, 1), a_{21} = 0 \) and \( a_{01} = 4 \).
References


