The characteristic polynomial and the matchings polynomial of a weighted oriented graph

Shi-Cai Gong *, Guang-Hui Xu

School of Science, Zhejiang A & F University, Hangzhou 311300, PR China

**Article Info**

Article history:
Received 11 May 2011
Accepted 27 December 2011
Available online 27 January 2012
Submitted by B. Shader

AMS classification:
05C20
05C31
15A18

Keywords:
Skew symmetric matrix
Oriented graph
Skew adjacency matrix
Characteristic polynomial
Matchings polynomial

**Abstract**

Let $G^o$ be a weighted oriented graph with skew adjacency matrix $S(G^o)$. Then $G^o$ is usually referred as the weighted oriented graph associated to $S(G^o)$. Denote by $\phi(G^o; \lambda)$ the characteristic polynomial of the weighted oriented graph $G^o$, which is defined as

$$\phi(G^o; \lambda) = \det(\lambda I_n - S(G^o)) = \sum_{i=0}^{n} a_i(G^o) \lambda^{n-i}.$$ 

In this paper, we begin by interpreting all the coefficients of the characteristic polynomial of an arbitrary real skew symmetric matrix in terms of its associated oriented weighted graph. Then we establish recurrences for the characteristic polynomial and deduce a formula on the matchings polynomial of an arbitrary weighted graph. In addition, some miscellaneous results concerning the number of perfect matchings and the determinant of the skew adjacency matrix of an unweighted oriented graph are given.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

Throughout the paper, graphs have no loops, all matrices are real and the weights of all arcs, or edges, of a graph are real and positive. The identity matrix is denoted by $I$ and the transpose of the matrix $A$ is $A^T$. Let $A$ be an $n \times n$ real matrix, then $A$ is symmetric if $A^T = A$; and $A$ is skew symmetric if

* Supported by National Natural Science Foundation of China (11171373 and 10871230), Natural Science Foundation of Department of Education of Anhui (KJ2010A092), Zhejiang Provincial Natural Science Foundation of China (Y7080364).

* Corresponding author.

E-mail addresses: scgong@zafu.edu.cn (S.-C. Gong), ghxu@zafu.edu.cn (G.-H. Xu).

0024-3795/ – see front matter © 2012 Elsevier Inc. All rights reserved.
$A^T = -A$. The characteristic polynomial of an $n \times n$ matrix $A$, denoted by $\phi(A, \lambda)$, is defined by

$$\phi(A, \lambda) = \det(\lambda I - A).$$

Let $G^\sigma$ be a simple weighted undirected graph with an orientation $\sigma$, which assigns to each edge a direction so that $G^\sigma$ becomes a weighted oriented graph, or a weighted digraph. (We will refer to an unweighted graph, oriented or not, which is just a weighted graph with weight of each edge, or arc, equals to 1.) The skew-adjacency matrix associated to the weighted oriented graph $G^\sigma$ with vertex set \{1, 2, \ldots, n\} is defined as the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$ whose $(i, j)$ entry satisfies:

$$s_{ij} = \begin{cases} 
\theta, & \text{if there has an arc with weight } \theta \text{ from } i \text{ to } j; \\
-\theta, & \text{if there has an arc with weight } \theta \text{ from } j \text{ to } i; \\
0, & \text{otherwise}.
\end{cases}$$

For an $n \times n$ real skew symmetric matrix $S$, the weighted oriented graph associated to $S$ is the oriented graph whose skew-adjacency matrix is $S$. Because $G^\sigma$ contains no loops, the skew-adjacency matrix $S(G^\sigma)$ of a weighted oriented graph is real skew symmetric. Thus each real skew symmetric matrix corresponds to a weighted oriented graph with fixed vertex labeling and vice versa. Therefore, we in the following do not distinguish the real skew symmetric matrix and its associated oriented graph. The characteristic polynomial of a weighted oriented graph $G^\sigma$ is defined by the characteristic polynomial of its skew adjacency matrix $S(G^\sigma)$, denoted by $\phi(S(G^\sigma), \lambda)$ or $\phi(G^\sigma, \lambda)$ briefly. Let $G$ be a weighted undirected graph containing $n$ vertices and $m$ edges. Denote by $\mathcal{G}(G)$ the set of all oriented graphs obtained from $G$ by giving an arbitrary orientation of all its edges. Because each of the $m$ edges in $G$ may be oriented in two different ways, it follows that the set $\mathcal{G}(G)$ has $2^m$ elements.

Let $G = (V(G), E(G))$ be a weighted graph on $n$ vertices. An $r$-matching in $G$ is a subset with $r$ edges such that every vertex of $V(G)$ is incident with at most one edge in it. The $r$-matching $M$ is a perfect matching (of $G$) if every vertex of $G$ is incident with exactly one edge in $M$. The weight of an $r$-matching is defined by the product of the weights of all edges in it. Denoted by $p(G, r)$ the sum of the weights of all $r$-matchings in $D$ and set $p(G, 0) = 1$. (In particular, $p(G, r)$ denotes the cardinality of all $r$-matchings in $G$ if such a graph $G$ is unweighted.) The matchings polynomial $\mu(G, \lambda)$ of a graph $G$ is defined by

$$\mu(G, \lambda) = \sum_{r \geq 0} (-1)^r p(G, r) \lambda^{n-2r}.$$

We refer to Cvetković et al. [3] for more terminology and notation not defined here.

The matchings polynomial is a crucial concept in Combinatorics, Statistical Physics and Theoretical Chemistry (see e.g. [6, 8–10, 16, 19]) and is also called as the acyclic polynomial [9, 19] or the reference polynomial [1]. The computation of the matchings polynomial of a general graph (even an unweighted graph) is NP-complete [12]. In general, to compute the matchings polynomial of an unweighted graph we should use the recurrence formulas relating the matchings polynomial of such a graph to the matchings polynomial of its subgraphs, and the connections between the matchings polynomial and the characteristic polynomial of the adjacency matrix of its underlying graph, which is the undirected graph obtained by removing the orientations of all its arcs; see for example [5].

Research applying the skew symmetric matrix theory to combinatorial theory can be traced back to 1947 when Tutte [20] derived his famous characterization of the graphs with no perfect matchings. Tutte’s result motivates a lot of work on the matchings polynomial and enumerating perfect matchings of graphs in terms of its skew adjacency matrix, see for example [13–15, 17, 21–24]. Below we recall some results concerning the matchings polynomial of a graph. A formula due to Godsil [4] relating the matchings polynomial of a general unweighted graph to the matchings polynomial of a certain associated tree is given. Unfortunately, this tree is too large in general to be of any help in concrete computations. Let $G$ be an unweighted graph with $m$ edges and adjacency matrix $A(G) = (a_{ij})_{n \times n}$. If $S$ is a subset of $E(G)$, let $A_S$ be the matrix such that $(A_S)_{ij} = a_{ij}$ if $(v_i, v_j) \notin S$ and $(A_S)_{ij} = -a_{ij}$ if
\((v_i, v_j) \in S\), Godsil and Gutman [7] showed that
\[
\mu(G, \lambda) = \frac{1}{2^m} \sum_{S \subseteq E(G)} \det(\lambda I - A_S).
\]

In [5,12], the number of perfect matchings of an unweighted graph \(G\) with \(m\) edges is enumerated by
\[
\frac{1}{2^m} \sum_{O(G)} \det(S(G^o)), \quad (1.1)
\]
where the summation ranges over all oriented graphs in \(O(G)\). Then Yan et al. [21] investigate the matchings polynomial of an unweighted graph in terms of the characteristic polynomial of its skew adjacency matrix and generalize formula (1.1) as
\[
\mu(G, \lambda) = \frac{1}{2^m} \sum_{G^\sigma \in O(G)} \det(\lambda I + iS(G^\sigma)), \quad (1.2)
\]
where the summation ranges over all oriented graphs in \(O(G)\), \(m\) is the number of arcs contained in \(G^\sigma\) and \(i = \sqrt{-1}\).

Recently, many researchers have studied the spectral properties of skew symmetric matrices in terms of oriented graphs. IMA-ISU Research Group on Minimum Rank [11] study the problem the minimum rank of skew-symmetric matrices. In 2010, Adiga et al. [2] study the properties of the skew energy of an unweighted oriented graph, which is defined as the summation of the singular values of its skew adjacency matrix; In that paper, Adiga et al. posed several open problems on the skew adjacency matrix of an oriented graph. Then Shader and So [18] investigate the spectra of the skew adjacency matrix of an unweighted oriented graph.

The question below is posed by Adiga et al. in [2]:

Interpret all the coefficients of the characteristic polynomial \(\phi(G^\sigma, \lambda)\) in terms of the oriented graph \(G^\sigma\).

The present paper studies this problem for a general version. In Section 2, we interpret all the coefficients of the characteristic polynomial \(\phi(S, \lambda)\) of an arbitrary real skew symmetric matrix \(S\) in terms of its associated oriented graph and establish some recurrences for \(\phi(G, \lambda)\), obtaining some analogues of recursions for matchings polynomials. In Section 3, we deduce a formula on the matchings polynomial of an arbitrary weighted graph, which is a generalization for both of the formulas (1.1) and (1.2). In addition, some miscellaneous results concerning the number of the perfect matching and the determinant of skew adjacency matrices of an unoriented weighted graph are given in terms of its characteristic polynomial.

2. The coefficients of the characteristic polynomial of a weighted oriented graph

Let \(G = (V, E)\) be a graph with vertex set \(V = V(G) = \{v_1, v_2, \ldots, v_n\}\) and edge set \(E = E(G)\). Denote by \(N(v)\) the neighborhood of the vertex \(v\) in \(G\), by \(G \setminus e\) the graph obtained from \(G\) by deleting the edge \(e\) and by \(G \setminus v\) the graph obtained from \(G\) by removing the vertex \(v\) together with all edges incident to it. A walk \(W\) of length \(k\) from \(u\) to \(v\) in \(G\) is a sequence of \(k + 1\) vertices starting \(u\) and ending \(v\) such that consecutive vertices are adjacent. If all vertices in a walk are distinct, then such a walk is called a path of \(G\), denoted by \(P\). Let \(P = v_1v_2 \cdots v_k\) be a path with \(k \geq 3\). Then \(P\) together with the edge \((v_k, v_1)\) (if there exists) is called a cycle of \(G\). The complete graph and the path of order \(n\) are usually denoted by \(K_n\) and \(P_n\), respectively.

For convenience, in terms of defining subgraph, degree and matching of an oriented graph, we focus only on its underlying graph. Certainly, each subgraph of an oriented graph is also referred as an oriented graph and preserve the orientation of each arc, even if we do not indicate specially. Moreover,
we will briefly use notations $C_n, K_n$ and $P_n$ to denote the oriented cycle, the oriented complete, and the oriented path on $n$ vertices, respectively, if no conflict exists there.

Let $G^{\sigma}$ be a weighted oriented graph on $n$ vertices with skew adjacency matrix $S = [s_{ij}]_{n \times n}$ and $C = u_1 u_2 \cdots u_k u_1 (k \geq 3)$, an oriented cycle of $G^{\sigma}$. The sign of the cycle $C$, denoted by $\text{sgn}(C)$, is defined by

$$\prod_{i=1}^{k} \text{sgn}(s_{u_iu_{i+1}}) \pmod{k}.$$ 

Let $\tilde{C} = u_1 u_k \cdots u_2 u_1$ be the oriented cycle by inverting the order of the vertices along the cycle $C$. Then one can find that

$$\text{sgn}(\tilde{C}) = \begin{cases} -\text{sgn}(C), & \text{if } k \text{ is odd;} \\ \text{sgn}(C), & \text{if } k \text{ is even}. \end{cases}$$

Hence, for an even oriented cycle, we can simply refer it as an evenly oriented cycle or an oddly oriented cycle according its sign is positive or not, regardless the order of its vertices. Note that, to involve the sign of an oriented cycle with length odd, we need indicate its vertex order. For another definition for an evenly (or oddly) oriented cycle, one can also see e.g. [13,14,17].

An oriented graph is called an “elementary oriented graph” if such an oriented graph is

(a) $K_2$ or
(b) a cycle with length even.

The weight of an elementary oriented graph $H$ is defined as the square of the weight of the unique arc if $H$ is $K_2$; or the product of all weights of those arcs if $H$ is an even cycle.

An oriented graph $H$ is called a “basic oriented graph” if each component of $H$ is an elementary oriented graph. The weight of a basic oriented graph, denoted by $\mathcal{W}(H)$, is the product of all weights of those elementary oriented graphs contained in it.

To interpret the coefficients of the characteristic polynomial $\phi(G^{\sigma}, \lambda)$ of the oriented graph $G^{\sigma}$, the following well known result is needed.

**Lemma 2.1** [5, Lemma 2.1.1]. Let $X$ and $Y$ be any two $n \times n$ matrices. Then $\det(X + Y)$ is equal to the sum of the determinants of the $2^n$ matrices obtained by replacing each subset of the columns of $X$ by the corresponding subset of the columns of $Y$.

**Lemma 2.2** [5, Lemma 2.1.2]. Let $A$ be any $n \times n$ matrix and

$$\phi(A, \lambda) = \det(\lambda I - A) = \sum_{i=0}^{n} (-1)^i a_i \lambda^{n-i}.$$ 

Then $a_i$ is equal to the sum of the principal minors of $A$ with order $i$.

**Theorem 2.3.** Let $G^{\sigma}$ be a weighted oriented graph on $n$ vertices with skew adjacency matrix $S(G^{\sigma})$ and its characteristic polynomial

$$\phi(G^{\sigma}, \lambda) = \sum_{i=0}^{n} (-1)^i a_i \lambda^{n-i} = \lambda^n - a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + (-1)^{n-1} a_{n-1} \lambda + (-1)^n a_n.$$ 

Then $a_i = 0$ if $i$ is odd; and

$$a_i = \sum_{\mathcal{C}} (-1)^{\mathcal{C}} 2^\mathcal{C} \mathcal{W}(\mathcal{C}) \quad \text{if } i \text{ is even},$$
where the summation is over all basic oriented subgraphs \( \mathcal{H} \) of \( G^e \) having \( i \) vertices and \( c^+ \), \( c \) are respectively the number of evenly oriented even cycles and even cycles contained in \( \mathcal{H} \).

**Proof.** By Lemma 2.2, \( a_i \) is equal to the summation of all principal minors of \( S(G^e) \) with order \( i \). By the fact that the determinant of each skew symmetric matrix with odd order is zero, then \( a_i = 0 \) if \( i \) is odd. Therefore, the first part follows.

Let \( A = (a_{ij}) \) be a principal minor of \( S(G^e) \) with order \( i \) and the oriented graph associated to \( A \) be \( H \). Then

\[
\text{det}(A) = \sum_{\tau \in \text{Sym}(i)} (-1)^{p(\tau)} \prod_{i=1}^{t} a_{i,i},
\]

with summation taken over all permutations

\[
\tau = \begin{pmatrix} 1 & 2 & \cdots & i \\ 1\tau & 2\tau & \cdots & i\tau \end{pmatrix}
\]

in the symmetric group \( \text{Sym}(i) \), where \( p(\tau) \) denotes the parity of \( \tau \). Then the contribution of a term

\[
S_\tau = (-1)^{p(\tau)} a_{1,1} a_{2,2\tau} \cdots a_{i,i},
\]

in the summation (2.1) is nonzero if and only if all pairs \((1, 1\tau), (2, 2\tau), \ldots, (i, i\tau)\) are arcs of \( H \). The permutation \( \tau \) can be partitioned as a product

\[
\tau = (11\tau \cdots \cdots)(\cdots \cdots \cdots)
\]

of disjoint cycles. Hence, if \( S_\tau \neq 0 \), the subgraph, of \( H \), corresponding to each cycle of \( \tau \) is either \( K_2 \) or an oriented cycle with length no less than 3, since \( G^e \) as well as \( H \) contains no loops.

Let now \( C_1, C_2, \ldots, C_k \) be all cycles of \( \tau \) and their associated oriented subgraphs, of \( H_1, H_2, \ldots, H_k \), respectively. Note that the determinant of \( A \) is independent to the labeling of the vertices in its associated oriented graph \( H \), then we below always refer the vertices of each oriented subgraph \( H_1 \) of \( H \) is labeled by a natural sequence. Moreover, without loss of generality, we suppose that

\[
C_1 = (12 \cdots t-1 t) = \begin{pmatrix} 1 & 2 & \cdots & t-1 & t \\ 2 & 3 & \cdots & t & 1 \end{pmatrix}
\]

with length \( t \geq 2 \). According the length of the cycle \( C_1 \), we discuss the contribution, to \( \text{det}(A) \), of the subgraph \( H_1 \) below.

If \( t \) is odd, then there has another permutation, say \( \tau' \), which is obtained by replacing just the subset of \( C_1 \) by its inverse, and hence having the same permutation partition as \( \tau \), since the cycle \((1 t t-1 \cdots 2) =: C_1 \) has the same parity as that of \( C_1 \). However, \( \text{sgn}(H_1) = -\text{sgn}(\bar{H}_1) \), \( \mathcal{W}(H_1) = \mathcal{W}(\bar{H}_1) \) and hence the contribution of the sum of permutations \( \tau \) and \( \tau' \) is zero, where \( \bar{H}_1 \) is the oriented cycle corresponding to cycle \( C_1 \).

If \( t = 2 \), then \( C_1 = (121) \) and thus the contribution of the cycle \( C_1 \) is the square of the weight of such an arc, i.e., the weight of \( H_1 \), since

\[
(-1)^{p(C_1)} S_{12}^2 S_{21}^2 = S_{12}^2 = \mathcal{W}(H_1).
\]

If \( t \) is even and \( t > 2 \), then there has another permutation, say \( \tau' \), which is obtained by replacing just the subset of \( C_1 \) by its inverses, say \( C_1 \), and hence having the same permutation partition as \( \tau \). Note that \( \text{sgn}(H_1) = \text{sgn}(\bar{H}_1) \) and \( \mathcal{W}(H_1) = \mathcal{W}(\bar{H}_1) \), where \( \bar{H}_1 \) is the oriented cycle corresponding to the cycle \( C_1 \). Consequently, the contribution of \( C_1 \) is

\[
(-1)^{p(C_1)} \prod_{i=1}^{t} S_{i,i+1} + (-1)^{p(\bar{C}_1)} \prod_{i=1}^{t} S_{i+1,i} = -2 \text{sgn}(C_1) \mathcal{W}(C_1) \pmod{t}.
\]
Similarly, we can discuss the contribution of each other cycle, and thus \( \det(A) \) has the form
\[
\sum_{\mathcal{H}} (-1)^{c^+} 2^c \mathcal{W}(\mathcal{H})
\]
where the summation is over all basic oriented graphs \( \mathcal{H} \) of \( D \) having \( i \) vertices and \( c^+, c \) are respectively the number of evenly oriented even cycles and even cycles contained in \( \mathcal{H} \). Hence, the result follows. \( \Box \)

If the oriented graph \( G^\sigma \) is restricted to be unweighted, then the weight of each arc is 1 and thus \( \mathcal{W}(\mathcal{H}) = 1 \) for any basic oriented subgraph with even order, regardless such a basic oriented subgraph contains even cycles or not. Therefore, Theorem 2.3 can be simplified as follows.

**Corollary 2.4.** Let \( G^\sigma \) be an unweighted oriented graph on \( n \) vertices with the characteristic polynomial
\[
\phi(G^\sigma, \lambda) = \sum_{i=0}^{n} (-1)^i a_i \lambda^{n-i} = \lambda^n - a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + (-1)^{n-1} a_{n-1} \lambda + (-1)^n a_n.
\]
Then \( a_i = 0 \) if \( i \) is odd; and
\[
a_i = \sum_{\mathcal{H}} (-1)^{c^+} 2^c \text{ if } i \text{ is even},
\]
where the summation is over all basic oriented subgraphs \( \mathcal{H} \) of \( G^\sigma \) having \( i \) vertices and \( c^+, c \) are respectively defined as Theorem 2.3.

By the method parallel to the proof of Theorem 2.1.5 in [5], we can obtain recursions for the characteristic polynomial of a weighted oriented graph, each of those can be considered as an analogue for the recursion concerning the matchings polynomial of a graph in Theorem 2.1.5 of [5].

**Theorem 2.5.** The characteristic polynomial of weighted oriented graphs satisfy the following identities:

(a) \( \phi(G \cup H, \lambda) = \phi(G, \lambda) \phi(H, \lambda) \);

(b) \( \phi(D, \lambda) = \phi(D \setminus e, \lambda) + s_{uv}^2 \phi(D \setminus uv, \lambda) \) if \( e = uv \) is an arc not contained in any even cycle;

(c) \( \phi(D, \lambda) = \lambda \phi(D \setminus u, \lambda) + \sum_{v \in \mathcal{N}(u)} s_{uv}^2 \phi(D \setminus uv, \lambda) \) if \( u \) is a vertex that is not contained in any even cycle;

(d) \( \frac{d}{d\lambda} \phi(D, \lambda) = \sum_{i \in V(D)} \phi(D \setminus i, \lambda) \)

where \( G \cup H \) denotes the union of two disjoint weighted oriented graphs.

**Proof.**

(a) If \( A \) and \( B \) are square matrices, not necessarily of the same order, then
\[
\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A) \det(B).
\]
Hence the result follows.

(b) If \( e \) is contained in some basic oriented graph, then \( e \) must form an elementary oriented graph \( K_2 \) by itself, since \( e \) is not contained in any even cycle. Consequently, all basic oriented graphs \( \mathcal{H} \) of \( D \) having \( i \) vertices can be divided into two parts: those that do not contain \( e \) and those that contain the arc \( e \) as an elementary oriented graph. The former can be considered as the coefficient of \( \lambda^{n-i} \) in \( \phi(D \setminus e, \lambda) \) and the latter can be considered as the coefficient of \( \lambda^{n-i} \) in \( \phi(D \setminus uv, \lambda) \) multiplied by the square of the weight of the arc \( e \).
(c) Similarly, if \( u \) is contained in some basic oriented subgraph, then \( u \) must be contained in some elementary oriented graph \( K_2 \). Consequently, all basic oriented subgraphs \( \hat{G} \) of \( D \) having \( i \) vertices can be divided into two parts: those that do not contain the vertex \( u \) and those that contain \( u \) together with a neighbor \( v \) of it as the elementary oriented graph \( K_2 \). The former is the coefficient of \( \lambda^{n-i-1} \) in \( \phi(D \setminus u, \lambda) \) and the later can be considered as the coefficient of \( \lambda^{n-i} \) in \( \phi(D \setminus uv, \lambda) \) multiplied by the square of the weight of the arc \((u, v)\).

(d) Note that
\[
\phi(S(D), \lambda + h) - \phi(S(D), \lambda) = \det((\lambda + h)I - S(D)) - \det(\lambda I - S(D)).
\]

Set \( X \) and \( Y \) equal to \( \lambda I - S(D) \) and \( hI \) respectively. Applying Lemma 2.1 to expand \( \det((\lambda + h)I - S(D)) \) and \( \det(\lambda I - S(D)) \) with constant term equal to \( \det(X) \) and the linear term in \( h \) is the sum of the determinants of the matrices obtained by replacing the \( i \)th row of \( X \) with the \( i \)th row of \( Y \), for \( i = 1, 2, \ldots, n \). But the \( i \)th such determinant is
\[
h \det(\lambda I - S(D \setminus i)) = h\phi(D \setminus i)
\]
and so the coefficient of \( h \) in the right hand side of above equality is \( \sum_{i \in V(D)} \phi(D \setminus i, \lambda) \). Since the coefficient of \( h \) in the polynomial \( \phi(S(D), \lambda + h) - \phi(S(D), \lambda) \) is the derivative of \( \phi(S(D), \lambda) \), the result is proved. \( \square \)

3. The matchings polynomial of a weighted oriented graph

As an application of Theorem 2.3, we generalize both of the formulas (1.1) obtained by Godsil and Gutman [5] and (1.2) obtained by Yan et al. [21]. Then we interpret that it is valid to compute the matching polynomials of graphs containing small number of even cycles.

Let \( G^\sigma \) be a weighted oriented graph on \( n \) vertices. We define an \( n \times n \) matrix related to \( G^\sigma \) as \( S^*(G^\sigma) = [s^*_ij] \), where
\[
s^*_ij = \begin{cases} 
\sqrt{\theta}, & \text{if there has an arc with weight } \theta \text{ from } i \text{ to } j; \\
-\sqrt{\theta}, & \text{if there has an arc with weight } \theta \text{ from } j \text{ to } i; \\
0, & \text{otherwise.}
\end{cases}
\]

Obviously, for each entry \( s^*_ij \) in \( S^*(G^\sigma) \),
\[
s^*_ij = \text{sgn}(sij)\sqrt{|sij|},
\]
where \( sij \) is the entry corresponding to \( s^*_ij \) in \( S(G^\sigma) \). In particular, \( S^*(G^\sigma) = S(G^\sigma) \) if \( G^\sigma \) is unweighted. Hence, by the method parallel to the proof of Theorem 2.3, the following result can be obtained immediately.

**Lemma 3.1.** Let \( G^\sigma \) be a weighted oriented graph on \( n \) vertices and the matrix \( S^*(G^\sigma) \) is defined as (3.1). Suppose that
\[
\det(\lambda I - S^*(G^\sigma)) = \sum_{i=0}^{n} (-1)^i a^*_i \lambda^{n-i} = \lambda^n - a^*_1 \lambda^{n-1} + a^*_2 \lambda^{n-2} + \cdots + (-1)^{n-1} a^*_{n-1} \lambda + (-1)^n a^*_n.
\]
Then \( a^*_i = 0 \) if \( i \) is odd; and
\[
a^*_i = \sum_{\hat{G}} (-1)^{i-1} c^+ \sqrt{\det(S^\hat{G})} \text{ if } i \text{ is even,
}\]
where the summation is over all basic oriented subgraphs \( \hat{G} \) of \( G^\sigma \) having \( i \) vertices, \( c^+ \) and \( c \) are respectively defined as Theorem 2.3.

Now we give a formula on the matchings polynomial of an arbitrary weighted oriented graph.
Theorem 3.2. Let $G^σ$ be a weighted oriented graph containing $n$ vertices and $m$ arcs. Then

$$
\mu(G^σ, λ) = \frac{1}{2^m} \sum_{G^o ∈ O(G)} \det(λI + i S^*(G^o)),
$$

where the matrix $S^*(G)$ is defined as (3.1) and the summation ranges over $O(G)$ and $i = \sqrt{-1}$.

**Proof.** We first show that $p(G^σ, r)$ equals to the coefficient of $λ^{n−2r}$ in polynomial

$$
\frac{1}{2^m} \sum_{D^o ∈ O(G)} \det(λI - S^*(G^o)).
$$

From Lemma 3.1, the coefficient $a_{2r}^o$ of $λ^{n−2r}$ in polynomial $\det(λI - S^*(G^o))$ of a given oriented graph $G^o$ has the form

$$
\sum_{\mathcal{H}} (-1)^c 2^c \sqrt{\mathcal{W}(\mathcal{H})}
$$

where the summation is over all basic oriented subgraphs $\mathcal{H}$ in $G^σ$ having $2r$ vertices, $c^+$ and $c$ are respectively defined as Theorem 2.3.

Let now $\mathcal{H}$ be a basic oriented subgraph in $G^σ$ containing $2r$ vertices and denote by $\mathcal{H}^o$ the corresponding basic oriented subgraph in $G^o ∈ O(G)$. If $\mathcal{H}$ contains an oriented cycle $C_1$ with length more than 2, then the length of such a cycle must be even and hence there has equal number of re-oriented oriented graphs among $O(G)$ such that the re-oriented cycle $C_1$ in the corresponding basic oriented graph $\mathcal{H}^o$ are evenly oriented and oddly oriented, respectively, and preserving the orientations of all other arcs in $\mathcal{H}$, since each oriented graph $G^o$ is a random re-oriented oriented graph, obtained by re-orienting each arc in $G^o$ independently of the others with probability 1/2 in either direction. On the other hand, note that the contribution, to the term (2.1), of $C_1$ is $2\sqrt{\mathcal{W}(C_1)}$ if $C_1$ is an oddly oriented cycle, and $−2\sqrt{\mathcal{W}(C_1)}$ otherwise by Lemma 3.1. Consequently,

$$
\sum_{G^o ∈ O(G)} (-1)^c 2^c \sqrt{\mathcal{W}(\mathcal{H}^o)} = 0
$$

holds for each basic oriented graph which contains even cycles with length more than 2, where $c^+$ and $c$ are respectively the number of evenly oriented even cycles and even cycles in the corresponding basic oriented subgraph $\mathcal{H}^o$ in $G^o$.

If each elementary oriented subgraph of $\mathcal{H}$ is $K_2$, i.e., $\mathcal{H}$ is an $r$-matching of $G^σ$. Then $\mathcal{H}^o$ is also a basic oriented subgraph all of whose elementary oriented subgraphs are $K_2$ and thus, from the proof of Theorem 2.3, the contribution to $a_{2r}$ is $\sqrt{\mathcal{W}(\mathcal{H})}$, the weight of such an $r$-matching, regardless the orientation of each of those arcs. Consequently, for each basic oriented subgraph $\mathcal{H}$ whose elementary oriented graphs are all $K_2$,

$$
\frac{1}{2^m} \sum_{G^o ∈ O(G)} \sqrt{\mathcal{W}(\mathcal{H}^o)}
$$

equals the weight of the $r$-matching contained in $\mathcal{H}$.

Therefore, combining with (3.2), the contribution to the coefficient of $λ^{n−2r}$ in polynomial

$$
\frac{1}{2^m} \sum_{G^o ∈ O(G)} \det(λI - S^*(G^o))
$$

are all coming from those basic oriented subgraphs all of whose elementary oriented graphs are $K_2$, i.e., the summation of the weights of all $r$-matchings contained in $G^σ$. Then the result follows from the fact that the coefficient of $λ^{n−2r}$ in $\det(λI + i S^*(G^o))$ equals $(-1)^r$ multiplied by coefficient of $λ^{n−2r}$ in $\det(λI - S^*(G^o))$ for each oriented graph $G^o$ in $O(G)$, where $i$ satisfies $i^2 = −1$. □
Lemma 3.3. Suppose that $G^\sigma$ is a connected oriented graph and $T$ is any spanning tree of $G$. Let $G^0 \in \mathcal{O}(G)$ be any re-orientation of $G^\sigma$ and let $T^0$ be the corresponding re-orientation of $T^\sigma$. Then there is a sequence of reversals of $G^0$ that re-orients $G^0$ so that the resulting re-orientation of $T^0$ agrees with the orientation of $T^\sigma$.

Let $G^\sigma$ be an oriented graph with $n$ vertices and $m$ arcs and $T$ be any spanning tree of $G$. Applying Lemma 3.3, we can fix the orientations of all arcs of $T^\sigma$ in each re-oriented oriented graph $G^0$, by giving reversals of $G^0$ at some of its vertices repeatedly if necessary. Hence we can simplify Theorem 3.2 as follows, in which we only need to consider the $2^{m-n+1}$ oriented graphs in $\mathcal{O}(G)$ that have the same spanning oriented tree $T^\sigma$ as $G^\sigma$.

Theorem 3.4. Let $G^\sigma$ be a connected oriented weighted graph with $n$ vertices and $m$ arcs and $T$ be any spanning tree of $G$. Denote by $\mathcal{O}(G \setminus T)$ the set of all oriented graphs obtained from $G$ by giving arbitrary orientations of all arcs in $G \setminus T$ and preserving the orientations of all arcs in $T^\sigma$. Then

$$\mu(G, \lambda) = \frac{1}{2^{m-n+1}} \sum_{G^0 \in \mathcal{O}(G \setminus T)} \det(\lambda I + iS^*(G^0)),$$

where the summation ranges over all oriented graphs in $\mathcal{O}(G \setminus T)$ and $i = \sqrt{-1}$.

In addition, as a consequence of Theorem 2.3, we have the following result, which can also be considered as a generalization of Theorem 8(1) in [21].

Corollary 3.5. Let $G^\sigma$ be an oriented graph, weighted or not, on $n$ vertices and the matrix $S^*(G^\sigma)$ related to $G^\sigma$ be defined as (3.1). If $G$ contains no even cycles, then

$$\det(\lambda I - S^*(G^\sigma)) = \sum_{r \geq 0} p(G^\sigma, r) \lambda^{n-2r},$$

that is

$$\mu(G^\sigma, \lambda) = \det(\lambda I + iS^*(G^\sigma)),$$

where $p(G^\sigma, r)$ denotes the summation of the weight of all $r$-matchings in $G^\sigma$.

Proof. Note that $G^\sigma$ contains no evenly oriented cycles, then each basic oriented subgraph of $G^\sigma$ with order $2r$ is consisted of $r$ elementary oriented subgraphs $K_2$, which forms an $r$-matching of $G^\sigma$. Moreover, the contribution of each basic oriented subgraph of $G^\sigma$ with order $2r$, to the coefficient...
\( \lambda^{n-2r} \) is the weight of such an \( r \)-matching, regardless the orientations of all arcs contained in it. Hence the result follows. □

Applying Corollary 3.5, we can by further simplify Theorem 3.4 as follows.

**Theorem 3.6.** Let \( G^\sigma \) be a connected weighted oriented graph with \( n \) vertices and \( m \) arcs, and \( G_1 \) be any spanning subgraph of \( G \). Suppose that the matrix \( S^*(G^\sigma) \) related to \( G^\sigma \) is defined as (3.1) and the number of edges contained in \( G_1 \) is \( l \). Denote by \( \mathcal{D}(G \setminus G_1) \) the set of all oriented graphs obtained from \( G^\sigma \) by preserving the orientations of all arcs in \( G_1 \) and giving arbitrary orientations of all other arcs. If \( G_1 \) contains no even cycles, then

\[
\mu(G, \lambda) = \frac{1}{2^{m-l}} \sum_{G^\sigma \in \mathcal{D}(G \setminus G_1)} \det(\lambda I + iS^*(G^\sigma)),
\]

where the summation ranges over all oriented graphs in \( \mathcal{D}(G \setminus G_1) \) and \( i = \sqrt{-1} \).

In 1947, Tutte [20] established a relation between the perfect matching of an unweighted oriented graph \( G^\sigma \) and the determinant of its skew adjacency matrix \( S(G^\sigma) \), showing that if \( G^\sigma \) has no perfect matching then \( \det(S(G^\sigma)) = 0 \). However, the converse is not true! From the proof of Theorem 2.3 we can obtain some miscellaneous results on \( \det(S(G^\sigma)) \neq 0 \) of an unweighted oriented graph \( G^\sigma \).

**Theorem 3.7.** Let \( G^\sigma \) be an unweighted oriented graph on \( n \) vertices with skew adjacency matrix \( S(G^\sigma) \). Suppose that \( G^\sigma \) contains no evenly oriented even cycles and has a perfect matching. Then

\[ \det(S(G^\sigma)) \neq 0. \]

**Proof.** Obviously \( n \) is even. Let \( M \) be a perfect matching of \( G^\sigma \). Then \( M \) can be considered as a basic oriented subgraph with order \( n \), and then its contribution to \( \det(S(G^\sigma)) \) is 1 as \( M = \frac{n}{2}K_2, \frac{n}{2} \) elementary oriented subgraphs \( K_2 \). On the other hand, note that \( G^\sigma \) contains no evenly oriented even cycles, then the contribution of all basic oriented subgraphs with order \( n \) of \( G^\sigma \) to \( \det(S(G^\sigma)) \) is positive (if there exist). Hence the result follows. □

**Theorem 3.8.** Let \( G^\sigma \) be an unweighted oriented graph on \( n \) vertices with skew adjacency matrix \( S(G^\sigma) \). Suppose that the number of perfect matchings of \( G^\sigma \) is odd. Then

\[ \det(S(G^\sigma)) \neq 0. \]

**Proof.** The result follows from the fact that the contribution of each basic oriented subgraph whose elementary oriented subgraphs are all \( K_2 \) is 1, and the contribution of each basic oriented subgraph which contains at least one cycle with length greater than 2 is an even integer. □

**Acknowledgments**

The authors gratefully acknowledge the comments by the anonymous referees and Professor Jia-Yu Shao of Tongji University of China for his careful reading of this paper and for his most valuable advice.

**References**