On the Asymptotic Behavior of Nonlinear Semigroups and the Range of Accretive Operators*

SIMEON REICH

Department of Mathematics, University of Southern California,
Los Angeles, California 90007

Submitted by Ky Fan

INTRODUCTION

Let $C$ be a closed convex subset of a Banach space $E$, $T: C \to C$ a nonexpansive mapping, $A \subset E \times E$ an accretive operator that satisfies the range condition, and $S$ the nonexpansive nonlinear semigroup generated by $-A$. Assume that the norm of $E$ is uniformly Gâteaux differentiable and that the norm of $E^*$ is Fréchet differentiable. It has been known [20, 22] that if $C$ and $\text{cl}(\text{R}(A))$ are (sunny) nonexpansive retracts of $E$, then the strong $\lim_{n \to \infty} T^n x/n = -v_1$ and the strong $\lim_{t \to \infty} S(t)x/t = -v_2$, where $v_1$ and $v_2$ are the points of least norm in $\text{cl}(R(I - T))$ and $\text{cl}(R(A))$, respectively. However, the question whether these results are true without the restriction on $C$ and $\text{cl}(D(A))$ has remained open. In Section 3 we present a positive solution to this problem.

In fact, more general results are proved—see Theorems 3.3 and 3.4. In the proofs we use the following theorem: If $E$ is (UG) and an accretive $A \subset E \times E$ satisfies the range condition, then $d(0, \text{cl}(R(A))) = d(0, R(A))$. This result (Theorem 2.3) is established in Section 2, where we also modify an idea of Kohlberg and Neyman [13] to show that the same theorem is true for smooth, uniformly convex $E$. See Theorem 2.6 and the remark after Theorem 3.4. These theorems provide a positive answer to a question of Pazy [17, p. 239].

In Section 2 we also present examples which show that Theorem 2.3 is not true for all Banach spaces (even if $A$ is $m$-accretive), nor is it true for accretive operators that do not satisfy the range condition (even if $E$ is Hilbert). In addition, we show that in the setting of the theorem, $\text{cl}(R(A))$ is not convex in general, even if $E$ is Hilbert. (It is convex if $A$ is $m$-accretive.)

Section 3 also contains several related theorems, e.g., on the asymptotic behavior of resolvents and infinite products of resolvents. In some cases the

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conclusions of Theorems 3.3 and 3.4 can be sharpened. See Theorem 3.7. More results are presented in Section 4.

The first results in this direction were established by Crandall (see [3, p. 166]) and Pazy [17] in Hilbert space. See also [7, 15, 18]. A special result in Banach spaces [2] has an interpretation in the theory of stochastic games [32]. For more recent developments in Banach spaces see [1, 28]. Some of the theorems of the present paper were announced in [30].

1. Preliminaries

Let $E$ be a real Banach space, and let $I$ denote the identity operator. Recall that a subset $A$ of $E \times E$ with domain $D(A)$ and range $R(A)$ is said to be accretive if $|x_1 - x_2| \leq |x_1 - x_2 + r(y_1 - y_2)|$ for all $[x_i, y_i] \in A$, $i = 1, 2$, and $r > 0$. The resolvent $J_r: (I + rA)^{-1} \rightarrow D(A)$ and the Yosida approximation $A_r: (I + rA) \rightarrow R(A)$ are defined by $J_r = (I + rA)^{-1}$ and $A_r = (I - J_r)/r$. We denote the closure of a subset $D$ of $E$ by $\text{cl}(D)$, its closed convex hull by $\text{clco}(D)$, and its distance from a point $x$ in $E$ by $d(x, D)$. We also define $\|D\| = d(0, D)$. We shall say that $A$ satisfies the range condition if $R(I + rA) \supset \text{cl}(D(A))$ for all $r > 0$. In this case, $-A$ generates a nonexpansive nonlinear semigroup $S: [0, \infty) \times \text{cl}(D(A)) \rightarrow \text{cl}(D(A))$ by the exponential formula [8]: $S(t)x = \lim_{n \rightarrow \infty} (I + (t/n)A)^{-n}x$.

Recall that the norm of $E$ is said to be Gâteaux differentiable (and $E$ is said to be smooth) if $\lim_{t \rightarrow 0} (\|x + ty\| - \|x\|)/t$ exists for each $x$ and $y$ in $U = \{x \in E: \|x\| = 1\}$. It is said to be uniformly Gâteaux differentiable if for each $y$ in $U$, this limit is approached uniformly as $x$ varies over $U$. The norm is said to be Fréchet differentiable if for each $x$ in $U$ this limit is attained uniformly for $y$ in $U$. We shall write that $E$ is (UG) and (F), respectively. Finally, the norm is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth) if the limit is attained uniformly for $[x, y] \in U \times U$. Since $E$ is uniformly smooth if and only if its dual $E^*$ is uniformly convex, $E$ is (UG) if $E^*$ is uniformly convex and $E^*$ is (F) if $E$ is uniformly convex. The converse implications are false. In fact, there are spaces $E$ such that $E$ is (UG) and $E^*$ is (F), but $E$ is not even isomorphic to a uniformly convex space. A discussion of these and related concepts may be found in [10].

The duality map from $E$ into the family of nonempty subsets of $E^*$ is defined by

$$J(x) = \{x^* \in E^*: (x, x^*) - \|x\|^2 - \|x^*\|^2\}.$$ 

It is single valued if and only if $E$ is smooth. An operator $A \subset E \times E$ is accretive if and only if for each $x_i \in D(A)$ and each $y_i \in A x_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $(y_1 - y_2, j) \geq 0$. 

114 SIMEON REICH


2. The Minimum Property

A closed subset $D$ of a Banach space $E$ is said to have the minimum property [17] if $d(0, \text{clco}(D)) = d(0, D)$. In this section we show that if $E$ is "nice" and an accretive $A \subset E \times E$ satisfies the range condition, then $\text{cl}(R(A))$ has the minimum property. This provides a positive answer to a question of Pazy [17, p. 239]. We also present several counterexamples.

We begin with a known result (see the proof of [28, Proposition 5.2]), the proof of which is included here for completeness.

**Lemma 2.1.** Let $E$ be an arbitrary Banach space, and $A \subset E \times E$ an accretive operator that satisfies the range condition. Then for each $x$ in $\text{cl}(D(A))$, $\lim_{t \to \infty} |J_Ax/t| = d(0, R(A))$.

**Proof.** Denoting $d(0, R(A))$ by $d$, we have on the one hand

\[ \lim \inf_{t \to \infty} |A_t x| \geq d \]

because $A_t x \in R(A)$. On the other hand, given $\epsilon > 0$, there is $[y, z] \in A$ such that $|z| \leq d + \epsilon$. Since $|A_t x| \leq |A_t x - A_t y| + |A_t y| \leq 2|x - y|/t + |z|$, $\limsup_{t \to \infty} |A_t x| < d$, and the result follows.

**Remark.** As a matter of fact, $|A_t x|$ decreases as $t \to \infty$ to $d$.

The next lemma is also essentially known (cf. [9]).

**Lemma 2.2.** If a Banach space $E$ is (UG), then $J : E \to E^*$ is uniformly continuous on bounded subsets of $E$ from the strong topology of $E$ to the weak-star topology of $E^*$.

**Proof.** If the result were not true, there would be sequences $\{x_n\}$ and $\{z_n\}$, a point $y_0$, and a positive $\epsilon$ such that $|x_n| = |z_n| = |y_0| = 1$, $z_n - x_n \to 0$, and $(y_0, J(z_n)) - J(x_n)) \geq \epsilon$ for all $n$. Let $a_n = (|x_n + t y_0| - |x_n| - t(y_0, J(z_n)))/t$ and $b_n = (|z_n - t y_0| - |z_n| + t(y_0, J(z_n)))/t$. If $t > 0$ is sufficiently small, then both $a_n$ and $b_n$ are less then $\epsilon/2$. On the other hand, we have

\[
 a_n \geq ((x_n + t y_0, J(z_n)) - (x_n + t y_0, J(x_n)))/t
 = (y_0, J(z_n) - J(x_n)) + (x_n, J(z_n) - J(x_n))/t,
\]

and

\[
 b_n \geq ((z_n - t y_0, J(x_n)) - (z_n - t y_0, J(z_n)))/t
 - (y_0, J(z_n) - J(x_n)) - (z_n, J(z_n) - J(x_n))/t.
\]

Hence $a_n + b_n \geq 2(y_0, J(z_n) - J(x_n)) + (x_n - z_n, J(z_n) - J(x_n))/t \geq 2\epsilon - 2|x_n - z_n|/t$. We arrive at a contradiction by choosing $t = 2|x_n - z_n|/\epsilon$ for sufficiently large $n$. 
Remark. The converse of this lemma is also true. Indeed, let $|y_0| = 1$, $\epsilon > 0$, $z_i = (x + ty)/|x + ty|$, and $w_i = (x - ty)/|x - ty|$. If $t > 0$ is sufficiently small, then both $(y_0, J(z_i) - J(x))$ and $(y_0, J(x) - J(w_i))$ are less than $\epsilon/2$ for all $|x| = 1$. Since $(y_0, J(z_i)) = (x + ty - x, J(z_i)/t) \geq (|x + ty| - |x|)/t$ and $(y_0, J(w_i)) = (x - ty - x, J(w_i))/(-t) \leq (|x - ty| - |x|)/(-t)$, the result follows.

**Theorem 2.3.** Let $E$ be a Banach space, and let $A \subset E \times E$ be an accretive operator. If $E$ is $(UG)$ and $A$ satisfies the range condition, then $\text{cl}(R(A))$ has the minimum property.

**Proof.** If $z \in Ay$, $x \in \text{cl}(D(A))$, and $t > 0$, then $(z - Ax, J((y - Jx)/t)) \geq 0$. Let a subnet of $j_i = J((y - Jx)/t)$ converge weak-star to $j$ as $t \to \infty$. Clearly

$$|j| \leq \liminf_{t \to \infty} |j_i| = d(0, R(A)) = d$$

by Lemma 2.1. We also have

$$\lim_{t \to \infty} (Ax, J((y - Jx)/t)) = d^2.$$ 

Therefore $(z, j) \geq d^2$. By Lemma 2.2, $j$ does not depend on $y$ and $z$. Thus $(w, j) \geq d^2$ for all $w \in \text{clco}(R(A))$. Hence

$$|w| d \geq |w| |j| \geq (w, j) \geq d^2,$$

and the result follows.

We show now that Theorem 2.3 remains true if the assumption that $E$ is $(UG)$ is replaced by the assumption that $E$ is uniformly convex (equivalently $E^*$ is $(UF)$) and smooth. This is done by modifying an idea of Kohlberg and Neyman [13]. We need several preliminary results.

**Lemma 2.4.** Let $E$ be a Banach space. For $0 < \epsilon \leq 2$, let $\gamma(\epsilon) = \inf\{|1 - (y, j)|: |x| = |y| = 1, |x - y| \geq \epsilon, j \in Jx\}$. If $E$ is uniformly convex, then $\gamma(\epsilon)$ is positive.

**Proof.** Let $\delta$ be the modulus of convexity of $E$. If $|x| = |y| = 1$, $|x - y| \geq \epsilon$, and $j \in Jx$, then $|(x + y)/2| \leq 1 - \delta(\epsilon)$ and $\frac{1}{2} + \frac{1}{2}(y, j) = ((x + y)/2, j) \leq 1 - \delta(\epsilon)$. Hence $\gamma(\epsilon) \geq 2\delta(\epsilon)$.

Remark. The converse of Lemma 2.4 is also true. Indeed, let $|x| = |y| = 1$, $|x - y| \geq \epsilon$, $z = (x + y)/|x + y|$, and $j \in J(z)$. If $|x + y| < 2 - \epsilon/2$, then $|x + y|/2 > \epsilon/4$. If, on the other hand, $|x + y| \geq 2 - \epsilon/2$, then $|x - z| = |x - y - (2 - |x + y|) x/|x + y| \geq \frac{1}{2}(\epsilon - \epsilon/2) = \epsilon/4$. Since
\[ |y - z| \text{ is also greater or equal to } \varepsilon/4, \text{ we obtain in this case } 1 - |x + y|/2 = \frac{1}{2}(1 - (x, j) + 1 - (y, j)) \geq \gamma(\varepsilon/4). \text{ Thus } \delta(\varepsilon) \geq \min\{\varepsilon/4, \gamma(\varepsilon/4)\}.

**Theorem 2.5.** Let \( E \) be a uniformly convex Banach space and \( A \subseteq E \times E \) an accretive operator that satisfies the range condition. Then for each \( x \) in \( \text{cl}(D(A)) \), the strong \( \lim_{t \to \infty} J_{x/t} \) exists.

**Proof.** By Lemma 2.1 we may assume that \( d = d(0, R(A)) \) is positive. Given \( \varepsilon > 0 \), there is \([y, z] \in A\) such that \( |z| \leq d(1 + \gamma(\varepsilon/2)) \), where \( \gamma \) is defined in Lemma 2.4. Since \([J_{x}, A_{x}] \in A\) and \( A \) is accretive, there is \( j_{t} \in J((y - J_{t}x)) \) such that \((z - A_{t}x, j_{t}) \geq 0\). Therefore

\[
\left( \frac{z}{|z|}, \frac{j_{t}}{|y - J_{t}x|} \right) \geq \left( \frac{A_{x}}{|z|}, \frac{j_{t}}{|y - J_{t}x|} \right) \frac{d}{|z|} \geq \frac{1}{1 + \gamma(\varepsilon/2)}. \]

Consequently, \( (z/|z|, j_{t}/|y - J_{t}x|) > 1 - \gamma(\varepsilon/2) \) for \( t \geq t_{1}(\varepsilon) \). By Lemma 2.4 this implies that \( |z/|z| - (y - J_{t}x)/|y - J_{t}x|| < \varepsilon/2 \) for \( t \geq t_{1}(\varepsilon) \). Since \( \lim_{t \to \infty} |J_{t}x| = \infty \), we also have \( |(y - J_{t}x)/|y - J_{t}x| + J_{t}x/|J_{t}x| - \varepsilon/2 | \) for all \( t \geq t_{2}(\varepsilon) \). Hence \( J_{t}x/|J_{t}x| \) is Cauchy and converges strongly to \( w \). By Lemma 2.1, the strong \( \lim_{t \to \infty} J_{x/t} = dw \).

We can now present a variant of Theorem 2.3.

**Theorem 2.6.** Let \( E \) be a Banach space and let \( A \subseteq E \times E \) be an accretive operator. If \( E \) is uniformly convex and smooth and \( A \) satisfies the range condition, then \( \text{cl}(R(A)) \) has the minimum property.

**Proof:** If \( z \in Ay, x \in \text{cl}(D(A)), \) and \( t > 0 \), then \((z - A_{t}x, J((y - J_{t}x)/t)) \geq 0\). Denote \( \lim_{t \to \infty} J_{x/t} \) (which exists by Theorem 2.5) by \( -v \). Then \( |v| = d(0, R(A)) = d \) by Lemma 2.1. Thus \((z - v, J(v)) \geq 0\). It follows that \((w, J(v)) \geq |v|^2 = d^2 \) for all \( w \) in \( \text{clco}(R(A)) \). Hence the result.

**Remark.** In fact, the unique element of least norm in \( \text{clco}(R(A)) \) already belongs to \( \text{cl}(R(A)) \).

Recall that an accretive operator \( A \subseteq E \times E \) is called \( m \)-accretive if \( R(I + A) = E \). (It then follows that \( R(I + rA) = E \) for all positive \( r \).) For \( m \)-accretive \( A \), Theorems 2.3 and 2.6 can be improved (cf. [20, Theorem 2.6]).

**Theorem 2.7.** Let \( E \) be a Banach space and let \( A \subseteq E \times E \) be \( m \)-accretive. If \( E \) is either (UG) or uniformly convex and smooth, then \( \text{cl}(R(A)) \) is convex.
Proof. For each $y$ in $E$, define $B \subseteq E \times E$ by $Bx = Ax - y$. $B$ is also $m$-accretive. Thus Theorems 2.3 and 2.6 yield $\|\text{clco}(R(B))\| = \|\text{cl}(R(B))\|$. In other words, $d(y, \text{clco}(R(A))) = d(y, \text{cl}(R(A)))$, and each $y \in \text{clco}(R(A))$ belongs to $\text{cl}(R(A))$.

We now provide several examples which show that the results of this section are quite sharp.

We begin with an example that appears in [8, p. 295]. Let $E$ be $\mathbb{R}^2$ with the maximum norm. Let $g: [-1, 1] \rightarrow [-1, 1]$ be continuous and nonincreasing. Assume that $g(-1) = 1$ and that $g(1) = -1$. Define $A \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ by $A(a, b) = (-1, 1)$ if $b > a$, $A(a, b) = (1, -1)$ if $b < a$, and $A(a, b) = \{(x, g(x)) : -1 \leq x \leq 1\}$ if $a = b$. Then $A$ is $m$-accretive, but $\text{cl}(R(A))$ is not convex if $g$ is not the identity. Thus Theorem 2.7 is not true for all Banach spaces. Now let $x_0$ be the unique fixed point of $g$. Since $\|\text{cl}(R(A))\| = |x_0|$ and $\|\text{clco}(R(A))\| = 0$, we see that $A$ does not even have the minimum property, unless $x_0 = 0$. Thus Theorems 2.3 and 2.6 are not true in all Banach spaces, even if $A$ is $m$-accretive.

Both theorems are not true if $A$ does not satisfy the range condition, even if $E$ is Hilbert. To see this, let $E = \mathbb{R}^2$ and $A = \{(x, y), (y, -x)\}$: $x^2 + y^2 = 1$.

Theorem 2.7 is not true if $A$ is not $m$-accretive, even if it satisfies the range condition and $E$ is Hilbert. Thus Theorems 2.3 and 2.6 cannot be improved. To see this, let $E = \mathbb{R}^2$, define $P$ on $\{(x, y) : x + y \geq 0\}$ by $P((x, y)) = (\max(x, 0), \max(y, 0)) + (a, a)$, where $a \geq 0$, and let $A = I - P$. We have $A(-1, 1) = (-1 - a, -a)$, $A(1, -1) = (-a, -1 - a)$, but $(A(-1, 1) + A(1, -1))/2 = (-\frac{1}{2} - a, -\frac{1}{2} - a)$ does not belong to $R(A) = \text{cl}(R(A))$.

3. Asymptotic Behavior

Let $C$ be a closed convex subset of a Banach space $E$, $T: C \rightarrow C$ a nonexpansive ($\|Tx - Ty\| \leq |x - y|$ for all $x$ and $y$ in $C$) mapping, $A \subseteq E \times E$ an accretive operator that satisfies the range condition, and $S: [0, \infty) \times \text{cl}(D(A)) \rightarrow \text{cl}(D(A))$ the nonexpansive nonlinear semigroup generated by $-A$. Assume that $E$ is (UG) and that $E^*$ is (F). It has been known [20, 22, 26] that if $C$ and $\text{cl}(D(A))$ are (sunny) nonexpansive retracts of $E$, then the strong $\lim_{n \rightarrow \infty} T^n x/n = -v_1$ and the strong $\lim_{t \rightarrow \infty} S(t) x/t = -v_2$, where $v_1$ and $v_2$ are the points of least norm in $\text{cl}(R(I - T))$ and $\text{cl}(R(A))$, respectively. However, the question whether this is true without the restriction on $C$ and $\text{cl}(D(A))$ has remained open [21, Problem 7; 23, Problem 4]. In this section we present a positive solution to this problem. Several related results are also included.

In addition to the results of Section 2, we shall need the following two
lemmata. The first is essentially known (cf. [9, p. 296; 12, p. 555]), and the second follows from the proof of Theorem 2.5.

**Lemma 3.1.** Let $E$ be a Banach space. Then $E^*$ is (F) if and only if for any convex set $K \subset E$, every sequence $\{x_n\}$ in $K$ such that $|x_n|$ tends to $d(0, K)$ converges.

**Lemma 3.2.** Let $E$ be a uniformly convex Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, $d = d(0, R(A))$, and $x \in \text{cl}(D(A))$. Let $\lim_{t \to \infty} J_t x/t = -v$. Then for each $\varepsilon > 0$, there is $\delta > 0$ such that if $z \in R(A)$ and $|z| < d + \delta$, then $|z - v| < \varepsilon$.

**Remark.** The strong $\lim_{t \to \infty} J_t x/t$ exists by Theorem 2.5. It follows that $v$ is the unique point of least norm in $\text{cl}(R(A))$.

**Theorem 3.3.** Let $E$ be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, and $S$ the semigroup generated by $-A$. Assume either that $E$ is (UG) and $E^*$ is (F), or that $E$ is uniformly convex. Then for each $x$ in $\text{cl}(D(A))$, $\lim_{t \to \infty} S(t) x/t = \lim_{t \to \infty} J_t x/t = -v$, where $v$ is the point of least norm in $\text{cl}(R(A))$.

**Proof.** Assume first that $E$ is (UG) and $E^*$ is (F). Let $d = d(0, R(A)) = d(0, \text{clco}(R(A)))$ by Theorem 2.3. We always have $\lim \sup_{t \to \infty} |x - S(t)x| \leq d$ and $\lim_{t \to \infty} |J_t x/t| = d$ by Lemma 2.1. Since $(x - S(t)x)/t$ belongs to $\text{clco}(R(A))$, we also have $(x - S(t)x)/t \geq d$ for all $t$. Thus $\lim_{t \to \infty} |(x - S(t)x)/t| = d$. The result now follows by Lemma 3.1.

Assume now that $E$ is uniformly convex. Given $\varepsilon > 0$, let $z \in Ay$ satisfy $|z| < d + \delta$, where $\delta$ is determined by Lemma 3.2. Since $|A_j x| \leq \|AJ_j x\| \leq |A_j x|$, we have for $1 \leq i \leq n$, $|A_{un} J_{un} y| \leq \|A_{un} J_{un} y\| \leq |A_{un} J_{un}^{-1} y|$. Therefore $|A_{un} J_{un} y - v| < \varepsilon$ for all $0 \leq i \leq n - 1$. By Lemma 3.2, $|A_{un} J_{un} y - v| < \varepsilon$ for all $0 \leq i \leq n - 1$. Consequently, $|y - S(t)y)/t - v| < \varepsilon$ for all $n$, and $|(y - S(t)y)/t - v| < \varepsilon$ for all $t$. Thus $\lim \sup_{t \to \infty} |S(t) x/t + v| = \lim \sup_{t \to \infty} |S(t) y/t + v| \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the result follows.

The existence of $\lim_{t \to \infty} J_t x/t$ can be used in the study of certain explicit and implicit iterative methods. See [16, 29]. Theorem 3.3 extends previous results of Morosanu [15] and Pazy [19] for monotone operators in Hilbert space. Their methods are different from ours.

Theorem 3.3 also improves upon [29, Theorem 1]. If $E$ is (UG), reflexive, and strictly convex, then $S(t) x/t$ and $J_t x/t$ converge weakly as $t \to \infty$ to $-v$.

**Theorem 3.4.** Let $C$ be a closed convex subset of a Banach space $E$, and let $T : C \to C$ be nonexpansive. Let the sequence $\{x_n : n = 0, 1, 2, \ldots\}$ be defined by $x_{n+1} = c_n Tx_n + (1 - c_n) x_n$, where $x_0 \in C$ and $\{c_n\}$ is a real
sequence such that $0 < c_n < 1$ and $a_n = \sum_{i=0}^{n} c_i \to n \to \infty$. Assume either that $E$ is (UG) and $E^*$ is (F), or that $E$ is uniformly convex. Then the strong \( \lim_{n \to \infty} x_{n+1}/a_n = -v \), where $v$ is the point of least norm in $\text{cl}(R(I - T))$.

**Proof.** If $E$ is (UG) and $E^*$ is (F), then the result follows from Theorem 2.3 and [20, Theorem 2.3]. Now let $E$ be uniformly convex. $I - T$ is accretive and satisfies the range condition. Given $c > 0$, let $y_0 \in C$ satisfy $|y_0 - Ty_0| < d(0, R(I - T)) + \delta$, where $\delta$ is determined by Lemma 3.2. Let \( \{y_n\} \) be defined by $y_{n+1} = c_n Ty_n + (1 - c_n) y_n$, $n \geq 0$. Since $|y_{n+1} - Ty_{n+1}| \leq |y_n - Ty_n|$, $|y_n - Ty_n - v| < \epsilon$ for all $n$. Therefore $|(y_0 - y_{n+1})/a_n - v| < \epsilon$. The result follows because $|x_{n+1} - y_{n+1}| \leq |x_0 - y_0|$.

If $c_n = 1$ for all $n$, then Theorem 3.4 can be deduced from Theorem 3.3. Kohlberg and Neyman [13] have established Theorem 3.4 for uniformly convex $E$ in case $c_n = 1$ for all $n$. Our proof is based on a modification of their idea.

**Corollary 3.5.** Let $E$ be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, and $S$ the semigroup generated by $-A$. Assume either that $E$ is (UG) and $E^*$ is (F), or that $E$ is uniformly convex. If $\text{cl}(D(A))$ is convex, then the point of least norm in $\text{cl}(R(A))$ is the point of least norm in $\text{cl}(R(I - S(1)))$.

**Proof.** Let $T: \text{cl}(D(A)) \to \text{cl}(D(A))$ be defined by $Tx = S(1)x$. Then $\lim_{t \to \infty} S(t)x/t = \lim_{n \to \infty} T^n x/n$.

There are examples that show that Theorems 3.3 and 3.4 are not true in all Banach spaces. For example, let $E = l^1$ and $T(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$, or $E = c_0$ and $T(x_1, x_2, \ldots) = (1 + |x|, x_1, x_2, \ldots)$.

Let $A \subset E \times E$ be an accretive operator that satisfies the range condition, and let \( \{r_n\} \) be a positive sequence. Given $x_0 \in \text{cl}(D(A))$, define an "infinite product of resolvents" [31,4] by $x_{n+1} = J_{r_{n+1}} x_n$, $n \geq 0$.

**Theorem 3.6.** Let $E$ be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, and \( \{r_n\} \) an infinite product of resolvents. Assume that $\sum_{i=1}^{\infty} r_i = \infty$. Suppose either that $E$ is (UG) and $E^*$ is (F), or that $E$ is uniformly convex. Then the strong \( \lim_{n \to \infty} x_n/(\sum_{i=1}^{n} r_i) = -v \), where $v$ is the point of least norm in $\text{cl}(R(A))$.

**Proof.** This result follows from previous ideas (cf. [27, Theorem 1]) and the fact that $\{ |A_{r_{n+1}} x_n| \}$ is decreasing:

$$|A_{r_{n+1}} x_n| = |A_{r_{n+1}} J_{r_n} x_{n-1}| \leq \|AJ_{r_n} x_{n-1}\| \leq |A_{r_n} x_{n-1}| \leq \cdots \leq |A_{r_1} x_0| \leq \|Ax_0\|.$$
In some cases the conclusions of Theorems 3.3, 3.4, and 3.6 can be sharpened. For example, [1, Corollary 2.3; 6, Corollary 1.5] are now seen to be true for all closed convex $C$. Here is a general result in this direction.

**Theorem 3.7.** Let $E$ be a uniformly convex Banach space, $A \subseteq E \times E$ an accretive operator that satisfies the range condition, $C$ a closed convex subset of $E$, $T : C \to C$ a nonexpansive mapping, $v$ the point of least norm in $\text{cl}(R(A))$, and $x \in \text{cl}(D(A))$.

(a) If $A = a(I - T)$, where $a > 0$, then $\lim_{t \to \infty} dS(t) x/dt = -v$.

(b) If $A = I - T$, where $T$ is strongly nonexpansive, then $\lim_{n \to \infty} (T^n x - T^{n+1} x) = v$.

(c) If $A = I - T$, $x_{n+1} = c_n T x_n + (1 - c_n) x_n$, and $\sum_{n=0}^{\infty} c_n (1 - c_n) = \infty$, then $\lim_{n \to \infty} (x_n - T x_n) = v$.

(d) If $x_{n+1} = J_{r_{n+1}} x_n$ and $r_n \to 0$, then $A_{r_{n+1}} x_n \to v$.

(e) If $x_{n+1} = J_{r_{n+1}} x_n$, the modulus of convexity of $E$ satisfies $\delta(\varepsilon) \leq k \varepsilon^p$ for some $p \geq 2$ and $k > 0$, and $\sum_{n=1}^{\infty} r_n^p = \infty$, then again $A_{r_{n+1}} x_n \to v$.

**Proof.** (a) and (b) follow because $\lim_{t \to \infty} |dS(t) x/dt| = |v|$ and $\lim_{n \to \infty} |T^n x - T^{n+1} x| = |v|$ (see [1, Theorem 4.3; 6, Proposition 1.2]). To prove (c), let $\{x_n\}$ and $\{y_n\}$ be defined by $x_{n+1} = (1 - c_n) x_n + c_n T x_n$ and $y_{n+1} = (1 - c_n) y_n + c_n T y_n$. Let $\delta$ denote the modulus of convexity of $E$. We have $|x_{n+1} - y_{n+1}| \leq (1 - 2 \min(c_n, 1 - c_n)) \delta(|Ax_n - Ay_n||x_n - y_n|)$ $|x_n - y_n|$. Hence $2c_n (1 - c_n) \delta(|Ax_n - Ay_n||x_n - y_n|) |x_n - y_n| \leq |x_n - y_n| - |x_{n+1} - y_{n+1}|$. Since $\delta(\varepsilon)/\varepsilon$ is an increasing function of $\varepsilon$, it follows that

$$2c_n (1 - c_n) \delta\left(\frac{|Ax_n - Ay_n|}{|x_n - y_n|}\right) |x_0 - y_0| \leq |x_n - y_n| - |x_{n+1} - y_{n+1}|.$$

Summing from $n = 0$ to $\infty$, and noting that $|Ax_n| - |Ay_n| \leq |Ax_n - Ay_n|$, we see that $\lim_{n \to \infty} |Ax_n| = d$. The result follows. (d) and (e) follow from the proof of [27, Theorem 2].

In the setting of this theorem, $A$ is zero free if and only if

$$\lim_{t \to \infty} |S(t) x| = \infty$$

for all $x$, and if and only if $\lim_{n \to \infty} |x_n| = \infty$ for all initial values. There are examples that show that the statements of Theorem 3.7 are no longer true in the more general settings of Theorems 3.3, 3.4, and 3.6. (For (b), (c), and (a) consider the example in [11] and its continuous version, and for (d) and (e) consider the one in [4, p. 331].)

**Remark.** Prof. A. M. Gleason has kindly informed me that he too has a proof of Theorem 3.4 in case $E$ is uniformly convex and $c_n = 1$ for all $n$. 

4. ADDITIONAL RESULTS

In this section we present several consequences of the theorems of the
previous sections, as well as some related results.

PROPOSITION 4.1. Let $E$ be a uniformly convex Banach space,
$A \subset E \times E$ an accretive operator that satisfies the range condition,
and $S$ the semigroup generated by $-A$. If $\text{cl}(D(A))$ is convex, then

(a) $0 \in R(A)$ if and only if $S$ is bounded;
(b) $0 \in \text{cl}(R(A))$ if and only if $\lim_{t \to \infty} |S(t)x|/t$ is positive
    for each $x \in \text{cl}(D(A));$
(c) $0 \in \text{cl}(R(A))$, but $0 \notin R(A)$ if and only if $S$ is unbounded
    and $S(t)x/t \to 0$ for each $x \in \text{cl}(D(A))$.

Proof. Since $\text{cl}(D(A))$ is convex and $E$ is uniformly convex, $S$ is bounded
if and only if it has a fixed point. Since $A$ satisfies the range condition, a
point is a zero of $A$ if and only if it is a fixed point of $S$. This proves (a); (b)
and (c) follow from Theorem 3.3.

PROPOSITION 4.2. Let $E$ be a Banach space and let $A \subset E \times E$ be m-
accretive. Assume either that $E$ is (UG), reflexive and strictly convex,
or that $E$ is uniformly convex and smooth. If $P: E \to \text{cl}(R(A))$ is the nearest point
map, then $I - P$ is nonexpansive.

Proof. $P$ exists because $\text{cl}(R(A))$ is convex (Theorem 2.7). If $y \in E$ and
$B \subset E \times E$ is defined by $Bx = Ax - y$, $x \in D(A)$, then $B$ is $m$
accretive and

$$J^B_t x = J^A_t (x + ty).$$

Let $w$ be the weak $\lim_{t \to \infty} J^B_t x/t$. (See the remark after
Theorem 3.3; the limit is a strong one if $E$ is uniformly convex.) Define
$Q: E \to E$ by $Qy = y - w$. Then $|Qy - y| \leq |z - y|$ for all $z \in \text{cl}(R(A))$
and $Qy \in \text{cl}(R(A))$. Thus $Q = P$. Let $B_i$ correspond to $y_i$, $i = 1, 2$. Then

$$|(y_1 - Py_1) - (y_2 - Py_2)| \leq \liminf_{t \to \infty} |J^B_t x - J^B_t x|/t \leq \liminf_{t \to \infty} |J^A_t (x + ty_1) - J^A_t (x + ty_2)|/t \leq |y_1 - y_2|.$$

In case $A = I - T$, where $T: E \to E$ is nonexpansive, $E$ is (UG) and $E^*$
is (F), Proposition 4.2 is due to Bruck [5]. It restricts (outside Hilbert space)
$\text{cl}(R(A))$ for an $m$-accretive $A$. A restriction on $\text{cl}(D(A))$ was established in
An alternative proof of Proposition 4.2 can be based on the fact that if \( A \) is \( m \)-accretive, then \( R(A) = R(A_r) \) for all \( r > 0 \).

Our next result is valid in all Banach spaces.

**Proposition 4.3.** Let \( E \) be an arbitrary Banach space, \( C \) a closed convex subset of \( E \), \( T: C \to C \) a nonexpansive mapping, and \( x \in C \). Then
\[
\lim_{n \to \infty} \| T^n x / n \| = d(0, \text{cl}(R(I - T))).
\]

**Proof.** Let \( S \) be the semigroup generated by \(-A = T - I\). Given \( \varepsilon > 0 \), let \( y \in C \) satisfy \( \| y - T y \| < d + \varepsilon \).

Since \( \| dS(t) y / dt \| \) is nonincreasing and \( -dS(t) y / dt \in R(A) \),
\[
d \leq \lim_{t \to \infty} \| dS(t) y / dt \| < d + \varepsilon.
\]

By [1, Theorem 4.3], \( \lim_{t \to \infty} \| dS(t) x / dt \| = \lim_{t \to \infty} \| (S(t) x) / t \| = \lim_{t \to \infty} \| dS(t) y / dt \| \), so that \( \lim_{t \to \infty} \| dS(t) x / dt \| = \lim_{t \to \infty} \| S(t) x / t \| = d \). The result now follows because \( \| S(n) x - T^n x \| \leq \sqrt{n} \| x - T x \| \).

**Remark.** This result is true even if \( C \) is not convex, provided \( T: C \to C \) satisfies the range condition. It may have applications in the setting of [2]. Compare also Lemma 2.1.

Another fact which is true in all Banach spaces is related to an example of Kohlberg and Neyman [13]. Let \( E \) be an arbitrary Banach space and let an accretive \( A \in E \times E \) satisfy the range condition. Suppose that \( J_{x/n} x / t_n \rightarrow v_1 \) and \( J_{y/s} y / s_n \rightarrow v_2 \). We have \( \| (y - J_i x) / t \| \leq \| (y - J_i x) / t + r(z - A_i x) \| \) for all \( r > 0 \) and \( \| y, z \| \in A \). Therefore \( |v_1| \leq |v_1 + r(z - v_1)| \) for all \( z \in \text{cl}(R(A)) \). In particular, \( |v_1| \leq |v_1 + r(v_2 - v_1)| \) for all \( r > 0 \). In other words, there is \( j \in J(v_1) \) such that \( (v_2 - v_1, j) > 0 \). Thus \( d^2 \geq (v_2, j) \geq |v_1|^2 = d^2 \) by Lemma 2.1, so that \( J(v_2) \). Hence \( (v_1, j) = (v_2, j) = d^2 \) and \( |j| = d \). We conclude that \( v_1 \) and \( v_2 \) belong to the same face of \( B(0, d) \).

**Proposition 4.4.** Let \( E \) be a reflexive Banach space, \( T: E \to E \) an affine nonexpansive mapping, and \( x \in E \). Then the strong \( \lim_{n \to \infty} T^n x / n = z \), where \( |z| = d(0, \text{cl}(R(I - T))) \).

**Proof.** Define \( L: E \to E \) by \( Lx = Tx - y \), where \( y = T(0) \). Then \( L \) is linear and \( T^n x = L^n x + \sum_{i=0}^{n-1} L_i y \). Thus \( \{T^n x / n\} \) converges strongly to a fixed point \( z \) of \( L \) by the mean ergodic theorem. We also have \( \sum_{i=0}^{n-1} L_i' x = x - L^n x + \sum_{i=0}^{n-1} L_i' y \). Hence \( (1/n) \sum_{i=0}^{n-1} L_i' y \leq \| x - L^n x / n + x - Lx + y \| \) and \( |z| \leq \| x - Lx + y \| \). In other words, \( |z| \leq \| x - Lx - y \| = \| x - T x \| \) for all \( x \in E \). Since \(-z \) belongs to \( \text{cl}(R(I - T)) \), the result follows.

The first example mentioned after Corollary 3.5 shows that Proposition 4.4 is no longer true if \( E \) is not reflexive.

Now let \( C \) be a closed convex subset of a uniformly convex Banach space \( E \), and let \( T: C \to C \) be a (nonlinear) nonexpansive mapping. For \( x \in C \), let
$S_n x = (\sum_{i=0}^{n-1} T^i x)/n$. Suppose that $\{S_n x\}$ has a bounded subsequence. It is an open question whether this implies that $T$ has a fixed point. (This is known to be true if $E$ is Hilbert.) We observe that Theorem 3.4 shows that at least $0 \in \text{cl}(R(I - T))$ in this case. Indeed, since $S_n x = (\sum_{i=0}^{n-1} i(T^i x/i))/n$, the strong limit $\lim_{n \to \infty} (2S_n x)/(n - 1) = -v$, where $v$ is the point of least norm in $\text{cl}(R(I - T))$. Thus $\lim_{n \to \infty} S_n x/n = -v/2$ and the result follows. (This observation arose during a conversation with Bruck.)

Another application of Theorem 3.4 occurs in the following setting. Let $T_1$ and $T_2$ be two nonexpansive self-mappings of a closed convex subset $C$ of a Banach space $E$. Assume either that $E$ is (UG) and $E^*$ is (F), or that $E$ is uniformly convex. Consider the iteration $x_{2n} = (T_2 T_1)^n x_0$, $x_{2n+1} = T_1 x_{2n}$, $n \geq 0$. Let $T_3 : C \times C \to C \times C$ be defined by $T_3(x, y) = (T_1 y, T_2 x)$. $T_3$ is nonexpansive with respect to the norm $|(x, y)| = (|x|^2 + |y|^2)^{1/2}$. We also have $T_3^n(x_1, x_0) = (x_{2n+1}, x_{2n})$. Therefore $\lim_{n \to \infty} x_{2n}/n = -v_1$, $\lim_{n \to \infty} x_{2n+1}/n = -v_2$, and $\lim_{n \to \infty} (x_{2n+1}, x_{2n})/2n = -u$, where $v_1, v_2$ and $u = (u_1, u_2)$ are the points of least norm in $\text{cl}(R(I - T_2 T_1))$, $\text{cl}(R(I - T_1 T_2))$, and $\text{cl}(R(I - T_3))$, respectively. (The device of using $T_3$ is due to Lapidus [14].) Clearly $u_1 = v_2/2$ and $u_2 = v_1/2$. We conclude that

$$\inf\{|x - T_1 y|^2 + |y - T_2 x|^2 : (x, y) \in C \times C\}$$

$$= \frac{1}{4} (\inf\{|x - T_1 T_2 x|^2 : x \in C\} + \inf\{|x - T_2 T_1 x|^2 : x \in C\}).$$

If $T_1$ and $T_2$ are strongly nonexpansive, then so are $T_1 T_2$ and $T_2 T_1$ [6, Proposition 1.1]. Consequently, in this case we also have $\lim_{n \to \infty} (x_{2n} - x_{2n+2}) = v_1$ and $\lim_{n \to \infty} (x_{2n+1} - x_{2n+3}) = v_2$ by Theorem 3.7(b). Finally, we mention a result for the quasi-autonomous Cauchy problem

$$u'(t) + Au(t) \ni f(t), \quad 0 < t < \infty$$

$$u(0) = x_0.$$  

Here $A$ is an accretive operator that satisfies the range condition and $f \in L^1_{\text{loc}}(0, \infty; E)$. Suppose that this problem has a limit solution $u$ for each $x_0 \in \text{cl}(D(A))$. If $E$ is (UG), $E^*$ is (F), and $\lim_{t \to \infty} (1/t) \|f\|_{L^1(0, t; E)} = 0$, then the proofs of Theorem 3.3 and [25, Theorem 1.1] show that the strong limit $\lim_{t \to \infty} u(t)/t = -v$, where $v$ is the point of least norm in $\text{cl}(R(A))$. This includes the original result of Crandall (mentioned in [3, p. 166]), where $E$ is Hilbert and $A$ is $m$-accretive (equivalently, maximal monotone).

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Note added in proof. 1. We have recently (partially) improved upon some of the theorems of the present paper. We have, for example, established the following results (cf. Theorems 2.7 and 3.3): (a) Let \( E \) be a Banach space and let \( A \subset E \times E \) be \( m \)-accretive. If \( E^* \) is strictly convex, then \( \text{cl}(\text{R}(A)) \) is convex. (b) Let \( E \) be a Banach space and let \( A \subset E \times E \) an accretive operator that satisfies the range condition, and \( S \) the semigroup generated by \(-A\). If \( E \) is smooth and \( E^* \) is \( (F) \), then the strong \( \lim_{t \to \infty} S(t)x/t \) exists for each \( x \) in \( \text{cl}(D(A)) \).

2. For an application of Theorems 2.3 and 2.6, see the preprint by M. M. Israel, Jr. and the author entitled “Asymptotic behavior of solutions of a nonlinear evolution equation.”

3. Let \( E \) be a uniformly convex Banach space, \( A \subset E \times E \) an accretive operator that satisfies the range condition, and \( S \) the semigroup generated by \(-A\). According to Theorem 2 of a paper by A. T. Plant entitled “The differentiability of nonlinear semigroups in uniformly convex spaces,” the strong \( \lim_{t \to \infty} (x - J,t)x/t \) and \( \lim_{t \to \infty} (x - S(t)x)/t \) exist and are equal for each \( x \) in \( D(A) \). Thus the behavior of \( J \) and \( S(t) \) is similar near the origin as well as at infinity. Plant’s idea also leads to a proof of Theorem 2.5.

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