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Testing the validity of the "value of a prevented fatality" (VPF) used to assess UK safety measures

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ABSTRACT

The "value of a prevented fatality" (VPF), the maximum amount that it is notionally reasonable to pay for a safety measure that will reduce by one the expected number of preventable premature deaths in a large population, is published by the UK Department for Transport (DfT). The figure, updated for changes in GDP per head, is used by the DfT, the Health and Safety Executive and other UK regulatory bodies as well as very widely in the process, nuclear and other industries as the standard by which to judge how much to spend to reduce harm to humans. The paper tests the validity of the 1999 study on which the VPF is based and finds that that study fails numerous tests of its validity. It is concluded that there is no evidential base for the VPF that has been used for many years in the UK and is still in standard use today. Given the difficulties evident in the interpretation of survey results, an urgent re-appraisal is needed of alternative statistical methodologies that can allow robust regulatory and industry safety decision making and, vitally, give adequate protection to the UK public and to those working in the UK's transport, process, nuclear and other industries.

Keywords: Safety; Health and safety; Value of a prevented fatality; VPF; Department for Transport; Health and Safety Executive

1. Introduction

The UK Government's Department for Transport publishes the "value of a prevented fatality" (VPF), the maximum amount that it is notionally reasonable to pay for a safety measure that will reduce by one the expected number of premature deaths in a large population associated with a transport process or system (Department for Transport, 2013). Updated for increases in GDP per head, the figure is assumed to be the same for all people in the UK, irrespective of age or gender, an assumption that has been questioned elsewhere (e.g. Nathwani et al., 1997, 2008; Pandey and Nathwani, 2003; Pandey et al., 2006; Sunstein, 2004a,b; Thomas et al., 2006a,b, 2010; Thomas and Vaughan, 2013). Nevertheless the figure is important, as it is used extensively in the UK as a reference both by Government departments and by the Health and Safety Executive. Its adoption by the HSE has led to its widespread use in the UK process, nuclear and other industries as a standard for judging how much should be spent on protection measures aimed at reducing risks to life.

The UK Health and Safety Executive (HSE) has, in fact, built the Department for Transport's VPF figure into the core of its decision making process. In its statement (Health and Safety Executive, 2001) of "the protocols and procedures we follow to ensure that the process of decision-making, including risk assessment and risk management, is perceived as valid", the HSE states, in paragraph 103:

> "When an option produces the benefit of preventing fatalities, this requires putting a monetary value on achieving a reduction in the risk of death. For example, for the purpose of conducting CBAs [cost-benefit analyses], we currently take as a benchmark that the value for preventing a fatality (VPF) is about £1 000 000 (2001 figure).... This figure derives from the value used by the Department of Transport, Local

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Government and the Regions (DTLR) for the appraisal of new road schemes."

Thus the DfT VPF is the benchmark by which the HSE judges schemes to protect human life in the process, nuclear and other industries. Regarding death from cancer as causing people particular dread, HSE doubles the VPF figure when considering death from this disease (HSE, 2001, Appendix 3):

"HSE takes the view that it is only in the case where death is caused by cancer that people are prepared to pay a premium for the benefit of preventing a fatality and has accordingly adopted a VPF twice that of the roads benchmark figure."

While the Office for Nuclear Regulation has been spun out recently from the Health and Safety Executive, its requirements remain unchanged, as shown in the UK's answers to questions posed by other countries at the 2011 International Convention on Nuclear Safety (Office for Nuclear Regulation, 2011):

"This value is based on the value for preventing a fatality (VPF) applied by the UK Government when appraising road safety measures, and is set at £1m per life at 2001 prices"

It is noted in the same document that the "factor of 2 for 'dread' of cancer" is currently "under review".

The adoption by the regulators of the process and nuclear industries of the Department for Transport's VPF figure means that that figure has become *de facto* a benchmark across those industries.

A history of the development of the UK VPF figure is given in Wolff and Orr (2009), Appendix 1, where those authors conclude that

"it appears that the Carthy study is now the primary source of VPF figures, adjusted for inflation and changes in GDP."

The Carthy study (Carthy et al., 1999) consists of an opinion survey, conducted in 1997, in which 167 respondents were asked about the amounts that they would be prepared to pay to reduce the risk of death or injury in a road accident. The results were analysed using utility theory to produce a VPF figure, and this has been updated each year in line with the growth in GDP per head. Despite the fact that it has been many years since the survey was made, a 2011 report for the Department for Transport, with authors in common with the Carthy study, recommended "against any early new full scale WTP [willingness to pay] study" (Spackman et al., 2011). Thus the opinion survey of the 167 people in 1997 remains the evidential base for the VPF used by the Government, regulators and many industries in the UK today, including the process and nuclear industries.

The Carthy study interprets survey data using a method that may be described as the "two-injury chained" method, which depends on a fairly involved application of utility theory. Believing that people could not be expected to answer accurately on how much they would spend to reduce further an already low risk of a fatal injury, Carthy et al. decided to seek an estimate of the VPF indirectly, by asking people to consider a lesser injury first, with which they considered people would be better able to cope. They introduced the two-injury chained model, where the response to a serious injury of the second type, e.g. a fatal injury, is deduced after first eliciting from the person a statement of how much he would spend to reduce the probability of a lesser injury of type 1. Further questioning is then used to establish the ratio of the individual's desired spending to reduce the chances of the more serious, type 2 injury to his spending to reduce the probability of the type 1 injury. Multiplying the two values together gives the person's marginal rate of substitution (MRS) of wealth in place of non-injury probability for the type 2 injury, a process Carthy et al. call "chaining". When the type 2 injury is fatal, the individual's MRS is his personal VPF (see Eq. (A.36)), and averaging over the individuals in the sample produces the sample VPF.

The validation of the chained method is regarded by Wolff and Orr (2009) as "severely challenging":

"the method needs to make assumptions about the shape of an average person's utility function, connecting their attitudes to small risks with their attitudes to the much larger risks that are presented to the subject. Yet the testing of any such assumption, and hence the validation of the method, presents severe challenges. In order to test the chained method it appears necessary to elicit individual willingness to pay for very small changes in risk so they can be compared with attitudes to larger changes. However, if it were possible to elicit such preferences then the chained method would not be necessary; it was introduced precisely because it has so far proven impossible to elicit such preferences in any reliable way. In short, if it were possible to validate the chained method, it would not be necessary."

However, it is shown in this paper that the method may indeed be subjected to a fair test of its validity using no more data than were available at the time to Carthy et al. To do so, however, it is necessary first to generalise the mathematics of the two-injury chained model (Appendix A) and to explain the four utility functions used by Carthy et al.: Constrained Power, Logarithmic, Negative Inverse and Negative Exponential (Appendix B).

While attempting to retain many of the original symbols used by Carthy et al., it has been found necessary to extend the notation in places, particularly through the use of clarifying subscripts and superscripts.

2. Data

The authors are grateful to Professor Michael Jones-Lee for providing them with a copy of the document, "Individual Responses & VOSLS – PEG Study", HSE/Peg/Nov 1997/SC, that contains responses from the people taking part in the Carthy survey as well as the calculated marginal rate of substitution (MRS) values for the Constrained Power, Logarithmic and Negative Exponential utility functions, although, anomalously, not for the Negative Inverse utility function.

We examine the data and the processes that have been applied to them to show that there are inconsistencies in the results that are derived and that the number of responses that can reasonably be used in defining a VPF is less than the authors propose. One of the early findings is that there are discrepancies in the application of the various utility functions, with values produced by one utility function being labelled and used as if they came from a different one.

The 167 people in the Carthy survey were each given a numeric identifier in the range 1–169, with numbers 62 and 63 being unassigned. The calculations have been repeated and checked in detail with those in HSE/Peg/Nov 1997/SC (see Tables 1 and 2).

The variable, $m_{Di}^{(2)}$, is defined as individual i's MRS of wealth in place of non-fatal-injury probability – his personal

Table 1 – Comparison of computations of VPF in Carthy et al. and in this paper: 2-part chaining.

Utility function	Mean (£)	n	Sample standard deviation (£)	Standard error (£)
Carthy et al.				
Constrained Power	3.41E+06	151	2.16E+07	1.76E+06
Logarithmic	2.98E+06	151	1.99E+07	1.62E+06
Negative Inverse	2.74E+06	151	1.95E+07	1.59E+06
Negative Exponential	2.62E+06	151	1.94E+07	1.58E+06
This paper				
Constrained Power	3.40E+06	151	2.17E+07	1.76E+06
Logarithmic	2.95E+06	146	2.02E+07	1.67E+06
Negative Inverse	2.75E+06	150	1.97E+07	1.60E+06
Negative Exponential	2.69E+06	151	1.93E+07	1.57E+06

VPF – calculated by the two-injury chained method, applied once. Here the bracketed superscript, (2), indicates that "2 part chaining" from a lesser to a greater injury has been used, while the subscript, D, indicates that the greater injury is death in this case. Injury X is the lesser injury on which Carthy et al. choose to base their main calculation of $m_{Di}^{(2)}$ (see Section 3 for a definition of the injuries considered by Carthy et al.). The process involves, inter alia, the individual stating both the maximum acceptable price (MAP), x_{Xi}, he is prepared to pay to avert injury X, and the minimum acceptable compensation (MAC), y_{Xi}, he would require to make up for his receiving injury X. A utility function is then used to find from x_{Xi} and y_{Xi} the individual's MRS, m_{Xi}, of wealth in place of non-injury-X probability, as described in Appendix B. Since m_{Xi} is derived in a one-stage process rather than in a two-stage process, m_{Xi} may be written $m_{Xi}^{(1)}$ in this case.

The two-injury chaining model may be applied successively in a pairwise manner to injuries of increasing seriousness. Carthy et al. use the term "3 part chaining" to describe a process based on three injuries of monotonically increasing magnitudes: from the least serious W through X and then on to D (D for death). The two-injury chaining method is applied first to generate the MRS associated with injury X from the baseline MRS for injury W, and then the MRS just calculated for injury X is used to estimate the MRS for fatal injury, D. Hence respondents are asked to state their MAP, x_{Wi}, and their MAC, y_{Wi}, for an injury W that is less serious than injury X, which allows $m_{Wi} = m_{Wi}^{(1)}$ to be calculated. 2-part chaining is then to calculate $m_{Xi}^{(2)}$ from $m_{Wi}^{(1)}$ and then reapplied to calculate $m_{Di}^{(3)}$ from $m_{Xi}^{(2)}$. $m_{Di}^{(3)}$ is then the individual i's personal VPF calculated by "3-part chaining".

Examination of the data contained in HSE/Peg/Nov 1997/SC reveals that, in deriving the sample VPF using 2-part chaining from injury X to fatal injury D, Carthy et al. sometimes applied common values for $m_{Di}^{(2)}$ across the three utility functions for which data were made available. This substitution affects a significant fraction, about 20%, of the sample: 34 out of the 167 cases. Specifically the personal VPF coming from the Negative Exponential utility function is imposed on the Logarithmic utility function for respondent 13 and on both the Constrained Power utility function and the Logarithmic utility function for the following respondents: 6, 7, 12, 24, 27, 39, 58, 59, 70, 77, 85, 86, 88, 91, 97, 99, 100, 107, 108, 109, 113, 114, 122, 123, 128, 130, 145, 148, 152, 161, 163, 164 and 165. This anomalous substitution does not appear to have been acknowledged by the authors, nor justified.

In some instances the result found using the Negative Exponential utility function appears to have been used because no valid number would have been returned by the actual utility function specified. For example, respondent 114 gave $y_{X114} = x_{X114} = 500$, where y_{X114} is respondent 114s MAC for receiving injury X, while x_{X114} is his MAP for a system providing protection against injury X. From Eq. (B.20) derived in Appendix B, the equality, $y_{X114} = x_{X114}$, yields an incomputable, infinite value for the MRS, m_{X114}, associated with injury X, and this leads in turn to an incomputable, infinite value for $m_{D114}^{(2)}$ when the Logarithmic utility function is used. By contrast, HSE/Peg/Nov 1997/SC suggests that individual 114's personal VPF, $m_{D114}^{(2)}$, found using the Logarithmic utility, is £5045, which is the result found from using the Negative Exponential utility function.

The equality, $y_{Xi} = x_{Xi}$, occurs also in the data collected for respondents 13, 108, 128 and 130. The present authors have excluded the resulting numbers for $m_{Di}^{(2)}$, as they are not computable for the Logarithmic utility function (see Eq. (B.20)), so that the number of usable results, n, is recorded in Table 1 as $n = 146$, rather than $n = 151$, as declared by Carthy et al. for that utility function. There is a further, important issue concerning

Table 2 – Comparison of computations of VPF in Carthy et al. and in this paper: 3-part chaining.

Utility function	Mean (£)	n	Sample standard deviation (£)	Standard error (£)
Carthy et al.				
Constrained Power	3.30E+07	150	2.69E+08	2.20E+07
Logarithmic	2.50E+07	150	1.96E+08	1.60E+07
Negative Inverse	1.80E+07	150	1.22E+08	1.00E+07
Negative Exponential	1.40E+07	150	9.92E+07	8.10E+06
This paper				
Constrained Power	3.30E+07	150	2.63E+08	2.15E+07
Logarithmic	2.66E+07	138	2.00E+08	1.70E+07
Negative Inverse	1.76E+07	150	1.26E+08	1.03E+07
Negative Exponential	1.45E+07	150	9.86E+07	8.05E+06

the cases where the MAC for injury, X, is strictly less than the MAP: $y_{Xi} < x_{Xi}$, and this will be discussed in Section 5. For the present, however, and to give better comparability with the declared figures of Carthy et al., the present authors have retained results wherever they are computable.

In a similar fashion, it was found that common figures for the personal VPF, $m_{Di}^{(3)}$, found from 3-part chaining had been applied in HSE/Peg/Nov 1997/SC across all the reported utility functions for 32 cases, namely for respondents 7, 12, 13, 16, 19, 24, 58, 59, 65, 70, 77, 79, 84, 85, 86, 88, 99, 101, 112, 113, 122, 123, 124, 126, 130, 148, 150, 152, 159, 161, 163 and 169. Once again the figure found using the Negative Exponential utility function appears to have been substituted in place of the figure (or lack of one) coming from the Constrained Power and Logarithmic utility functions, without acknowledgement or justification.

For the Logarithmic utility function, it was found necessary in this paper to exclude 10 cases where $y_{Wi} = x_{Wi}$: no value of $m_{Di}^{(3)}$ was computable for respondents 13, 19, 65, 77, 84, 86, 112, 126, 130 and 148. Additional exclusions were needed for respondent 108, since $y_{W108} = \infty$, and respondent 114, since $x_{W114} = 0$ (see Eq. (B.20)). These twelve incomputable cases account for the fact that "n" for the $m_{Di}^{(3)}$ values is listed by the present authors for the Logarithmic utility function as 138 rather than the 150 declared by Carthy et al. (see Table 2).

The sample standard deviations associated with the VPF for the various utility functions were not given in Carthy et al., but that omission has been remedied in Tables 1 and 2. Fig. 1 shows a histogram of the calculated personal VPF values, $m_{Di}^{(2)}$, under the Constrained Power utility function using the two-injury chained method. The spread of personal VPFs is very extensive, from £850 to £264 M. However, a lognormal probability distribution, using a mean of £3.40 M and a standard deviation of £21.7 M, as listed in Table 1, gives a good representation of the reported data throughout this very large range, as Fig. 1 demonstrates.

3. Testing the validity of the two-injury chained method

3.1. Fundamental test

Carthy et al. consider 3 injuries, listed below in increasing order of severity:

injury W: 2–3 days in hospital, with full recovery after 3–4 months;
injury X: 2 weeks hospitalisation, with full recovery taking 18 months;
injury D: fatal injury (D for death).

Because X is more severe than W, the two-injury chained model explained in Appendix A allows an indirect estimate of person i's MRS of wealth in place of non-injury probability for injury X, m_{Xi}, to be found from his MRS in place of non-injury probability for injury W, m_{Wi}. Consider now injury D: being fatal, it is more severe than either X or W. Hence an indirect estimate of person i's MRS associated with injury D, m_{Di}, may be found from either his MRS associated with injury X, m_{Xi}, or his MRS associated with injury W, m_{Wi}.

In fact, Carthy et al. recommend a VPF based on using an individual's responses to injury X to produce an indirect estimate of his MRS associated with fatal injury D – his personal VPF. Thus person i's personal VPF, $m_{Di}^{(2)}$, is based on Eq. (1), derived and explained in Appendix A as Eq. (A.20):

$$m_{Di}^{(2)} = \frac{1-\theta_X}{\Pi_{Xi}-\theta_X} m_{Xi}^{(1)}(x_{Xi}, y_{Xi}) \tag{1}$$

The values for x_{Xi}, y_{Xi} and Π_{Xi} are elicited by questioning from the respondent: x_{Xi} is the MAP that individual i says he will pay for a measure that will avert injury, X, while y_{Xi} is the MAC he says he requires for enduring injury X. Meanwhile Π_{Xi} is his stated indifference probability of failure for an operation to counter an otherwise fatal injury that will produce a swift recovery if successful but will lead to death if unsuccessful, given that the alternative is an operation with failure probability, θ_X, that will leave him with injury X if successful and death if unsuccessful. Carthy et al. pre-set the value of θ_X at 10^{-3}.

The value of the MRS associated with injury X, $m_{Xi}^{(1)}$, used in Eq. (1) depends on which of the four utility functions put forward by Carthy et al. is used. Thus for a Constrained Power utility function (Eq. (B.12) of Appendix B, with $k = X$):

$$m_{Xi}^{(1)} = x_{Xi}\frac{\ln(x_{Xi}+y_{Xi}) - \ln x_{Xi}}{\ln 2} \tag{2}$$

while for a Logarithmic utility function (Eq. (B.20) of Appendix B, with $k = X$):

$$m_{Xi}^{(1)} = \frac{x_{Xi}y_{Xi}}{y_{Xi}-x_{Xi}} \ln\frac{y_{Xi}}{x_{Xi}} \tag{3}$$

and for a Negative Inverse utility function (Eq. (B.27) of Appendix B, with $k = X$):

$$m_{Xi}^{(1)} = \frac{2x_{Xi}y_{Xi}}{x_{Xi}+y_{Xi}} \tag{4}$$

and, finally, for a Negative Exponential utility function:

$$m_{Xi}^{(1)} = \frac{e^{\beta_i x_{Xi}}-1}{\beta_i} \tag{5}$$

where $\beta_i = (\ln\phi_{Xi})/x_{Xi}$, in which $\phi_{Xi}(x_{Xi}, y_{Xi})$ is the non-unitary solution of

$$\phi_{Xi} - 2 + \phi_{Xi}^{-(y_{Xi}/x_{Xi})} = 0 \tag{6}$$

See Eqs. (B.31), (B.33), (B.36) and (B.38).

While Carthy et al. do not present any detailed evidence to validate their two-injury chained model, it is, in fact, possible to test its validity immediately using Eq. (7) based on Eq. (A.21):

$$m_{Xi}^{(2)} = \frac{1-\theta_W}{\Pi_{Wi}-\theta_W} m_{Wi}^{(1)}(x_{Wi}, y_{Wi}) \tag{7}$$

The parameters, x_{Wi} and y_{Wi}, defined fully analogously with x_{Xi} and y_{Xi}, are the respondent's stated MAP and MAC with respect to injury W. Meanwhile Π_{Wi} is the individual's indifference probability of failure for an operation against bodily harm that will result in injury X if untreated, where that operation will bring about a rapid recovery if successful but will leave him with injury X if unsuccessful, given that the alternative is an operation with failure probability, θ_W, that will leave him with injury W if successful and injury X if unsuccessful. Carthy et al. fixed θ_W at 10^{-2}.

It may be seen that Eq. (7) is exactly analogous to Eq. (1), the basis of the VPF value recommended by Carthy et al., and is

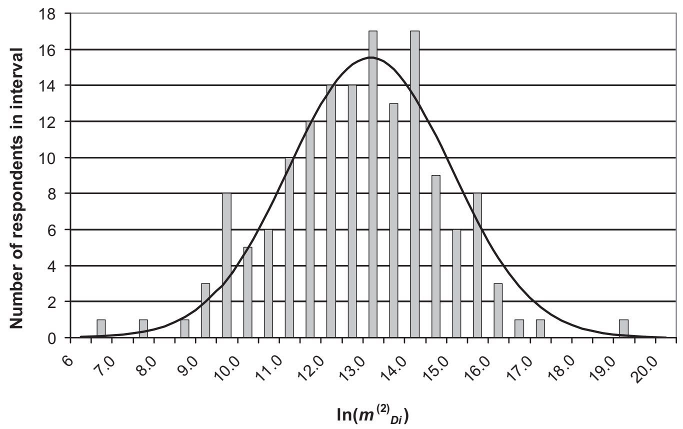

Fig. 1 – Histogram for $\ln(m^2_{Di})$: results of Carthy et al. compared with a lognormal distribution with mean = sample mean and variance = sample variance – Constrained Power utility function.

therefore an excellent vehicle for a test of their method's validity. In this regard, it is fortunate that Carthy et al. collected a full set of data: x_{Wi}, y_{Wi}, Π_{Wi} for each respondent, thus allowing a calculation of the MRS associated with injury X, m_{Xi}, from Eq. (7). The value of the MRS associated with injury W, $m^{(1)}_{Wi}$, in Eq. (7) is given by (cf. Eqs. (2)–(6)):

$$m^{(1)}_{Wi} = x_{Wi}\frac{\ln(x_{Wi}+y_{Wi}) - \ln x_{Wi}}{\ln 2} \quad \text{for a Constrained Power utility function} \tag{8}$$

$$m^{(1)}_{Wi} = \frac{x_{Wi}y_{Wi}}{y_{Wi}-x_{Wi}} \ln\frac{y_{Wi}}{x_{Wi}} \quad \text{for a Logarithmic utility function} \tag{9}$$

$$m^{(1)}_{Wi} = \frac{2x_{Wi}y_{Wi}}{x_{Wi}+y_{Wi}} \quad \text{for a Negative Inverse utility function} \tag{10}$$

$$m^{(1)}_{Wi} = \frac{e^{\beta_i x_{Wi}}-1}{\beta_i} \quad \text{for a Negative Exponential utility function} \tag{11}$$

where $\beta_i = (\ln\phi_{Wi})/x_{Wi}$, in which $\phi_{Wi}(x_{Wi}, y_{Wi})$ is the non-unitary solution of $\phi_{Wi} - 2 + \phi_{Wi}^{-(y_{Wi}/x_{Wi})} = 0$.

In addition to the four values of $m^{(2)}_{Xi}$ found from Eq. (7) there is available an independent set of estimates, $m^{(1)}_{Xi}$, one for each utility function, from Eqs. (2) to (6). It is clear that for the chained approach based on the two-injury chained model to be valid, the individual's MRS value associated with injury X should be the same or at least close, irrespective of which of the two methods is used. It should be found that $m^{(2)}_{Xi} = m^{(1)}_{Xi}$ or, equivalently, $m^{(2)}_{Xi}/m^{(1)}_{Xi} = 1$.

Unfortunately, far from matching each other closely, the m_{Xi} values calculated from the direct, one-stage method and the two-injury chained method are completely different. Table 3 shows that for each of the four utility functions, the ratio, $m^{(2)}_{Xi}/m^{(1)}_{Xi}$, is not restricted to the vicinity of 1.0 but ranges from near zero to about 100, with an average value of approximately 6, subject to the very large sample standard deviation of about 15.

Another perspective is given by Fig. 2, which shows the plot of $m^{(2)}_{Xi}$ versus $m^{(1)}_{Xi}$ under the Constrained Power utility function for the 158 data points available. The straight line plotted on the log–log graph is the best-fit straight line through the origin, $m^{(2)}_{Xi} = 8.2442m^{(1)}_{Xi}$, very different from the equation $m^{(2)}_{Xi} = m^{(1)}_{Xi}$ that the data should be supporting if the chaining method were to be validated. Moreover, the degree of linear correlation is almost non-existent: while the square of the correlation coefficient, R^2 should obey $R^2 = 1.0$ or be close, the actual value is $R^2 = 0.0719$.

The plots for the other utility functions show similar scatter, and the results are summarised in the linear equations found from minimising the squared error:

Constrained Power utility function: $m^{(2)}_{Xi} = 8.2442m^{(1)}_{Xi}$, $R^2 = 0.0719$
Logarithmic utility function: $m^{(2)}_{Xi} = 6.1887m^{(1)}_{Xi}$, $R^2 = 0.0443$
Negative Inverse utility function: $m^{(2)}_{Xi} = 4.0335m^{(1)}_{Xi}$, $R^2 = 0.0195$
Negative Exponential utility function: $m^{(2)}_{Xi} = 3.3146m^{(1)}_{Xi}$, $R^2 = 0.0079$

There is no disguising the fact that the slope violates by a very large margin its requirement to be equal to 1.0. Moreover, the plotted variables are barely linearly correlated at all: $R^2 < 0.08$ in each of the four cases. It is clear than 2-part chaining has failed comprehensively this fundamental test of the method's validity.

Based on data collected for the Carthy study, it has to be concluded that the chained method is devoid of validity.

3.2. *The "internal consistency" check advanced by Carthy et al.*

Carthy et al. presented their version of the fundamental validation test of Section 3.1 when they combined Eqs. (1) and (7) to give the individual's VPF, $m^{(3)}_{Di}$, based on 3-part chaining as:

$$m^{(3)}_{Di} = \frac{1-\theta_X}{\Pi_{Xi}-\theta_X}\frac{1-\theta_W}{\Pi_{Wi}-\theta_W}m^{(1)}_{Wi}(x_{Wi}, y_{Wi}) \tag{12}$$

and compared this with the individual's VPF, $m^{(2)}_{Di}$, based on 2-part chaining as Eq. (1), repeated below for ease of comparison:

$$m^{(2)}_{Di} = \frac{1-\theta_X}{\Pi_{Xi}-\theta_X}m^{(1)}_{Xi}(x_{Xi}, y_{Xi}) \tag{1}$$

They called this an "internal consistency" check, saying that "too wide a divergence would be cause for concern".

The only difference between comparing $m^{(2)}_{Xi}$ from Eq. (7) with $m^{(1)}_{Xi}$ found directly and comparing $m^{(3)}_{Di}$ from Eq. (12) with

Table 3 – Statistics for $m^{(2)}_{Xi}/m^{(1)}_{Xi}$.

Utility function	n	Mean	Sample standard deviation	Minimum	Maximum
Constrained Power	158	5.94	14.58	0.16	109.6
Logarithmic	145	6.30	16.17	0.16	114.5
Negative Inverse	159	5.89	14.66	0	102.8
Negative Exponential	158	5.86	14.24	0.13	99.0

$m^{(2)}_{Di}$ from Eq. (1) is that each of the former values are multiplied by $\rho_i = (1-\theta_X)/(\Pi_{Xi}-\theta_X)$ to produce the latter:

$$\begin{aligned} m^{(3)}_{Di} &= \rho_i m^{(2)}_{Xi} \\ m^{(2)}_{Di} &= \rho_i m^{(1)}_{Xi} \end{aligned} \quad i = 1, 2, 3\ldots \tag{13}$$

Hence, provided ρ_i, i = 1, 2, 3... is computable and non-zero, it will cancel when these two equations are divided:

$$\frac{m^{(3)}_{Di}}{m^{(2)}_{Di}} = \frac{m^{(2)}_{Xi}}{m^{(1)}_{Xi}} \quad i = 1, 2, 3\ldots \tag{14}$$

Thus the "internal consistency" check may be regarded as a version of the fundamental test of the two-injury chained methodology, slightly abbreviated because of the incomputability of about 10% of the ρ_i values.

The requirement for validity of the two-injury chained method is that $m^{(3)}_{Di}/m^{(2)}_{Di} = 1$, at least approximately, a condition that is completely equivalent to $m^{(2)}_{Xi}/m^{(1)}_{Xi} = 1$. The statistics for the ratios, $m^{(3)}_{Di}/m^{(2)}_{Di}$, given in Table 4 are, of course, almost identical to those of Table 3. The small differences arise from the elimination of the 16 cases where ρ_i is incomputable, a situation occurring when a respondent sets his indifference probability at $\Pi_{Xi} = \theta_X$. While a few of these instances coincide with other computability problems affecting the $m^{(2)}_{Xi}$ vs. $m^{(1)}_{Xi}$ assessment also, the overall effect is that the "internal consistency" check will have fewer comparisons to draw upon than the fundamental test detailed in Section 3.1. 144 comparisons may be made if one of the Constrained Power, Negative Inverse or Negative Exponential utility functions is used and 133 when the Logarithmic utility function is chosen as the vehicle for interpreting the survey results. But far from producing the result needed to validate the chained method, namely $m^{(3)}_{Di}/m^{(2)}_{Di} = 1.0$ (or a close approximation), Table 4 shows that the ratio, $m^{(3)}_{Di}/m^{(2)}_{Di}$, has a sample mean of about 6 and a sample standard deviation of about 15.

Fig. 3 shows a plot of the 144 comparisons possible when the Constrained Power utility function is used to interpret the opinion-survey data. The degree of scatter is similar to that of Fig. 2 when due allowance is made for the 1000-fold increase in scale compared with Fig. 2. The data points should reflect the equation: $m^{(3)}_{Di} = m^{(2)}_{Di}$, with an R^2 value of 1.0, or at least provide a close approximation. However, when $m^{(3)}_{Di}$ is plotted against $m^{(2)}_{Di}$, the best linear matches are

Constrained Power utility function: $m^{(3)}_{Di} = 3.6458 m^{(2)}_{Di}$, $R^2 = 0.0767$

Logarithmic utility function: $m^{(3)}_{Di} = 2.9032 m^{(2)}_{Di}$, $R^2 = 0.0742$

Negative Inverse utility function: $m^{(3)}_{Di} = 2.2313 m^{(2)}_{Di}$, $R^2 = 0.1024$

Negative Exponential utility function: $m^{(3)}_{Di} = 1.9145 m^{(2)}_{Di}$, $R^2 = 0.1218$

As with m_{Xi}, the two m_{Di} values are barely linearly correlated, whichever utility function is used. An intuitive feel for the sort of disparity involved is provided in Table 5, which shows the personal VPFs, m_{Di}, values for respondents 30, 31, 32, 33 and 34. The table shows that not only does each of the respondents appear to disagree dramatically with his neighbours on the value of the VPF, he appears also to have a major disagreement with himself.

Surprisingly, Carthy et al. make no acknowledgement of the internal inconsistencies their two-injury chained method generates. Those studying Fig. 3 and Tables 4 and 5 might

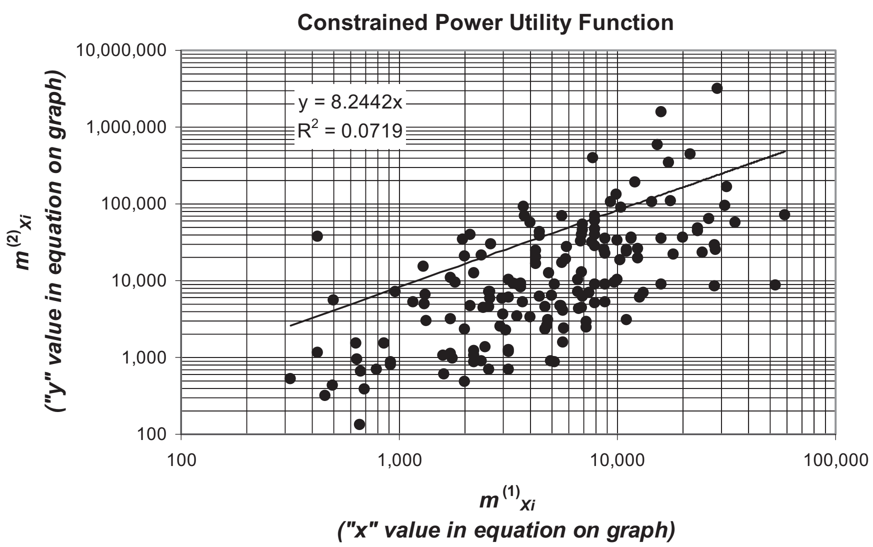

Fig. 2 – m_{Xi} values from chained approach, $m^{(2)}_{Xi}$, versus more directly calculated values, $m^{(1)}_{Xi}$: Constrained Power utility function.

Table 4 – Statistics for $m_{Di}^{(3)}/m_{Di}^{(2)}$.

Utility function	n	Mean	Sample standard deviation	Minimum	Maximum
Constrained Power	144	6.30	15.21	0.16	109.6
Logarithmic	133	6.73	16.82	0.16	114.5
Negative Inverse	144	6.30	15.33	0.16	102.8
Negative Exponential	144	6.23	14.84	0.16	99.0

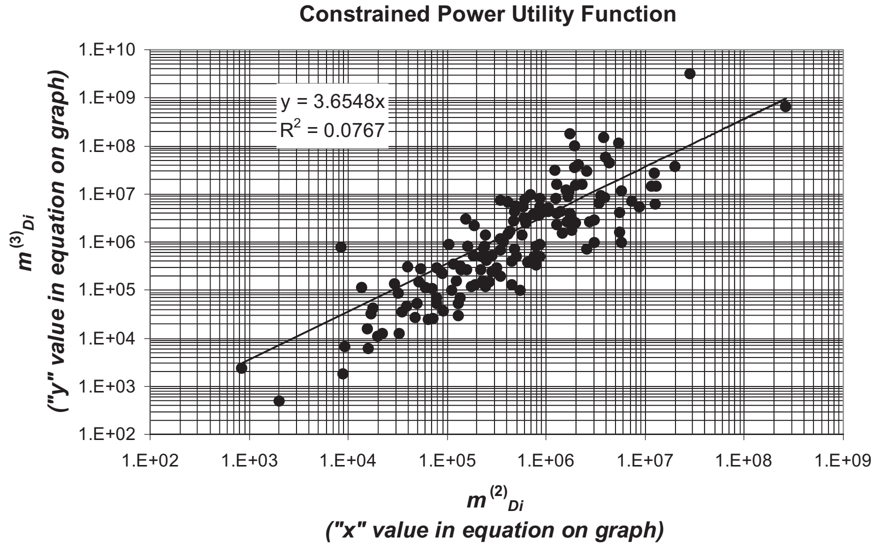

Fig. 3 – Personal VPF found from 3-part chaining, $m_{Di}^{(3)}$, against that found from 2-part chaining, $m_{Di}^{(2)}$: Constrained Power utility function.

find it difficult to understand just how wide the divergence would need to be before becoming acknowledged as a “cause for concern”. Carthy et al. content themselves with suggesting that

> “the indirect approach involves a three-part chaining process and is therefore clearly more vulnerable to compounding of errors than the more direct approach”

The suggestion is misleading, since the “internal consistency” check of Carthy et al. is actually equally a test of 2-part chaining as of 3-part chaining, for, as has been shown, the poor match between $m_{Di}^{(3)}$ and $m_{Di}^{(2)}$ is caused not by the extension from 2-part to 3-part chaining, which involves merely the use of a common multiplier (see Eq. (13)), but by the deviation of the ratio, $m_{Xi}^{(2)}/m_{Xi}^{(1)}$, from unity, where $m_{Xi}^{(1)}$ is found directly and $m_{Xi}^{(2)}$ is found through the 2-part chaining process.

The “internal consistency” check was available to Carthy et al. at the time their paper was written, putting them in a position to make a test of the validity of the two-injury chained method. As has just been shown, their two-injury chained method fails this test comprehensively.

Table 5 – Comparison of the VPF value (m_{Di}) from 2-part chaining and 3-part chaining for respondents 30, 31, 32, 33 and 34. Constrained Power utility function.

Respondent	VPF from 2-part chaining (£)	VPF from 3-part chaining (£)
30	618,569	7,713,783
31	104,726	892,040
32	707,034	9,487,290
33	131,948	29,093
34	79,387	68,020

3.3. *Comparison with a random simulation*

To put Fig. 3 into further perspective, a simulation was carried out of the comparison of the personal VPFs, $m_{Di}^{(2)}$ and $m_{Di}^{(3)}$, arising from 2-part and 3-part chaining, when the Constrained Power utility function was used as the vehicle to interpret the survey data. Random number generators were used first to simulate the VPFs, $M_{Di}^{(2)}$, found from 2-part chaining and then to simulate a corresponding set of personal VPFs, $M_{Di}^{(3)}$, found from 3-part chaining. In line with the number of computable results resulting from the application of the Constrained Power utility function, 144 comparison pairs were generated.

The enormous spread of personal VPFs, $m_{Di}^{(2)}$, reported (from £850 to £264 M) was accommodated by conditioning the random numbers using the lognormal distribution found to provide a good match to the results from the Constrained Power utility function (see Fig. 1). The $M_{Di}^{(2)}$ figures were then multiplied by a random multiplier, R_i, to give a simulation of the personal VPFs, $M_{Di}^{(3)}$, found from 3-part chaining:

$$M_{Di}^{(3)} = R_i M_{Di}^{(2)} \tag{15}$$

The 3-orders-of magnitude spread in the observed values of the personal multiplier, R_i, from 0.16 to 109.6, was modelled by once again using a lognormal probability distribution to condition the random numbers generated. Fig. 4 shows the match between the recorded number of respondents associated with given intervals of $\ln(m_{Di}^{(3)}/m_{Di}^{(2)})$ and those predicted by a lognormal probability distribution with a mean of 6.30, and a standard deviation 15.21, as listed in the top line of Table 4.

It is clear that the lognormal distribution captures the general outline and spread of the recorded values, if not their

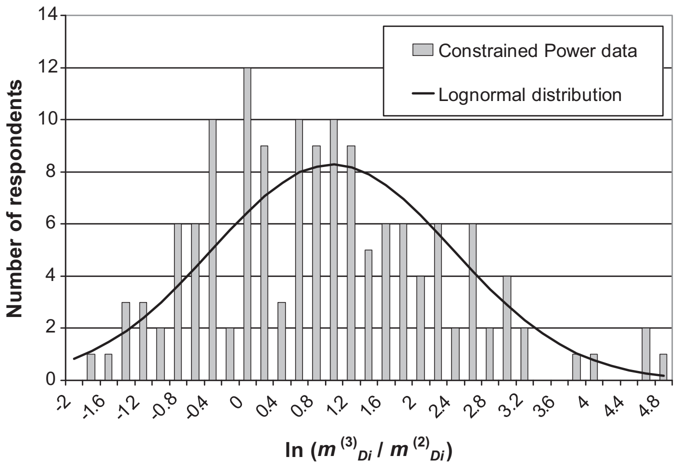

Fig. 4 – Histogram for ln $(m^{(3)}_{Di}/m^{(2)}_{Di})$: results of Carthy et al. compared with a lognormal distribution with mean = sample mean and variance = sample variance – Constrained Power utility function.

full granularity. The match is good enough to allow the values reported by Carthy et al. to be compared against those coming from a random simulation over the same range.

Fig. 5 presents the simulation results under one realisation, indicating that simulation based on random number generators can reproduce approximately the very wide spread across both axes observed in the graph of $m^{(3)}_{Di}$ versus $m^{(2)}_{Di}$ given in Fig. 3. A credible visual match between Figs. 3 and 5 is revealed. Moreover, the best-fit straight-line passing through the origin for the simulated data of Fig. 5 has the equation, $m^{(3)}_{Di} = 3.735m^{(2)}_{Di}$, close to the survey result of Carthy et al., where the best-fit line is $m^{(3)}_{Di} = 3.6548m^{(2)}_{Di}$.

The correlation for the random simulation is, however, significantly better than for the recorded results: $R^2 = 0.3331$ as compared with $R^2 = 0.0767$ for the observed data. The higher linear correlation between the randomly generated results than between the observed data of Carthy et al. is characteristic of many of the simulation realisations. This may be because of the greater regularity of the lognormal distribution as compared with the actual distribution for the multiplier, R_i, as illustrated by in Fig. 4.

It is, of course, at least questionable whether the spread of personal VPFs, $m^{(2)}_{Di}$, in the UK population in 1997 could have been as wide as the 5 orders-of-magnitude range from £850 to £264 M reported by Carthy et al., as reflected in the mean, £3.4 M, and standard deviation, £21.7 M, of the simulated VPF under the Constrained Power utility function. But, whether or not the range, £ 850 ≤ $m^{(2)}_{Di}$ ≤£ 264, 100, 000 is reasonable, it is certain that the ratio of $m^{(3)}_{Di}$ to $m^{(2)}_{Di}$ (exactly equal to the ratio of $m^{(2)}_{Xi}$ to $m^{(1)}_{Xi}$) should not have deviated appreciably from unity, and that the square of the correlation coefficient should not have fallen much below 1.0. The fact that a random realisation incorporating a lognormal distribution for the multiplier, R_i, with a mean of 6.3 and standard deviation of 15.21, is able to provide a good visual match to the graph of Fig. 3 (even showing a better correlation value) provides an intuitively understandable illustration of how far the results reported by Carthy et al. fall short of satisfying the validity test for the 2-part chaining principle. This is surely evidence of the "wide divergence" feared, but apparently not discovered by Carthy et al.

4. The wealths of the respondents

The wealth, w_i, of each respondent emerges as an intermediate result that allows the individual VPFs, $m^{(1)}_{Di}$, to be calculated for the Constrained Power, Logarithmic and Negative Inverse utility functions (see Appendix B). If the utility function methodology is correct, the wealth of each respondent will be predicted accurately. An incorrect prediction of wealth will mean that the utility function methodology used by Carthy et al. is flawed, which will mean that it cannot be relied upon for the purposes of estimating the respondent's personal VPF.

Two values of wealth may be calculated per individual for each of these utility functions, corresponding to his MAP and MAC for each of the injuries, W and X. The figures should, of course, coincide, but it is found that they are not even close. Thus the utility function methodology relied upon by Carthy et al. must be judged immediately to be flawed. Thus a further test of the method's validity is failed.

For the 112 respondents with positive wealths deduced under the Logarithmic utility function from the responses to both injury W and injury X, the best linear match between the wealth under injury W and the wealth found under injury X

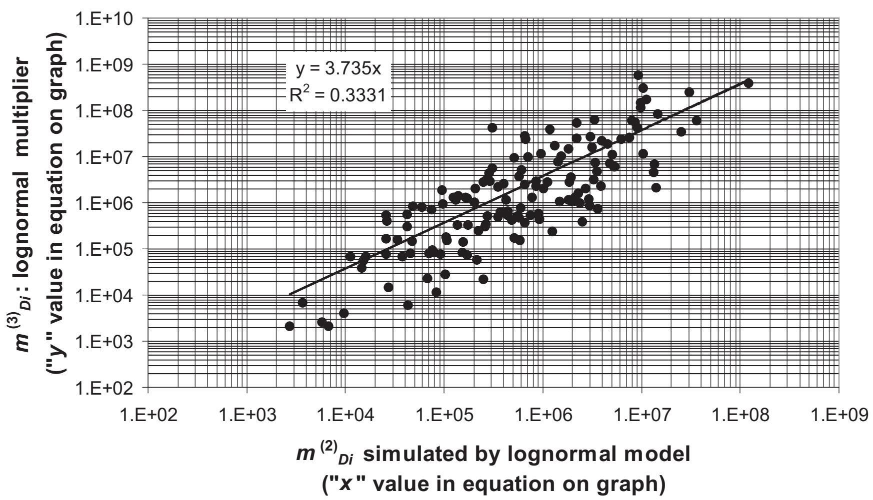

Fig. 5 – Simulation of the Constrained Power utility function results: $m^{(2)}_{Di}$ obeys a lognormal distribution; $m^{(3)}_{Di}$ is found using a log normally distributed multiplier.

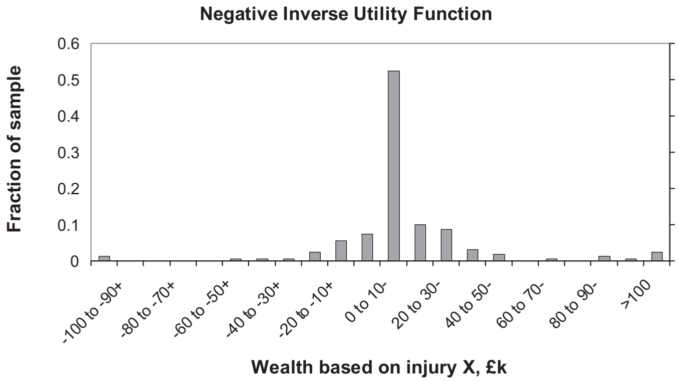

Fig. 6 – Histogram of wealths under injury *X* for the Negative Inverse utility function (notation: 50− indicates 49,999, while 50+ indicates 50,001).

(where "under" is shorthand for "based on the responses for") is:

$$w_{Wi} = 0.0984 w_{Xi} \quad R^2 = 0.1591 \tag{17}$$

It should be borne in mind that for the methodology to have validity, the equation should be $w_{Wi} = w_{Xi}$ or a close approximation, while the R^2 value should be unity or nearly so. It is clear from Eq. (17) that the actual values come nowhere near those requirements.

Since the wealth under the Negative Inverse utility function is always twice that for the Logarithmic utility function (see Eqs. (B.16) and (B.24)), Eq. (17) will apply to the Negative Inverse for the same 112 respondents. It is clear that the wealths, which should be identical, are hardly correlated.

The poor linear correlation improves for the 161 respondents where wealths may be calculated under injury W and under injury X for the Constrained Power utility function:

$$w_{Wi} = 0.2452 w_{Xi}, \quad R^2 = 0.5751 \tag{18}$$

but the slope of the line is only a quarter what it should be, were the method to be valid. The respondent's predicted wealths under injuries W and X are clearly markedly different when, of course, they should be the same.

Fig. 6 gives a histogram for wealth under injury X for the Negative Inverse utility function. Meanwhile Fig. 7 plots the wealth histogram for the Constrained Power utility function under injury X. While Fig. 6 suggests that a significant number of the respondents are heavily in debt, this conclusion is contradicted by Fig. 7, in another inconsistency.

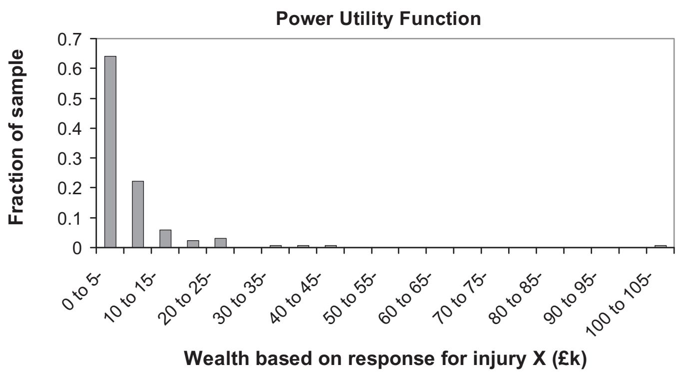

Fig. 7 – Histogram of wealths under injury *X* for the Constrained Power utility function (notation: 50− indicates 49,999).

Moreover, the low level of wealth predicted for the respondents is striking, even when the wealths under injury X (generally much higher than those under injury W) are considered. The average wealths under this injury are: £5252 for the Constrained Power utility function, £3568 for the Logarithmic utility function and £7136 for the Negative Inverse utility function. These figures are less than 10% of the average net wealth of UK adults (aged 18 and over) in 1997, given by Matheson and Summerfield (2000) as £78,300 in 1998 £s.

This discrepancy in wealth is particularly important because the VPF can be expected to be strongly increasing in wealth. For example under the Logarithmic utility function preferred by the UK Treasury, a person without heirs and no interest in charity could be expected to have the following utility function

$$m_{Di} \approx w_i \ln w_i \tag{19}$$

which is clearly strongly increasing in wealth, w_i (see Eq. (A.9) and the discussion following).

Section 3 has demonstrated already that the method used by Carthy et al. to produce a VPF is invalid. This conclusion is reinforced by the method failing to come even close to predicting the same wealth under injury X as under injury W for the same individual. But if we were to set these failures aside for the moment, and take the methods on trust up to this point, it is clear that even then the VPF figures reported by Carthy et al. would have been biased low and therefore incorrect as a result of the selection of respondents with low wealths.

If the wealths of the respondents in the surveys could be shown actually to have been much higher than the predicted wealths generated by from the utility functions used by Carthy et al. then this new inconsistency between predicted and actual wealths would cause their methods to fall once more.

Further problems associated with wealth, including negative wealths, are discussed in the next section.

5. The problems when an individual's MAC is less than or equal to his MAP

Some respondents stated a MAC value less than or equal to their MAP value, which causes a particular problem for the methods of Carthy et al., as impossible or incalculable values are implied either for the individual's wealth or for the utility of his wealth. The exploration of this situation for the different utility functions suggests that many of the responses have to be discarded.

5.1. *Logarithmic utility function*

The Logarithmic utility function is defined by Eq. (B.13), repeated below:

$$U_i(w_i) = \ln w_i \tag{B.13}$$

An obvious problem arises when an individual's MAC, y_{ki}, is equal to his MAP, x_{ki}: $y_{ki} = x_{ki}$ since, from Eq. (B.16) the individual's wealth, w_i, is calculated to be infinite, which is impossible. Similarly, $y_{ki} = \infty$, is inadmissible on grounds of impossibility. Using Eq. (B.16), wealth would appear to become negative when an individual's MAC is less than his MAP:

$y_{ki} < x_{ki}$, and become zero when $x_{ki} = 0$. But since no logarithm can be taken in either case, the Logarithmic utility function cannot exist (Eq. (B.13)). This means that, while it may be possible to calculate a zero or negative wealth from Eq. (B.16), the theory on which its validity is based has collapsed.

Thus the evidence of the following 36 respondents concerning injury X is ruled out on the grounds that $y_{Xi} \le x_{Xi}$: 6, 7, 8,10, 12, 13, 24, 27, 39, 58, 59, 65, 70, 77, 78, 85, 86, 88, 90, 91, 97, 99, 100, 103, 107, 113, 114, 122, 128, 130, 145, 148, 149, 152, 161 and 163. In addition, $y_{X108} = \infty$, meaning that m_{X108} is not computable.

When considering calculating $m_{Di}^{(2)}$ values, based on 2-part chaining, it needs to be borne in mind that no computable value of $\rho_i = (1 - \theta_X)/(\Pi_{Xi} - \theta_X)$ exists when $\Pi_{Xi} = \theta_X$, as was the case with 16 respondents, specifically respondents 3, 8, 10, 46, 78, 83, 90, 93, 95, 103, 115, 118, 137, 149, 153 and 155. After taking out the 6 respondents who appear in more than one list based on injury X, namely 8, 10, 78, 90, 103 and 149, this implies that $36 + 1 + 16 - 6 = 47$ of the 167 responses cannot be interpreted using the 2-part chained approach based on the Logarithmic utility function. This leaves only 120 of the original 167 responses as meaningful for any attempt to estimate a VPF based on 2-part chaining using the Logarithmic utility function.

As noted above, Carthy et al. allowed themselves the freedom to substitute results derived from the Negative Exponential utility function into the column of the Logarithmic utility on a good many occasions. This might raise the thought that the Negative Exponential utility function would be robust against cases when $y_{ki} \le x_{ki}$, but this is not so, as will now be shown.

5.2. *Negative Exponential utility function*

The Negative Exponential utility function obeys Eq. (B.28), repeated below:

$$U_i(w_i) = -e^{-\beta_i w_i} \quad \beta_i > 0 \tag{B.28}$$

It shown in Appendix B that the only admissible values for the ratio of MAC to MAP, $c_{ki} = y_{ki}/x_{ki}$, are those in the range $1 < c_{ki} \le \infty$, while ϕ_{ki} is constrained to $1 < \phi_{ki} \le 2$. Interestingly, the response $y_{X108} = \infty$ (or at least $y_{X108} \to \infty$) can be coped with, since $\phi_{X108} \to 2$.

Thus for the Negative Exponential utility function, the same set of 36 respondents listed in Section 5.1 for whom $y_{Xi} \le x_{Xi}$ are still ruled out. This means that estimating a VPF using 2-part chaining and the Negative Exponential utility function can be based validly on only 121 out of the 167 respondents.

5.3. *Negative Inverse utility function*

The Negative Inverse utility function takes the form of Eq. (B.22), repeated here:

$$U_i(w_i) = -\frac{1}{w_i} \tag{B.22}$$

The wealth is given by Eq. (B.24), so that $w_i = \infty$ when $y_{Xi} = x_{Xi}$, which is unrealistic – no-one has infinite wealth. Meanwhile $y_{Xi} < x_{Xi}$ implies negative wealth, which leads the utility function to behave in an anomalous and unrealistic way. For the utility of having a negative wealth of $-P$ pounds, $P > 0$, viz. owing P pounds, will always be greater than the utility of owning P pounds:

$$U(-P) = \frac{1}{P} > U(P) = -\frac{1}{P} \tag{20}$$

For this reason the Negative Inverse function can be used for utility calculations only when the wealth stays strictly positive.

Note that in Eq. (B.27) as $y_{ki} \to \infty$ so $m_{ki} \to 2x_{ki}$. Hence the response $y_{X108} \to \infty$ can be accommodated. But the 36 respondents with $y_{Xi} \le x_{Xi}$ are ruled out, and the 2-part chaining calculation must proceed on the basis of only 121 respondents.

5.4. *Constrained Power utility function*

The Power utility function is given by Eq. (B.1), repeated below

$$U_i(w_i) = w_i^{1-\varepsilon_i} = w_i^{s_i} \tag{B.1}$$

This utility function, too, is designed only for non-negative wealths, but Carthy et al. circumvent the problem of negative wealths by constraining the individual's wealth to be the same as his MAP:

$$w_i = x_{ki} \tag{B.6}$$

Hence the name, "Constrained Power utility function".

The assumption behind Eq. (B.6) is that the person will be prepared to give up all his wealth to avoid an injury, be it injury X or the less serious injury W. The rationale behind this assumption is not made clear, and it introduces an immediate contradiction, since, in general, the person is prepared to spend more to avert a more serious injury, so that $x_{Xi} > x_{Wi}$, implying that Eq. (B.6) cannot be true. This is a built-in logical failure: the individual's wealth is assumed to change depending on which injury he is contemplating, an impossibility.

Moreover, the assumption invites the further questions: if every individual is prepared to give up all his wealth to avoid an injury "of lesser severity" and "involving no permanent injury", how much wealth will each person be willing to give up to avoid death? and if much more, where is that money coming from?

Nevertheless, artificial and contradictory though it might be, the constraint of Eq. (B.6) does ensure that the Constrained Power utility function does not encounter negative wealths. So is the problem solved? It turns out that the artifice preventing negative wealths merely transfers the problem elsewhere. Noting that $s_i = 1 - \varepsilon_i$, where ε_i is risk-aversion, we may rewrite Eq. (B.11) as:

$$\varepsilon_i = \frac{1}{\ln(1 + y_{ki}/x_{ki})}\left(\ln\left(1 + \frac{y_{ki}}{x_{ki}}\right) - \ln 2\right) \tag{21}$$

revealing that the individual will be risk-neutral, with $\varepsilon_i = 0$, if $y_{ki} = x_{ki}$ and risk-seeking, $\varepsilon_i < 0$, if $y_{ki} < x_{ki}$. Neither descriptor is likely to be a realistic characterisation of someone taking decisions on serious injury, involving hospitalisation and a recovery period of 3–4 months in the one case and fully 18 months in the other. More probable risk-aversions are likely to be between 0.8 and 0.9, as found by Thomas et al. (2010), or approaching unity, as recommended by the UK Treasury (2011), when the Power utility function is transformed into the Logarithmic utility function.

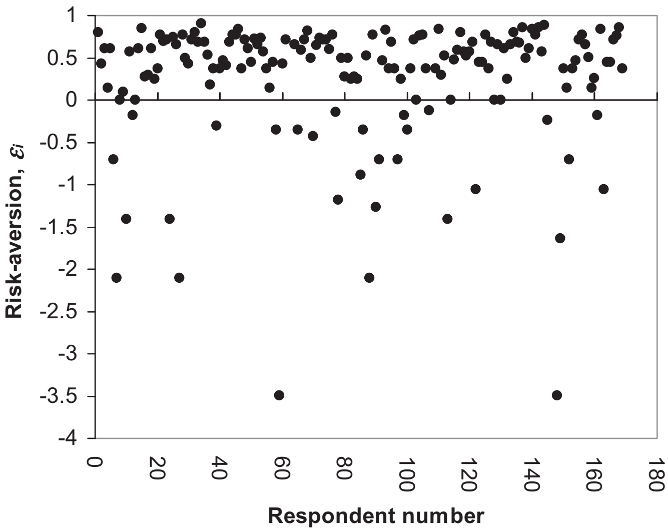

Fig. 8 – Risk-aversions, ε_i, for the Constrained Power utility function.

Fig. 8 shows the risk-aversions calculated using Eq. (21) for the Constrained Power utility function applied to the responses concerning injury X. The anomalous behaviour for the 36 cases listed in Section 5.1, where $y_{ki} \le x_{ki}$, shows up in zero or negative risk-aversions, indicating risk-neutral or risk-seeking decision making of a reckless kind.

It may be concluded that the results associated with these idiosyncratic risk-aversions cannot be relied upon, suggesting that the analysis method cannot apply to the aforementioned 36 respondents with $y_{ki} \le x_{ki}$. Moreover, the case of $y_{X108} = \infty$ cannot be accommodated as $m_{X108} \to \infty$ and cannot be computed. Hence, once again the 2-part chaining calculation will be constrained to proceed on the basis of only 120 respondents.

6. Effects of removing the cases where MAP ≤ MAC

As shown above, many responses that have been included in the Carthy study need to be dropped, including some where Carthy et al. avoided exclusion by erroneously substituting results from one utility function into those of another.

6.1. *The effect on calculated average wealth*

Removing the inadmissible cases discussed in Section 5 on the larger, injury-X-based estimates of wealth has the effect of decreasing the average wealth under the Constrained Power utility function from £5252 to £3873, but increases it from £3568 to £8516 under the Logarithmic utility function and from £7136 to £17,032 under the Negative Inverse utility function. Thus the estimates of average wealth of the survey respondents start to diverge, indicating a further problem, since it is an impossibility for a respondent's instantaneous wealth to be multiple-valued.

But note that all the revised averages of wealth remain far below the actual wealth of an average UK adult at the time of the survey (£78,300, 1998 £s).

6.2. *The effect on the "internal consistency" check*

The "internal consistency" check has to work with fewer data points than the fundamental test of the two-injury chained method of Section 3.1, but poses a similar test for the validity of 2-part chaining, as discussed in Section 3.2. In fact, stripping out the inadmissible cases discussed in Section 5 reduces the number of feasible comparisons to just 102 under all four utility functions – the evidence of 65 out of the original 167 respondents has to be ignored. But no improvement is seen in the "internal consistency" check when applied to this smaller number of valid cases. The best linear matches are almost unchanged at:

Constrained Power utility function: $m_{Di}^{(3)} = 3.6747 m_{Di}^{(2)}$ $R^2 = 0.0724$

Logarithmic utility function: $m_{Di}^{(3)} = 2.9053 m_{Di}^{(2)}$ $R^2 = 0.0707$

Negative Inverse utility function: $m_{Di}^{(3)} = 2.2355 m_{Di}^{(2)}$ $R^2 = 0.0983$

Negative Exponential utility function: $m_{Di}^{(3)} = 1.915 m_{Di}^{(2)}$ $R^2 = 0.1177$

The slope remains well away from the value of 1.0 required for validity, and the linear correlation is again almost non-existent.

The mean VPFs, standard deviations and standard errors remain similar but slightly higher (see Table 6).

7. Censoring of the data

One feature of Carthy et al. is their ambivalence between the mean and the median as a measure for consolidating different people's varying valuations deduced from an opinion survey: "Estimates of the VOSL for road risks – taken as the mean or median of m_D... – are reported" is the almost throw-away introduction to Section 3.3 (VOSL, "value of a statistical life" is used synonymously with VPF).

However, it has been shown (Thomas, 2014) that the median violates the conditions for structural view independence, which requires that the algorithm for consolidating human views into a single figure should contain no in-built structural bias that would render some people's views less important than others. In the case of the median, attention is paid to the view of one person only or the views of two persons at most: it is the ultimate in double-sided "trimming". The effect of someone else in the sample changing his view, possibly by a lot, will be zero, unless his view happens to supplant the old median. Indeed, out of the general, nonlinear, increasing and differentiable transformations that may be applied to the consolidation of views, only the linear transformation satisfies the requirement of structural view independence and leads to the sample mean. Use of the sample mean brings the analyst the considerable advantage that he will be able to refute any charge of lack of objectivity or bias in his consolidation of the sample views into a single figure.

In the case under examination, the distribution of the VPF found from 2-part chaining using the Constrained Power utility function is revealed by simple calculations to be approximately lognormal (see Fig. 1 and the end of Section 2), which implies that the ratio of the mean to the median is $e^{\sigma^2/2}$. Thus the mean VPF must be bigger than the median VPF whenever $\sigma^2 > 0$, that is to say wherever there is any variation whatsoever between the opinions of the respondents. It will be much bigger when there are large divergences in personal VPFs. In the present instance, the ratio of mean to median is about 6.5. This ratio is reduced a little by the decision of Carthy et al., for the purposes of calculating the median only, to restore the 16

Table 6 – Results from 2-part chaining when the infeasible cases discussed in Section 5 are removed.

Utility function	Mean (£)	n	Sample standard deviation (£)	Standard error (£)
Constrained Power	3.88E+06	120	2.42E+07	2.21E+06
Logarithmic	3.34E+06	120	2.22E+07	2.03E+06
Negative Inverse	3.02E+06	121	2.18E+07	1.98E+06
Negative Exponential	2.85E+06	121	2.15E+07	1.95E+06

respondents censored from their one-sided, "trimmed mean" calculation, but only to 6.2. So when Carthy et al. declare that

> "it is our view that policy recommendations should be based on the untrimmed medians and trimmed means",

they are granting themselves the freedom to reduce their recommended VPF by a factor of up to 6 below the mean VPF found from all computable results.

The mean VPF based on all computable results, £3.40 M under the Constrained Power utility function, is itself based on discarding the views of the 10% of the sample population who, when faced with the alternative possibility of even a low probability of death, choose to regard the effect of the lesser injury, X, as having a negligible effect on their enjoyment of life and so put $\Pi_X = \theta_X$ (see the discussion at the end of Appendix A.4). But Carthy et al. decide to carry out a further, unilateral "trimming" of the mean by disregarding the views of two further people: respondents 132 and 13, asserting that their opinions are "extreme outliers", although they advance no justification for this view. Table 7 shows the 5 highest and the 5 lowest personal VPFs under each of the utility functions used by Carthy et al. It is difficult to see the personal VPF of respondent 13 as an outlier when it is not even the second highest view under the Constrained Power utility function. Nor is it clear that the VPF of respondent 13 is "extremely" different from those of respondents 51 and 152 under the Negative Inverse and Negative Exponential utility functions.

Looking now at the bottom 5 VPFs, it is noteworthy that Carthy et al. do not regard as extreme the personal VPF of respondent 111, who apparently values human life at between £742 and £850 a factor of over 1000 below their final recommendation of £1 M – roughly the price of an everyday dining table and 4 chairs. But interestingly, in this they are correct: their mistake is to regard the personal VPF of respondent 132, £264 M, as an outlier.

Fig. 1 shows how well a lognormal distribution is able to model the reported VPFs results under 2-part chaining and the Constrained Power utility function, incorporating a mean and a standard deviation supplied by the Carthy survey. Running 400 simulations of groups of 167 respondents using this model allows the average maximum personal VPF, $\overline{\max_{167}(m_{Di}^{(2)})}$, to be calculated as £131.5 M, with a sample standard deviation of £182.6 M. Fig. 9 shows that the probability distribution is asymmetrical, with a long tail. 90% of the groups of 167 simulated produced a maximum personal VPF in the range: $£\,30.7M \leq \max_{167}(m_{Di}^{(2)}) \leq £\,356.8M$, which may be regarded as a 90% empirical probability interval for the maximum personal VPF in the sense that 5% of the 400 groups simulated returned a value of $\max_{167}(m_{Di}^{(2)})$ that was below £30.7 M while 95% of the groups produced a value of $\max_{167}(m_{Di}^{(2)})$ below £356.8 M. Clearly this range includes comfortably the personal VPF of £264 M of respondent 132. In fact, just under 9% of the simulations based on the evidence of Carthy et al. yielded a maximum personal VPF of more than £264 M.

Meanwhile the average minimum personal VPF, $\overline{\min_{167}(m_{Di}^{(2)})}$ turned out to be £3840, with a sample standard deviation of £2605. On the evidence of 400 simulations, the 90% empirical probability interval for the minimum personal VPF, $\min_{167}(m_{Di}^{(2)})$, in a survey of 167 people based on the two-injury chained model was between £805 and £8779, which includes the personal VPF of £850 of respondent 111.

Thus both the low and high ends of the reported range for personal VPFs, £850 and £264,100,000, are included in their respective 90% empirical probability intervals, derived from the evidence of the Carthy survey. Therefore there would appear to be no reason to see either of the ends of the reported range for personal VPFs as other than credible outputs of the two-injury chained model. There is thus no justification for any "trimming", which Carthy et al. implemented always in the direction of reducing the VPF.

Since no trimming is justified, one might perhaps expect the VPF figure coming from the Carthy study to be based on Table 1 of this paper, or, after accounting for the inadmissible nature of some of the data, as discussed in Section 5, in Table 6. The recommended VPF might then have been based the average of the four mean VPFs listed in Table 6, that is to say about £3.3 M rather than £1 M.

But this does not capture the very large degree of person to person variation suggested by the Carthy study. Interestingly, Carthy et al. attempt to quantify this by considering possible variations in the MRS of wealth in place of non-probability of injury X using a geometric approach to the utility function. After (unjustifiably) trimming out the views of respondents

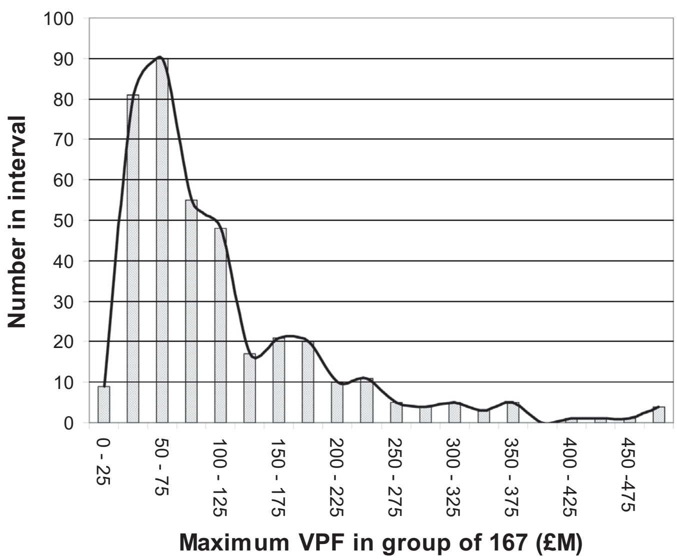

Fig. 9 – Histogram for the maximum personal VPF in a group of 167 respondents based on 400 simulations.

Table 7 – Top 5 and bottom 5 personal VPFs: all computable $m_{Di}^{(2)}$.

Constrained Power		Logarithmic		Negative Inverse		Negative Exponential	
Respondent no.	Personal VPF (£)	Respondent no.	Personal VPF (£)	Respondent no.	Personal VPF (£)	Respondent no.	Personal VPF (£)
Top 5							
132	264.1 M	132	243.0 M	132	239.8 M	132	236.6 M
51	28.89 M	51	21.23 M	13	19.98 M	13	19.98 M
13	19.98 M	152	13.85 M	51	14.68 M	152	12.83 M
76	13.16 M	112	10.31 M	152	13.32 M	51	11.53 M
112	12.68 M	76	9.45 M	112	9.22 M	112	8.35 M
Bottom 5							
84	8923	99	9099	67	7280	67	5775
99	8644	84	8211	114	5046	114	5046
114	5045	140	6520	140	3013	140	2206
79	2020	79	1664	79	1515	79	1391
111	850	111	774	111	758	111	742

132 and 13, in addition to the 16 respondents whose views lead to the failure of their model, they come up with an upper bound of £5.76 M. However, apparently not regarding their analysis as sufficiently robust to stand on its own, Carthy et al. then dismiss their figure for lack of prior precedent: "to the best of our knowledge, no-one has argued for a VOSL for road risks in the UK in excess of £5.0 × 10^6".

But an alternative and easy procedure presents itself immediately, namely to estimate the upper bound of the 90% confidence interval for the average VPF as the mean VPF plus 1.645 × the standard error. Using only the figures listed in Table 2 of Carthy et al. (repeated in Table 1 of this paper), the average across all utility functions for the upper bound of the 90% confidence interval for the mean VPF is £5.63 M. Using the figures listed in Table 6 of this paper, that figure rises to £6.63 M. These simply calculated values are strikingly close to the estimate of £5.76 M dismissed by Carthy et al. So while Carthy et al. might not have been arguing for a VPF in excess of £5 M, a reasonable interpretation of their results points in that direction.

8. Discussion

Believing that people could not be relied upon to say how much they would be prepared to spend to reduce the already low probability of a fatal injury (because of a presumed inability to make such a computation), Carthy et al. invented the two-injury chained model, whereby the response to a serious injury (a class that includes fatal injuries) is deduced in a two-stage process. In the first stage, the questioner tries to elicit how much the person would be prepared to spend to reduce the probability of a less serious injury. Then, in the second stage, an attempt is made to find the respondent's acceptable ratio of how much should be spent to reduce the chances of the more serious injury to the comparable spend against the lesser injury by asking him to estimate an indifference probability. The method depends on a fairly involved application of utility theory, at the end of which the two stages are linked in a process called "2-part chaining" to produce the personal VPF of the respondent.

The proposition is at least questionable ab initio that people should be better equipped to give accurate monetary values and indifference probabilities associated with non-fatal injuries if they could not perform the arguably simpler task of putting a value on avoiding a fatality. What is clear is that the views of respondents in Part 1 of the study (Beattie et al., 1998) did not conform to the expectations of Carthy et al., who felt that there was:

> "a failure on the part of many respondents to take adequate account of the magnitude of the risk reduction in CV [contingent valuation] questions aimed at estimating individual marginal rates of substitution of wealth for risk."

Even so, Carthy et al. felt it necessary in Part 2 of their study to reject numerous results that did not meet their conceptions, as embodied in the two-injury chained model they developed.

But whatever the hopes and expectations held out for the new technique, using only data from the Carthy survey, Section 3.1 of this paper shows that the two-injury chained method fails by a very large margin the fundamental test necessary for it to be valid. It follows that the recommendations of the Carthy study have no valid evidential base.

Moreover, although Carthy et al. did not perform the fundamental test of Section 3.1, largely equivalent results were available to them from the "internal consistency" check that they performed, as explained in Section 3.2. Of course the two-injury chained method fails that internal consistency check as comprehensively as it failed the fundamental test.

Section 3.3 shows how random number generators can be used to simulate the values of the personal VPFs, $m_{Di}^{(2)}$ and $m_{Di}^{(3)}$, arising from 2-part and 3-part chaining when the Constrained Power utility function is used to interpret the survey results. Whether or not the spread of personal VPFs in the UK population in 1997 was as wide as the 5 orders-of-magnitude range reported by Carthy et al. based on 2-part chaining, it is certain that the ratio of $m_{Di}^{(3)}$ to $m_{Di}^{(2)}$ should not have deviated appreciably from unity, and that the square of the correlation coefficient should not have fallen much below 1.0. But a random realisation incorporating a lognormal distribution for the multiplier, $R_i = M_{Di}^{(3)}/M_{Di}^{(2)}$, with a mean of 6.3 and standard deviation more than twice that figure, was able to provide not only a good visual match to the reported results for $m_{Di}^{(3)}$ and $m_{Di}^{(2)}$ but also a better linear correlation. The fact that a random simulation is able to perform better casts a further shadow on the credibility of the results reported by Carthy et al.

If, for the sake of argument, judgement on the invalidity of the two-injury chained method is suspended temporarily, we should expect the wealths predicted for the same person to be the same under injury X as under injury W. They turn out to be different by a factor of between 4 and 10 for each of the three utility functions for which wealth can be calculated: Constrained Power, Logarithmic and Negative Inverse. Thus another test of the methodology of the Carthy study is failed.

If all the tests failed up to this point are put to one side, an examination of the average wealths under injury X (which are much higher than those under injury W) shows that these are far too low to be representative of the UK population in 1997 – less than 10% of the actual average wealth in that year. Because the VPF is strongly increasing in wealth, this implies that the figure for VPF produced by Carthy et al. will be biased low and must be incorrect on these grounds alone.

The methodology used by Carthy et al. means that they are obliged to declare a significant number of cases incomputable. They suggest 16 incomputable results for their calculation of personal VPFs from 2-part chaining, associated with the 16 respondents who set their indifference probability, Π_{Xi}, at $\Pi_{Xi} = \theta_X = 10^{-3}$, implying that, when faced with an alternative possibility of death, such respondents decide that their enjoyment of life will not be adversely affected by injury X. Given the adaptability of human nature, it is not unreasonable that people should make such a judgement. In fact, the figure of 16 assumes the use of a different utility function on occasion to supply alternatives to otherwise incomputable parameters. Thus the true number of incomputable results rises to 21 rather than 16 when the Logarithmic utility function is used.

But further problems arise for all four utility functions used in Carthy et al. when the individual's minimum acceptable compensation (MAC) is less than or equal to his maximum acceptable price (MAP). As shown in Section 5, this causes the number of cases that are either incomputable or infeasible to rise to 56 or 57, no less than a third of the original sample of 167. The effect of removing the extra, infeasible cases is to raise the VPF calculated using the 2-part chaining by about 10%. But there is no improvement in the "internal consistency" check, and no confidence can be expressed in these (slightly different) results any more than in the originals.

A feature of the Carthy et al. study is the freedom the authors allow themselves to censor their results beyond the nominally 16 incomputable cases by excluding further cases when their personal VPF is high – one-sided "trimming". But, as shown in Fig. 1, the Carthy results for personal VPFs are modelled well by a lognormal distribution with a very large spread. Simulations carried out on the basis of the representative lognormal distribution suggest that the 90% empirical probability interval for the maximum personal VPF in a sample of 167 is between £30.7 M and £356.8 M under the Constrained Power utility function, a range that easily includes the highest personal VPF reported by Carthy et al., namely £264.1 M, meaning that there is no case for censoring the opinion of respondent 132, whose personal VPF this is. There is certainly no case for censoring the opinion of respondent 13, whose personal VPF is £19.98 M, a figure exceeded by the maximum personal VPF in all but 4 out of the 400 simulations (the lowest maximum VPF in all 400 simulations was £14.32 M). Given that Carthy et al. decide to exclude the view of respondent 13 (although, surprisingly, not the opinion of respondent 51), there would appear to be a 99% chance that they would regard the maximum personal VPF as an outlier and hence a candidate for censorship in whichever group of 167 people they chose to survey.

Although the detailed analysis has been applied to the results as analysed under the Constrained Power utility function, the general similarity of the personal VPFs under all four utility functions suggests that there is no case for censoring the opinion of any respondent, no matter which utility function is used.

When the censored opinions are restored, the mean VPF based on all computable and feasible results from the Carthy study emerges as £3.27 M. The average VPF is, however, subject to a wide 90% confidence interval for the mean: between £0.061 M and £6.63 M, when averaged across all four utility functions. These figures are of somewhat academic interest because they are based on a method shown to have failed several validity tests comprehensively.

9. Conclusions

The two-injury chained model proposed by Carthy et al. has been subjected to several tests of its validity. Failure in just one test would have been enough to invalidate the model, but, in fact, numerous tests have been failed comprehensively. Therefore no reliance can be placed on the Carthy study, in which so many flaws and inconsistencies have been identified both in the main text and in the appendices of this paper.

This finding has particular significance for the UK because, as noted by Wolff and Orr, the Carthy study provides the underpinning for the VPF figure used by the Department for Transport, by the Health and Safety Executive and by other UK regulators. The fact that the VPF figure in such widespread use has been shown to be devoid of a valid evidential base raises questions about the many decisions that have been made based upon it in the past 15 years, relating to the safety of road and rail development and to safety in the process, nuclear and other industries in the UK.

Looking to the future, the uncovering of the serious flaws in the method used to estimate the VPF currently used is important since that figure forms the basis by which cost-effectiveness continues to be judged for many health, safety and environmental protection measures. While the goal must be to allocate resources in an equitable and consistent manner, the severe methodological shortcomings identified here may well have led to the true value of the UK's VPF being underestimated. This would imply that the safety of UK citizens has not been and is not being protected properly.

If it is desired to continue with the stated preference approach to valuing human life, then the statement in Spackman et al. (2011) recommending "against any early new full scale WTP [willingness to pay] study" requires reconsideration. A key requirement for those who might undertake a new opinion survey is that they are fully seized with the fundamental importance of careful analysis and painstaking interpretation of survey evidence.

However, given the difficulties evident in the interpretation of survey results, active consideration should be given to methods of valuing human life in the UK that offer an alternative to stated preference techniques. An urgent re-appraisal is needed of other statistical methodologies that can allow robust regulatory and industry safety decision making and, vitally, will ensure that UK workers and public receive adequate protection from industrial and transport hazards.

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Appendix A. The application of the two-injury chained model

A.1. *The two-injury chained model*

The analysis presented in Carthy et al. is based on expected utility theory. Let w_i be the starting wealth of an individual, i, and $U_i(w_i)$ be his utility of wealth when he is in good health. Possible forms of his utility function, $U_i(\cdot)$, will be discussed in Appendix B. Now consider an injury, k, that will reduce the individual's utility of wealth to $I_{ki}(w_i)$, $k=1, 2, \ldots$, with more serious injuries having a higher index value, so that $U_i(w_i) \geq I_{k_1 i}(w_i) \geq I_{k_2 i}(w_i)$ for $k_2 > k_1$. A simple model for the utility in the injured condition is provided by Carthy et al. (1999):

$$I_{ki}(w_i) = U_i(w_i) - a_{ki} \quad a_{ki} \geq 0, \quad k = 1, 2, \ldots \tag{A.1}$$

where a_{ki} may be termed the injury offset, particular to the injury and to the individual. Eq. (A.1) is claimed by Carthy et al. to apply provided the injuries are not too severe: "e.g. those involving permanent disability", suggesting that they would regard it as no longer valid when the injury is fatal. But note that the specific form of this equation is critical to the analysis in Carthy et al. because it renders equal the derivatives of utility in the (non-fatally) injured and non-injured states:

$$\frac{dI_{ki}(w_i)}{dw_i} = \frac{dU_i(w_i)}{dw_i} \quad k = 1, 2, \ldots \tag{A.2}$$

This is no longer the case if Eq. (A.1) is replaced by $I_{ki}(w_i) = \gamma_{ki} U_i(w_i)$; $0 \leq \gamma_{ki} < 1$, on the face of it preferable and at least equally plausible, when $dI_{ki}(w_i)/dw_i = \gamma_{ki} dU_i(w_i)/dw_i$. This would prevent the cancellation in Eq. (A.5) necessary for the approach of Carthy et al. to work, indicating a weakness in the method.

The main vehicle used by Carthy et al. is the two-injury chained model, under the assumption that respondents will be capable of making more accurate economic choices concerning the less serious injury. It is supposed that, over some period of time, an individual faces 3 mutually exclusive possibilities: (i) he may incur injury 1, with probability q_1, or (ii) he may incur injury 2, with probability q_2 or (iii) he may continue in full health. Carthy et al. ignore the possibility that the individual will fall ill or die from other causes during the interval, effectively assuming that the probability of disease or death from other causes is negligible at all ages. Since in reality the possibility of both becomes ever more likely the greater the age of the individual, this assumption constitutes a further weakness in the two-injury chained model.

Pursuing the approach of Carthy et al., the individual's utility of wealth will depend on the random events of injury and death and hence will be a random variable, $Z_i(w_i)$, with an expected value, z_i, given by:

$$z_i = E[Z_i(w_i)] = (1 - q_1 - q_2)U_i(w_i) + q_1 I_{1i}(w_i) + q_2 I_{2i}(w_i) \tag{A.3}$$

It may be possible for the individual to reduce the probability, q_k, of injury k, $k=1, 2$, over some period by expending money on protection, leading to a fall in wealth, w_i. Alternatively, he might accept a higher injury probability but receive compensation, with the result that his wealth, w_i, rises. Thus changes to q_k will be accompanied by related changes in w_i, and the limiting condition that the expected utility stays the same, viz. $z_i = \text{constant}$, implies that $\partial z_i/\partial q_k = 0$; $k=1, 2$.

Carrying out the partial differentiation of Eq. (A.3) with respect to q_k and setting the result to zero gives

$$\frac{\partial z_i}{\partial q_k} = \left(\left(1 - \sum_{k=1}^{2} q_k\right)\frac{dU_i}{dw_i} + \sum_{k=1}^{2} q_k \frac{dI_{ki}}{dw_i}\right)\frac{\partial w_i}{\partial q_k} + I_{ki} - U_i = 0 \quad k = 1, 2 \tag{A.4}$$

This may be solved for the rate of change, $\partial w_i/\partial q_k$, of wealth, w_i, with the probability, q_k, of injury k, at constant expected utility.

Now the probability, p_k, of not receiving injury k is $p_k = 1 - q_k$, so that $dp_k/dq_k = -1$ and $\partial w_i/\partial p_k = -\partial w_i/\partial q_k$. It is reasonable to assume that a person would be prepared to trade some of his wealth, w_i, so as to increase his probability of not receiving injury k. An indifference curve should then exist, where the person's utility stays constant. The expression, $-\partial w_i/\partial p_k\big|_{z_i=const}$, quantifies the trade-off at a general point (p_k, w_i) on the indifference curve: wealth is decreased by a positive small amount, δw_i, from w_i to $w_i - \delta w_i$ in order that non-injury probability, p_k, should increase to $p_k + \delta p_k$, where δp_k is small and positive. The negative of the partial differential of wealth with respect to non-injury probability may be described as the marginal rate of substitution, m_{ki}, of non-injury probability, p_k, in place of wealth, w_i: $m_{ki} = -\partial w_i/\partial p_k\big|_{z_i=const}$. Since $\partial w_i/\partial p_k = -\partial w_i/\partial q_k$, it follows that

$$m_{ki} = \left.\frac{\partial w_i}{\partial q_k}\right|_{z_i=const} = \frac{U_i(w_i) - I_{ki}(w_i)}{\left(1 - \sum_{k=1}^{2} q_k\right)\frac{dU_i}{dw_i} + \sum_{k=1}^{2} q_k \frac{dI_{ki}}{dw_i}} \quad k = 1, 2 \tag{A.5}$$

Incidentally, Carthy et al. describe their version of m_{ki} as "the individual's MRS, m_I, of wealth for risk of injury", which is an incorrect usage unless they define "risk of injury" as the probability of non-injury, which seems unreasonable.

Putting $k=1$ in Eq. (A.2) and then substituting the result into Eq. (A.5) yields:

$$m_{ki} = \left.\frac{\partial w_i}{\partial q_k}\right|_{z_i=const} = \frac{U_i(w_i) - I_{ki}(w_i)}{(1 - q_2)\frac{dU_i}{dw_i} + q_2 \frac{dI_{2i}}{dw_i}} \quad k = 1, 2 \tag{A.6}$$

where the cancellation of the denominator terms in the probability, q_1, of the lesser injury will occur by Eq. (A.2), valid if and only if $I_{1i}(w_i)$ is equal $U_i(w_i)$ less the injury offset.

If the more serious injury, injury 2 in the two-injury chained model, is also insufficiently severe to cause permanent disability, then Eq. (A.1) may be applied to injury 2 also. The result of Eq. (A.2) with k set to 2 may now be inserted into Eq. (A.6) to cancel the terms in q_2 as well, leading to:

$$m_{ki} = \frac{U_i(w_i) - I_{ki}(w_i)}{dU_i/dw_i} = \frac{a_{ki}}{dU_i/dw_i} \quad k = 1, 2 \tag{A.7}$$

When injury 2 is serious enough to cause permanent disability, however, the more general form of Eq. (A.6) needs to be retained. Thus Eq. (A.6) needs to be used when considering mortal injuries. Examination of that equation when the more severe injury, injury 2, is fatal, shows that the marginal rate of substitution of non-death-probability, $p_2 = 1 - q_2 = 1 - q_D$, in place of wealth, $m_{2i} = m_{Di}$, will be a function of both wealth, w_i, and death-probability, q_D:

$$m_{Di} = m_{Di}(w_i, q_D) \tag{A.8}$$

The dependence of m_{Di} on the individual's wealth is clearly very important, although this fact does not appear to be brought out in Carthy et al. The strength of this dependence may be seen by replacing Eq. (A.1) by the form, $I_{2i}(w_i) = I_{Di}(w_i) = \gamma_{Di} U_i(w_i)$, as discussed below Eq. (A.2), which is at least equally plausible and regarded as realistic alternative by Carthy et al. (see their Note 5). Here γ_{Di}: $0 \le \gamma_{Di} \le 1$ is a constant for the individual, indicating the fraction of utility of wealth retained at death, so that $dI_{2i}(w_i)/dw_i = dI_{Di}(w_i)/dw_i = \gamma_{Di} dU_i(w_i)/dw_i$. Using the logarithmic utility function adopted by the UK Treasury (Treasury, 2011), $U_i(w_i) = \ln w_i$ and $dU_i(w_i)/dw_i = 1/w_i$. Substituting into Eq. (A.6) gives:

$$\begin{aligned} m_{Di}(w_i, q_D) &= \frac{w_i(1 - \gamma_{Di}) \ln w_i}{1 - (1 - \gamma_{Di}) q_D} \\ &\approx \lambda_i w_i \ln w_i \quad \text{when } q_D << 1 \end{aligned} \tag{A.9}$$

where $\lambda_i = 1 - \gamma_{Di}$ is a constant for the individual. It is thus clear that m_{Di} is strongly increasing in w_i. In cases where the person has no close relatives and no interest in charitable giving, it is possible that $\gamma_{Di} \to 0$ so that $\lambda_i \to 1.0$. In this case the marginal rate of substitution of non-death-probability in place of wealth would simplify to $m_{Di} = w_i \ln w_i$.

Returning to the general two-injury chained model, suppose that for an individual of wealth, w_i, the parameter, m_{1i}, is estimated when the injury 2 probability is $q_2 = q_{21}$, while the parameter, m_{2i}, is estimated when the injury probability has the value: $q_2 = q_{22}$. Using Eq. (A.6), the ratio of the marginal rates of substitution for individual i may be found as

$$\frac{m_{2i}}{m_{1i}} = \frac{U_i(w_i) - I_{2i}(w_i)}{U_i(w) - I_{1i}(w_i)} \frac{(1 - q_{22}) \frac{dU_i}{dw_i} + q_{22} \frac{dI_{2i}}{dw_i}}{(1 - q_{21}) \frac{dU_i}{dw_i} + q_{21} \frac{dI_{2i}}{dw_i}} \tag{A.10}$$

Clearly this ratio will be approximately equal to the ratio of utility differences if both $q_{21} \ll 1$ and $q_{22} \ll 1$, and strictly equal if the probability of incurring injury 2 is the same during the measurement of m_{1i} and m_{2i}:

$$\frac{m_{2i}}{m_{1i}} \approx \frac{U_i(w_i) - I_{2i}(w_i)}{U_i(w) - I_{1i}(w_i)} \quad \text{if } q_{21} << 1 \text{ and } q_{21} << 1 \text{ or } \quad \text{if } q_{22} = q_{21} \tag{A.11}$$

A.2. *Finding the marginal rate of substitution of wealth in place of non-injury probability for injury 1*

The two-injury chained model relies on finding the value of the injury offset, a_{1i}, for the lesser injury, injury 1. As neither injury W nor injury X leads to permanent disability, Eq. (A.7) is valid for marginal rate of substitution of wealth in place of non-injury probability for each injury with k set either to W or to X. Hence each is a possible candidate for the role of injury 1.

In considering these injuries, X and W, respondents were asked to estimate the maximum acceptable price (MAP, £) they would pay to avert the injury and the minimum acceptable compensation (MAC, £) they would take as compensation for enduring the injury. The MAP of individual i associated with injury k, x_{ki}, will be reached when its payment will reduce the utility of the healthy individual to the level he would experience without paying it and suffering the injury in consequence:

$$U_i(w_i - x_{ki}) = I_{ki}(w_i) \quad k = W, X \tag{A.12}$$

In an analogous way, the MAC, y_{ki}, will be reached when the utility of the injured person has risen by virtue of the increase in wealth to the level it would have been in the absence of both injury and compensation:

$$U_i(w_i) = I_{ki}(w_i + y_{ki}) \quad k = W, X \tag{A.13}$$

Using Eq. (A.1) in Eq. (A.12) and (A.13), we achieve the equation pair:

$$\left.\begin{aligned} U_i(w_i - x_{ki}) &= U_i(w_i) - a_{ki} \\ U_i(w_i + y_{ki}) - a_{ki} &= U_i(w_i) \end{aligned}\right\} \quad k = W, X \tag{A.14}$$

from which a_{ki} may be eliminated to give

$$U_i(w_i) = \frac{U_i(w_i - x_{ki}) + U_i(w_i + y_{ki})}{2} \quad k = W, X \tag{A.15}$$

Eq. (A.15) is an equation pair that may be solved for the individual's wealth, w_i, once values of x_{ki} and y_{ki} are available and the form of the utility function has been specified. The values of wealth, w_i, found from responses to injuries W and X should, of course, be the same, a proposition is tested in Section 4 of the main text.

The constant, a_{ki}, may be found from equation pair (A.14) by back substitution. The value of dU_i/dw_i depends on the utility function chosen and is derived in Appendix B as a function of x_{ki} and y_{ki} for the four forms used in Carthy et al. A knowledge of a_{ki} and dU_i/dw_i allows the marginal rate of substitution, m_{ki}, of non-injury probability, $p_k = 1 - q_k$, in place of wealth, w_i, to be calculated from Eq. (A.7) for $k = $ W, X.

A.3. *Using the "standard gamble" to estimate the ratio of utility differences in the two-injury chained model*

Suppose that a person has incurred an injury for which two treatments, α and β, are assumed to be available. If successful, treatment α will give the patient an outcome equivalent to the lesser injury, injury 1, so that his utility of wealth will be $I_{1i}(w_i)$. But it has a probability of failure, θ_1, where θ_1 carries the label of the lesser injury, injury 1, in its subscript. If unsuccessful, treatment α will lead to an outcome equivalent to the more severe injury, injury 2, so that the individual's utility of wealth will be $I_{2i}(w_i)$. Hence the expected utility when treatment α is chosen will be:

$$E(U|_\alpha) = (1 - \theta_1) I_{1i}(w_i) + \theta_1 I_{2i}(w_i) \tag{A.16}$$

Meanwhile treatment β is assumed to have a probability of failure, Π_{1i}, to be chosen by respondent i. In keeping with the convention introduced by Carthy et al., Π_{1i} is labelled with injury 1, in its first subscript, analogously to θ_1. If treatment β is successful, it will lead to a rapid return to full health, but, if unsuccessful, it will lead to an outcome equivalent to the more severe injury, injury 2, with utility of wealth, $I_{2i}(w_i)$. Hence the expected utility when treatment β is chosen will be:

$$E(U|_\beta) = (1 - \Pi_{1i})U_i(w_i) + \Pi_{1i}I_{2i}(w_i) \tag{A.17}$$

Treatment β, is to be preferred to treatment α provided its failure probability is regarded by individual, i, as sufficiently low. Each respondent was asked to specify the probability of failure, Π_{1i}, at which he would be equally content to have treatment α or treatment β administered to him. Π_{1i} is thus an indifference probability.

Indifference between treatments α and β at the respondent's choice of probability, Π_{1i}, is taken to imply that the respondent's expected utilities are equal: $E(U|_\alpha) = E(U|_\beta)$. Hence:

$$(1 - \theta_1)I_{1i}(w_i) + \theta_1 I_{2i}(w_i) = (1 - \Pi_{1i})U_i(w_i) + \Pi_{1i}I_{2i}(w_i) \tag{A.18}$$

This may be re-arranged into the form:

$$\frac{U_i(w_i) - I_{2i}(w_i)}{U_i(w_i) - I_{1i}(w_i)} = \frac{1 - \theta_1}{\Pi_{1i} - \theta_1} \tag{A.19}$$

Note that $I_{2i}(w_i) \le I_{1i}(w_i) \le U(w_i)$ requires the indifference probability, Π_{1i}, to lie in the range: $\theta_1 \le \Pi_{1i} \le 1$. When $\Pi_{1i} = 1$, Eq. (A.19) degenerates to $I_{2i}(w_i) = I_{1i}(w_i)$, implying that the person feels that his utility is reduced equally by both the lesser and the more severe injury. Meanwhile $\Pi_{1i} = \theta_1$ implies, from Eq. (A.18), that that person believes that his enjoyment of life when healthy will be unaffected by the lesser injury, injury 1, in the sense that $U_i(w_i) = I_{1i}(w_i)$. Given the adaptability of humans, this last is a possible and reasonable viewpoint (and one held by a sizeable fraction, 10%, of the sample when the two-injury chained model has a fatal injury as injury 2). But it causes particular problems for the method of Carthy et al. because it causes the ratio on the right-hand side of Eq. (A.19) to return an infinite value.

A.4. *Finding the marginal rate of substitution of wealth in place of non-injury probability for injury 2: testing the two-injury chained model*

The marginal rate of substitution, m_{ki}, of wealth in place of non-injury probability, $p_k = 1 - q_k$, may be found as explained at the end of Appendix A.2 for injuries k = W, X, based on MAP and MAC data, x_{ki}, y_{ki}; k = W, X. Casting either injury W or injury X in the role of injury 1 in the two-injury chained model, this estimate may be used in conjunction with Eqs. (A.11) and (A.19) to find the marginal rate of substitution of wealth in place of non-injury probability for a more severe injury, injury 2.

In particular, the two-injury chained model may be used to determine the marginal rate of substitution of wealth in place of non-injury probability for a fatal injury, by using the fatal injury, denoted by D for death, as injury 2. Carthy et al. chose to use injury X as injury 1 in their main calculation. Putting X as the first injury, X = 1, and D as the second injury, D = 2, in Eqs. (A.11) and (A.19) and combining the results gives:

$$m_{Di} = \frac{1 - \theta_X}{\Pi_{Xi} - \theta_X} m_{Xi} \tag{A.20}$$

where Carthy et al. set $\theta_X = 10^{-3}$, a figure said to be "not unrealistic for a treatment involving surgery and anaesthesia".

A particularly interesting case occurs when injury W is chosen as injury 1 and injury X as injury 2. In this case, putting W = 1 and X = 2 in Eqs. (A.11) and (A.19) and combining gives:

$$m_{Xi} = \frac{1 - \theta_W}{\Pi_{Wi} - \theta_W} m_{Wi} \tag{A.21}$$

where Carthy et al. set $\theta_W = 10^{-2}$. (No justification is advanced in Carthy et al. for choosing a figure for θ_W that is ten times higher than that for θ_X, but the number seems to reflect the thinking that a treatment failure resulting in a less severe outcome – 2 weeks in hospital and an 18-month recovery time as opposed to death – might occur much more often.)

The value of m_{Xi} from Eq. (A.21) is based on MAP and MAC data associated with injury W and on the standard gamble probability, Π_{Wi}. Eq. (A.21) is exactly analogous to Eq. (A.20) and may be used to test the method, since an independent value of m_{Xi} is available, based on directly MAP and MAC data for injury X. The values produced of m_{Xi} for each individual using the two independent methods should be the same. Using $m_{Xi}^{(2)}$ the MRS for injury X found by the two-stage chaining process and $m_{Xi}^{(1)}$ the MRS for the same injury found directly from a one-stage process, the following equation should be valid: $m_{Xi}^{(2)} = m_{Xi}^{(1)}$.

A.5. *Population averages: the value of a prevented injury 2*

On finding the VPF Carthy et al. say only:

> "Estimates of the VOSL for road risks – taken as the mean or median of m_D, computed from equation (27) – are reported"

where VOSL is the "value of statistical life", synonymous with VPF, and their Eq. (27) corresponds to Eq. (A.20). No further mathematical justification is given for this assertion.

Moreover, Carthy et al. seem to regard the median as occupying a similar status to the mean. But it has been shown that this is not the case: only arithmetic averaging complies with the requirement that the opinion of each person in the sample should be given equal weight (see Thomas, 2014), and the median is not a valid measure for measuring human valuations.

A justification for taking the mean of the m_{Di} values as the VPF may be attempted as follows. Consider a population of size, N. Making the assumption that the initial probability, denoted by the superscript, (1), for injury 2, $q_2^{(1)}$, is binomial, the expected number of type 2 injuries, G_1, over the given interval will be: $E(G_1) = q_2^{(1)}N$. An analogous equation will pertain when a safety measure has been installed to reduce q_2 to $q_2^{(2)}$, where the superscript (2) indicates a probability for injury 2 that has been reduced by the installation of a safety measure. Thus the expected difference in numbers of type 2 injuries may be found by subtraction as

$$E(G_1) - E(G_2) = (q_2^{(1)} - q_2^{(2)})N \tag{A.22}$$

where G_2 is the number of type 2 injuries after the safety measure has been installed.

When the expected difference in the number of type 2 injuries is unity, *viz.* $E(G_1) - E(G_2) = 1$, then

$$q_2^{(2)} = q_2^{(1)} - \frac{1}{N} \tag{A.23}$$

The maximum change in wealth individual i, with wealth, w_i, will be prepared to undergo in order to change his probability of the type 2 injury from an initial value, $q_2^{(1)}$, to a final value, $q_2^{(2)}$, is found by integrating the marginal rate of substitution, m_{2i}, of the probability of non-injury 2, $p_2 = 1 - q_2$, in place of wealth, w_i, at constant expected utility of wealth between these probability limits:

$$\Delta w_{2i}(w_i, p_2^{(1)}, p_2^{(2)}) = \int_{p_2=p_2^{(1)}}^{p_2^{(2)}} \left.\frac{\partial w_i}{\partial p_2}\right|_{z_i=\text{const}} dp_2$$

$$= \int_{q_2=q_2^{(1)}}^{q_2^{(2)}} \left.\frac{\partial w_i}{\partial q_2}\right|_{z_i=\text{const}} dq_2 = \int_{q_2=q_2^{(1)}}^{q_2^{(2)}} m_{2i}(w_i, q_2) dq_2 \tag{A.24}$$

where the limits have been changed between in the penultimate step, noting that $dp_2 = -dq_2$, while $\partial w_i/\partial p_2 = -\partial w_i/\partial q_2$, as shown in Appendix A.1. Moreover, in the general case, m_{2i} may be a function of both w_i and q_2 (see Eqs. (A.6) and (A.9)). Eq. (A.24) will yield a negative value, indicating a reduction in wealth when $q_2^{(2)} < q_2^{(1)}$, corresponding to a protective action. Thus the maximum acceptable price (MAP), $v_{2i}(w_{2i}, q_2^{(1)}, q_2^{(2)})$, that the individual is prepared to pay for this probability reduction will be $v_{2i} = -\Delta w_{2i}$, which may be found by reversing the order of the limits in Eq. (A.24). It follows from using Eq. (A.24), that the maximum acceptable price, v_{2i}, that an individual, i, with wealth, w_i, will be prepared to countenance for a reduction in probability that will reduce by one the expected number of type 2 injuries over the chosen period will be

$$v_{2i}(w_i, q_2^{(1)}) = \int_{q_2=q_2^{(1)}-\frac{1}{N}}^{q_2^{(1)}} m_{2i}(w_i, q_2) dq_2 \approx \frac{1}{N} m_{2i}(w_i, q_2^{(1)}) \tag{A.25}$$

Since the population as a whole will be prepared to pay a total of $v_2 = \sum_{i=1}^{N} v_{2i}$ for the safety measure that will reduce its expected number of type 2 injuries by unity, it is possible to calculate the value of a prevented injury 2, v_2, as:

$$v_2 = \sum_{i=1}^{N} v_{2i} \approx \sum_{i=1}^{N} \frac{1}{N} m_{2i}(w_i, q_2^{(1)}) = \frac{1}{N} \sum_{i=1}^{N} m_{2i}(w_i, q_2^{(1)}) \tag{A.26}$$

When the population is large and if $f(w)$ is the probability density for wealth in the population, treated as a continuum, we may replace Eq. (A.26) by

$$v_2(q_2^{(1)}) \approx \int_{w=0}^{\infty} f(w) m_2(w, q_2^{(1)}) dw \tag{A.27}$$

which indicates a remaining dependence on the initial injury 2 probability, $q_2^{(1)}$. However, provided the measurement of marginal rate, m_2, of substitution of wealth in place of type 2 non-injury probability, p_2, is made when the initial injury 2 probability, $q_2^{(1)}$, is small, then $m_2(w, 0) \approx m_2(w, q_2^{(1)})$. See Eq. (A.9). This allows the evaluation of a single value of prevented injury 2, when its probability is already small:

$$v_2 \approx \int_{w=0}^{\infty} f(w) m_2(w, 0) dw \quad \text{when } q_2^{(1)} \approx 0 \tag{A.28}$$

If wealth is divided into equal intervals of width, $\Delta w = w_{max}/K$, where w_{max} is the maximum wealth of any individual in the population and K is large so that Δw is small, then Eq. (A.28) may be written

$$\begin{aligned} v_2 &\approx \int_0^{\Delta w} f(w) m_2(w, 0)\, dw + \cdots + \int_{w_{k-1}}^{w_k} f(w) m_2(w, 0) dw \\ &\quad + \cdots + \int_{w_{K-1}}^{w_K} f(w) m_2(w, 0) dw \\ &\approx f(0)\, m_2(0, 0) \Delta w + \cdots + f(w_{k-1})\, m_2(w_{k-1}, 0) \Delta w \\ &\quad + \cdots + f(w_{K-1}) m_2(w_{K-1}, 0) \Delta w \end{aligned} \tag{A.29}$$

where $w_k = w_{k-1} + \Delta w$. Now suppose the respondents in the sample have been chosen so that the probability density for wealth in the sample matches that in the target population (for example the UK adult population). Assuming a large sample size, N_s, this implies that the number of people, n_k, in the kth sample interval, between wealths w_{k-1} and w_k, will be

$$\frac{n_k}{N_s} \approx \int_{w_{k-1}}^{w_k} f(w) dw \approx f(w_{k-1}) \Delta w \tag{A.30}$$

where $\sum_{k=1}^{K} n_k = N_s$. Substituting from Eq. (A.30) into the second line of Eq. (A.29) gives:

$$v_2 \approx \frac{1}{N_s}(n_1 m_2(0, 0) + \ldots + n_k m_2(w_{k-1}, 0) + \cdots + n_K m_2(w_{K-1}, 0)) \tag{A.31}$$

Consider further the kth wealth interval, which contains n_k people with wealths, $\omega_{k-1}^{(m)}$, $m = 1, 2, \ldots, n_k$, where $w_{k-1} \le \omega_{k-1}^{(m)} \le w_{k-1} + \Delta w$ and each person's wealth is assumed to be distinct (valid when the measuring scale is very fine). If Δw is very small, then $\omega_k^{(m)} \approx w_{k-1}$ for all $m = 1, 2, \ldots, n_k$, which implies that $m_2(\omega_{k-1}^{(m)}, 0) \approx m_2(w_{k-1}, 0)$ for all m. Hence

$$\sum_{m=1}^{n_k} m_2(w_{k-1}, 0) = n_k m_2(w_{k-1}, 0) \approx \sum_{m=1}^{n_k} m_2(\omega_{k-1}^{(m)}, 0) \tag{A.32}$$

As a result, Eq. (A.31) may be approximated by:

$$v_2 \approx \frac{1}{N_s}\left(\sum_{m=1}^{n_1} m_2(\omega_0^{(m)}, 0) + \cdots + \sum_{m=1}^{n_k} m_2(\omega_{k-1}^{(m)}, 0) + \cdots + \sum_{m=1}^{n_K} m_2(\omega_{K-1}^{(m)}, 0)\right) \tag{A.33}$$

Relabeling the wealth of each individual on a single scale, w_i, $i = 1, 2, \ldots, N_s = \sum_{k=1}^{K} n_k$, according to:

$$w_{\left(\sum_{j=1}^{k-1} n_j\right)+m} = \omega_{k-1}^{(m)} \quad k = 1, 2, \ldots, \quad K \text{ and } m = 1, 2, \ldots, n_k \tag{A.34}$$

allows Eq. (A.33) to be written as the arithmetic average of the marginal rate of substitution of type 2 non-injury probability in place of wealth, viz. the sample mean:

$$v_2 \approx \frac{1}{N_s}\sum_{i=1}^{N_s} m_2(w_i, 0) = \frac{1}{N_s}\sum_{i=1}^{N_s} m_{2i} \tag{A.35}$$

The value of a prevented fatality, v_D, is found by setting the type 2 injury as a fatal injury, hence setting $D = 2$ in Eq. (A.35):

$$v_D \approx \frac{1}{N_s}\sum_{i=1}^{N_s} m_D(w_i, 0) = \frac{1}{N_s}\sum_{i=1}^{N_s} m_{Di} \tag{A.36}$$

This equation shows that m_{Di} may be regarded as the personal VPF of individual i.

It needs to be remembered that the validity of Eq. (A.36) rests critically on the sample population reflecting very closely the probability density for wealth of the target population as a whole. This includes the requirement that the wealth intervals in the sample population should be small. In this context, Carthy et al. (1999) say of their study that it

> "involved a quota sample of 167 respondents selected by professional market research organisations on the basis of gender, age and social class quotas supplied by the research team to reflect national breakdowns for Great Britain."

but do not give further details of the quotas. However, the wealth value, w_i, for each respondent emerges as an integral parameter used in the analysis procedure. Hence a check of this figure against national figures provides a legitimate and important test for the validity of the VPF for the UK recommended by Carthy et al.

A.6. *Accounting for the increasing wealth of the country*

Carthy et al. (1999) state at the end of their paper:

> "In the light of the findings reported in this paper, the UK Department of the Environment, Transport and the Regions has elected to maintain its current WTP-based monetary value for the prevention of a road fatality at £902,500 in 1997 prices, with the figure to be updated annually in line with inflation and the rate of growth of real output per capita."

However Chilton et al. (2002) updated this statement to:

> "In the light of the findings in [Beattie et al., 1998 and Carthy et al., 1999] the ... DTER [Department for Transport, Environment and the Regions] elected to increase its WTP-based roads VPF to some £1.05 million in 1998 prices."

Thus Wolff and Orr (2009), after noting that "the resulting figure was very close to the existing one", concluded:

> "it appears that the Carthy study is now the primary source of VPF figures, adjusted for inflation and changes in GDP."

This suggests that the VPF used in the UK, $v_D(t)$, for year, t, is related to the VPF for 1998 by:

$$v_D(t) = v_D(1998)\frac{G(t)}{G(1998)} \tag{A.37}$$

where G(t) is the GDP per head in year, t, in £s of that year.

Appendix B. Derivation of wealth and the marginal rate of substitution of non-injury probability in place of wealth for the utility functions used in Carthy et al. (1999)

It has been argued elsewhere (Thomas, 2010) that human decision making is modelled most realistically by assuming that the decision-maker's risk-aversion stays constant during the course of a decision on monetary matters. This argument reduces the number of families of viable utility functions to one, the Power utility function, to which family the Logarithmic utility function belongs as a limiting case.

Carthy et al. use the Power utility (albeit in constrained form) and the Logarithmic utility in their analysis, as well as two further utility functions, the Negative Inverse and the Negative Exponential. All will be analysed here, and the influence that the individual's wealth has on the VPF valuations brought out.

B.1. *Constrained Power utility*

The Power utility function, has the basic form:

$$U_i(w_i) = w_i^{1-\varepsilon_i} = w_i^{s_i} \tag{B.1}$$

where $s_i = 1 - \varepsilon_i$ and ε_i is the dimensionless risk-aversion of individual i, defined by $\varepsilon_i = -w_i U_i''(w_i)/U_i'$. Carthy et al. (1999) assign $n = 1/s_i$ and describe the Constrained Power utility as the "nth root" utility function as a consequence. Their method assumes that $s_i = 1 - \varepsilon_i$ is particular to each individual and is to be determined from the MAP for injury, k, x_{ki}, and the MAC, y_{ki}. Substituting from Eq. (B.1) into Eq. (A.15) gives

$$(w_i - x_{ki})^{s_i} + (w_i + y_{ki})^{s_i} = 2w_i^{s_i} \tag{B.2}$$

Meanwhile, from equation pair (A.14), the individual's injury offset, a_{ki}, for injury, k, is

$$a_{ki} = w_i^{s_i} - (w_i - x_{ki})^{s_i} \tag{B.3}$$

Furthermore, differentiating Eq. (B.1) with respect to wealth, w_i, gives

$$\frac{dU_i}{dw_i} = s_i w_i^{s_i - 1} \tag{B.4}$$

Substituting from Eqs. (B.3) and (B.4) into Eq. (A.7) gives the first estimate of marginal rate of substitution of non-injury probability in place of wealth for injury, k:

$$m_{ki} = \frac{w_i^{s_i} - (w_i - x_{ki})^{s_i}}{s_i w_i^{s_i - 1}} \tag{B.5}$$

The marginal rate of substitution is thus a function of the two unknowns, the exponent, s_i, and the wealth, w_i, (no data on wealth are made available in Carthy et al., 1999) in addition to the MAP, x_{ki}, which is to be measured. One of the dependencies may be eliminated by bringing in the MAC, y_{ki}, which is the second value to be measured, but one unknown will be left. One natural course of action would have been to set a value for risk-aversion, ε_i, common to all in the same way that risk-aversion is set to unity for everyone when the Logarithmic utility function is used. This value of ε_i would fix the exponent: $s_i = 1 - \varepsilon_i$. However, Carthy et al. make the assumption that the injury offset, a_{ki}, should take the highest value possible that does not make the utility of wealth after injury negative. That is to say that the utility of wealth after the injury should be made zero, $I_{ki}(w_i) = 0$, no matter how insignificant the injury and no matter how wealthy the individual is. Thus from Eqs. (A.12) and (B.1), $U_i(w_i - x_{ki}) = (w_i - x_{ki})^{s_i} = 0$. It follows that the individual's wealth is given by

$$w_i = x_{ki} \tag{B.6}$$

which eliminates the wealth as an unknown. Reflecting the fact that Carthy et al. have chosen to impose the constraint of Eq. (B.6) on their Power utility function, it will be referred to as the Constrained Power utility function.

The rationale for Eq. (B.6) offered by Carthy et al. is

> "as α [$=\alpha_{ki}$] decreases it transpires that the value of n [$=1/s_i$] required to accommodate any particular values of $\hat{x}$ [$=x_{ki}$] and $\hat{y}$ [$=y_{ki}$] gets larger. Thus, for example, with $\hat{y} = 5\hat{x}$, and setting $\alpha = U(\bar{w} - \beta)$, $0.25U(\bar{w} - \beta)$ and $0.01U(\bar{w} - \beta)$ [$a_{ki} = U_i(w_i)$, $0.25U_i(w_i)$ and $0.01U_i(w_i)$] entails values of 2.6, 7.6 and 160 respectively. Since in this case larger values of n yield implausibly *low* values for the Pratt-Arrow coefficient of relative risk aversion [= risk aversion, ε_i], it seems appropriate to set α at the upper end of its range of admissible values. In view of this, and in the interests of analytical tractability, it seems most straightforward to set $\alpha = U(\bar{w} - \beta)$ [$a_{ki} = U_i(w_i)$]".

where the italics are those of Carthy et al., while the square brackets indicate the notation used in the current paper.

To explore further, put $a_{ki} = r_{ki}U(w_i)$, $0 \le r_{ki} < 1$ to accommodate the values $r_{ki} = 1.0$, 0.25 and 0.01 instanced by Carthy et al. The equation pair (A.14) is now transformed into:

$$\begin{aligned} U_i(w_i - x_{ki}) &= (1 - r_{ki})U_i(w_i) \\ U_i(w_i + y_{ki}) &= (1 + r_{ki})U_i(w_i) \end{aligned} \quad k = W, X \tag{B.7}$$

When r_{ki} is assumed to be unity, the top line of equation pair (B.7) gives: $U_i(w_i - x_{ki}) = (w_i - x_{ki})^{s_i} = 0$, which implies Eq. (B.6). The development of Eqs. (B.9)–(B.11) then allows s_i to be calculated as 0.387, so that $n = 2.58$ and $\varepsilon_i = 0.61$. It appears to be the fact that $\varepsilon_i = 0.61$ when $r_{ki} = 1.0$ and when y_{ki}/x_{ki} takes the reasonably representative value of 5.0 that encourages Carthy et al. to regard $r_{ki} = 1.0$ as a reasonable value, presumably on the grounds that they regard $\varepsilon_i = 0.61$ as a reasonably good figure for risk-aversion, although that is not made clear.

Nor is it clear how the values of $n = 7.6$ and $n = 160$ could be derived based only on the ratio $y_{ki}/x_{ki} = 5$. The development below indicates that the ratio of the individual's MAP to his wealth, x_{ki}/w_i, is also needed. Dividing the lower equation by the upper and using Eq. (B.1) gives, after taking logs and re-arranging:

$$s_i = \ln\left(\frac{1 + r_{ki}}{1 - r_{ki}}\right) \div \ln\left(\frac{1 + \frac{y_{ki}}{x_{ki}}\frac{x_{ki}}{w_i}}{1 - \frac{x_{ki}}{w_i}}\right) \quad k = W, X \tag{B.8}$$

The solution of Eq. (B.8) requires not only r_{ki} and y_{ki}/x_{ki} but also a value for x_{ki}/w_i. Furthermore, it is clear from Eq. (B.8) that Eq. (B.6), $w_i = x_{ki}$, will lead to an incomputable value of s_i and hence n. Given the values quoted by Carthy et al. of $r_{ki} = 0.25$, $n = 7.6 = 1/s_i$ and $r_{ki} = 0.01$, $n = 160 = 1/s_i$, a back solution of Eq. (B.8) gives $x_{ki}/w_i = 0.89$ in the first case and $x_{ki}/w_i = 0.80$ in the second.

Now note that risk-aversion is given by $\varepsilon_i = 1 - s_i = 1 - 1/n$. This means that the values of "the Pratt-Arrow coefficient of relative risk aversion" or risk-aversion, ε_i, corresponding to $n = 7.6$ and $n = 160$ are $\varepsilon_i = 0.87$ and $\varepsilon_i = 0.99$ respectively, both of which lie between the value, $\varepsilon_i = 0.61$, which Carthy et al. seem to regard as reasonable, and the value $\varepsilon_i = 1.0$ that characterises the Logarithmic utility function, which Carthy et al. must regard as reasonable since they use it in one of their analysis streams. Thus neither $\varepsilon_i = 0.87$ nor $\varepsilon_i = 0.99$ should have been regarded by Carthy et al. as "implausibly low". Therefore the primary justification for setting α "at the upper end of its range of admissible values" disappears. What remains seems to be that the combination of $r_{ki} = 1.0$ and $y_{ki}/x_{ki} = 5$ produces $\varepsilon_i = 0.61$, apparently regarded as a reasonable value, plus the "interests of analytical tractability". The full spread of risk-aversions, ε_i, is given in Fig. 8, which shows that the constraint of Eq. (B.6) does not prevent some very low and highly negative values of risk-aversion being observed.

Substituting Eq. (B.6) into Eq. (B.5) gives:

$$m_{ki} = \frac{w_i}{s_i} \quad k = W, X \tag{B.9}$$

Meanwhile Eq. (B.2) degenerates to:

$$(w_i + y_{ki})^{s_i} = 2w_i^{s_i} \quad k = W, X \tag{B.10}$$

Taking logs and rearranging gives the exponent, s_i, as

$$s_i = \frac{\ln 2}{\ln(w_i + y_{ki}) - \ln w_i} \left[= \frac{\ln 2}{\ln(1 + y_{ki}/x_{ki})} \right] \quad k = W, X \tag{B.11}$$

allowing m_{ki} to be given as

$$m_{ki} = w_i \frac{\ln(w_i + y_{ki}) - \ln w_i}{\ln 2} \left[= x_{ki} \frac{\ln(x_{ki} + y_{ki}) - \ln x_{ki}}{\ln 2} \right] \quad k = W, X \tag{B.12}$$

for the Constrained Power utility function.

B.2. *Logarithmic utility function*

The Logarithmic utility function was introduced by Daniel Bernoulli (Bernoulli, 1738) and follows from the precept that "the utility resulting from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed", or $dU_i/dw_i = 1/w_i$, integration of which produces:

$$U_i(w_i) = \ln w_i \tag{B.13}$$

This form may be shown to be a limiting case of the Power utility when the person's risk-aversion is set to unity, $\varepsilon_i = 1$ (see e.g. Thomas, 2010), and it is the utility function recommended by the UK Treasury (2011).

Substituting into Eq. (A.15) gives

$$\ln(w_i - x_{ki}) + \ln(w_i + y_{ki}) = 2\ln w_i \tag{B.14}$$

so that

$$\ln((w_i - x_{ki})(w_i + y_{ki})) = \ln w_i^2 \tag{B.15}$$

Exponentiating and rearranging then gives the individual's wealth as

$$w_i = \frac{x_{ki}y_{ki}}{y_{ki} - x_{ki}} \tag{B.16}$$

Thus wealth becomes infinite if $x_{ki} = y_{ki}$. Meanwhile, from the first equation in equation pair (A.14)

$$\ln(w_i - x_{ki}) = \ln w_i - a_{ki} \tag{B.17}$$

Rearranging and using Eq. (B.16) gives the injury offset as

$$a_{ki} = \ln\frac{w_i}{w_i - x_{ki}}\left[= \ln\left(\frac{x_{ki}y_{ki}}{(y_{ki} - x_{ki})}\frac{1}{\left(\frac{x_{ki}y_{ki}}{y_{ki}-x_{ki}} - x_{ki}\right)}\right) = \ln\frac{y_{ki}}{x_{ki}}\right] \tag{B.18}$$

Differentiating Eq. (B.13) with respect to w_i produces

$$\frac{dU_i}{dw_i} = \frac{1}{w_i} \tag{B.19}$$

Substituting from Eqs. (B.18) and (B.19) into Eq. (A.7) gives the marginal rate of substitution of non-injury probability in place of wealth as

$$m_{ki} = w_i \ln\frac{w_i}{w_i - x_{ki}}\left[= \frac{x_{ki}y_{ki}}{y_{ki} - x_{ki}}\ln\frac{y_{ki}}{x_{ki}}\right] \tag{B.20}$$

It may be noted that m_{ki} becomes infinite for $x_{ki} = y_{ki}$.

B.3. *Homogeneous utility function: Negative Inverse utility function*

Carthy et al. describe as "homogeneous" the utility function:

$$U_i(w_i) = -w_i^{-n} \quad \text{for } n > 0 \tag{B.21}$$

and then choose the single value, $n = 1$ for reasons of "analytical tractability", meaning that only the Negative Inverse utility function is actually considered:

$$U_i(w_i) = -\frac{1}{w_i} \tag{B.22}$$

Substituting from Eq. (B.22) into Eq. (A.15) gives:

$$-\frac{1}{w_i - x_{ki}} - \frac{1}{w_i + y_{ki}} = -\frac{2}{w_i} \tag{B.23}$$

After rearranging and cancelling terms, the individual's wealth, w_i, emerges as:

$$w_i = \frac{2x_{ki}y_{ki}}{y_{ki} - x_{ki}} \tag{B.24}$$

Comparing (B.24) with Eq. (B.16), it is clear that for the same values of maximum acceptable price, x_{ki}, and minimum acceptable compensation, y_{ki}, the wealth associated with the Negative Inverse utility function is predicted to be twice the wealth of the Logarithmic utility function.

Meanwhile, solving equation pair (A.14) for a_{ki} and using the utility function as defined by Eq. (B.22) gives the injury offset as:

$$a_{ki} = \frac{1}{w_i - x_{ki}} - \frac{1}{w_i} = \frac{x_{ki}}{w_i^2 - w_i x_{ki}}\left[= \frac{(x_{ki} - y_{ki})^2}{2x_{ki}y_{ki}(x_{ki} + y_{ki})}\right] \tag{B.25}$$

where the last step involves the use of Eq. (B.24). Meanwhile differentiating Eq. (B.22) with respect to w gives:

$$\frac{dU_i}{dw_i} = \frac{1}{w_i^2}\left[= \frac{(y_{ki} - x_{ki})^2}{4x_{ki}^2 y_{ki}^2}\right] \tag{B.26}$$

where Eq. (B.24) has been used in the second step. Substituting from Eqs. (B.25) and (B.26) into Eq. (A.7) gives the marginal rate of substitution of non-injury probability in place of wealth as

$$m_{ki} = \frac{w_i x_{ki}}{w_i - x_{ki}}\left[= \frac{2x_{ki}y_{ki}}{x_{ki} + y_{ki}}\right] \tag{B.27}$$

B.4. *Negative exponential utility function*

In this case

$$U_i(w_i) = -e^{-\beta_i w_i} \quad \beta_i > 0 \tag{B.28}$$

Substituting into Eq. (A.15) gives, after minor rearrangement:

$$e^{-\beta_i w_i}e^{\beta_i x_{ki}} + e^{-\beta_i w_i}e^{-\beta_i y_{ki}} = 2e^{-\beta_i w_{ki}} \tag{B.29}$$

Multiplying throughout by $e^{\beta_i w_i}$ for any wealth, w_i, gives

$$e^{\beta_i x_{ki}} + e^{-\beta_i y_{ki}} = 2 \tag{B.30}$$

which implies that all further results will apply whatever the individual's wealth. The degenerate solution, $\beta_i = 0$ for all x_{ki} and y_{ki}, is ruled out by the condition $\beta_i > 0$, but an alternative solution may be found by first using the substitution:

$$c_{ki} = \frac{y_{ki}}{x_{ki}} \tag{B.31}$$

in Eq. (B.30):

$$e^{\beta_i x_{ki}} + e^{-\beta_i c_{ki} x_{ki}} = e^{\beta_i x_{ki}} + (e^{-\beta_i x_{ki}})^{c_{ki}} = 2 \tag{B.32}$$

where $c_{ki} \geq 0$.

Using the further substitution, $\phi_{ki} = e^{\beta_i x_{ki}}$, yields:

$$\phi_{ki} - 2 + \frac{1}{\phi_{ki}^{c_{ki}}} = 0 \tag{B.33}$$

This equation will normally need to be solved for ϕ_{ki} numerically, but it is advisable to examine the properties of the equation further before doing so.

One solution that will apply for all c_{ki} is $\phi_{ki} = 1$, but since $\phi_{ki} = e^{\beta_i x_i}$, or, equivalently, $\beta_i = (\ln\phi_{ki})/x_{ki}$, this solution implies $\beta_i = 0$. Such a beta-value would render $U_i(w_i)$ in Eq. (B.28) independent of wealth, w_i, and therefore inadmissible as a utility function. Hence the solution $\phi_{ki} = 1$ may be disregarded in this application.

Turning to particular values of c_{ki}, if $c_{ki} = 0$, then Eq. (B.33) yields the single solution $\phi_{ki} = 1$, while $c_{ki} = 1$ turns Eq. (B.33) into a quadratic equation, with $\phi_{ki} = 1$ for both solutions. Hence values of $c_{ki} = 0$ and $c_{ki} = 1$ are of no interest.

For the values of c_{ki} that are intermediate between 0 and 1, viz. $0 < c_{ki} < 1$, the first solution is $\phi_{ki} = 1$, which has already been explained to be of no interest. Furthermore, it can be shown by numerically mapping the function, $\phi_{ki} - 2 + 1/\phi_{ki}^{c_{ki}}$, over the space defined by $0 < \phi_{ki} < 1$ and $0 < c_{ki} < 1$ that the second solution will obey the open condition $0 < \phi_{ki} < 1$. This implies that

$\beta_i = (\ln\phi_{ki})/x_{ki} < 0$. But substituting a negative value of β_i into Eq. (B.28) implies that an individual's utility, $U_i(w_i)$, will decrease with increasing wealth, an inadmissible condition for a utility function.

Thus it has been shown that values of c_{ki} at or below unity violate the constraints of the application at hand, so that the range of possible c_{ki} is restricted to $1 < c_{ki} \le \infty$.

A lower, open limit of $\phi_{ki} > 1$ may be established from the foregoing by assuming continuity, viz. $\phi_{ki} \to 1$ as $c_{ki} \to 1$ from above. Moreover, it is clear from Eq. (B.33) that no $\phi_{ki} > 2$ is possible if $1 < c_{ki} \le \infty$. Thus ϕ_{ki} is constrained to stay within the bounds: $1 < \phi_{ki} \le 2$, which narrows down the search for a numerical solution.

Solving equation pair (A.14) for a_{ki} and substituting from Eq. (B.28) into the result gives

$$\begin{aligned} a_{ki} &= -e^{-\beta_i w_i} + e^{-\beta_i (w_i - x_{ki})} = e^{-\beta_i w_i}(e^{\beta_i x_{ki}} - 1) \\ &= -U_i(w_i)(\phi_{ki} - 1) \end{aligned} \tag{B.34}$$

Meanwhile, differentiating Eq. (B.28) with respect to w_i gives:

$$\frac{dU_i}{dw_i} = \beta_i e^{-\beta_i w_i} = -\frac{\ln\phi_{ki}}{x_{ki}} U_i(w_i) \tag{B.35}$$

since $\phi_{ki} = e^{\beta_i x_i} \Leftrightarrow \beta_i = (\ln\phi_{ki})/x_{ki}$. Thus, substituting from Eqs. (B.34) and (B.35) into Eq. (A.7) gives the marginal rate of substitution of non-injury probability in place of wealth as

$$m_{ki} = \frac{(\phi_{ki} - 1)}{\ln\phi_{ki}} x_{ki} \tag{B.36}$$

which may be calculated once Eq. (B.33) has been solved for ϕ_{ki}.

Using l'Hôpital's rule, Eq. (B.36) reduces to

$$m_{ki} \to x_{ki} \text{ as } \phi_{ki} \to 1 \tag{B.37}$$

but since $\phi_{ki} = 1$ has been shown to lead to the inadmissible value $\beta_i = 0$, the result $m_{ki} = x_{ki}$ cannot be used.

Eq. (B.36) may, of course, be written in terms of β_i as:

$$m_{ki} = \frac{e^{\beta_i x_{ki}} - 1}{\beta_i} \tag{B.38}$$

As noted above, it is not possible to use the values of x_{ki} and y_{ki} to find the wealth for an individual if his behaviour is represented by this utility function.

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