Some new inequalities related to the invariant means and uniformly bounded function sequences

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Abstract

Agnew gave the necessary and sufficient conditions for the triangular regular matrices to yield

$$\limsup_{n} G(A_n(\mathcal{F}); D) \leq \limsup_{k} G(f_k; D)$$

for all uniformly bounded real function sequences $\mathcal{F} = (f_k)$. Recently, Duman studied some variants of this inequality by replacing the operator limsup with st-limsup. In this note, we present some new inequalities involving invariant means like the above mentioned results.

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1. Introduction

Let $\ell_\infty$ and $c$ be the Banach spaces of real bounded and convergent sequences with the usual supremum norm. Let $\sigma$ be a one-to-one mapping of $\mathbb{N}$, the set of all positive integers, into itself such that $\sigma^p(n) \neq n$ for all $n$ and $p \geq 1$; where $\sigma^p(n)$ is the $p$th iterate of $\sigma$ at $n$ [9]. A continuous linear functional $\phi$ on $\ell_\infty$ is called an invariant mean or a $\sigma$-mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all $k$,

(ii) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$

and

(iii) $\phi(x) = \phi(\sigma x)$ for all $x \in \ell_\infty$, where $\sigma x = (x_{\sigma(n)})$.

Let $V_\sigma$ be the set of bounded sequences all of whose $\sigma$-means are equal. It is known [10] that

$$V_\sigma = \left\{ x \in \ell_\infty : \lim_{p} t_{pk}(x) = 0 \text{ uniformly in } k, s = \sigma\text{-lim } x \right\}$$

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where \[ t_{pk}(x) = \frac{1}{p+1} \left( x_k + x_{\sigma(k)} + \cdots + x_{\sigma^n(k)} \right), \quad t_{-1,k}(x) = 0. \]

It is also known that \( \sigma \)-lim \( x = s \) if and only if \( q_\sigma(x) = -q_\sigma(-x) = s \) where \( q_\sigma(x) = \limsup \rho \sup_k t_{pk}(x) \). Let \( Z \) be the set of all sequences with \( \sigma \)-limit zero. Then, by (iii), it is clear that \( (\sigma x - x) \in Z \) for all \( x \in \ell_\infty \).

Let \( E \) be a subset of \( \mathbb{N} \). The natural density \( \delta \) of \( E \) is defined by

\[ \delta(E) = \lim_n \frac{1}{n} \left| \{ k \leq n : k \in E \} \right| \]

where \( |\{ k \leq n : k \in E \}| \) denotes the number of elements of \( E \) not exceeding \( n \). A sequence \( x = (x_k) \) is said to be statistically convergent to a number \( \ell \) if \( \delta(\{ k : |x_k - \ell| \geq \varepsilon \}) = 0 \) for every \( \varepsilon \). In this case we write \( \text{st-lim} x = \ell \) [6].

By \( st \) we denote the space of all statistically convergent sequences. Fréyd and Orhan [7] have introduced the notions of statistical boundedness, statistical-limit superior (st-limsup) and statistical-limit inferior (st-liminf).

Let \( A = (a_{nk}) \) be an infinite matrix of real numbers and \( x = (x_k) \) a real sequence such that \( Ax = (A_n(x)) = (\sum k a_{nk} x_k) \) exists for each \( n \). Then the sequence \( Ax = (A_n(x)) \) is called an \( A \)-transform of \( x \). For two sequence spaces \( E \) and \( F \) we say that the matrix \( A \) maps \( E \) into \( F \) if \( Ax \) exists and belongs to \( F \) for each \( x \in E \). By \( (E, F) \) we denote the set of all matrices which map \( E \) into \( F \). If \( E \) and \( F \) are equipped with the limits \( E \)-lim and \( F \)-lim, respectively, \( A \in (E, F) \) and \( F \)-lim \( A_n(x) = E \)-lim \( x_k \) for all \( x \in E \), then we say that \( A \) regularly maps \( E \) into \( F \) and write \( A \in (E, F)_{\text{reg}} \).

The characterizations of the classes \((c, V_\sigma)_{\text{reg}}, (V_\sigma, V_\sigma)_{\text{reg}}\) and \((\text{st} \cap \ell_\infty, V_\sigma)_{\text{reg}}\) were given in [10,2,3], respectively.

Let \( D \) be a subset of \( \mathbb{R} \), the set of real numbers, and \( \mathcal{F} = (f_k) \) be a sequence of real valued functions defined on \( D \). Then, we write \( G(f_k; D) = \sup_{x \in D} f_k(x) \), and \( A_n(\mathcal{F}) \) for the \( A \)-transform of \( \mathcal{F} \) is given by

\[ A_n(\mathcal{F}) = \sum_k a_{nk} f_k(x) \]

provided that the series on the right-hand side converges for each \( n \in \mathbb{N} \) and for all \( x \in D \).

In [1], Agnew characterized a class of triangular matrices for which the inequality

\[ \limsup_n G(A_n(\mathcal{F}); D) \leq \limsup_k G(f_k; D) \]

holds for every uniformly bounded function sequence \((f_k)\). Some variants of Agnew’s result were investigated in [5] by replacing the operator \( \limsup \) with \( \text{st-lim sup} \).

In the present work, we have proved some new inequalities related to the uniformly bounded function sequences to fulfill some gaps which exist in this topic.

2. The main results

Firstly, we should begin with some lemmas which are needed in proving our theorems.

**Lemma 2.1** ([8, Theorem 2]). \( \sigma \)-Core of \( Ax \subset K \)-core of \( x \) if and only if \( A \) is \( \sigma \)-regular and \( \sigma \)-uniformly positive.

**Lemma 2.2** ([8, Theorem 3]). \( \sigma \)-Core of \( Ax \subset \sigma \)-core of \( x \) if and only if \( A \) is \( \sigma \)-regular and \( \sigma \)-uniformly positive.

**Lemma 2.3** ([4, Lemma 1 and Remark]). Let \( P : X \to \mathbb{R} \) and \( Q : X \to \mathbb{R} \) be sublinear functionals. Then, \( \{X, P\} \subset \{X, Q\} \) if and only if \( P \leq Q \). Additionally, \( \{X, P\} = \{X, Q\} \) if and only if \( P = Q \).

**Lemma 2.4** ([3, Theorem 2.3]). \( \sigma \)-core \((Ax) \subset \text{st-core} \( (x) \) for all \( x \in \ell_\infty \) if and only if \( A \in (\text{st} \cap \ell_\infty, V_\sigma)_{\text{reg}} \) and \( A \) is \( \sigma \)-uniformly positive.

Throughout the work, for brevity, we will write

\[ a(p, n, k) = \frac{1}{p+1} \sum_{i=0}^{p} a_{\sigma^i(n), k} \quad \text{and} \quad t_{pn}[A_n(\mathcal{F})] = \sum_k a(p, n, k) f_k(x). \]
Theorem 2.5. Let \( \|A\| := \sup_n \sum_k |a_{nk}| < \infty \). Then, for every uniformly bounded function sequence \( \mathcal{F} = (f_k) \) on \( D \), one has

\[
\limsup_p \sup_n G\{t_{pn}[A_n(\mathcal{F})]; D\} \leq \limsup_k G(f_k; D)
\]

(2.1) if and only if \( A \in (c, V_\sigma)_{\text{reg}} \) and

\[
\lim_p \sum_k |a(p, n, k)| = 1 \quad \text{uniformly in } n.
\]

(2.2)

Proof. Firstly, suppose that (2.1) holds for every uniformly bounded function sequence \( \mathcal{F} = (f_k) \) on \( D \). Define \( \mathcal{F} = v \) on \( \hat{D} \), where \( v = (v_k) \) is a bounded sequence of real numbers. Then, clearly, \( \mathcal{F} \) is uniformly bounded on \( D \), \( G(f_k; D) = v \) and \( G(t_{pn}[A_n(\mathcal{F})]; D) = t_{pn}(v) \). Also, since \( \|A\| < \infty \) and \( v \in \ell_\infty \), it follows that \( Av \in \ell_\infty \). Hence, we observe from (2.1) that

\[
\limsup_p \sup_n t_{pn}(Av) \leq \limsup_v.
\]

Therefore, the part of the necessity of theorem follows by Lemma 2.1.

Conversely, let \( A \in (c, V_\sigma)_{\text{reg}} \) and the condition (2.2) hold. To establish the sufficiency of conditions, we observe that if \( \mathcal{F} = (f_k) \) is any uniformly bounded function sequence on \( D \), then \( \{f_k(x)\} \in \ell_\infty \) for every \( x \in D \), i.e., there exists a positive number \( K \) such that \( |f_k(x)| \leq K \) for every \( x \in D \) and for all \( k \in \mathbb{N} \). Also, for any \( \varepsilon > 0 \), there exists a \( k_0 \in \mathbb{N} \) such that

\[
G(f_k; D) < \limsup_k G(f_k; D) + \varepsilon = L(f_k) + \varepsilon
\]

whenever \( k \geq k_0 \). For any real \( \lambda \), let us write \( \lambda^+ = \max\{0, \lambda\}, \lambda^- = \max\{-\lambda, 0\} \). Then \( \lambda = \lambda^+ - \lambda^- \).

Now, for any \( x \in D \), we can write

\[
t_{pn}[A_n(\mathcal{F})] = \sum_k a(p, n, k) f_k(x)
\]

\[
= \sum_{k < k_0} a(p, n, k) f_k(x) + \sum_{k \geq k_0} a(p, n, k)^+ f_k(x) - \sum_{k \geq k_0} a(p, n, k)^- f_k(x)
\]

\[
\leq K \sum_{k < k_0} |a(p, n, k)| + (L(f_k) + \varepsilon) \sum_k |a(p, n, k)| + K \sum_k (|a(p, n, k)| - a(p, n, k)).
\]

(2.3)

Applying the operator \( \limsup_p \sup_n \) to (2.3) and using the conditions of the class \( (c, V_\sigma)_{\text{reg}} \) with the condition (2.2), we have

\[
\limsup_p \sup_n G\{t_{pn}[A_n(\mathcal{F})]; D\} \leq L(f_k) + \varepsilon.
\]

Since the sequence \( \mathcal{F} = (f_k) \) and \( \varepsilon \) are arbitrary, this completes the proof. \( \square \)

Theorem 2.6. Let \( \|A\| < \infty \). Then, for every uniformly bounded function sequence \( \mathcal{F} = (f_k) \) on \( D \), one has

\[
\limsup_p \sup_n G\{t_{pn}[A_n(\mathcal{F})]; D\} \leq \limsup_k G(t_{pk}(f_k); D)
\]

(2.4) if and only if \( A \in (V_\sigma, V_\sigma)_{\text{reg}} \) and the condition (2.2) holds.

Proof. Suppose that (2.4) holds for every uniformly bounded function sequence \( \mathcal{F} = (f_k) \) on \( D \). By the same argument as was used in the proof of Theorem 2.5, we immediately get from (2.4) that

\[
\limsup_p \sup_n t_{pn}(Av) \leq \limsup_p \sup_k t_{pk}(v).
\]

Therefore, the necessity of the conditions follows by Lemma 2.2.
Conversely, let $A \in (V_\sigma, V_\sigma)_{\text{reg}}$ and the condition (2.2) hold. Then, since $(V_\sigma, V_\sigma)_{\text{reg}} \subset (c, V_\sigma)_{\text{reg}}$ (see [2]), the inequality (2.1) holds for any uniformly bounded function sequence $F = (f_k)$. Now, let $M$ be the set of all uniformly bounded function sequences $g = (g_k)$ defined on $D$ such that $(g_k(x)) \in Z$. Since $V_\sigma \subset \ell_\infty$, $M$ is not empty. Then, we get from (2.1) that

$$
\inf_{g \in M} \limsup_{p} \sup_{n} G\{t_{pn}[A(f_k(x) + g_k(x))]; D\} \leq \inf_{g \in M} \limsup_{g} G\{f_k(x) + g_k(x); D\}
$$

$$
= W(f_k).
$$

(2.5)

On the other hand, since $A \in (V_\sigma, V_\sigma)_{\text{reg}}, t_{pn}[A_n(g_k(x))] \in Z$. So, we have that

$$
\inf_{g \in M} \limsup_{p} \sup_{n} G\{t_{pn}[A(f_k(x) + g_k(x))]; D\} \geq \inf_{g \in M} \limsup_{p} \sup_{n} G\{t_{pn}[A(f_k(x))]; D\}
$$

$$
+ \inf_{g \in M} \limsup_{p} \sup_{n} G\{t_{pn}[A(g_k(x))]; D\}
$$

$$
= \limsup_{p} \sup_{n} G\{t_{pn}[A(f_k(x))]; D\}
$$

$$
= \limsup_{p} \sup_{n} G\{t_{pn}[A_n(F)]; D\}.
$$

(2.6)

Finally, by combining (2.5) with (2.6), we conclude that

$$
\limsup_{p} \sup_{n} G\{t_{pn}[A_n(F)]; D\} \leq W(f_k).
$$

Further, by Lemma 2.3, $W(f_k) = \limsup_{p} \sup_{k} G(t_{pk}(f_k); D)$ and so the proof is completed. □

For the next theorem, it is convenient to give the definition of $\text{st-limsup}$ given in [7]: Let $x = (x_k)$ be a real number sequence and $B_x = \{b \in \mathbb{R} : \delta(k : x_k > b) \neq 0\}$. Then, $\text{st-limsup} x = \sup B_x$ if $B_x \neq \emptyset$; otherwise it is equal to $-\infty$.

**Theorem 2.7.** Let $\|A\| < \infty$. Then, for every uniformly bounded function sequence $F = (f_k)$ on $D$, one has

$$
\limsup_{p} \sup_{n} G\{t_{pn}[A_n(F)]; D\} \leq \text{st-limsup}_{k} G(f_k; D)
$$

if and only if $A \in (\text{st} \cap \ell_\infty, V_\sigma)_{\text{reg}}$ and the condition (2.2) holds.

**Proof.** Suppose that (2.7) holds for every uniformly bounded function sequence $F = (f_k)$ on $D$. Then, by the same argument as was used in above theorems, one can easily see that

$$
\limsup_{p} \sup_{n} G\{t_{pn}[A_n(F)]; D\} \leq \text{st-limsup}(v).
$$

So, the proof of the necessity follows from Lemma 2.4.

Conversely, assume that $A \in (\text{st} \cap \ell_\infty, V_\sigma)_{\text{reg}}$ and (2.2) holds. Let $F = (f_k)$ be any uniformly bounded function sequence on $D$. Then, there exists a positive number $K$ such that $|f_k(x)| \leq K$ for every $x \in D$ and for all $k \in \mathbb{N}$. Let $\text{st-limsup}_{k} G(f_k; D) = H$. Then, for any given $\varepsilon > 0$, the set $E = \{k : G(f_k; D) > H + \varepsilon\}$ has zero density (see [7]) and $G(f_k; D) \leq H + \varepsilon$ whenever $k \notin E$. Now, for any $x \in D$, we can write

$$
t_{pn}[A_n(F)] = \sum_{k} a(p, n, k) f_k(x)
$$

$$
= \sum_{k \in E} a(p, n, k) f_k(x) + \sum_{k \notin E} a(p, n, k)^+ f_k(x) - \sum_{k \notin E} a(p, n, k)^- f_k(x)
$$

$$
\leq K \sum_{k \in E} |a(p, n, k)| + (H + \varepsilon) \sum_{k} |a(p, n, k)| + K \sum_{k} (|a(p, n, k)| - a(p, n, k)).
$$

(2.8)

So, by applying the operator $\limsup_{p} \sup_{n}$ to (2.8), we conclude that

$$
\limsup_{p} \sup_{n} G\{t_{pn}[A_n(F)]; D\} \leq H + \varepsilon.
$$

Since $\varepsilon$ and $F$ are arbitrary, the proof is completed. □
References


