Note on powers of 2 in sumsets

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**Abstract**

Let \( n \geq 2 \) be an integer. Let \( A \) be a subset of \([0, n]\) with \( 0, n \in A \). Assume the greatest common divisor of all elements of \( A \) is 1. Let \( k \) be an odd integer and \( s = \frac{k-1}{2} \). Then, we prove that when \( 3 \leq k \leq 11 \) and \(|A| \geq 7s + 3 + 2(s + 1)(7s + 4)(n - 2) + 2\), there exists a power of 2 which can be represented as a sum of \( k \) elements (not necessarily distinct) of \( A \). But when \( k \geq 13 \), the above constraint should be changed to \(|A| \geq \frac{s + 1}{s^2 + 2s + 2}(n - 2) + 2\).

In the present paper, we generalize the results of Pan and Lev, and obtain a non-trivial progress towards a conjecture of Pan.

**1. Introduction**

For two sets \( A, B \) of integers, let

\[ A + B = \{a + b : a \in A, \ b \in B\} \]

and

\[ A - B = \{a - b : a \in A, \ b \in B\}. \]

The \( h \)-fold sumset of \( A \) is defined by

\[ hA = \{a_1 + a_2 + \cdots + a_h : a_1, \ a_2, \ldots, a_h \in A\}. \]

A set \( A \) of non-negative integers is called normal if \( 0 \in A \) and the greatest common divisor of all elements of \( A \) is 1. The problem of integer powers of 2 has been investigated by many authors, e.g., [1–6]. In particular, Lev obtained the following important result in [5]:

*Let \( A \subseteq \{0, n\} \) and \(|A| \geq \frac{1}{2}n + 1. \) Then \( 2A \) contains a power of 2.*

Later, Abe extended Lev’s result in [1] as follows.

*Let \( m \geq 2 \) be an integer. Let \( A \) be a subset of \([0, n]\) with \( 0 \in A \), and assume that \(|A| > \frac{1}{2}n + 1. \) Then \( mA \) contains a power of \( m \).*

Recently, Pan generalized the results of Lev and Abe. The following theorem is one of the main results in [7].
Theorem 1.1 ([7, Theorem 1]). Let \( k, m, n \geq 2 \) be integers. Let \( A \) be a normal subset of \([0, n]\) satisfying
\[
|A| > \frac{1}{l+1} \left( \left( 2 - \frac{k}{lm} \right) n + 2l \right),
\]
where \( l = \lfloor k/m \rfloor \). If \( m \geq 3 \), or \( m = 2 \) and \( k \) is even, then \( kA \) contains a power of \( m \).

Moreover, Pan conjectured that the conclusion is still true when \( m = 2 \) and \( k \) is odd.

On the other hand, Lev proved the following in [8].

Theorem 1.2 ([8, Theorem 9]). Let \( A \) be a finite set of integers with \( \min A \coloneqq 0 \). Write \( n \coloneqq \max A \) and \( x \coloneqq |A| \) and suppose that \( x \geq \frac{17}{54} n + 2 \). Then the sumset \( 5A \) contains a power of 2, unless all elements of \( A \) are divisible by 3.

In this paper, we obtain the following result.

Theorem 1.3. Let \( n \geq 2 \) be an integer. Let \( A \) be a normal subset of \([0, n]\) with \( n \in A \). Let \( k \) be an odd integer and \( s = \frac{k-1}{2} \). Then \( kA \) contains a power of 2 if one of the following conditions holds:

1. \( 3 \leq k \leq 11 \) and \( |A| \geq \frac{7k+3}{(k+1)(7k+4)}(n-2) + 2 \);
2. \( k \geq 13 \) and \( |A| \geq \frac{2k+1}{n^2 + 2n + 2}(n-2) + 2 \).

Note that, when \( k = 5 \),
\[
\frac{7 \cdot 2 + 3}{(2 + 1)(7 \cdot 2 + 4)}(n-2) - 2 = \frac{17}{54} n + \frac{74}{54} n < \frac{17}{54} n + 2.
\]

So clearly, Theorem 1.3 implies Lev’s result. However, our result is still weaker than Pan’s conjecture. For example, when \( k = 3 \),
\[
\frac{7 \cdot 1 + 3}{(1 + 1)(7 \cdot 1 + 4)} = \frac{5}{11} > \frac{1}{2 + 1} \left( 2 - \frac{3}{2 \cdot 2} \right).
\]

2. Proof of Theorem 1.3

The following two lemmas will be needed in the proof of Theorem 1.3.

Lemma 2.1 ([8, Lemma 8]). Let \( n \) be a positive integer and suppose that \( B, C \subseteq [0, n] \) are integer sets, satisfying \( |B| + |C| \geq n + 2 \). Then the sumset \( B + C \) contains a power of 2.

Lemma 2.2 ([9, Corollary 1]). A is a normal subset of \([0, n]\) with \( n \in A \), and \( |A| \geq \frac{7k+3}{(k+1)(7k+4)}(n-2) + 2 \), \( l = \lfloor (n-1)/(|A| - 2) \rfloor - 1 \). Then for any non-negative integer \( h \), we have
\[
|hA| \geq \begin{cases} 
B_h(|A|) & \text{if } h \leq l, \\
B_h(|A| + (h-l)n) & \text{if } h \geq l,
\end{cases}
\]
where \( B_h(x) = \frac{1}{2} h(h+1)(x-2) + h + 1 \).

(1) We first prove that \( kA \) contains a power of 2 when \( 3 \leq k \leq 11 \) and \( |A| \geq \frac{7k+3}{(k+1)(7k+4)}(n-2) + 2 \).

From Theorem 1.1, we can see that \( 2sA \) contains a power of 2 if \( |A| > \frac{n+2k}{s+1} \). Thus we only need to prove the result for \( |A| \leq \frac{n+2s}{s+1} \). We will prove this by contradiction.

To do this, assume that the conclusion in Theorem 1.3 does not hold, i.e., \( kA \) does not contain a power of 2. Then
\[
l = \left\lfloor \frac{n - 1}{|A| - 2} \right\rfloor - 1 \geq \left\lfloor \frac{n - 1}{\frac{n+2k}{s+1} - 2} \right\rfloor - 1 = s + \left\lfloor \frac{s+1}{n-2} \right\rfloor \geq s + 1,
\]
where \( [x] \coloneqq \min\{z \in \mathbb{Z} : z \geq x\} \).

Letting \( x = |A| \), from Lemma 2.2, we get
\[
|sA| \geq \frac{1}{2} s(s+1)(x-2) + s + 1. \tag{2.1}
\]
\[
|(s+1)A| \geq \frac{1}{2} (s+1)(s+2)(x-2) + s + 2. \tag{2.2}
\]
Setting $B = (s + 1)A \cap [0, sn]$ and $C = sA$, we know that the sumset $B + C$ does not contain a power of 2 by our assumption. Then, by the proof of Lemma 2.1 (see [8, Lemma 8]), we get

$$|(s + 1)A \cap [0, sn]| \leq sn + 1 - |sA|.$$ 

So we have

$$|(s + 1)A \cap [sn, (s + 1)n]| = |(s + 1)A| - |(s + 1)A \cap [0, sn]| + 1$$

$$\geq |(s + 1)A| + |sA| - sn,$$

(2.3)

and

$$|(sn, (s + 1)n) \setminus (s + 1)A| = n + 1 - |(s + 1)A \cap [sn, (s + 1)n]|$$

$$\leq (s + 1)n - |(s + 1)A| - |sA| + 1.$$ 

(2.4)

Fix a positive integer $r$ with $sn < 2' < 2sn$. (The equalities $2' = sn$ and $2' = 2sn$ are ruled out by the assumption that $kA$ does not contain a power of 2.) Now we consider the following two cases.

Case 1: $(s + 1)n < 2' < 2sn$. (Also the equalities are ruled out by the assumption.)

We can assume that $un < 2' < (u + 1)n$, $s + 1 \leq u \leq 2s - 1$, $u \in \mathbb{Z}$. For any $a \in A$ we have $2' - a \not\in 2sA$, and hence

$$|A| + ||2' - n, 2'| \cap 2sA| \leq n + 1.$$ 

(2.5)

Since $(u + 1)A \subseteq 2sA$, we have

$$||2' - n, 2'| \cap 2sA| \geq ||2' - n, 2'| \cap (u + 1)A|$$

$$= ||2' - n, un| \cap (u + 1)A| + ||un, 2'| \cap (u + 1)A| - 1.$$ 

(2.6)

Since $|(u - 1)n, 2' - n| \cup uA) + |n| \leq |(un, 2') \cap (u + 1)A|$, by (3.6) we have

$$||2' - n, 2'| \cap 2sA| \geq ||2' - n, un| \cap uA| + |(u - 1)n, 2' - n| \cap uA| - 1$$

$$= ||(u - 1)n, un| \cap uA| - 1.$$ 

(2.7)

Since

$$(sn, (s + 1)n) \cap (s + 1)A + ((u - 1 - s)n) \subseteq |(u - 1)n, un| \cap uA,$$

by (2.5), (2.7), (2.8), we can get

$$|A| + |(sn, (s + 1)n) \cap (s + 1)A| - 1 \leq n + 1.$$ 

Then by (2.1), (2.2) and (2.3) we have that

$$(s^2 + 2s + 2)(x - 2) + 2s + 3 \leq (s + 1)n,$$

$$x < \frac{s + 1}{s^2 + 2s + 2} (n - 2) + 2,$$

which contradicts

$$x \geq \frac{7s + 3}{(s + 1)(7s + 4)} (n - 2) + 2,$$

when $1 \leq s \leq 5$.

Case 2: $sn < 2' < (s + 1)n$.

For any $a \in sA$, we have that $2' - a \not\in (s + 1)A$. Consequently,

$$||0, 2^{l-1}| \cap sA| + ||2^{l-1}, 2'| \cap (s + 1)A| \leq 2^{l-1} + 1,$$

$$||0, 2^{l-1}| \cap sA| + ||2^{l-1}, sn| \cap sA| + ||sn, 2'| \cap (s + 1)A| \leq 2^{l-1} + 2,$$

$$|sA| + ||sn, 2'| \cap (s + 1)A| \leq 2^{l-1} + 2,$$

$$|sA| + (2' - sn + 1 - ||sn, 2'| \setminus (s + 1)A|) \leq 2^{l-1} + 2.$$ 

By (2.4), we have

$$2^{l-1} \leq sn - |sA| + ||sn, 2'| \setminus (s + 1)A| + 1$$

$$\leq (2s + 1)n - |(s + 1)A| - 2|sA| + 2.$$ 

(2.9)
From (2.1) and (2.2) we get
\[ 2^{r-1} \leq (2s+1)n - \frac{(s+1)(3s+2)}{2}(x-2) - 3s - 2 \leq \frac{(7s^2 + 7s + 2)}{2(7s + 4)}n + \frac{(7s + 3)(3s + 2)}{(7s + 4)} - 3s - 2 < \frac{(7s^2 + 7s + 2)}{2(7s + 4)}n \leq \frac{(2s + 1)}{4}n, \]
since
\[ x \geq \frac{7s + 3}{(s + 1)(7s + 4)}(n - 2) + 2. \]
So we have
\[ 2sn < 2^{r+1} < (2s+1)n. \]

We notice that if \( a \in sA \), then \( 2^{r+1} - a \not\in (s+1)A \). Thus
\[ ||2^{r+1} - (s+1)n, sn| \cap sA| \leq ||2^{r+1} - sn, (s+1)n| \setminus (s+1)A|. \quad (2.10) \]

Take \( D = [0, 2^{r+1} - (s+1)n - 1] \cap sA \). Since \( D + D \) does not contain a power of 2, by Lemma 2.1, we know that
\[ 2|D| \leq 2^{r+1} - (s+1)n. \]

So
\[ |D| \leq 2^{r} - \frac{s+1}{2}n. \]

From (2.10) it follows that
\[ ||2^{r+1} - sn, (s+1)n| \setminus (s+1)A| \geq |sA| - |D| \geq |sA| - 2^{r} + \frac{s+1}{2}n. \quad (2.11) \]

Since
\[ [2^{r+1} - sn, (s+1)n| \setminus (s+1)A \subseteq [2^{r}, (s+1)n| \setminus (s+1)A, \]
we have
\[ 2||sn, (s+1)n| \setminus (s+1)A| = 2||sn, 2^{r}| \setminus (s+1)A| + 2[2^{r}, (s+1)n| \setminus (s+1)A| - 2 \geq 2||sn, 2^{r}| \setminus (s+1)A| + ||2^{r+1} - sn, (s+1)n| \setminus (s+1)A|. \quad (2.12) \]

Then by (2.4), (2.9) and (2.11), we obtain that
\[ 2((s+1)n - ||(s+1)A| - |sA| + 1) \geq 3|sA| - \frac{3s - 1}{2}n - 2, \]
\[ \frac{7s + 3}{2}n \geq \frac{(7s + 4)(s + 1)}{2}(x - 2) + 7s + 5, \]
\[ x < \frac{7s + 3}{(s + 1)(7s + 4)}(n - 2) + 2. \]

This is a contradiction.
(2) When \( k \geq 13 \), by the proof of (1), it is easy to see that if
\[ |A| \geq \frac{s+1}{s^2 + 2s + 2}(n - 2) + 2, \]
then \( kA \) contains a power of 2.

The proof is complete.

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References